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A FINITE DIMENSION IMPLEMENTATION OF HUM EXACT CONTROLLABILITY METHOD FOR THE ONE DIMENSION WAVE EQUATION

APPLICATION EN DIMENSION FINIE DE LA METHODE HUM DE CONTROLABILITE EXACTE POUR L'EQUATION DES ONDES UNIDIMENSIONELLE

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Abstract

We deal with the problem of implementation of exact controllers of distributed parameter systems without damping. Specifically, we study a model of a vibrating simple string. We propose a practical closed loop implementation of HUM controls and we also analyze the structural instability of the proposed implementation under parameter perturbations on system's data.

Résumé

On considère ici le problème de l'application réel de contrôleurs exacts de systèmes aux paramètres distribués sans amortissement. Spécifiquement, on étudie un modèle de la vibration d'une corde simple. On propose une schéma à boucle fermé des contrôles donnés par la méthode HUM et on analyse l'instabilité structurale du système en boucle fermé, face à des perturbations dans les données du système.

Keywords: *exact controllability, HUM method, overspilling, structural instability, wave equation, distributed parameter.*

AMS Classification: Primary: 49B50, 49B22 Secondary: 49E25, 49A22

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1 Introduction

We consider the exact controllability problem of a distributed parameter system which evolves according to a linear partial differential equation.We specifically study the one dimensional wave equation. The principal results of this work are described as follows:

- We introduce a practical implementation of a finite dimension exact control with the properties:

    1. The synthesis is carried out by applying the Hilbert Uniqueness Method (HUM) proposed by J.-L. Lions in [16].
    2. The implementation scheme comprises a finite number of oscillators which play a double role
        (a) the oscillators identify some of the harmonic components of the system in an exact form (total absence of aliasing)
        (b) the oscillators synthesize the control signal that must be applied to the system to eliminate the same chosen harmonic components.
    3. After a time $2T$ (where T is the normal period of vibration of the string), the closed loop scheme eliminates the harmonic components identified by the system ***observer-controller***.
    4. The closed loop scheme does not modify the remaining harmonic components.

- The properties of structural instability of the system are analyzed.

    1. By simulation (using MATLAB system), we show that when the natural frequency ω of the closed loop system is perturbed by an additional quantity $\epsilon \times \omega$, it appears a modulated spurious oscillation with the following features:
        (a) the frequency of the spurious oscillation is of type $\epsilon \times \omega$
        (b) this oscillation increases exponentially with a factor of the form $(1 + c\epsilon^4)^t$.
    2. This instability is proved by analytical computations – made with the MATHEMATICS programming language – of the eigenvalues of the transition matrix of the closed loop system.

2 The Controlled System

2.1 Evolution equations

We study a system which evolves according to a linear partial differential equation. The state y of the system depends on $x \in \Omega = (0,1)$, and time t. We control the system using a *control function* v that is applied at the point $x = 1$ of the boundary.

We consider the wave equation with boundary conditions of Dirichlet type at $x = 0$.

$$\begin{cases} \ddot{y} - \dfrac{\partial^2 y}{\partial x^2} = 0 & \text{in } Q\,, \\ y(x,0) = y^0(x) & \forall x \in \Omega\,, \\ \dot{y}(x,0) = y^1(x) & \forall x \in \Omega\,, \\ y(0,t) = 0 & \forall t \in (0,\tau)\,, \\ y(1,t) = v(t) & \forall t \in (0,\tau)\,, \end{cases} \tag{1}$$

where $Q = \Omega \times (0,\tau) = (0,1) \times (0,\tau)$, $\tau > 0$.

We will also study the following perturbed system (where the perturbation is a function of the parameter $\epsilon > 0$)

$$\begin{cases} \ddot{y} - (1+\epsilon)\dfrac{\partial^2 y}{\partial x^2} = 0 & \text{in } Q\,, \\ y(x,0) = y^0(x) & \forall x \in \Omega\,, \\ \dot{y}(x,0) = y^1(x) & \forall x \in \Omega\,, \\ y(0,t) = 0 & \forall t \in (0,\tau)\,, \\ y(1,t) = v(t) & \forall t \in (0,\tau)\,. \end{cases} \tag{2}$$

Using the change of variables:

$$z(x,t) = y(x,t) - v(t)\,x\,, \tag{3}$$

we arrive to the following system with distributed control and zero boundary conditions

$$\begin{cases} \ddot{z} - (1+\epsilon)\dfrac{\partial^2 z}{\partial x^2} = -x\,\ddot{v}(t) & \text{in } Q\,, \\ z(x,0) = y^0(x) - v(0)\,x & \forall x \in \Omega\,, \\ \dot{z}(x,0) = y^1(x) - \dot{v}(0)\,x & \forall x \in \Omega\,, \\ z(0,t) = 0 & \forall t \in (0,\tau)\,, \\ z(1,t) = 0 & \forall t \in (0,\tau)\,. \end{cases} \tag{4}$$

Using harmonic decomposition techniques we obtain the solution of (4) as the serie

$$z(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin(n\pi x)\,. \tag{5}$$

We define

$$\omega_n^\epsilon = n\pi\sqrt{(1+\epsilon)}\,, \qquad w_k = k\pi\,. \tag{6}$$

The coefficients $a_n(t)$ verify the following ordinary differential equation

$$\begin{cases} \ddot{a}_n + (\omega_n^\epsilon)^2 a_n = -x_n\,\ddot{v}\,, \\ a_n(0) = y_n^0 - v(0)\,x_n\,, \\ \dot{a}_n(0) = y_n^1 - \dot{v}(0)\,x_n\,. \end{cases} \tag{7}$$

Here, x_n, y_n^0 and y_n^1 are the Fourier coefficients of the functions x, $y^0(x)$ and $y^1(x)$, i.e.

$$y^0(x) = \sum_{n=1}^{\infty} y_n^0 \sin(n\pi x)\,, \qquad y^1(x) = \sum_{n=1}^{\infty} y_n^1 \sin(n\pi x)\,, \tag{8}$$

where

$$\begin{cases} y_n^0 = \frac{1}{2}\displaystyle\int_0^1 y^0(x)\,\sin(n\pi x)\,dx\,, \\ y_n^1 = \frac{1}{2}\displaystyle\int_0^1 y^1(x)\,\sin(n\pi x)\,dx\,; \end{cases} \tag{9}$$

and

$$x = \sum_{n=1}^{\infty} x_n \sin(n\pi x)\,, \qquad x_n = \frac{2(-1)^{n+1}}{n\pi}\,. \tag{10}$$

2.2 Free evolution

It is easy to check that

$$a_n(t) = a_n(0)\cos(\omega_n^\epsilon t) + \frac{\dot{a}_n(0)}{\omega_n^\epsilon}\sin(\omega_n^\epsilon t), \tag{11}$$

is the solution of the homogeneous equation associated to (7):

$$\begin{cases} \ddot{a}_n + (\omega_n^\epsilon)^2 a_n = 0\,, \\ a_n(0) = y_n^0\,. \\ \dot{a}_n(0) = y_n^1\,. \end{cases} \tag{12}$$

We introduce the notation

$$\begin{cases} A_n(t) = \begin{pmatrix} a_n(t) \\ \dfrac{\dot{a}_n(t)}{\omega_n^\epsilon} \end{pmatrix}, \\ E_n(t) = \begin{pmatrix} \cos(\omega_n^\epsilon t) & \sin(\omega_n^\epsilon t) \\ -\sin(\omega_n^\epsilon t) & \cos(\omega_n^\epsilon t) \end{pmatrix}. \end{cases} \tag{13}$$

From (11) we have that

$$A_n(t) = E_n(t)\cdot A_n(0)\,. \tag{14}$$

The normal period of oscillation of the string is $T = 2$. In consequence, at the end of the normal period we have

$$A_n(T) = E_n(T)\cdot A_n(0)\,, \tag{15}$$

where

$$E_n(T) = \begin{pmatrix} \cos(2n\pi\epsilon) & \sin(2n\pi\epsilon) \\ -\sin(2n\pi\epsilon) & \cos(2n\pi\epsilon) \end{pmatrix}. \tag{16}$$

In case $\epsilon = 0$, we have:

$$E_n(T) = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{17}$$

2.3 Forced evolution

We now show the solution of the system (7), which represents the action of the control on the system

$$
\begin{cases}
\ddot{a}_n + (\omega_n^\epsilon)^2 a_n = -x_n \ddot{v}\,, \\
a_n(0) = y_n^0 - v(0)\, x_n\,, \\
\dot{a}_n(0) = y_n^1 - \dot{v}(0)\, x_n\,.
\end{cases}
\tag{18}
$$

To find that solution, we consider the Fourier decomposition of the control v, i.e. $\forall t \in [0, T]$

$$
v(t) = \sum_{k=1}^{\infty} v_k^1 \cos(w_k t) + v_k^2 \sin(w_k t)\,. \tag{19}
$$

We denote each harmonic component of v by v_k, where

$$
v_k(t) = v_k^1 \cos(w_k t) + v_k^2 \sin(w_k t)\,. \tag{20}
$$

We consider now only an isolated generic component v_k and, by replacing (20) in (18), we obtain the following equation

$$
\ddot{a}_n + (\omega_n^\epsilon)^2\, a_n = x_n\, w_k^2\, \left(v_k^1 \cos(w_k t) + v_k^2 \sin(w_k t)\right). \tag{21}
$$

It is easy to check - considering null initial conditions - that the solution of (21) is given by

$$
a_n(t) = \frac{x_n\, w_k^2}{\omega_n^\epsilon} \int_0^t \sin(\omega_n^\epsilon(t-s))\, \left(v_k^1 \cos(w_k s) + v_k^2 \sin(w_k s)\right)\, ds\,. \tag{22}
$$

Hence, $\dot{a}_n(\cdot)$ has the form:

$$
\dot{a}_n(t) = x_n\, w_k^2 \int_0^t \cos(\omega_n^\epsilon(t-s))\, \left(v_k^1 \cos(w_k s) + v_k^2 \sin(w_k s)\right)\, ds\,. \tag{23}
$$

Taking $t = T$, from (22) and (23) we obtain

$$\left\{\begin{aligned}
a_n(T) &= \frac{x_n\, w_k^2}{\omega_n^\epsilon}\left(\frac{v_k^1}{2\pi}(1-\cos(2\pi n\epsilon))\left(\frac{1}{n\,(1+\epsilon)+k}+\frac{1}{n\,(1+\epsilon)-k}\right)\right. \\
&\quad \left.+\frac{v_k^2}{2\pi}\sin(2\pi n\epsilon)\left(\frac{1}{n\,(1+\epsilon)+k}+\frac{-1}{n\,(1+\epsilon)-k}\right)\right), \\
\frac{\dot a_n(T)}{\omega_n^\epsilon} &= \frac{x_n\, w_k^2}{\omega_n^\epsilon}\left(\frac{v_k^1}{2\pi}\sin(2\pi n\epsilon)\left(\frac{1}{n\,(1+\epsilon)+k}+\frac{1}{n\,(1+\epsilon)-k}\right)\right. \\
&\quad \left.+\frac{v_k^2}{2\pi}(1-\cos(2\pi n\epsilon))\left(\frac{-1}{n\,(1+\epsilon)+k}+\frac{1}{n\,(1+\epsilon)-k}\right)\right).
\end{aligned}\right. \tag{24}$$

We define

$$fs(k,n)=\frac{1}{n\,(1+\epsilon)+k}+\frac{1}{n\,(1+\epsilon)-k}=\frac{2\,n\,(1+\epsilon)}{(n\,(1+\epsilon)+k)\,(n\,(1+\epsilon)-k)}, \tag{25}$$

$$fr(k,n)=\frac{1}{n\,(1+\epsilon)+k}-\frac{1}{n\,(1+\epsilon)-k}=\frac{-2\,k}{(n\,(1+\epsilon)+k)\,(n\,(1+\epsilon)-k)}. \tag{26}$$

Then, (24) can be written in a compact form as

$$A_n(T)=S_{nk}\cdot V_k\,, \tag{27}$$

being

$$S_{nk}=\frac{x_n\,w_k^2}{2\,\pi\,\omega_n^\epsilon}\begin{pmatrix} fs(k,n)\,(1-\cos(2\pi n\epsilon)) & fr(k,n)\,\sin(2\pi n\epsilon) \\ fs(k,n)\,\sin(2\pi n\epsilon) & -fr(k,n)\,(1-\cos(2\pi n\epsilon)) \end{pmatrix}, \tag{28}$$

$$V_k=\begin{pmatrix} v_k^1 \\ v_k^2 \end{pmatrix}. \tag{29}$$

We conclude that if the input signal (20) is applied in $(0,T)$, to get the forced evolution, we must add a term of the form $S_{nk}\cdot V_k$, to the term (15) corresponding to the free evolution.

Considering that the control applied to the system vanishes at $t = T$, and taking into account (2), (4) and (7), to obtain the value of $a_n(T+)$ we must also add to (15) the term

$$x_n\,(I-E_n)\cdot V_k\,. \tag{30}$$

Finally, using the superposition principle, and considering all the components of v, we get the complete expression:

$$A_n(T) = E_n(T) \cdot A_n(0) + \sum_{k=1}^{\infty} \left(S_{nk} + x_n \left(I - E_n \right) \right) \cdot V_k \,. \tag{31}$$

2.4 The Exact control

For the case $T = 2$, the control $v(t)$ given by HUM method has the form (see [6], [16])

$$v(t) = \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{-y_n^1}{2n\pi} \cos(w_n t) + \frac{y_n^0}{2} \sin(w_n t) \right) \,. \tag{32}$$

This implies

$$V_k = \frac{(-1)^{k+1}}{2} J \cdot A_k(0) \,, \tag{33}$$

where J is the matrix

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \,. \tag{34}$$

For $\epsilon = 0$, using (6), (13), (33) and (34) in (31), we obtain

$$A_n(T) = E_n(T) \cdot A_n(0) - \frac{1}{2} \sum_{k=1}^{\infty} (-1)^k \, S_{nk} \cdot J \cdot A_k(0) \,, \tag{35}$$

but also, for $\epsilon = 0$

$$S_{nk} = 2\,(-1)^{n+1} \, J \cdot \delta_{nk} \tag{36}$$

and

$$E_n(T) = I \,. \tag{37}$$

In consequence, as $I + J^2 = 0$, we have

$$A_n(T) \equiv 0 \,. \tag{38}$$

So, we have checked that the signal (32) synthesizes an exact controller.

This is the ideal case, corresponding to total exact observation and complete control. In the following section we will analyze the action of a more realistic control.

3 The Observer-Controller System

3.1 Fourier Analysis in Real Time

For the case $T = 2$, the control $v(t)$ given by the *HUM* method is

$$v(t) = \sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{-y_n^1}{2n\pi}\cos(w_n t) + \frac{y_n^0}{2}\sin(w_n t)\right). \tag{39}$$

Generally, this control is impossible to implement with a practical procedure, because no physical or computing device can bring all coefficients (y_n^1, y_n^0). In fact, only a finite number of components will be reconstructed by any implementable observer device.

In our problem, we assume we know $y(\xi, t)$, $\forall t \in (0, \infty)$. Theoretically, if the point ξ is strategic (condition that in this problem is equivalent to ξ irrational, see [7]), at the end of a time $t \geq T$ we will be able to reconstruct all the initial conditions, i.e. we could know $A_n(0)$, $\forall n = 1, 2, \ldots$.

In practice, as the available computing power is always finite, the computation must be restricted to a finite number of harmonic components. Here, we will design an implementable procedure to compute a partial estimate of the state comprising NC harmonic components, which in turns will provide a truncated approximation of (39).

Foundations of the method

For the case of free evolution, the observation in $(0, T)$ has the form

$$y(\xi, t) = \sum_{k=1}^{\infty}(\sin(k\pi\xi), 0)\cdot E_k(t)\cdot A_k(0). \tag{40}$$

By multiplying (40) by $E_n(-t)\cdot\begin{pmatrix}1\\0\end{pmatrix}$, we get the vector $b(t)$, being

$$b(t) = \sum_{k=1}^{\infty}\begin{pmatrix}\left(a_n(0)\cos(k\pi t) + \frac{\dot a_n(0)}{\omega_n^\epsilon}\sin(k\pi t)\right)\cos(n\pi t)\\ \left(a_n(0)\cos(k\pi t) + \frac{\dot a_n(0)}{\omega_n^\epsilon}\sin(k\pi t)\right)\sin(n\pi t)\end{pmatrix}\sin(k\pi\xi).$$

Integrating $b(t)$ in $(0, T)$, we obtain (for the case of null perturbation, i.e. $\epsilon = 0$)

$$\int_0^T b(t)\,dt = \begin{pmatrix} a_n(0) \\ \dfrac{\dot{a}_n(0)}{\omega_n^\epsilon} \end{pmatrix} = \frac{2}{T \sin(n\pi\xi)} \begin{pmatrix} \int_0^T y(\xi, t) \cos(\omega_n^\epsilon t)\,dt \\ \int_0^T y(\xi, t) \sin(\omega_n^\epsilon t)\,dt \end{pmatrix}. \tag{41}$$

These formulas imply that the components $a_n(0)$ and $\dot{a}_n(0)$ can be computed by convolution of the output $y(\xi, t)$ with the kernels $\sin(\omega_n^\epsilon t)$ and $\cos(\omega_n^\epsilon t)$.

This last result hints us a practical procedure to compute a finite number of coefficients A_n, because the operation of convolution is implicitly performed by any dynamic device whose impulsive response is the kernel $\sin(\omega_n^\epsilon t)$ or $\cos(\omega_n^\epsilon t)$.

Specifically, our procedure consists in injecting the observation signal $y(\xi, t)$ into an oscillating second order system whose characteristic frequency is w_k, i.e. we consider the differential equation

$$\ddot{\alpha}_k + w_k^2 \alpha_k = y(\xi, t)\,. \tag{42}$$

At the time T, the state of the system is (up to a constant factor) the original coefficient $A_n(0)$ multiplied by the matrix J.

3.2 The Oscillators as Observers

The implementable practical procedure consists in allowing the system to evolve freely in $(0, T)$ and during that time, processing the observation signal at each oscillator that compose the filtering device.

The filtering device comprises NC oscillators without dam ping, whose common input is the output of the real system (the observation signal $y(\xi, t)$). In consequence, denoting the state of each oscillator by α_k, we have that α_k is governed by an equation of the following type:

$$\ddot{\alpha}_k + w_k^2 \alpha_k = y(\xi, t) = \sum_{n=1}^{\infty} \sin(n\pi\xi) \left(a_n(0) \cos(\omega_n^\epsilon t) + \frac{\dot{a}_n(0)}{\omega_n^\epsilon} \sin(\omega_n^\epsilon t) \right) . \tag{43}$$

It is easy to check that the solution of (43) is given by

$$\alpha_k(t) = \frac{1}{w_k} \int_0^t \sin(w_k(t-s)) \sum_{n=1}^{\infty} \sin(n\pi\xi) \left(a_n(0) \cos(\omega_n^\epsilon s) + \frac{\dot{a}_n(0)}{\omega_n^\epsilon} \sin(\omega_n^\epsilon s) \right) ds . \tag{44}$$

From (44), we obtain for $\dot{\alpha}_k(\cdot)$ the following expression

$$\dot{\alpha}_k(t) = \int_0^t \cos(w_k(t-s)) \sum_{n=1}^{\infty} \sin(n\pi\xi) \left(a_n(0) \cos(\omega_n^\epsilon s) + \frac{\dot{a}_n(0)}{\omega_n^\epsilon} \sin(\omega_n^\epsilon s) \right) ds . \tag{45}$$

Taking $t = T$ in (44) and (45) we obtain

$$\left\{ \begin{aligned} \alpha_k(T) &= \frac{1}{2\pi w_k} \sum_{n=1}^{\infty} \sin(n\pi\xi) \left(\frac{\dot{a}_n(0)}{\omega_n^\epsilon} \sin(2\pi n\epsilon) \left(\frac{1}{n(1+\epsilon)+k} - \frac{1}{n(1+\epsilon)-k} \right) \right. \\ &\qquad \left. + a_n(0)(1-\cos(2\pi n\epsilon)) \left(\frac{1}{n(1+\epsilon)-k} - \frac{1}{n(1+\epsilon)+k} \right) \right) , \\ \frac{\dot{\alpha}_k(T)}{w_k} &= \frac{1}{2\pi w_k} \sum_{n=1}^{\infty} \sin(n\pi\xi) \left(a_n(0) \sin(2\pi n\epsilon) \left(\frac{1}{n(1+\epsilon)+k} + \frac{1}{n(1+\epsilon)-k} \right) \right. \\ &\qquad \left. + \frac{\dot{a}_n(0)}{\omega_n^\epsilon} (1-\cos(2\pi n\epsilon)) \left(\frac{1}{n(1+\epsilon)+k} + \frac{1}{n(1+\epsilon)-k} \right) \right) . \end{aligned} \right. \tag{46}$$

In matrix terms, we have

$$\Phi_k(T) = \sum_{n=1}^{\infty} F_{kn} \cdot A_n(0) , \tag{47}$$

where

$$F_{kn} = \frac{\sin(n\pi\xi)}{2\pi w_k} \begin{pmatrix} -fr(k,n)(1-\cos(2\pi n\epsilon)) & fr(k,n) \sin(2\pi n\epsilon) \\ fs(k,n) \sin(2\pi n\epsilon) & fs(k,n)(1-\cos(2\pi n\epsilon)) \end{pmatrix} , \tag{48}$$

$$\Phi_k(t) = \begin{pmatrix} \alpha_k \\ \\ \dfrac{\dot{\alpha}_k}{w_k} \end{pmatrix} = \begin{pmatrix} v_k^1 \\ \\ v_k^2 \end{pmatrix} . \tag{49}$$

From here, it is easy to check – for the case $\epsilon = 0$, i.e. null perturbation – that at time T the output of the k^{th} component of the filtering device is the signal $J \cdot A_k(0)$, $\forall\, k = 1, \ldots, NC$.

3.3 The Oscillators as Controllers

The filter set – as an auxiliary dynamical system – not only brings at time T the partial estimation $J \cdot A_n(0)$, $\forall n = 1, \dots, NC$, but also it generates in $[T, 2T]$ a truncated control signal.

Specifically, in $[T, 2T]$, we allow the oscillators evolve freely – disconnecting the input $y(\xi, t)$ to the filtering device – and that evolution generates the real control signal $\hat{v}(t)$ (a truncated version of (39)), being:

$$\hat{v}(t) = \sum_{n=1}^{NC} (-1)^{n+1} \left(\frac{-y_n^1}{2n\pi} \cos(w_n t) + \frac{y_n^0}{2} \sin(w_n t) \right) . \tag{50}$$

This control signal is injected to the system – during the interval $[T, 2T]$ – to steer it to the null state.

This dual operation is shown in Figure 1. The detailed procedure is the following:

1. In $[T, 2T]$ we let each one of the filter oscillators evolve freely (without input signal); then, their output has the following evolution

$$\alpha_k(t) = \alpha_k(T) \cos(w_k t) + \frac{\dot{\alpha}_k(T)}{w_k} \sin(w_k t) . \tag{51}$$

2. We multiply the output of each oscillator of the filter by the constant

$$\gamma_k = \frac{-1}{x_k \, \sin(k\pi\xi)} . \tag{52}$$

3. We sum the resulting signals as it is shown on Figure 1. In consequence, for the real synthesized control we have the following expression, $\forall t \in [T, 2T]$

$$\hat{v}(t) = \sum_{k=1}^{NC} \gamma_k \, \alpha_k(t) . \tag{53}$$

Finally, using (47), (49), (51), (53) we obtain – in terms of the initial conditions of the system (coefficients $A_n(0)$, $n = 1, 2, \dots$) – that the control has the form

$$\hat{v}(t) = \sum_{k=1}^{NC} \gamma_k \, E^k(t) \sum_{n=1}^{\infty} F_{kn} \cdot A_n(0) , \tag{54}$$

where

$$E^k(t) = \Big(\cos(w_k t) , \ \sin(w_k t) \Big) . \tag{55}$$

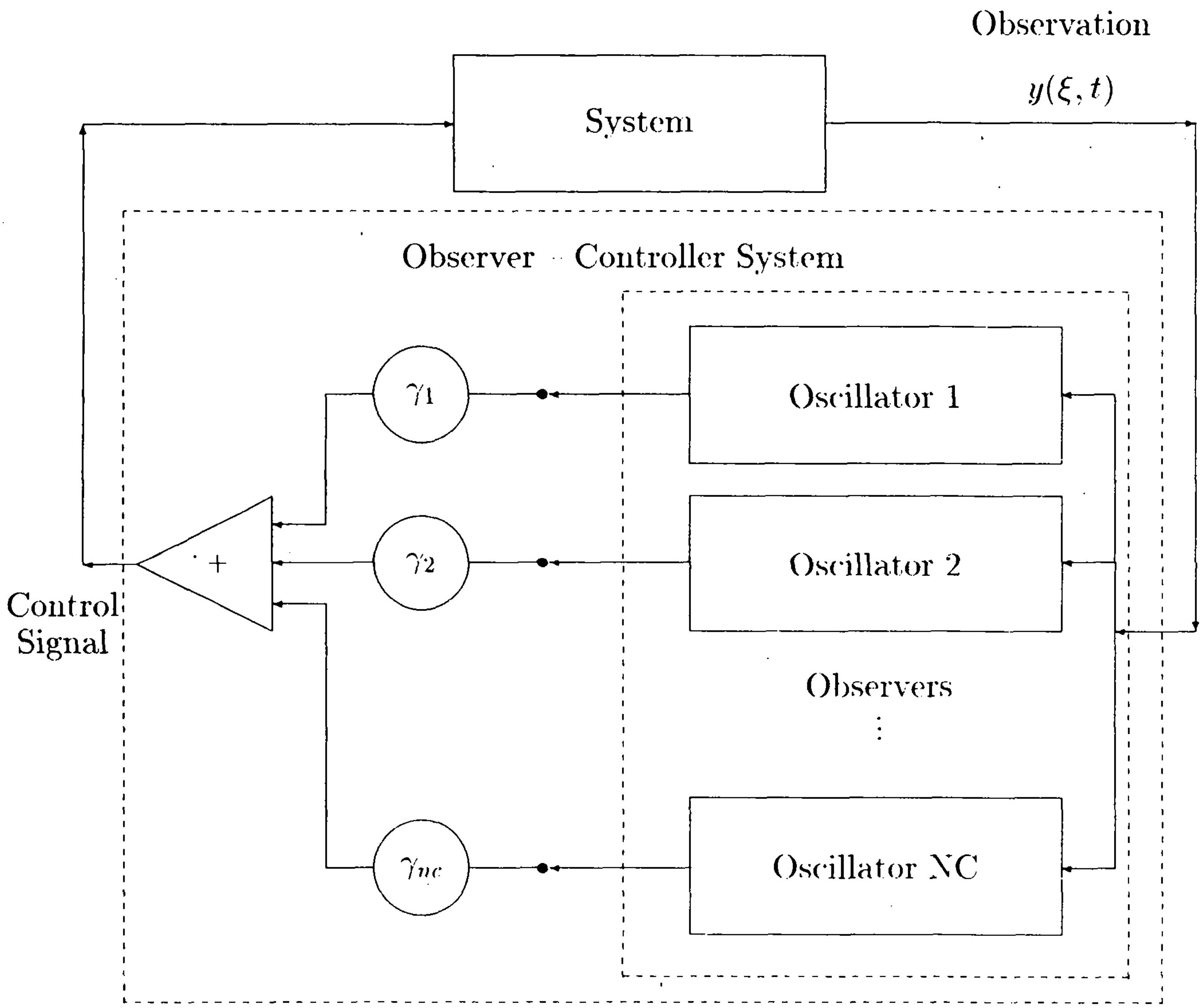

Figure 1: The closed loop system

3.4 Equation of evolution for the closed loop system

To get the closed loop evolution, we use the expression (31) and (54), suitably transformed because now the control is applied during the interval [T, 2T]. As the real control is given by (53), then, $\forall k > NC$ it is $V_k = 0$ and $\forall k = 1, \ldots, NC$,

$$V_k = \gamma_k \sum_{m=1}^{\infty} F_{km} \cdot A_m(0) \,. \tag{56}$$

For the complete expression of the filter, and considering the natural evolution of the system (31), we have

$$A_n(2T+) = E_n(2T) \cdot A_n(0) + \sum_{k=1}^{NC} \gamma_k \left(S_{nk} + x_n \left(I - E_n(T)\right)\right) \sum_{m=1}^{\infty} F_{km} \cdot A_m(0) \,. \tag{57}$$

To analyze the evolution of the system at the discrete times $2\nu T$, $\forall \nu \in \mathbb{N}$, we define the operator $Q : \ell^2 \mapsto \ell^2$ such that

$$A_{(\cdot)}(0-) \mapsto A_{(\cdot)}(2T+) \,. \tag{58}$$

The term Q_{nm} (of the infinite matrix Q) has the following expression

$$Q_{nm} = \left(\sum_{k=1}^{NC} \gamma_k \; \left(x_m(I - E_n(T)) + \; S_{nk}\right) F_{km} \right) + E_n(2T) \cdot \delta_{nm} \,. \tag{59}$$

It is easy to see that for $\epsilon = 0$, Q has the following form

$$Q = \begin{pmatrix} \bigcirc_{NC\times NC} & \bigcirc_{NC\times 2} & \cdots & \cdots \\ \bigcirc_{2\times NC} & I & \bigcirc_{2\times 2} & \cdots \\ \vdots & \bigcirc_{2\times 2} & I & \bigcirc_{2\times 2} \\ \vdots & \vdots & \bigcirc_{2\times 2} & \ddots \end{pmatrix}, \tag{60}$$

where $I_{2\times 2}$ is the identity matrix and $\bigcirc_{2\times 2}$ is the null matrix.

Therefore, the closed loop system annihilates the first NC harmonics and leaves the remaining components unchanged. This behaviour holds for the case $\epsilon = 0$, corresponding to an exact calibration of the *observer-controller* device. The real case is analyzed in section 5.

4 Examples

The utilized model corresponds to a truncation of the original system in the first NA harmonics. The observer and the control act on the first NC harmonics. The observation point is $x = 1/\sqrt{2}$.

In the figures it is only shown the evolution of the output of the system in the control interval $[T, 2T]$ (the identification interval $[0, T]$ has been eliminated from the figures). We have considered NA=1, NC=1 on Figure 2 and NA=3, NC=3 on Figure 3. In these examples, we have computed the values of $y(x, t)$ corresponding to a partition of the spatial domain $[0, 1]$ in NX=145 subintervals and the time interval $[0, 2]$ in NT=46 subintervals.

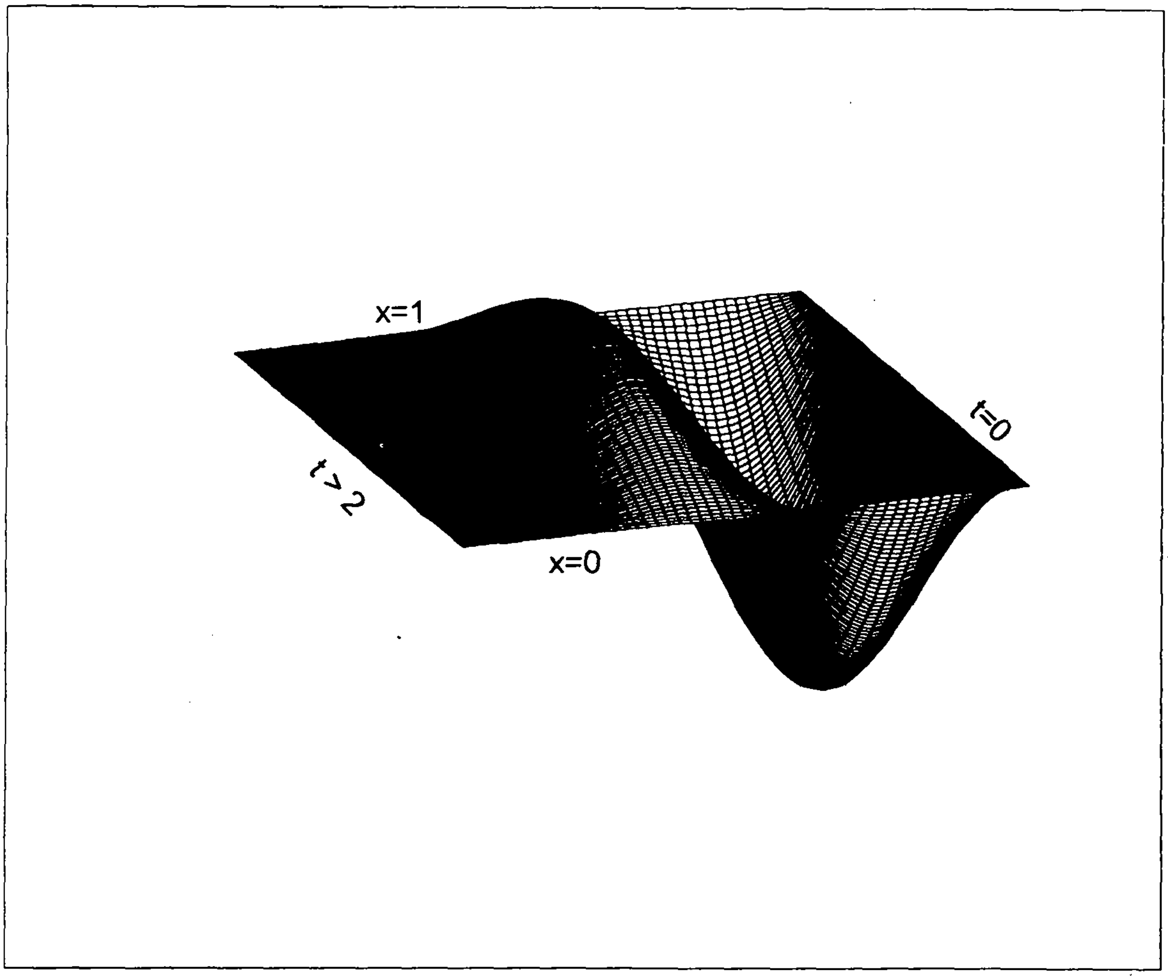

Figure 2: Evolution of the system. NA=1, NC=1.

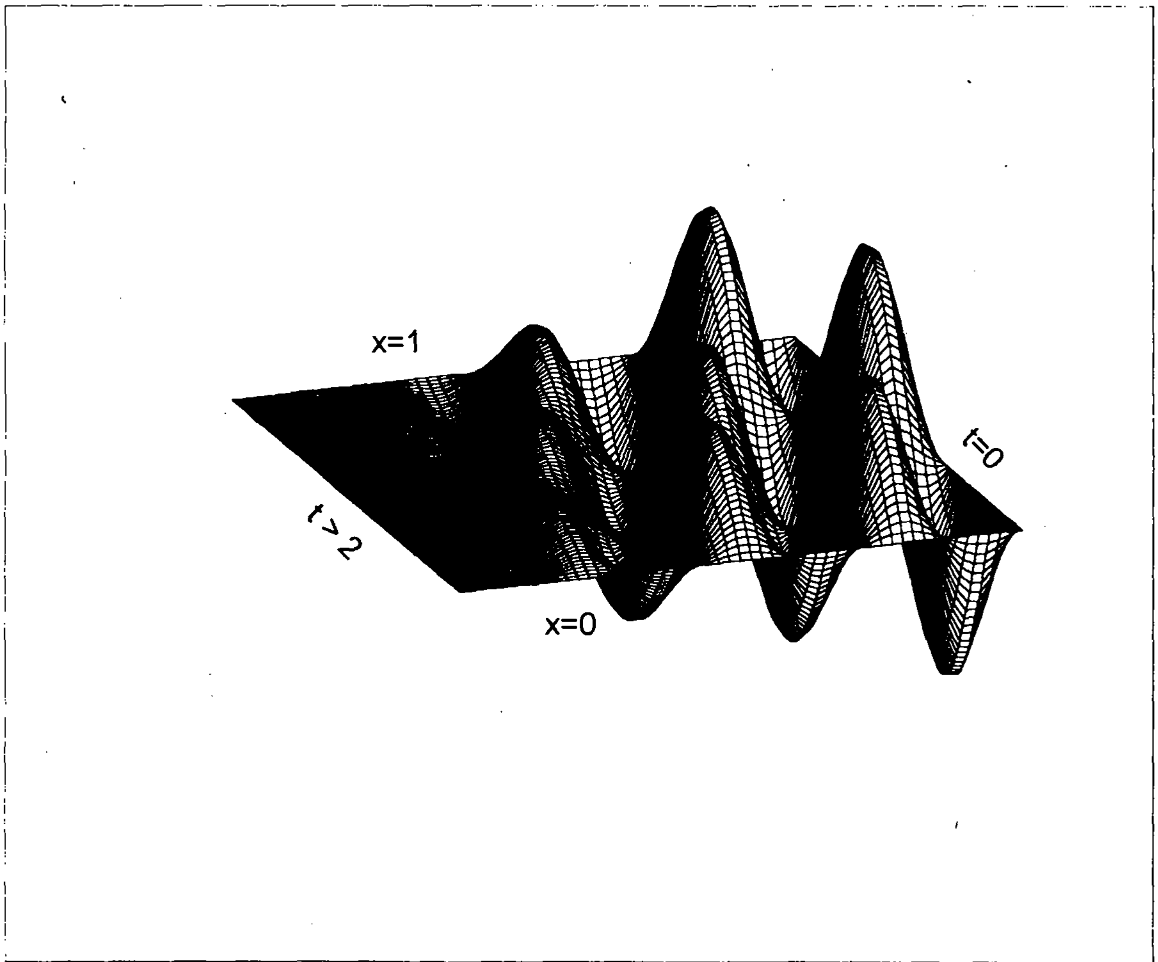

Figure 3: Evolution of the system. NA=3, NC=3.

5 Structural instability

In order to eliminate any residual no null state of the system, we can apply the same control procedure at the intervals:

$$[2T, 4T], \ [4T, 6T], \ldots, [2\,v\,T, 2\,(v+1)\,T], \quad \forall v \in \mathbb{N}$$

In that case, considering the state of the system at the discrete instants $2vT+$, we have that the discrete time evolution is given by the transition operator Q, i.e.

$$A_n(2\,v\,T+) = Q^{v} \cdot A_n(0).$$

It is easy to check that for any perturbation $\epsilon \neq 0$ the procedure is unstable, because $\forall \epsilon \neq 0$ some eigenvalues of Q have modulus $\lambda = 1 + \rho$, with ρ positive.

The instability factor ρ is a slowly increasing function of ϵ. The following tables show the behaviour of ρ in terms of ϵ (the eigenvalues have been computed on the truncation of Q to the first NA components).

5.1 Tables of unstable eigenvalues

ρ	$\epsilon = 10^{-2}$	$\epsilon = 10^{-3}$	$\epsilon = 10^{-4}$	$\epsilon = 10^{-5}$	$\epsilon = 10^{-6}$	$\epsilon = 10^{-7}$
NA=2	$1.137e^{-5}$	$1.2733e^{-9}$	$1.2752e^{-13}$	$1.2753e^{-17}$	$1.2753e^{-21}$	$1.2753e^{-25}$
NA=3	$1.1373e^{-5}$	$1.2733e^{-9}$	$1.2752e^{-13}$	$1.2753e^{-17}$	$1.2753e^{-21}$	$1.2753e^{-25}$
NA=4	$1.1372e^{-5}$	$1.2733e^{-9}$	$1.2752e^{-13}$	$1.2753e^{-17}$	$1.2753e^{-21}$	$1.2753e^{-25}$
NA=5	$1.1372e^{-5}$	$1.2733e^{-9}$	$1.2752e^{-13}$	$1.2753e^{-17}$	$1.2753e^{-21}$	$1.2753e^{-25}$

Table 1: Factor of instability ρ versus ϵ. NC=1

ρ	$\epsilon = 10^{-2}$	$\epsilon = 10^{-3}$	$\epsilon = 10^{-4}$	$\epsilon = 10^{-5}$	$\epsilon = 10^{-6}$	$\epsilon = 10^{-7}$
NA=3	$4.2751e^{-5}$	$6.2924e^{-9}$	$6.36635e^{-13}$	$6.3689e^{-17}$	$6.3694e^{-21}$	$6.3695e^{-25}$
NA=4	$4.2749e^{-5}$	$6.2924e^{-9}$	$6.36635e^{-13}$	$6.3689e^{-17}$	$6.3694e^{-21}$	$6.3695e^{-25}$
NA=5	$4.2745e^{-5}$	$6.2924e^{-9}$	$6.36635e^{-13}$	$6.3689e^{-17}$	$6.3694e^{-21}$	$6.3695e^{-25}$

Table 2: Factor of instability ρ versus ϵ. NC=2

ρ	$\epsilon = 10^{-1}$	$\epsilon = 10^{-2}$	$\epsilon = 10^{-3}$	$\epsilon = 10^{-4}$	$\epsilon = 10^{-5}$
NA=4	0.26596	0.000574	$1.1132e^{-7}$	$1.1313e^{-11}$	$1.1325e^{-15}$

Table 3: Factor of instability ρ versus ϵ. NC=3

ρ	$\epsilon = 10^{-1}$	$\epsilon = 10^{-2}$	$\epsilon = 10^{-3}$	$\epsilon = 10^{-4}$	$\epsilon = 10^{-5}$	$\epsilon = 10^{-6}$
NA=5	0.49803	0.00162	$4.2072e^{-7}$	$4.2891e^{-11}$	$4.2935e^{-15}$	$4.2939e^{-19}$

Table 4: Factor of instability ρ versus ϵ. NC=4

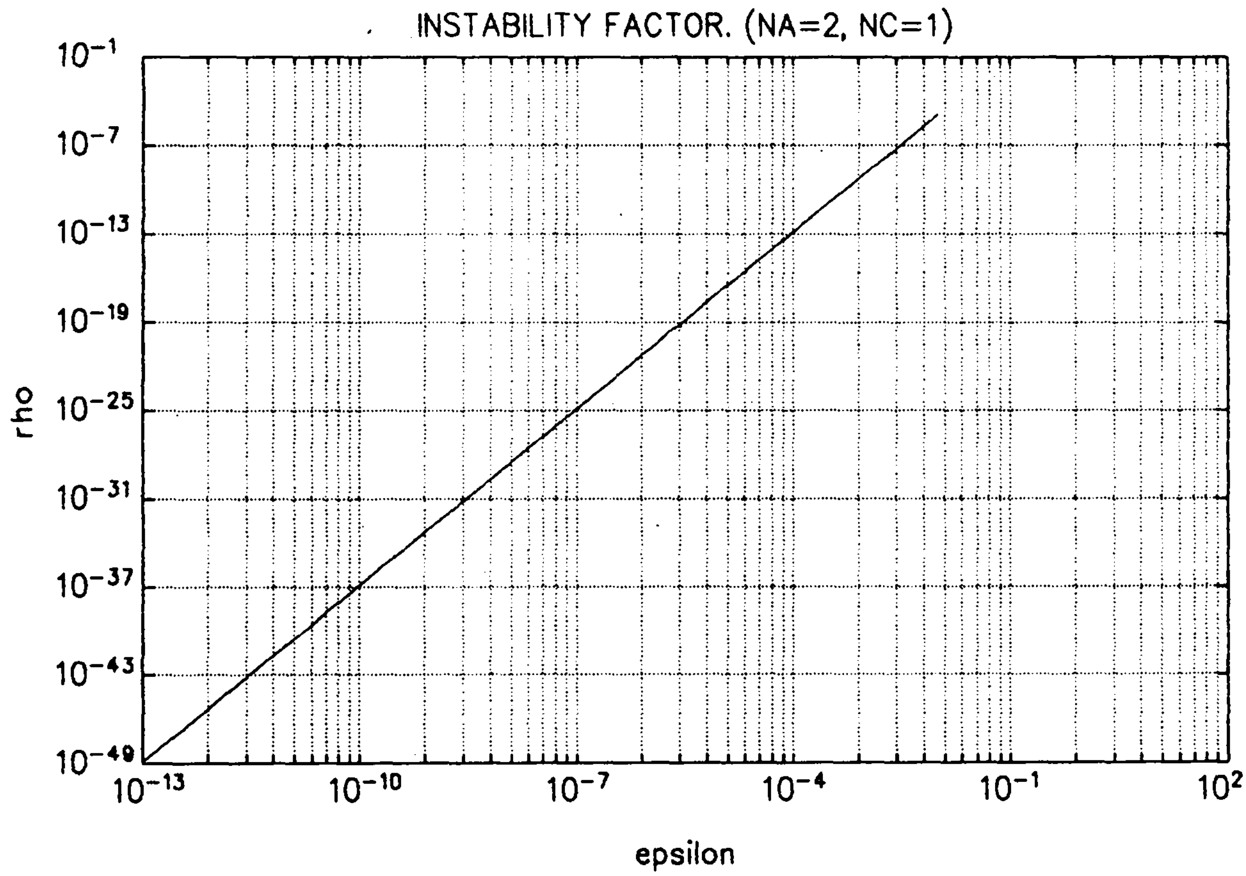

Figure 4: Function ρ versus ε. NA=2, NC=1.

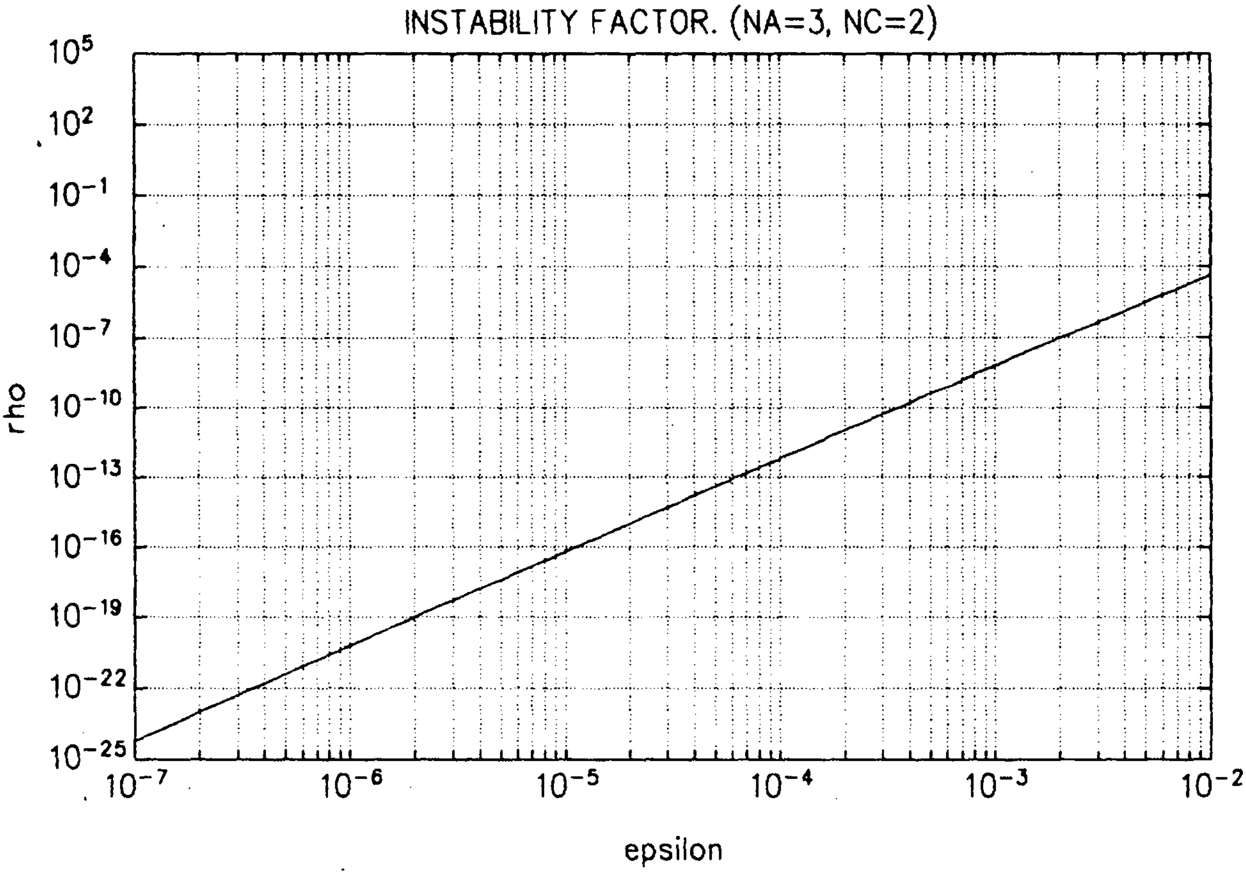

Figure 5: Function ρ versus ε. NA=3, NC=2.

5.2 Analytical results

It is possible to do analytical calculations we have done them with the MATHEMATICA programming language which proves that the unstable eigenvalues of Q are functions of ϵ verifying the following law

$$|\lambda(Q)| = (1 + O(\epsilon^4)).$$

Additional description of this behaviour will be contained in [11].

6 Conclusions

In this work we have found the following results for a practical implementation of *HUM* method of exact controllability:

- For the one-dimensional wave equation it is possible to implement in a practical way the exact controllability method. It annihilates a finite number of harmonic components of the closed loop system, while leaving unaltered the remaining part of the system.

- The proposed implementation is not robust with respect to perturbations on the plant data. For any small variations of systems's parameters, there appears spurious unstable oscillations in the closed loop operation.

The trade-off *exact controllability vs. closed loop instability* has been previously analyzed in some papers (see e.g. [2], [3], [6], [9]); in particular, this paper continues our work initially described in [6]. It seems an unavoidable phenomenon in the absence of damping and in that sense, it opens two roads to overcome it. The first one is to analyze how much damping is necessary to add to the system to stabilize it; the second one is to analyze how much computing power is necessary to use, to get the equivalent result to an artificial damping. These issues are the subject of some research in course [11].

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References

[1] Balakrishnan A.V., *Applied Functional Analysis*, Springer Verlag, New York, 1976.

[2] Balas M.J., *Feedback Control of Flexible Systems*, IEEE Trans. on A.C., AC-23, pp. 673-679, 1978.

[3] Balas M.J., *Trends in Large Space Structure Control Theory, Fondest Dreams, Wildest Hopes*, IEEE Trans. on A.C., AC-27, pp. 522-535, 1982.

[4] Bensoussan A., Da Prato G., Delfour M., Mitter S., *Representation and Control of Infinite Dimensional Systems*, Vol. I, Birkhäuser, Boston, 1993.

[5] Bensoussan A., Da Prato G., Delfour M., Mitter S., *Representation and Control of Infinite Dimensional Systems*, Vol. II, Birkhäuser, Boston, 1993.

[6] Braslavsky J.H., González R.L.V., Serón M.M., *Sobre una Implementación Práctica del Método HUM de Controlabilidad Exácta*, Anales de la IV Reunión de Trabajo en Procesamiento de la Información y Control, RPIC' 91, pág. 322-328, CONEA, Buenos Aires, 1991.

[7] El Jai A., Pritchard A.J., *Capteurs et Actionneurs dans l'Analyse des Systèmes Distribués*, Masson, Paris, 1986.

[8] Gasquet C., Witomski V., *Analyse de Fourier et Applications*, Masson, Paris, 1990.

[9] Gibson J.S., *A Note on Stabilization of Infinite Dimensional Linear Oscillators by Compact Linear Feedback*, SIAM Journal on Control and Optimization, Vol. 18, pp. 311-316, 1980.

[10] Glowinsky R., Li C., *On the Numerical Implementation of the Hilbert Uniqueness Method for the Exact Boundary Controllability of the Wave Equation*, C.R. Acad. Sci. Paris, Tomo 311, Série 1. pp. 135-142, 1990.

[11] González R.L.V., Reyero G.F., *Analytical Results on a Finite Dimension Implementation of HUM Exact Controllability Method for the One Dimension Wave Equation*, work in preparation.

[12] Haraux A., *Quelques Méthodes et Résultats Récents en Théorie de la Contrôlabilité Exacte*, Rapport de Recherche, N°1317, INRIA, Rocquencourt, Francia, 1990.

[13] Lagnese J.E., Leugering G., Schmidt E.J.P.G., *Modeling, Analysis and Control of Dynamic Elastic Multi-Link Structures*, Birkhäuser, Boston, 1994.

[14] Lions J.-L., *Contrôle Optimal des Systèmes Gouvernés par des Équations aux Dérivées Partielles*, Dunod, Paris, 1968.

[15] Lions J.-L., *Exact Controllability, Stabilization and Perturbations for Distributed Systems*, SIAM Review, Vol. 30, N°1, pp. 1-68, 1988.

[16] Lions J.-L., *Contrôlabilité Exacte, Stabilization et Perturbations des Systèmes Distribués*, Vol. 1-2, Masson, Paris, 1988.

[17] Renardy M., Rogers R.C., *An Introduction to Partial Differential Equations*, Springer-Verlag, New York, 1993.

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