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Abstract: Considering measure theory in which the semifield of positive real numbers is replaced by an idempotent semiring leads to the notion of idempotent measure introduced by Maslov. Then, idempotent measures or integrals with density correspond to supremums of functions for the partial order relation induced by the idempotent structure. In this paper, we give conditions under which an idempotent measure has a density and show by many examples that they are often satisfied. These conditions depend on the lattice structure of the semiring and on the Boolean algebra in which the measure is defined. As an application, we obtain a necessary and sufficient condition for a family of probabilities to satisfy the large deviation principle as defined by Varadhan.

Key-words: Idempotent semiring, Dioid, Max-plus algebra, Continuous lattice, Idempotent measure, Optimization, Large deviations.

(Résumé : tsvp)

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Densités des mesures idempotentes et grandes déviations

Résumé : Si l'on considère la théorie de la mesure dans laquelle le demi-corps des réels positifs est remplacé par un demi-anneau idempotent, on obtient la notion de mesure idempotente introduite par Maslov. Les mesures ou intégrales idempotentes à densité correspondent alors à des supremums de fonctions pour la relation d'ordre partiel induite par la structure idempotente. Nous donnons ici des conditions pour qu'une mesure idempotente ait une densité et montrons par de nombreux exemples qu'elles sont souvent verifiées. Ces conditions portent à la fois sur la structure de treillis du demi-anneau et sur l'algèbre de Boole sur laquelle la mesure est définie. On trouve alors un critère pour qu'une famille de probabilités satisfasse au principe des grandes déviations tel qu'il est défini par Varadhan.

Mots-clé : Demi-anneau idempotent, Dioïde, Algèbre max-plus, Treillis continu, Mesure idempotente, Optimisation, Grandes déviations.

Introduction

A probability or a positive measure is in some loose sense a continuous morphism from a Boolean σ-algebra $(\mathcal{A}, \cup, \cap)$ of subsets of some set Ω, to the semifield $(\mathbb{R}^+, +, \times)$. If we replace $(\mathbb{R}^+, +, \times)$ by an idempotent semiring (or dioid) [4] $(\mathbb{D}, \oplus, \otimes)$, we obtain the notion of idempotent measure. This notion has been introduced by Maslov in [12] where idempotent integrals were also constructed. In particular, if we consider the semifield $\mathbb{R}_{\max} = (\mathbb{R} \cup \{-\infty\}, \max, +)$, measure or probability theory (resp. Wiener processes, linear second order elliptic equations) is replaced by optimization theory (resp. Bellman processes, particular Bellman equations) and some of the notions may be transferred from the first domain to the second one. Illustrations and utilizations of this correspondence may be found in Maslov [12], Maslov and Samborski [13], Del Moral, Thuillet, Rigal and Salut [7], Del Moral [6], Quadrat [14], Bellalouna [5], Akian, Quadrat and Viot [2, 3], and Akian [1].

Whereas Maslov tried to treat idempotent measures theory in general idempotent (or ordered) semiring $\mathbb{D}$ and measure space Ω, some improvements may be done : firstly, at least for the construction of measures, $\mathbb{D}$ does not need to be a metric space but only a dually continuous lattice which is an order property. Secondly, the existence of a density has not been clarified. This point has been neglected in most of the studies on this domain, except in [9, 10], where Kolokoltsov and Maslov prove the existence of a "density" for linear forms, which in some particular cases implies the existence of a density for idempotent measures. The present paper is essentially devoted to this last problem. In particular, using the technics of continuous lattices theory, general conditions under which an idempotent measure has a density are found.

Let us consider the dioid $\mathbb{R}_{\max}$ with zero (neutral element for the "addition" $\max$) $\mathbb{0} = -\infty$ and unit (neutral element for the "multiplication" $+$) $\mathbb{1} = 0$. Addition corresponds to finite maximization, then "integration" corresponds to taking infinite supremum. The equivalent of the Lebesgue measure on $(\Omega, \mathcal{A})$ where $\Omega = \mathbb{R}$ and $\mathcal{A}$ is the Borel sets algebra, is the "uniform idempotent measure" $\lambda(A) = \mathbb{1}$ for any $A \subset \Omega$. Then, the "integral" of a continuous function f is $\lambda(f) = \sup_{\omega \in \Omega} f(\omega)$. Now, the function $\mathbb{K}(A) = \sup_{\omega \in A} c(\omega)$ defines an idempotent measure with density c with respect to the "Lebesgue measure". Under conditions on c and f the integral of a measurable function f with respect to the measure $\mathbb{K}$, as defined by Maslov, is $\mathbb{K}(f) = \oint_\Omega f(\omega) \otimes \mathbb{K}(d\omega) = \sup_{\omega \in \Omega} f(\omega) + c(\omega)$. Then, the integral of a function with respect to a measure with density has a simple expression. We may then ask if there exists, as in the classical measure theory, (interesting) measures which have no density. As a first answer, let us note that most natural measures without density in the classical measure theory have a density in $\mathbb{R}_{\max}$. Indeed, the upper semi-continuous (u.s.c.) function

$$\delta_m(\omega) = \mathbb{1} \text{ if } \omega = m, \quad \delta_m(\omega) = \mathbb{0} \text{ otherwise}$$

is the density of the “Dirac measure” at point m :

$$\delta_m(f) = \sup_{\omega\in\Omega} f(\omega) + \delta_m(\omega) = f(m).$$

However, we may exhibit the following measure without density :

$$\mathbb{K}(A) = \operatorname*{ess\,sup}_{\omega\in\Omega} c(\omega),$$

where c is a continuous function and the essential supremum is taken with respect to the (classical) Lebesgue measure. This measure satisfies the conditions of Definition 21 below on $(\Omega, \mathcal{A})$. $\mathbb{K}$ has no density since $\mathbb{K}(\{\omega\}) = \mathbb{0} = -\infty$ for any $\omega \in \Omega$. However, the restriction of $\mathbb{K}$ to the algebra of open sets has c as density. Then $\oint_\Omega f(\omega) \otimes \mathbb{K}(d\omega) = \sup_{\omega\in\Omega} f(\omega) + c(\omega)$ for any lower semi-continuous (l.s.c.) function f [12]. The non existence of a density to $\mathbb{K}$ on the entire algebra of Borel sets is in general not relevant and every measure seems to have a density in a sufficiently large algebra of subsets.

From the previous examples, we see that the order relation $\leq$ plays an important role in the semiring $\mathbb{R}_{\max}$. More generally, if $(\mathbb{D}, \oplus, \otimes)$ is an idempotent semiring, the idempotent law $\oplus$ defines a partial order relation $\preceq$ such that $(\mathbb{D}, \preceq)$ is a sup-semilattice. This implies that properties of measures and integrals are related with lattice properties of $\mathbb{D}$ that we will use throughout this paper. We thus begin by recalling and extending in section 1 definitions, properties of continuous lattices. We follow the presentation of Gierz, Hoffman, Keimel, Lawson, Mislove and Scott [11], up to subsidiary extensions. Then idempotent measures are introduced in section 2. In section 3, we prove that any idempotent measure on a suitable algebra $\mathcal{A}$ of subsets of a space Ω has necessarily a density. This includes Polish spaces with the algebra $\mathcal{A}$ of their open sets. For the proof, we construct the maximal extension of the idempotent measure to the algebra of all subsets of Ω and prove that the value of this extension on singletons is a density of the initial measure. In section 4, we recall in a general context the theorem of Maslov which prove the uniqueness of idempotent integrals of “semi-measurable” functions. This theorem is a consequence of the construction of the idempotent integral of Maslov, that we generalize to semirings $\mathbb{D}$ which are continuous lattices. Moreover, in order to relate our results on density of idempotent measures with the existing ones on density of idempotent linear forms, we prove a “probabilistic” version of Riesz representation theorem.

Our approach (the restriction of idempotent measures to open sets) was initially motivated by the large deviation principle of Varadhan [15]. In this theory, one essentially tries to obtain asymptotics of probabilities families P_ε of the form : $\mathbb{K}(A) = \lim_{\varepsilon\to 0} \varepsilon \log P_\varepsilon(A)$, where $\mathbb{K}(A) = -\inf_{\omega\in A} I(\omega)$ with I a l.s.c. function. Thus $\mathbb{K}$ is a $\mathbb{R}_{\max}$-idempotent measure with density $-I$. Generalizing this concept of large deviation by using general idempotent measures, we give (in section 5) necessary and sufficient conditions for the large deviation principle to be satisfied and prove that when it exists, I may be calculated by using open sets only.

1 Continuous lattices

In this section, we give a short presentation of definitions, results and examples concerning continuous lattices. Apart some minor extensions (on locally complete and locally continuous lattices), these results may be found in [11].

Let us first recall classical terminology. L denotes a set endowed with a partial order $\preceq$.

Definition 1 *$(L, \preceq)$ is a* semilattice *(resp. a* sup-semilattice, *resp. a* lattice*) if every nonempty finite set admits a greatest lower bound or infimum (resp. a least upper bound or supremum, resp. an infimum and a supremum). It is said a complete lattice if every set (even nonempty) admits an infimum (or equivalently if every set admits a supremum). $\top$ denotes the top element or supremum of L and $\perp$ the bottom element or infimum of L.*

In previous definition, we use the convention that the greatest lower bound (resp. the least upper bound) of the emptyset is the top element (resp. the bottom element) of the lattice L. The least upper bound or supremum is denoted by $\sup$ or $\vee$ and the greatest lower bound or infimum by $\inf$ or $\wedge$.

In the following sections we apply lattices formalism to dioids as follows.

Example 2 Let $(\mathbb{D}, \oplus)$ be a commutative idempotent monoid (that is $\oplus$ is associative commutative and idempotent : $a \oplus a = a$ for any $a \in \mathbb{D}$) with $\mathbb{0}$ as neutral element. We denote by $\preceq$ the partial order relation associated with the idempotent $\oplus$ operation : $a \preceq b \Leftrightarrow a \oplus b = b$. Then $a \oplus b$ is the least upper bound of a and b and $a \succeq \mathbb{0}$ for any $a \in \mathbb{D}$. Thus $(\mathbb{D}, \preceq)$ is sup-semilattice with bottom element $\mathbb{0}$. Conversely, a partial order $\preceq$ such that $(\mathbb{D}, \preceq)$ is a sup-semilattice with minimal element $\mathbb{0}$ defines an idempotent commutative, associative law $\oplus$ with neutral element $\mathbb{0}$ on $\mathbb{D}$. ■

In the sequel, we do not impose the completeness to the monoid dioid $(\mathbb{D}, \oplus, \otimes)$, but only the following property.

Definition 3 *The lattice $(L, \preceq)$ is a locally complete if it satisfies one of the following equivalent conditions :*

1. *every nonempty set admits an infimum;*

2. *every upper bounded set admits a supremum;*

3. *there exists a complete lattice denoted $\overline{L}$ with top element $\top$, such that L is a sublattice of $\overline{L}$, $\overline{L} = L \cup \{\top\}$ and $\sup L = \top$.*

The following definition concerns the continuity of complete lattices.

Definition 4
- *$D \subset L$ is a directed set if and only if any finite subset of D has an upper bound in D.*
- *Let $\preceq^{op}$ be the opposite order of L : $a \preceq^{op} b \Leftrightarrow b \preceq a$. If $(L, \preceq)$ is a lattice, then L^{op} denotes the lattice $(L, \preceq^{op})$.*
- *D is a filtered set of L iff D is a directed set of L^{op}.*
- *The "way below" $\ll$ relation is defined by : $a \ll b$ if and only if for all directed set D of L, such that $b \preceq \sup D$, there exists $x \in D$ such that $a \preceq x$.*
- *The complete lattice L is said continuous iff*
$$x = \sup\{y \in L,\ y \ll x\} \quad \textit{for any} \quad x \in L. \tag{1}$$
- *L is said dually continuous iff L^{op} is continuous.*
- *I is a basis of the continuous lattice L iff I is a sup-subsemilattice of L containing $\perp$ and such that*
$$x = \sup\{y \in I,\ y \ll x\} \quad \textit{for any} \quad x \in L. \tag{2}$$

Remark 5 In the definition of a basis we impose I to be a sup-subsemilattice of L containing $\perp$, which is equivalent to the condition $(I, \oplus)$ is a submonoid of $(L, \oplus)$ if $\oplus$ is defined as in Example 2. ■

Remark 6 Suppose that L is complete, $L \setminus \top$ is a directed set and that $\top$ is the supremum of $L \setminus \top$ (this means that $L \setminus \top$ is locally complete and $L = \overline{L \setminus \top}$). Then, $\top \not\ll \top$ and $a \ll b$ in L implies $a \in L \setminus \top$. Moreover, if I is a basis of L, then $I \setminus \top$ is also a basis of L : it is a sup-subsemilattice and $x = \sup\{a \in I \setminus \top,\ a \ll x\}$, for any $x \in L$. ■

Example 7 In a totally ordered lattice L, $a \prec b$ or $a = \perp$ implies $a \ll b$ (by definition $a \prec b \Leftrightarrow (a \preceq b$ and $a \neq b)$). If $L = \overline{\mathbb{R}}$ with the order $\leq$, then $a \ll b$ is equivalent to ($a < b$ or $a = -\infty$), and L is continuous (and dually continuous). If $L = \overline{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, +\infty\}$, then

$a \ll b$ is equivalent to ($a \le b$ and $a \neq +\infty$) and L is continuous (and dually continuous). ■

Example 8 If L is a complete lattice, then L^n with the componentwise order relation is a complete lattice and $a = (a_1, \dots, a_n) \ll b = (b_1, \dots, b_n)$ in L^n iff $a_i \ll b_i$ for $i = 1, \dots n$. Therefore, L continuous implies L^n continuous. In particular $((\overline{\mathbb{R}})^n, \le)$ is a continuous and dually continuous lattice.

Now, by eliminating the top element of $\overline{\mathbb{R}}$, we obtain $L = \mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ which is a locally complete lattice such that $\overline{L}$ is continuous and dually continuous. We can also prove that L locally complete implies L^n locally complete. However, $\overline{L^n} = [-\infty, +\infty)^n \cup \{+\infty \stackrel{\text{def}}{=} (+\infty, \dots, +\infty)\}$ endowed with the term to term order relation $\le$ is a dually continuous but not a continuous lattice. Indeed, $a \ll b$ iff $a = \perp = (-\infty, \dots, -\infty)$ (because for instance $+\infty = \sup(L \times \{-\infty\} \times \dots \times \{-\infty\})$). For the opposite order, however, all behaves as in $(\overline{\mathbb{R}})^n$. ■

The previous example shows that if we consider different complete sublattices of the same complete lattice L, the way-below relation defined in these sublattices may be different ($\overline{(\mathbb{R}_{\max})^n}$ is a sublattice of $(\overline{\mathbb{R}})^n$). This comes from the fact that these sublattices are not necessarily stable by infinite $\sup$ of L. However, if we generalize the way below relation to locally complete lattices in the following manner, this type of boundary effect disappears.

Definition 9 *Let L be a locally complete lattice. The "way below" $\ll$ relation is defined in L by : $a \ll b$ if and only if for all upper bounded directed sets D of L, such that $b \preceq \sup D$, there exists $x \in D$ such that $a \preceq x$.*

Then $a \ll b$ in L is equivalent to $a \ll b$ in any complete sublattice of L containing a and b and of the form $[\perp, c]$ with $c \in L$ (or stable by infinite $\sup$ of L). Then, definitions of continuous lattices and basis may be used without change. Locally complete lattices which are continuous will be called locally continuous. Under this definition, $(\mathbb{R}_{\max})^n$ becomes a locally continuous lattice. Moreover, a locally complete lattice L is continuous iff every complete sublattice of L of the form $[\perp, c]$ (or stable by infinite $\sup$ of L) is continuous.

Let us note that if L is a locally complete lattice, we may extend the definition of the way-below relation to $\overline{L}$ by taking $a \ll b$ in $\overline{L}$ iff ($a \ll b$ in L or $a \in L$ and $b = \top$). This relation is not equal to those defined directly in the complete lattice $\overline{L}$. For instance, if $L = (\mathbb{R}_{\max})^n$, this way below relation is the restriction of those of $(\overline{\mathbb{R}})^n$. If L is continuous and if I is a basis of L then (1) and (2) are still valid for $x = \top$.

Example 10 Another usual example of complete lattice is the set $\mathcal{P}(X)$ of subsets of a set X with the $\subset$ order relation. The set of open sets $\mathcal{O}(X)$ (resp. the set of closed sets $\mathcal{C}(X)$)

of a topological space X (resp. the set of closed convex sets $\mathrm{Con}(X)$ of a topological vector space X) is also a complete lattice with bottom element $\emptyset$ and top element X, even if it is not a sublattice of $\mathcal{P}(X)$.

In $\mathcal{C}(X)$, $A \ll B$ iff $A = \emptyset$, thus $\mathcal{C}(X)$ and $\mathrm{Con}(X)$ are not continuous and $\mathcal{O}(X)$ is not dually continuous. In $\mathcal{O}(X)$, $A \ll B$ if $\overline{A} \subset B$ and $\overline{A}$ compact, which is often noted $A \subset\subset B$. If X is locally compact, the two conditions are equivalent and $\mathcal{O}(X)$ is continuous and $\mathcal{C}(X)$ dually continuous. Now, if K is a compact convex subset of a locally convex topological vector space X, then $\mathrm{Con}(K)$ is a dually continuous lattice. ■

The following characterization is the main ingredient of the proofs of section 3 on extensions and densities of idempotent measures.

Theorem 11 ([11, th. 2.3]) *For a complete lattice L, the following conditions are equivalent :*

1. *L is continuous.*
2. *Let $\{D(j),\ j \in J\}$ be a family of directed sets of L. Let M be the set of all functions $f : J \to D \stackrel{def}{=} \cup_{j\in J} D(j)$ with $f(j) \in D(j)$ for all $j \in J$. Then the following identity holds :*
$$\inf_{j\in J} \sup D(j) = \inf_{j\in J} \sup_{x\in D(j)} x = \sup_{f\in M} \inf_{j\in J} f(j). \tag{3}$$
3. *Let $\{D(j),\ j \in J\}$ be any family of subsets of L. Let N be the set of all functions $f : J \to$ fin$D$, the set of finite subsets of $D \stackrel{def}{=} \cup_{j\in J} D(j)$ with $f(j) \in$ fin$D(j)$ for all $j \in J$. Then the following identity holds :*
$$\inf_{j\in J} \sup D(j) = \sup_{f\in M} \inf_{j\in J} \sup f(j). \tag{4}$$

The previous theorem is still valid if L is only a locally complete lattice and if the sets $D(j)$ considered in points 2 and 3 are supposed upper bounded (that is $\sup D(j) \in L$).

The classical definition of lower semi-continuous (l.s.c.) functions with values in $\mathbb{R}$ may be generalized to functions with values in any lattice L.

Definition 12 *A function $f : \Omega \to L$ is said l.s.c. iff*

$$f(x) \preceq \liminf_{y\to x} f(y) \stackrel{def}{=} \sup_{U\in\mathcal{U},\, U\ni x} \inf_{y\in U} f(y),$$

where $\mathcal{U}$ is the set of open sets of the topological space Ω.

The Scott topology defined below allows to characterize (in a locally continuous lattice) the semi-continuity in terms of topology.

Definition 13 *Let L be a (locally) complete lattice, we say that $U \subset L$ is Scott open (open for the Scott topology) if it satisfies the two following conditions :*

1. $U = \uparrow U \stackrel{\text{def}}{=} \{x \in L,\ \exists\, y \in U,\ y \preceq x\}$.
2. $\sup D \in U$ *implies* $D \cap U \neq \emptyset$ *for all directed sets* $D \subset L$.

Proposition 14 *Let L be a (locally) continuous lattice.*

The Scott topology on L is the weakest topology such that the sets $\{x \in L,\ a \ll x\}$ are open.

A function $f : \Omega \to L$ is l.s.c. iff it is continuous for the Scott topology of L.

The Scott topology is clearly not separated (Hausdorff). If we want to define the continuity of a function with values in L in terms of topology, we have to consider the common refinement of the two Scott topologies defined for $\preceq$ and $\preceq^{op}$ partial orders on L. This topology will be called "bi-Scott" and works only on bi-continuous lattices (lattices which are both continuous and dually continuous). The Lawson topology defined below is stronger than the Scott and weaker than the bi-Scott topology and works well on lattices L which are only continuous.

Definition 15 *The Lawson topology denoted ΛL is defined as the common refinement of the Scott topology and the lower topology, that is the topology generated by sets $[a, \top]^c = \{x \in L,\ a \not\preceq x\}$.*

Proposition 16 *For a continuous lattice L, ΛL is a compact Hausdorff space.*

Moreover, L has a countable basis iff ΛL is a compact metric space.

Remark 17 If L is a bi-continuous lattice, then the bi-Scott topology is equal to the bi-Lawson topology that is the common refinement of the two Lawson topologies defined for $\preceq$ and $\preceq^{op}$ partial orders on L. Then, if both L and L^{op} have a countable basis, the bi-Scott topology is metrizable (but not necessarily compact). ■

2 Idempotent measures

Let $\mathcal{A}$ be a Boolean algebra or a Boolean σ-algebra of subsets of a set Ω. A probability P on $(\Omega, \mathcal{A})$ is such that

i) $P(A \cup B) = P(A) + P(B)$ if $A \cap B = \emptyset$ and $P(\emptyset) = 0$.

In addition, $P(A \cap B) = P(A) \times P(B)$ if A and B are independent. Thus a probability may be compared to a morphism from the "complemented" (in the sense that $A \in \mathcal{A}$ implies that $A^c \in \mathcal{A}$) semiring $(\mathcal{A}, \cup, \cap)$ to the symetrizable semifield $(\mathbb{R}^+, +, \times)$ (such that $P(\Omega) = 1$). Since the field structure of $\mathbb{R}$ allows to write $P(A^c) = 1 - P(A)$, the continuity of a probability can be equivalently defined by one of the two properties :

ii) $P(A_n) \underset{n \to +\infty}{\nearrow} P(A)$ if $A_n \underset{n \to +\infty}{\nearrow} A$ with A_n and A in $\mathcal{A}$,
iii) $P(A_n) \underset{n \to +\infty}{\searrow} P(A)$ if $A_n \underset{n \to +\infty}{\searrow} A$ with A_n and A in $\mathcal{A}$.

If we replace $(\mathbb{R}^+, +, \times)$ by an idempotent semiring $(\mathbb{D}, \oplus, \otimes)$, we loose "opposites" for the additive law $\oplus$ and as a consequence : a) the entire structure of Boolean algebra is no longer needed in order to get a "morphism", b) properties ii) and iii) are not equivalent, moreover iii) is rarely satisfied and is not preserved after extension of a probability P to a larger algebra (see Examples 18 and 20 below).

Example 18 Let us consider $\mathcal{A}$ the set of Borel sets of $\Omega = \mathbb{R}$ and let consider $P(A) = \sup_{\omega \in A} c(\omega)$ with c an upper semi continuous (u.s.c.) function from $\mathbb{R}$ to $\mathbb{R}_{\max}$. Then, P satisfies property i) where addition is replaced by the $\max$ operator and property ii). Indeed, we will see in section 3 that the restriction to open sets of any idempotent $\mathbb{R}_{\max}$-probability on $(\Omega, \mathcal{A})$ has this form. If P satisfies also property iii) on $\mathcal{A}$, then $P((a - 1/n, a + 1/n) \setminus \{a\})$ and $P([-n, n]^c)$ decrease towards $P(\emptyset) = \mathbb{0} = -\infty$. This implies that the set $\{x \in \mathbb{R},\ c(x) \geq b\}$ is finite for all $b \in \mathbb{R}$ and thus c has countable support (as atomic classical probabilities). ■

Definition 19 *A set $\mathcal{A}$ of subsets of a given set Ω is called a Boolean semi-algebra if it is a sublattice of $(\mathcal{P}(\Omega), \subset)$, that is if it contains Ω and $\emptyset$ and is stable by the finite union and intersection operations. It is called a semi-σ-algebra if in addition it is stable by countable union operation.*

Example 20 Let us consider the compact metric space $\Omega = [0, 1]$. The set $\mathcal{A}$ of closed sets is a Boolean semi-algebra. Now, if P is as in Example 18, then P satisfies condition iii) on $\mathcal{A}$. However, the semi-σ-algebra generated by $\mathcal{A}$ contains open sets for which property iii) is false in general. ■

Let us consider $(\mathbb{D}, \oplus, \otimes)$ an idempotent semiring with $\mathbb{0}$ and $\mathbb{1}$ as neutral elements for the $\oplus$ and $\otimes$ operations respectively. We denote by $\preceq$ the partial order relation associated with the idempotent $\oplus$ operation (see Example 2). We denote also by "$\sup$" or $\oplus$ (resp. by "$\inf$" or $\wedge$) the supremum (resp. the infimum) operation. In all this paper, we suppose that $\mathbb{D}$ is locally complete. Note that, if the top element $\top$ of $\overline{\mathbb{D}}$ does not belong to $\mathbb{D}$, the law $\otimes$ may be extended to $\overline{\mathbb{D}}$ so that $(\overline{\mathbb{D}}, \oplus, \otimes)$ becomes a semiring ($\top \otimes a = a \otimes \top = \top$ if $a \neq \mathbb{0}$ and $\top \otimes \mathbb{0} = \mathbb{0} \otimes \top = \mathbb{0}$). Examples of such idempotent semirings are $\mathbb{R}_{\max} = (\mathbb{R} \cup \{-\infty\}, \max, +)$, $(\mathbb{R}^+, \max, \times)$, $\mathbb{R}_{\min} = (\mathbb{R} \cup \{+\infty\}, \min, +)$, (with $+\infty$, resp. $+\infty$, resp. $-\infty$, as upper bounds), and also ${\mathbb{R}_{\max}}^n$, ${\mathbb{R}_{\min}}^n$,...

Definition 21 *An idempotent $\mathbb{D}$-measure on a Boolean semi-algebra $\mathcal{A}$ of subsets of Ω is a mapping $\mathbb{K}$ from $\mathcal{A}$ to $\overline{\mathbb{D}}$ such that :*

1. $\mathbb{K}(\emptyset) = \mathbb{0}$,
2. $\mathbb{K}(A \cup B) = \mathbb{K}(A) \oplus \mathbb{K}(B)$ *for any* $A,\ B$ *in* $\mathcal{A}$,
3. $\mathbb{K}(A_n) \underset{n\to+\infty}{\nearrow} \mathbb{K}(A)$ *if* $A_n \underset{n\to+\infty}{\nearrow} A$, $A_n \in \mathcal{A}\ \forall n \in \mathbb{N}$ *and* $A \in \mathcal{A}$ *(σ-additivity).*

An idempotent $\mathbb{D}$-measure $\mathbb{K}$ is said finite if $\mathbb{K}(\Omega) \in \mathbb{D}$ and is called an idempotent probability if $\mathbb{K}(\Omega) = \mathbb{1}$.

Remark 22 It follows immediately from the definition, that any idempotent measure $\mathbb{K}$ is monotone : $\mathbb{K}(A) \preceq \mathbb{K}(B)$ if $A \subset B$. Then, if $\mathbb{K}$ is a probability, it takes its values in the subset $[\mathbb{0}, \mathbb{1}] = \{x \in \mathbb{D}, \mathbb{0} \preceq x \preceq \mathbb{1}\} = \{x \in \mathbb{D},\ x \preceq \mathbb{1}\}$ of $\mathbb{D}$. ■

By the idempotency, we have

Proposition 23 *A mapping $\mathbb{K}$ from $\mathcal{A}$ to $\overline{\mathbb{D}}$ is an idempotent $\mathbb{D}$-measure on $\mathcal{A}$ iff*

$$\mathbb{K}(\cup_{i\in I} A_i) = \bigoplus_{i\in I} \mathbb{K}(A_i)$$

for any finite or countable family $\{A_i,\ i \in I\}$ of elements of $\mathcal{A}$.

Remark 24 Note that the second law $\times$ or $\otimes$ is only necessary in the construction of integrals or the definition of independency but not in the construction of measures. In particular the results of the following section depend only on the first law, thus on the lattice structure of $\mathbb{D}$. ■

An idempotent measure with values in $\mathbb{R}_{\max}$ (resp. $\mathbb{R}_{\min}$) will be called a gain (resp. cost) measure. It is finite if and only if $\mathbb{K}(\Omega) < +\infty$ (resp. $\mathbb{K}(\Omega) > -\infty$) and it is a gain (resp. cost) probability if $\mathbb{K}(\Omega) = 0$. Note that the order relation associated to the "$\min$"

law is the opposite of the classical order $\leq$ of $\mathbb{R}$. Therefore, even if we are more interested with cost measures, that is with minimization problems, it is easier to consider gain measures since monotony properties and extensions constructions coincide with those of the classical Probability theory.

An idempotent $\mathbb{D}$-probability space (also called a decision space) $(\Omega, \mathcal{A}, \mathbb{K})$ is composed of a nonempty set Ω, a semi-σ-algebra $\mathcal{A}$ of subsets of Ω and an idempotent $\mathbb{D}$-probability $\mathbb{K}$.

Let us introduce the notion of density of an idempotent $\mathbb{D}$-measure. Consider a function c from Ω into $\mathbb{D}$ and define for any subset A of Ω,

$$\mathbb{K}(A) = \sup\{c(\omega), \omega \in A\}. \tag{5}$$

It is easy to check that $\mathbb{K}$ is an idempotent $\mathbb{D}$-measure on $\mathcal{P}(\Omega)$.

Definition 25 *An idempotent measure $\mathbb{K}$ is said to have a density if (5) holds for some function c. In this case, any function c satisfying (5) is called a density of $\mathbb{K}$.*

3 Idempotent measures extensions and densities

In [12] Maslov shows that there might be several extensions of the same idempotent measure from a Boolean algebra to the least σ-algebra containing it. For instance, the "Lebesgue measure" on $(\Omega, \mathcal{A})$ with values in $\mathbb{R}_{\max}$, where $\Omega = \mathbb{R}$ and $\mathcal{A}$ is the algebra of finite unions of any intervals with rational bounds :

$$\mathbb{K}(A) = 0 \qquad \text{if } A \neq \emptyset, \qquad \mathbb{K}(\emptyset) = -\infty$$

may be extended to the Borel sets σ-algebra as follows :

$$\mathbb{K}(A) = \sup_{x \in A} c(x)$$

with 1) $c(x) \equiv 0$ (which leads to the maximal extension) or 2) $c(x) = 0$ when x is rational and $c(x) = -\infty$ (or any number less that 0) when x is irrational, and clearly densities 1) and 2) do not lead to the same value of $\mathbb{K}$. Indeed nonempty elements of $\mathcal{A}$ necessarily contain rationals.

However, the maximal extension always exists and plays an important role (see section 4). Here we recall the definition of the maximal extension which only involves the Boolean semi-algebra structure of the initial set $\mathcal{A}$ of subsets of Ω. Although this construction seems natural (it is equivalent to those of classical measure theory), it implicitly uses the dual continuity of the complete lattice $\overline{\mathbb{D}}$ or at least of the sublattice $[\mathbb{0}, \mathbb{K}(\Omega)]$. This same property will still be necessary to prove that $\mathbb{K}$ has a density.

Let us consider $\mathbb{K}$ an idempotent measure on a Boolean semi-algebra $\mathcal{A}$ of subsets of Ω. We denote by $\mathcal{G}$ the set of countable unions of elements of $\mathcal{A}$, $\mathcal{G}$ is then the least semi-σ-algebra containing $\mathcal{A}$. We define on $\mathcal{G}$ the extension $\mathbb{K}^+$ of $\mathbb{K}$:

$$\mathbb{K}^+(G) = \sup_n \mathbb{K}(A_n) \ \text{ if } \ G = \bigcup_{n\in\mathbb{N}} A_n \ \text{ with } \ A_n \in \mathcal{A}.$$

This definition is well posed, as by the σ-additivity of $\mathbb{K}$ and the stability of $\mathcal{A}$ by finite intersections and unions, the supremum is independent of the sets A_n and $\mathbb{K}^+$ is the unique extension of $\mathbb{K}$ to $\mathcal{G}$.

Now, for any subset A of Ω we define :

$$\mathbb{K}^*(A) = \inf_{G\in\mathcal{G}, G\supset A} \mathbb{K}^+(G).$$

Proposition 26 ([12]) *Suppose that $([\mathbb{0}, \mathbb{K}(\Omega)], \preceq)$ is a dually continuous lattice. Then $\mathbb{K}^*$ is the maximal extension of $\mathbb{K}$ to the set of all subsets of Ω.*

Proof. We recall the proof in order to point out the use of dual continuity. Let us first prove that $\mathbb{K}^*$ is maximal. For any semi-σ-algebra $\mathcal{B}$ containing $\mathcal{A}$ and any extension $\mathbb{K}'$ of $\mathbb{K}$ to $\mathcal{B}$ we have $\mathcal{B} \supset \mathcal{G}$ and $\mathbb{K}' = \mathbb{K}^+$ on $\mathcal{G}$. Then, for any $B \in \mathcal{B}$ and $G \in \mathcal{G}$ such that $G \supset B$ we have $\mathbb{K}'(B) \preceq \mathbb{K}'(G) = \mathbb{K}^+(G)$, thus $\mathbb{K}'(B) \preceq \mathbb{K}^*(B)$.

In order to prove that $\mathbb{K}^*$ is an idempotent measure on $\mathcal{P}(\Omega)$ we only have to prove that for any finite or countable family $\{A_i,\ i \in I\}$ of subsets of Ω, $\mathbb{K}^*(\cup_i A_i) = \sup_i \mathbb{K}^*(A_i)$. The monotony of $\mathbb{K}^*$ is evident from the definition. Then, $\mathbb{K}^*(\cup_i A_i) \succeq \sup_i \mathbb{K}^*(A_i)$. For the other inequality, we have

$$\begin{aligned} \sup_i \mathbb{K}^*(A_i) &= \sup_i \inf_{G\in\mathcal{G},\, G\supset A_i} \mathbb{K}^+(G) \\ &= \inf_{G_i\in\mathcal{G},\, G_i\supset A_i\ \forall i\in I} \sup_i \mathbb{K}^+(G_i) \\ &= \inf_{G_i\in\mathcal{G},\, G_i\supset A_i\ \forall i\in I} \mathbb{K}^+(\cup_i G_i) \\ &\succeq \mathbb{K}^*(\cup_i A_i), \end{aligned} \tag{6}$$

which leads to the requested equality. In (6), we have used an inversion formula of the $\sup$ and $\inf$ operations of the same type than (3) but for the opposite order $\succeq$ ($\{\mathbb{K}^+(G),\ G \in \mathcal{G},\ G \supset A_i\}$ is a filtered set), which holds in a dually continuous lattice only. As $\mathbb{K}$ takes its values in $[\mathbb{0}, \mathbb{K}(\Omega)]$, the dual continuity of this sublattice is only needed. ■

Let us note that as $\Omega \in \mathcal{A}$, $\mathbb{K}^*$ is necessarily a probability if $\mathbb{K}$ is so.

Example 27 If $\mathbb{K}$ is the "Lebesgue measure" on $\mathcal{A} : \mathbb{K}(A) = \mathbb{1}$ for $A \neq \emptyset$ and $\mathbb{K}(\emptyset) = \mathbb{0}$, then $\mathbb{K}^*$ is the "Lebesgue measure" on $\mathcal{P}(\Omega) : \mathbb{K}^*(A) = \mathbb{1}$ for any non empty set A. The function $c(\omega) \equiv \mathbb{1}$ is the density of $\mathbb{K}^*$ and the maximal density of $\mathbb{K}$. ■

Example 28 If $\mathcal{U}$ is the set of open sets of a topological space X, the lattice $(\mathcal{U}, \subset)$ is not dually continuous (see section 1). Let us consider the dioid $\mathbb{D} = (\mathcal{U}, \cup, \cap)$ with neutral elements $\mathbb{0} = \emptyset$ and $\mathbb{1} = X$ and take $\Omega = X$, $\mathcal{A} = \mathcal{U}$ and $\mathbb{K}(A) = A$ for any $A \in \mathcal{A}$. Clearly, $\mathbb{K}$ is an idempotent $\mathbb{D}$-probability, $\mathcal{G} = \mathcal{U}$ and thus $\mathbb{K}^+ = \mathbb{K}$. Now, for any subset A of Ω, $\mathbb{K}^*(A) = \mathring{A}$, the interior of A. Then, $\mathbb{K}^*$ is not an idempotent measure and is even not additive. ■

Example 29 Consider now the dioid $\mathbb{D} = (\mathcal{C}, \cup, \cap)$, where $\mathcal{C}$ is the set of closed sets of a compact subspace K of X and $\mathbb{K}(A) = \overline{A \cap K}$ for all $A \in \mathcal{A} = \mathcal{U}$. Clearly, $\mathbb{K}$ is an idempotent $\mathbb{D}$-probability (note that if $\mathcal{F}$ is a subset of $\mathcal{C}$, then $\sup \mathcal{F} = \overline{\cup_{F \in \mathcal{F}} F}$) and as $(\mathcal{C}, \subset)$ is dually continuous, then Proposition 26 shows that $\mathbb{K}^*$ is an idempotent measure. Indeed, we find by calculation $\mathbb{K}^*(A) = \overline{A \cap K}$. Moreover, the function $c^*(x) \stackrel{\text{def}}{=} \mathbb{K}^*(\{x\}) = \{x\} \cap K$ is the density of $\mathbb{K}^*$. ■

As $\mathbb{K}^*$ is defined on all subsets of Ω, we find a good candidate to the density function of $\mathbb{K} : c^*(\omega) = \mathbb{K}^*(\{\omega\})$. Let us denote

$$\overline{\mathbb{K}}(A) = \sup\{c^*(\omega), \omega \in A\}.$$

Since $\mathbb{K}^*$ is monotone, we have $\overline{\mathbb{K}}(A) \preceq \mathbb{K}^*(A)$ for any subset A of Ω.

Proposition 30 *If $\mathbb{K}$ has a density on $\mathcal{A}$, then c^* is the maximal density of $\mathbb{K}$ on $\mathcal{A}$.*

Proof. Let c be a density of $\mathbb{K}$. We have $c^*(\omega) = \inf_{G \in \mathcal{G}, G \ni \omega} \mathbb{K}^+(G) = \inf_{A \in \mathcal{A}, A \ni \omega} \mathbb{K}(A) \succeq c(\omega)$. Thus $\mathbb{K}(A) \preceq \overline{\mathbb{K}}(A) \preceq \mathbb{K}^*(A) = \mathbb{K}(A)$ for any A in $\mathcal{A}$. ■

Example 31 Consider the dioid and idempotent measure of Example 28. Then Proposition 30 shows that $\mathbb{K}$ has no density since $c^*(x) = \emptyset = \mathbb{0}$ would have been its maximal density and $\mathbb{K} \not\equiv \mathbb{0}$. ■

Proposition 30 implies that if $\mathbb{K}$ has a density c, then $\mathbb{K}^+$ has c^* or c as density on $\mathcal{G}$. However, in order to prove that c^* is a density of $\mathbb{K}^*$, we need the stability of $\mathcal{G}$ by any union operation (even not countable). This is the case if $\mathcal{A}$ is the set of open sets of a topological space Ω. In this case, we have

$$\begin{aligned}
\overline{\mathbb{K}}(A) &= \sup_{\omega\in A} c^*(\omega) \\
&= \sup_{\omega\in A} \inf_{G\in\mathcal{G}, G\ni\omega} \mathbb{K}^+(G) \\
&= \inf_{(G_\cdot\in\mathcal{G}^A, G_\omega\ni\omega \ \ \forall\omega\in A)} \sup_{\omega\in A} \mathbb{K}^+(G_\omega).
\end{aligned}$$

Note that this last equality is of the same type as (3) for the opposite order $\preceq^{op}$ and thus requires the dual continuity of the lattice $[\mathbb{0}, \mathbb{K}(\Omega)]$. Now, if $\mathbb{K}^+$ has a density and $\bigcup_{\omega\in A} G_\omega \in \mathcal{G}$, we deduce that $\sup_{\omega\in A} \mathbb{K}^+(G_\omega) = \mathbb{K}^+(\bigcup_{\omega\in A} G_\omega)$. Then, as $\bigcup_{\omega\in A} G_\omega \supset A$, we obtain $\mathbb{K}^+(\bigcup_{\omega\in A} G_\omega) \succeq \mathbb{K}^*(A)$ and $\overline{\mathbb{K}}(A) \succeq \mathbb{K}^*(A)$. As the other inequality is always true, $\mathbb{K}^*$ has c^* as density.

In conclusion :

Proposition 32 *If $[\mathbb{0}, \mathbb{K}(\Omega)]$ is a dually continuous lattice, Ω is a topological space, $\mathcal{A}$ is the set of open sets of Ω and $\mathbb{K}$ has a density on $\mathcal{A}$, then $\mathbb{K}^*$ has c^* as density on $\mathcal{P}(\Omega)$.*

Remark 33 If $\mathbb{K}$ is a gain (resp. cost) measure with density c on the set $\mathcal{A}$ of open sets of a topological space Ω, then c^* is the upper semi-continuous (u.s.c.) (resp. lower semi-continuous (l.s.c.)) envelope of c. Indeed, $c^*(\omega) = \inf_{A\ni\omega, A\in\mathcal{A}} \sup_{y\in A} c(y)$, which is the definition of the u.s.c. (or l.s.c., if $\preceq$ corresponds to $\geq$) envelope. ∎

We prove now, that under some conditions on the Boolean semi-algebra $\mathcal{A}$, any idempotent measure has a density.

Theorem 34 *Consider a Boolean semi-algebra $\mathcal{A}$ of subsets of Ω such that the following property holds :*

> *for any $A \in \mathcal{A}$ and any cover $A \subset \bigcup_{i\in I} A_i$ by elements of $\mathcal{A}$, there exists a countable subcover of A : $A \subset \bigcup_{i\in J} A_i$ ($J \subset I$ and J countable).*

Then, for any idempotent $\mathbb{D}$-measure $\mathbb{K}$ on $\mathcal{A}$ such that $[\mathbb{0}, \mathbb{K}(\Omega)]$ is a dually continuous lattice, c^* is a density of $\mathbb{K}$ in $\mathcal{A}$ (and a density of $\mathbb{K}^+$ in $\mathcal{G}$).

Proof. As $\mathcal{G}$ is the set of countable unions of elements of $\mathcal{A}$, $\mathcal{G}$ satisfies the same property as $\mathcal{A}$. Now we prove that c^* is a density of $\mathbb{K}^+$ in $\mathcal{G}$.

We still have $\mathbb{K}^+(G) = \mathbb{K}^*(G) \succeq \overline{\mathbb{K}}(G)$ for any $G \in \mathcal{G}$. On the other hand, using again property (3), we have for any $A \in \mathcal{G}$

$$\overline{\mathbb{K}}(A) = \inf_{(G_\cdot \in \mathcal{G}^A, G_\omega \ni \omega \ \ \forall \omega \in A)} \sup_{\omega \in A} \mathbb{K}^+(G_\omega).$$

We can extract from the cover $A \subset \bigcup_{\omega \in A} G_\omega$, with A and G_ω in $\mathcal{G}$, a countable subcover : $A \subset \bigcup_{i \in I} G_{\omega_i}$. As I is countable, $\bigcup_{i \in I} G_{\omega_i} \in \mathcal{G}$ and $\mathbb{K}^+(A) \preceq \mathbb{K}^+(\bigcup_{i \in I} G_{\omega_i}) = \sup_{i \in I} \mathbb{K}^+(G_{\omega_i}) \preceq \sup_{\omega \in A} \mathbb{K}^+(G_\omega)$. Then $\overline{\mathbb{K}}(A) \succeq \mathbb{K}^+(A)$ for any A in $\mathcal{G}$. ■

Corollary 35 *Consider a topological space Ω such that the set of open sets $\mathcal{A}$ satisfies the conditions of Theorem 34. Then any idempotent $\mathbb{D}$-measure $\mathbb{K}$ on $\mathcal{A}$, such that $[\mathbb{0}, \mathbb{K}(\Omega)]$ is a dually continuous lattice, has c^* as density on $\mathcal{A}$, and $\mathbb{K}^*$ has c^* as density on $\mathcal{P}(\Omega)$.*

Corollary 36 *Consider a set Ω and a Boolean semi-algebra $\mathcal{A}$ of Ω. Suppose that there exists a countable subset $\mathcal{B}$ of $\mathcal{A}$, such that one of the following equivalent conditions holds :*

- *for any $\omega \in A \in \mathcal{A}$, there exists $B \in \mathcal{B}$ such that $\omega \in B \subset A$ ($\mathcal{B}$ is a "basis of neighborhoods"),*
- *any set $A \in \mathcal{A}$ is an union of elements of $\mathcal{B}$: $A = \bigcup_{i \in I} B_i$.*

Then, $\mathcal{G}$ is stable by any union operation and thus defines a topology on Ω with a countable basis of neighborhoods. Moreover, $\mathcal{A}$ satisfies the assumptions of Theorem 34 and thus the conclusion of Corollary 35 holds.

Example 37 A separable (that is with a dense countable subset) metrizable space, and then a Polish space (complete separable and metrizable space) has a countable basis of neighborhoods. Thus, the conclusion of Corollary 35 holds. This includes any separable Banach space E endowed with the strong topology, thus almost all classical functional spaces : $L^p(\Omega)$ for $1 \leq p < +\infty$ and Ω an open set of $\mathbb{R}^n$, $W^{k,p}(\Omega)$.... ■

Example 38 Any Banach space E such that its dual space E' is separable has a countable basis of neighborhoods for the weak topology, and any dual space E' of a separable Banach space E has a countable basis of neighborhoods for the weak-* topology. Thus, the result holds for $L^p(\Omega)$ endowed with the weak topology if $1 < p < +\infty$, for $L^\infty(\Omega)$ endowed with the weak-$*$ topology.... ■

Example 39 If Ω is a topological space such that $\Omega = \bigcup_{n \in \mathbb{N}} C_n$ with C_n compact metrizable, then the set of open sets $\mathcal{A}$ satisfies the assumptions of Theorem 34 (any open set is the countable union of compact sets) and thus the conclusion of Corollary 35 holds. Example 38 may also be treated along these lines. ■

Example 40 We may found non-separable complete metric spaces in which the conclusion of Theorem 34 is false. Let us consider Ω a non-separable normed vector space (such as $L^\infty((0,1))$), and denote by $B(\omega, r)$ the open ball of center ω and radius r. For any idempotent semiring $\mathbb{D}$, we define on the set of open sets of Ω, the following idempotent $\mathbb{D}$-measure :

$$\begin{aligned}\mathbb{K}(A) &= \mathbb{0} \quad \text{if } \exists\ (\omega_n) \in \Omega^{\mathbb{N}} \text{ such that } A \subset \bigcup_{n\in\mathbb{N}} B(\omega_n, 1), \\ &= \mathbb{1} \quad \text{otherwise.}\end{aligned}$$

By the definition, we obtain $c^*(\omega) \preceq \mathbb{K}(B(\omega,1)) = \mathbb{0}$, thus $c^*(\omega) = \mathbb{0}$ for any $\omega \in \Omega$. Nevertheless, $\mathbb{K}(\Omega) = \mathbb{1}$ which implies that $\mathbb{K}$ has no density (otherwise c^* would have been a density). Indeed, if $\Omega \subset \bigcup_n B(\omega_n, 1)$, then, by linearity, we have for any positive integer m, $\Omega \subset \bigcup_n B(\frac{\omega_n}{m}, \frac{1}{m})$, which implies that the countable set $\{\frac{\omega_n}{m},\quad m \in \mathbb{N}^*,\ n \in \mathbb{N}\}$ is dense in Ω. This leads to a contradiction with the non-separability of Ω. ■

As the property imposed to the sets of $\mathcal{A}$ in Theorem 34 is satisfied by any countable union of compact sets if $\mathcal{A}$ is composed of open sets, we have the following corollary of Theorem 34.

Corollary 41 *Let Ω be a topological space such that Ω is a countable union of compact sets and $\mathcal{A}$ be the set of $\mathcal{F}_\sigma$ open sets, defined as the open sets which are countable unions of closed sets. Then, $\mathcal{A}$ is a semi-σ-algebra and $\mathcal{A}$ satisfies the conditions of Theorem 34. Then any idempotent $\mathbb{D}$-measure such that $[\mathbb{0}, \mathbb{K}(\Omega)]$ is a dually continuous lattice has c^* as density on $\mathcal{A}$.*

However, $\mathcal{A}$ is in general not stable by any infinite union operation, thus $\mathbb{K}^*$ may not have c^* as density, as shown in the following example. Let us note that $\mathcal{A}$ plays the same role as the Baire sets σ-algebra in classical probability theory : this is the semi-σ-algebra making continuous functions semi-measurable (see section 4).

Example 42 Let $\Omega = [0,1]^{\mathbb{R}} = \mathcal{F}(\mathbb{R}, [0,1])$ be endowed with the product (simple convergence) topology. Topological space Ω is compact but not metrizable. Thus even if in general an idempotent measure has a density on the $\mathcal{F}_\sigma$ open sets semi-σ-algebra $\mathcal{A}$, it may not have a density on the entire open sets σ-algebra $\mathcal{A}'$ and also its extension to all sets may not have a density. As an example, let us consider on $\mathcal{A}$ or $\mathcal{A}'$, the following idempotent $\mathbb{D}$-probability :

$$\begin{aligned} \mathbb{K}(A) &= \mathbb{0} \quad \text{if } \exists\,(x_n) \in \mathbb{R}^{\mathbb{N}} \text{ such that } A \subset \{f \in \Omega,\ \inf_n f(x_n) < 1/2\}, \\ &= \mathbb{1} \quad \text{otherwise.} \end{aligned}$$

Then by calculation we find for both $\mathcal{A}$ and $\mathcal{A}'$ semi-σ-algebras the same value of c^* on Ω :

$$\begin{aligned} c^*(f) &= \mathbb{0} \quad \text{if } \exists\, x_0 \in \mathbb{R} \text{ such that } f(x_0) < 1/2 \text{ or equivalently } \inf_{x\in\mathbb{R}} f(x) < 1/2, \\ &= \mathbb{1} \quad \text{otherwise.} \end{aligned}$$

Indeed $\{f \in \Omega,\ f(x) < 1/2\}$ is a $\mathcal{F}_\sigma$ open set for any $x \in \mathbb{R}$.

Consider now $U = \{f \in \Omega,\ \inf_{x\in\mathbb{R}} f(x) < 1/2\} = \cup_{x\in\mathbb{R}}\{f \in \Omega,\ f(x) < 1/2\}$. We have $c^*(f) = \mathbb{0}$ for any $f \in U$. U is an open set which is not a countable union of closed sets and also which is not included in a countable union of sets of the form $\{f \in \Omega, f(x) < 1/2\}$. Then, if we use the semi-σ-algebra of open sets $\mathcal{A}'$, $U \in \mathcal{A}'$ but $\mathbb{K}(U) = \mathbb{1} \neq \sup_{f\in U} c^*(f)$, then $\mathbb{K}$ has no density in $\mathcal{A}'$. If this time we use the semi-σ-algebra of $\mathcal{F}_\sigma$ open sets $\mathcal{A}$, $\mathbb{K}$ has necessarily c^* as density on $\mathcal{A}$, but $\mathbb{K}^*$ has no density on $\mathcal{P}(\Omega)$ or even on $\mathcal{A}'$: $\mathbb{K}^*(U) = \mathbb{1} \neq \sup_{f\in U} c^*(f)$. ■

4 Idempotent integration

In [12] Maslov gives a construction of idempotent integrals over semirings $\mathbb{D}$ that are metric spaces with particular properties of the distance. In this context, he proves the following theorem concerning the integration of semi-measurable functions.

Theorem 43 ([12]) *Consider $\mathbb{K}'$ an extension of a finite idempotent measure $\mathbb{K}$ to the least σ-algebra containing $\mathcal{A}$. The idempotent integrals with respect to $\mathbb{K}'$ and $\mathbb{K}^*$ of any (bounded) lower semi-measurable function taking its values in a separable subspace of $\mathbb{D}$ are equal.*

This result, which is a direct consequence of the construction of the integral, gives a justification to consider only idempotent measures with densities. We generalize here the construction of the idempotent integral to locally continuous lattices and then prove the Riesz representation theorem.

Theorem 43 was set when $\mathcal{A}$ is a Boolean algebra, but the Boolean semi-algebra structure is only needed. Moreover, $\mathbb{D}$ is supposed to be a metric space and to have the following property : for any $a \prec b \in \mathbb{D}$, there exists $c \in \mathbb{D}$ such that $a \prec c \prec b$. Then, a lower semi-measurable function is a function f from Ω to $\mathbb{D}$ such that the sets $\Omega(a) = \{\omega \in \Omega, a \prec f(\omega)\}$ are elements of $\mathcal{G}$ for any $a \in \mathbb{D}$.

In order to generalize this result to any locally continuous lattice $\mathbb{D}$, we have to replace $\prec$ by $\ll$ (way below) in the definition of $\Omega(a)$. In this case, the property "for all $a \ll b \in \mathbb{D}$, there exists $c \in \mathbb{D}$ such that $a \ll c \ll b$" is a consequence of the continuity of the locally complete lattice $\mathbb{D}$ [11]. Then, the separability can be replaced by the existence of a countable basis to the lattice $\mathbb{D}$. Note that, however the existence of a countable basis is equivalent, if $\mathbb{D}$ is a complete lattice, to the property that $\mathbb{D}$ endowed with the Lawson topology is a compact metric space, the metric does not need to be explicitly described.

In the construction by Maslov of idempotent integrals, $\mathbb{D}$ was not necessarily an idempotent semiring but only an ordered semiring with the law $\oplus$ compatible with the order $\preceq$, thus the classical measure theory and the idempotent measure theory may be treated together. Here, we treat idempotent measures with semi-algebras (see section 2) and semi-measurable functions, whereas classical probabilities or probabilities over symmetrizable ordered semirings (such as $(\mathbb{R}^+, +, \times)^n$) have to be treated with algebras and measurable functions. We thus restrict ourselves to the idempotent measure theory and generalize the construction of idempotent integrals to general locally continuous lattices, by using only properties of this structure. The generalization of the previous theorem will then be a consequence of this construction.

Remark 44 In a locally continuous lattice $\mathbb{D}$, the lower semi-continuity (l.s.c.) is equivalent to the continuity for the Scott topology generated by sets $\{x \in \mathbb{D},\ a \ll x\}$. Thus, semi-measurability is a natural generalization of semi-continuity in the same way as measurability is a generalization of continuity.

For a general lattice $\mathbb{D}$, the set of l.s.c. functions from Ω to $\mathbb{D}$ is a sup-semilattice. It is a lattice if $\mathbb{D}$ is locally continuous and it is a $\mathbb{D}$-semimodule (a module over a semiring) if the $\otimes$ operation is distributive with respect to infinite $\sup$. If now continuous functions are defined as functions which are both l.s.c. and upper semi-continuous (u.s.c.), the set of continuous functions from Ω to $\mathbb{D}$ is a $\mathbb{D}$-semimodule if $\overline{\mathbb{D}}$ is dually continuous and if $\otimes$ is distributive with respect to infinite $\sup$ and filtered $\inf$. It is a lattice if in addition $\mathbb{D}$ is locally continuous.

A generalization of the classical integration in ordered symmetrizable semirings would have consisted in defining measurable functions as functions f such that the sets $\{\omega \in \Omega,\ a \ll f(\omega)\}$ and $\{\omega \in \Omega,\ a \gg f(\omega)\}$ are measurable, for instance Borel sets. But this requires both the local continuity of $\mathbb{D}$ and the dual continuity of $\overline{\mathbb{D}}$. ■

Proposition 45 *Let $\mathcal{A}$ be a Boolean semi-algebra of subsets of Ω and $\mathcal{G}$ the semi-σ-algebra generated by $\mathcal{A}$. We denote by $\mathcal{L}(\Omega, \mathcal{A})$ the set of (finite) $\mathbb{D}$-linear combinations of characteristic functions $\mathbb{1}_A$ of sets $A \in \mathcal{A}$ and by $\mathcal{I}(\Omega, \mathcal{A})$ the set of functions from Ω to $\mathbb{D}$ which are nondecreasing limits of elements of $\mathcal{L}(\Omega, \mathcal{A})$.*

For a general semiring $\mathbb{D}$*,* $\mathcal{L}(\Omega,\mathcal{A})$ *is a* $\mathbb{D}$*-semi-algebra (an algebra over a semiring). If the* $\otimes$ *law is distributive with respect to upper bounded countable* $\sup$*, then* $\mathcal{I}(\Omega,\mathcal{A})=\mathcal{I}(\Omega,\mathcal{G})$ *is a* $\mathbb{D}$*-semi-algebra stable by countable upper bounded (by any function) supremum. If in addition* $\mathbb{D}$ *is a locally continuous lattice, then* $\mathcal{L}(\Omega,\mathcal{A})$ *and* $\mathcal{I}(\Omega,\mathcal{A})$ *are lattices.*

Proof. By construction, $\mathcal{L}(\Omega,\mathcal{A})$ is a semimodule thus a $\sup$-semilattice and it is a semi-algebra since $\mathcal{A}$ is stable by finite intersection. Again by construction, $\mathcal{I}(\Omega,\mathcal{A})$ is stable by countable upper bounded supremum and is equal to $\mathcal{I}(\Omega,\mathcal{G})$ since $\mathbb{1}_{\cup_{n\in\mathbb{N}}A_n}=\oplus_{n\in\mathbb{N}}\mathbb{1}_{A_n}$. The distributivity of $\otimes$ with respect to countable $\oplus$ implies that it is a semi-algebra.

In order to prove that $\mathcal{L}(\Omega,\mathcal{A})$ and $\mathcal{I}(\Omega,\mathcal{A})$ are lattices, we need a formula of the form : $(\oplus_i \lambda_i\otimes\mathbb{1}_{A_i})\wedge(\oplus_j \mu_j\otimes\mathbb{1}_{B_j})=\oplus_{i,j}(\lambda_i\wedge\mu_j)\otimes\mathbb{1}_{A_i\cap B_j}$. This holds if $\mathbb{D}$ is locally continuous and the sums are directed and upper bounded. But any sum $\oplus_i \lambda_i\otimes\mathbb{1}_{A_i}$ may be replaced by the sum of all terms $(\oplus_{i\in I}\lambda_i)\otimes\mathbb{1}_{\cap_{i\in I}A_i}$ for I finite whose values in any point form a directed set. ∎

Proposition 46 *Let us denote by* $\mathcal{S}(\Omega,\mathcal{G})$ *the set of semi-measurable functions with respect to* $\mathcal{G}$ *:* $\mathcal{S}(\Omega,\mathcal{G})=\{f:\Omega\mapsto\mathbb{D},\ \Omega_f(a)\in\mathcal{G}\ \forall a\in\mathbb{D}\}$ *where* $\Omega_f(a)=\{\omega\in\Omega,\ a\ll f(\omega)\}$*. For any semi-measurable function* f*, we also denote by* $\mathcal{G}(f)$ *the semi-*σ*-algebra generated by the sets* $\Omega_f(a)$ *for* $a\in\mathbb{D}$*.*

We have $\mathcal{I}(\Omega,\mathcal{G})\subset\mathcal{S}(\Omega,\mathcal{G})$ *and any function* f *of* $\mathcal{I}(\Omega,\mathcal{G})$ *is such that* $\mathcal{G}(f)$ *has a countable basis in* $\mathcal{G}$*, that is a countable subset* $\mathcal{B}$ *of* $\mathcal{G}$ *(not necessarily included in* $\mathcal{G}(f)$*) stable by finite intersection, such that the elements of* $\mathcal{G}(f)$ *are unions of elements of* $\mathcal{B}$*.*

Let us suppose now that $\mathbb{D}$ *is a locally continuous lattice and that* $\otimes$ *is distributive with respect to upper bounded infinite* $\sup$*. Then,* $\mathcal{I}(\Omega,\mathcal{G})$ *is exactly the set of functions* $f\in\mathcal{S}(\Omega,\mathcal{G})$ *such that* $\mathcal{G}(f)$ *has a countable basis. If* $\mathbb{D}$ *has a countable basis or* $\mathcal{A}$ *has a countable basis, then*

$$\mathcal{I}(\Omega,\mathcal{G})=\mathcal{S}(\Omega,\mathcal{G}).$$

Proof. Consider a function $f=\oplus_i \lambda_i\otimes\mathbb{1}_{A_i}\in\mathcal{G}$ where the sum is countable and directed (as in previous proof) and the sets $A_i\in\mathcal{A}$. Then, the set of A_i is stable by finite intersection and the set of λ_i by finite addition. For any $a\in\mathbb{D}$, $\Omega_f(a)=\cup_{i,\,a\ll\lambda_i}A_i\in\mathcal{G}$, thus $f\in\mathcal{S}(\Omega,\mathcal{G})$. In addition, $\Omega_f(a)\cap\Omega_f(b)=\Omega_f(a\oplus b)$, therefore $\mathcal{G}(f)$ is the set of countable unions of sets $\Omega_f(a)$ and thus is included in the set of unions of sets A_i which forms a countable basis of $\mathcal{G}(f)$.

Now, suppose that $\mathbb{D}$ is locally continuous and consider $f\in\mathcal{S}(\Omega,\mathcal{G})$ such that $\mathcal{G}(f)$ has a countable basis $\mathcal{B}$ in $\mathcal{G}$. Then as $f(\omega)=\sup\{a\in\mathbb{D},a\ll f(\omega)\}$ for any $\omega\in\Omega$, we obtain

$$f \quad = \quad \bigoplus_{a\in\mathbb{D}} a\otimes\mathbb{1}_{\Omega_f(a)}$$

$$\begin{aligned} &= \bigoplus_{a\in\mathbb{D}} a \otimes \Big(\bigoplus_{B\in\mathcal{B},\, B\subset\Omega_f(a)} \mathbb{1}_B \Big) \\ &= \bigoplus_{B\in\mathcal{B}} \lambda(B) \otimes \mathbb{1}_B \end{aligned}$$

with $\lambda(B) = \sup\{a \in \mathbb{D},\ \ B \subset \Omega_f(a)\}$. Then, as $\mathcal{B}$ is countable, $f \in \mathcal{I}(\Omega, \mathcal{A})$. In the previous equalities, we have used the distributivity of the $\otimes$ law with respect to infinite $\oplus$. However, if $\mathbb{D}$ has a countable basis, the countable distributivity is only needed.

Now if $\mathcal{A}$ has a countable basis, for any $f \in \mathcal{S}(\Omega, \mathcal{G})$, $\mathcal{G}(f)$ has a countable basis, thus $\mathcal{I}(\Omega, \mathcal{G}) = \mathcal{S}(\Omega, \mathcal{G})$.

If this time $\mathbb{D}$ has a countable basis, then the Scott topology has a countable basis and since $\mathcal{G}(f)$ is the inverse image of the Scott topology by the function f, $\mathcal{G}(f)$ has also a countable basis. ■

Let us note that in general $\mathcal{S}(\Omega, \mathcal{G})$ is not a semi-algebra (it is not stable by addition) except if $\mathbb{D}$ has a countable basis or if $\mathcal{A}$ is stable by any union operation. But this last property implies that $\mathcal{A}$ is a topology and $\mathcal{S}(\Omega, \mathcal{G})$ is in fact the set of l.s.c. functions.

Proposition 47 *Let us consider a semiring $\mathbb{D}$ such that $\otimes$ is distributive with respect to upper bounded infinite $\sup$ and let $\mathbb{K}$ be an idempotent $\mathbb{D}$-probability on $(\Omega, \mathcal{A})$ with extension $\mathbb{K}^+$ to $\mathcal{G}$.*

If $\mathbb{D}$ is a locally continuous lattice or $\overline{\mathbb{D}}$ is a dually continuous lattice, then there exists a unique $\mathbb{D}$-linear form $\mathbb{V}$ on $\mathcal{I}(\Omega, \mathcal{A})$, continuous on converging nondecreasing sequences (i.e. such that $\mathbb{V}(f_n) \nearrow_{n\to+\infty} \mathbb{V}(f)$ if $f_n \nearrow_{n\to+\infty} f$) and extending $\mathbb{K}$, in the sense that that $\mathbb{V}(\mathbb{1}_A) = \mathbb{K}(A)$ for any $A \in \mathcal{A}$.

In the two following cases, we have a general expression for $\mathbb{V}$:

- *If $\mathbb{D}$ is locally continuous, then*

$$\mathbb{V}(f) = \bigoplus_{a\in\mathbb{D}} a \otimes \mathbb{K}^+(\Omega_f(a)).$$

- *If $[\mathbb{0}, \mathbb{1}]$ is dually continuous and $\mathcal{A}$ has a countable basis, or more generally if $\mathbb{K}^+$ has a density c^*, then*

$$\mathbb{V}(f) = \bigoplus_{\omega\in\Omega} f(\omega) \otimes c^*(\omega).$$

Proof. Consider a $\mathbb{D}$-linear form $\mathbb{V}$ on $\mathcal{I}(\Omega, \mathcal{A})$, continuous on converging nondecreasing sequences and such that $\mathbb{V}(\mathbb{1}_A) = \mathbb{K}(A)$ for any A in $\mathcal{A}$. The continuity implies that $\mathbb{V}(\mathbb{1}_A) =$

$\mathbb{K}^+(A)$ for any $A \in \mathcal{G}$. Now if $f \in \mathcal{I}(\Omega,\mathcal{A})$, then $f = \oplus_i \lambda_i \otimes \mathbb{1}_{A_i}$ where the sum is countable and then $\mathbb{V}(f) = \oplus_i \lambda_i \mathbb{K}(A_i)$. If this expression only depends on f, that is if

$$\bigoplus_{i\in I} \lambda_i \otimes \mathbb{1}_{A_i} = \bigoplus_{j\in J} \mu_j \otimes \mathbb{1}_{B_j} \Rightarrow \bigoplus_{i\in I} \lambda_i \otimes \mathbb{K}(A_i) = \bigoplus_{j\in J} \mu_j \otimes \mathbb{K}(B_j) \tag{7}$$

for any countable sets I and J and for A_i and B_j in $\mathcal{A}$, then $\mathbb{V}$ may be defined in that way and if $\otimes$ is distributive with respect to upper bounded countable $\sup$, $\mathbb{V}$ satisfies the properties of the proposition. Before proving (7) for particular cases, let us note that it is equivalent to

$$\mu \otimes \mathbb{1}_B \preceq \bigoplus_{i\in I} \lambda_i \otimes \mathbb{1}_{A_i} \Rightarrow \mu \otimes \mathbb{K}(B) \preceq \bigoplus_{i\in I} \lambda_i \otimes \mathbb{K}(A_i) \tag{8}$$

for any countable set I. Moreover, we only need to prove (8) for "directed" sums (indeed, adding terms of the form $(\oplus_{i\in J} \lambda_j) \otimes \mathbb{1}_{\cap_{j\in J} A_j}$ with J finite to the first expression does not change the second expression).

Let us suppose $\otimes$ distributive with respect to upper bounded infinite $\sup$ and first prove that (8) holds in a locally continuous lattice $\mathbb{D}$. Note that in this case, for any $f \in \mathcal{I}(\Omega,\mathcal{G})$, $\mathcal{G}(f)$ has a countable basis denoted $\mathcal{B}$ and following the previous proof, we have

$$f = \bigoplus_{B\in\mathcal{B}} \lambda(B) \otimes \mathbb{1}_B$$

with $\lambda(B) = \sup\{a \in \mathbb{D},\ B \subset \Omega_f(a)\}$. Thus

$$\begin{aligned}
\mathbb{V}(f) &= \bigoplus_{B\in\mathcal{B}} \lambda(B) \otimes \mathbb{K}(B) \\
&= \bigoplus_{a\in\mathbb{D}} a \otimes \Big(\bigoplus_{B\in\mathcal{B},\, B\subset\Omega_f(a)} \mathbb{K}(B) \Big) \\
&= \bigoplus_{a\in\mathbb{D}} a \otimes \mathbb{K}^+(\Omega_f(a)).
\end{aligned}$$

Suppose now that $\mu \otimes \mathbb{1}_B \preceq \oplus_{i\in I} \lambda_i \otimes \mathbb{1}_{A_i}$, that is $\mu \preceq \oplus_{i\in I,\, \omega\in A_i} \lambda_i$ for any $\omega \in B$, with I countable and the values of the sum directed. Then, for any $a \ll \mu$, $B \subset \cup_{i\in I,\, a\preceq\lambda_i} A_i$, and

$$\begin{aligned}
a \otimes \mathbb{K}(B) &\preceq a \otimes \Big(\bigoplus_{i\in I,\, a\preceq\lambda_i} \mathbb{K}(A_i) \Big) \\
&\preceq \bigoplus_{i\in I,\, a\preceq\lambda_i} \lambda_i \otimes \mathbb{K}(A_i) \\
&\preceq \bigoplus_{i\in I} \lambda_i \otimes \mathbb{K}(A_i).
\end{aligned}$$

Taking the supremum over $a \ll \mu$ and using the infinite distributivity of $\otimes$, we obtain the (8). Note that in previous proof, infinite distributivity of $\otimes$ may be replaced by countable distributivity if $\mathbb{D}$ has a countable basis.

Let us prove now that (8) holds if $[\mathbb{0}, \mathbb{1}]$ is a dually continuous lattice. Let us suppose that $\mu \otimes \mathbb{1}_B \preceq \oplus_{i\in I} \lambda_i \otimes \mathbb{1}_{A_i}$, with I countable and the values of the sum directed. Denote by $\mathcal{B}$ the set composed of sets A_i, B and $A_i \cap B$ and by $\mathcal{G}'$ the semi-σ-algebra generated by $\mathcal{B}$. Then, $\mathcal{B}$ is countable basis of $\mathcal{G}'$ and by the results of previous section, $\mathbb{K}$ has a density c^* on $\mathcal{G}'$. Then using the infinite distributivity of $\otimes$,

$$\begin{aligned}
\mu \otimes \mathbb{K}(B) &= \bigoplus_{\omega \in B} \mu \otimes c^*(\omega) \\
&\preceq \bigoplus_{\omega \in B} \Big(\bigoplus_{i \in I,\, \omega \in A_i} \lambda_i \Big) \otimes c^*(\omega) \\
&\preceq \bigoplus_{i \in I} \lambda_i \otimes \Big(\bigoplus_{\omega \in A_i \cap B} c^*(\omega) \Big) \\
&\preceq \bigoplus_{i \in I} \lambda_i \otimes \mathbb{K}(A_i)
\end{aligned}$$

which proves (8). We may prove along the same lines

$$\mathbb{V}(f) = \bigoplus_{\omega \in \Omega} f(\omega) \otimes c^*(\omega).$$

But as c^* depends on the sets A_i, thus on f this does not lead to a general expression for $\mathbb{V}(f)$ except if c^* is a density of $\mathbb{K}^+$ in the entire algebra $\mathcal{G}$. ■

A semi-measurable function f is said integrable if $\mathbb{V}(f) \in \mathbb{D}$. The linear form $\mathbb{V}$ coincides with the integral defined by Maslov. It will then be denoted by

$$\mathbb{V}(f) = \oint_{\Omega} f(\omega) \otimes \mathbb{K}(d\omega).$$

It can be defined for any Boolean semi-algebra or semi-σ-algebra $\mathcal{A}$. In particular, for any extension $\mathbb{K}'$ of $\mathbb{K}$ to a larger semi-σ-algebra $\mathcal{A}'$, we may define an integral $\mathbb{V}'$ on the set of semi-measurable functions with respect to $\mathcal{A}'$. By uniqueness, $\mathbb{V}'$ coincides with $\mathbb{V}$ on the set of semi-measurable functions with respect to $\mathcal{A}$. This is the result of Theorem 43. If $\mathcal{A}'$ is the σ-algebra generated by $\mathcal{A}$ and $\mathbb{D} = \mathbb{R}_{\max}$, semi-measurable functions coincide with classical measurable functions and there are at least as many integrals $\mathbb{V}'$ as extensions $\mathbb{K}'$ of $\mathbb{K}$. If $[\mathbb{0}, \mathbb{1}]$ is a dually continuous lattice, we can also consider the maximal continuous linear form $\mathbb{V}^*$ on the set of all functions, such that $\mathbb{V}^*(\mathbb{1}_A) = \mathbb{K}(A)$ for any $A \in \mathcal{A}$. Then, $\mathbb{V}^*(\mathbb{1}_A) \preceq \mathbb{K}^*(A)$ for any set A, where $\mathbb{K}^*$ is the maximal extension of $\mathbb{K}$ to $\mathcal{P}(\Omega)$. As $f = \oplus_{a\in\mathbb{D}} a \otimes \mathbb{1}_{\Omega_{f}(a)}$ for any function f, $\mathbb{V}^*$ is lower than the integral associated with $\mathbb{K}^*$ and then coincides with it.

If $\mathbb{K}$ has a density c, then for any $f \in \mathcal{I}(\Omega, \mathcal{A})$:

$$\mathbb{V}(f) = \bigoplus_{\omega \in \Omega} f(\omega) \otimes c(\omega). \tag{9}$$

In this case, we will say that $\mathbb{V}$ has a density. Then, Theorem 43 or Propositions 47-46 together with the results of the previous section imply that for any l.s.c. function f, the integral of f with respect to a $\mathbb{R}_{\max}$-probability $\mathbb{K}$ defined on the open sets or Borel sets has the form :

$$\mathbb{V}(f) = \sup_{\omega\in\Omega} f(\omega) + c^*(\omega) \tag{10}$$

with c^* an u.s.c. function ($\mathbb{R}_{\max}$ is locally continuous with countable basis and $[-\infty, 0]$ is dually continuous).

In [9, 10] Kolokoltsov and Maslov prove that any continuous (for the uniform convergence topology) linear form from the set $\mathcal{C}_K(\Omega)$ of continuous functions with compact support from Ω to $\mathbb{D} = \mathbb{R}_{\max}$ has the form (10). As from any (non necessarily bounded) measure on $(\Omega, \mathcal{A})$, where $\mathcal{A}$ is the set of open sets, we can construct an integral which is a continuous linear form on $\mathcal{C}_K(\Omega)$, the existence of a density to this measure may have been deduced from (10). Conversely, (10) can be deduced from the existence of the density of any idempotent measure by using the Riesz representation theorem [12]. However, even if some generalizations of (10) may be done (see Kolokoltsov [8]), many restrictions on the semiring $\mathbb{D}$ and the topological space Ω are necessary in order to get (10) or the Riesz representation theorem, restrictions which are not needed to prove the existence of a density.

We give now a "probabilistic" version of the Riesz representation theorem. This approach allows to consider general functional spaces, when the "integration" point of view adopted in [9, 10, 8] imposes to the topological space Ω to be locally compact. We thus consider idempotent probabilities (or bounded measures) and linear forms on the set $\mathcal{C}(\Omega, \mathbb{D})$ of continuous functions (or upper bounded continuous functions). Note that in general, $\mathcal{C}(\Omega, \mathbb{D})$ is not a $\mathbb{D}$-semimodule except if $\overline{\mathbb{D}}$ dually continuous and $\otimes$ distributes with respect to infinite $\sup$ and filtered $\inf$. In the following result, Ω needs only to be normal with respect to $\mathbb{D}$ [12], where we adopt the following definition of normality.

Definition 48 *We say that the topological space is $\mathbb{D}$-normal, if for any disjoint closed sets F and G of Ω and for any $a \in \mathbb{D}$, there exists a continuous function f from Ω to $[\mathbb{0}, a] \subset \mathbb{D}$ such that f is equal to $\mathbb{0}$ on F and a on G.*

A generalization of the classical definition of normality would have been to satisfy the previous condition with $a = \mathbb{1}$ only. However, the proof of the Riesz theorem requires the previous condition in its general form. On the other hand, we may deduce the general normality condition from the restricted normality, when the product $\otimes$ is continuous (for the good topology) that is when $\otimes$ is distributive with respect to infinite $\sup$ and filtered $\inf$. If $\mathbb{D}$ is a connected by arcs topological space, in particular for $\mathbb{D} = \mathbb{R}_{\max}$, $\mathbb{R}_{\min}$ or $(\overline{\mathbb{R}}, \max, \min)$, the $\mathbb{D}$-normality follows from the classical normality. For instance, Ω may be any metric space and thus any functional space.

Theorem 49 ("Riesz representation theorem") *Suppose that $\mathbb{D}$ is a non trivial ($\neq \{\mathbb{0}\}$) locally continuous lattice with countable basis, such that $\otimes$ is distributive with respect to upper bounded countable $\sup$ and Ω is a $\mathbb{D}$-normal topological space.*

We suppose that $\mathcal{C}(\Omega,\mathbb{D})$ is a $\mathbb{D}$-semimodule and denote by $\mathcal{A}$ the minimal semi-σ-algebra of subsets of Ω leading continuous functions to be semi-measurable with respect to $\mathcal{A}$. Then, $\mathcal{A}$ is the set of $\mathcal{F}_\sigma$ open sets (where a $\mathcal{F}_\sigma$ set is defined as a countable union of closed sets) and the set $\mathcal{S}(\Omega,\mathbb{D})$ of semi-measurable functions with respect to $\mathcal{A}$ is equal to the set of functions from Ω to $\mathbb{D}$ which are nondecreasing limits of continuous functions.

Let $\mathbb{V}$ be a linear form on $\mathcal{C}(\Omega,\mathbb{D})$, continuous on converging nondecreasing sequences and such that $\mathbb{V}(\mathbb{1}) = \mathbb{1}$. Then, $\mathbb{V}$ may be extended in a continuous linear form on $\mathcal{S}(\Omega,\mathbb{D})$. $\mathbb{V}$ is exactly the idempotent integral

$$\mathbb{V}(f) = \oint_\Omega f(\omega) \otimes \mathbb{K}(d\omega) \tag{11}$$

corresponding to the idempotent $\mathbb{D}$-probability $\mathbb{K}$ defined on $\mathcal{A}$ by

$$\mathbb{K}(A) = \mathbb{V}(\mathbb{1}_A) = \sup_{f \in \mathcal{C}(\Omega,\mathbb{D}),\, f \preceq \mathbb{1}_A} \mathbb{V}(f)$$

An idempotent probability $\mathbb{K}$ such that (11) holds is unique. Thus, (11) sets up a bijective correspondence between continuous linear forms on $\mathcal{C}(\Omega,\mathbb{D})$ such that $\mathbb{V}(\mathbb{1}) = \mathbb{1}$ and idempotent $\mathbb{D}$-probabilities on $(\Omega,\mathcal{A})$.

Moreover, if $\otimes$ is distributive with respect to infinite $\oplus$ then $\mathbb{V}$ has density in the sense of (9) if and only if $\mathbb{K}$ has a density.

Proof. Let us prove that $\mathcal{A}$ is the set of $\mathcal{F}_\sigma$ open sets. By definition, $\mathcal{A}$ is the semi-σ-algebra generated by sets $\Omega_f(a)$, with f continuous. If f is continuous, it is l.s.c. thus $\Omega_f(a)$ is an open set. If I is a countable basis of $\mathbb{D}$, then for any $x \ll y$ in $\mathbb{D}$, there exists $z \in I$ such that $x \ll z \preceq y$ and the converse is also true. Then $\Omega_f(a) = \cup_{b \in I,\, a \ll b} C_f(b)$, where $C_f(b) = \{\omega \in \Omega,\ b \preceq f(\omega)\}$. Since $C_f(b)$ is a closed set for any $b \in \mathbb{D}$ and I is countable, $\Omega_f(a)$ is a $\mathcal{F}_\sigma$ open set. The set of $\mathcal{F}_\sigma$ open sets is a semi-σ-algebra, then $\mathcal{A}$ is included in it.

Suppose now that U is a $\mathcal{F}_\sigma$ open set, i.e. $U = \cup_n F_n$ with F_n closed. From the $\mathbb{D}$-normality of Ω, there exist continuous functions f_n from Ω into $[\mathbb{0},\mathbb{1}]$, such that $f_n = \mathbb{1}$ on F_n and $\mathbb{0}$ on U^c. Since $\mathbb{0} \neq \mathbb{1}$ ($\mathbb{D} \neq \{\mathbb{0}\}$), there exists $a \neq \mathbb{0}$ such that $a \ll \mathbb{1}$, then $\Omega_{f_n}(a) \supset F_n$ and $\Omega_{f_n}(a) \subset U$ for any n. Thus, $U = \cup_n \Omega_{f_n}(a) \in \mathcal{A}$.

By construction, the set $\mathcal{S}(\Omega,\mathbb{D})$ of semi-measurable functions with respect to $\mathcal{A}$ contains continuous functions and then nondecreasing limits of continuous functions. Conversely, if $f \in \mathcal{S}(\Omega,\mathbb{D})$ and $a \in \mathbb{D}$, then $\Omega_f(a)$ is an open set such that there exist closed sets $F_{n,a}$ with $\Omega_f(a) = \cup_n F_{n,a}$. Then, there exists continuous functions $f_{n,a}$ with values in $[\mathbb{0}, a]$, such that $f_{n,a} = a$ on $F_{n,a}$ and $\mathbb{0}$ on $\Omega_f(a)^c$. This implies $\oplus_n f_{n,a} = a \otimes \mathbb{1}_{\Omega_f(a)}$. Now, if I is a countable basis, $f = \oplus_{a\in I} a \otimes \mathbb{1}_{\Omega_f(a)} = \oplus_{a\in I,\, n\in\mathbb{N}} f_{n,a}$ and then is the nondecreasing limit of continuous functions.

Suppose that $\mathbb{V}$ is a linear form on $\mathcal{C}(\Omega,\mathbb{D})$ continuous on converging nondecreasing sequences, and consider $f \in \mathcal{S}(\Omega,\mathbb{D})$. If $f_n \nearrow_{n\to+\infty} f$ with $f_n \in \mathcal{C}(\Omega,\mathbb{D})$, then $\mathbb{V}(f) = \lim_{n\to+\infty} \mathbb{V}(f_n)$. As $\mathcal{C}(\Omega,\mathbb{D})$ is stable by finite $\oplus$ (by assumption) and $\wedge$ (by the continuity of $\mathbb{D}$) and $\mathbb{V}$ is continuous on nondecreasing sequences, this formula is independent of the sequence (f_n) and thus define a $\mathbb{D}$-linear form on $\mathcal{S}(\Omega,\mathbb{D})$ which is continuous on converging nondecreasing sequences. As a consequence, $\mathbb{V}(f) = \sup_{g\preceq f,\, g\in\mathcal{C}(\Omega,\mathbb{D})} \mathbb{V}(g)$.

Now, if $A \in \mathcal{A}$, then $\mathbb{1}_A \in \mathcal{S}(\Omega,\mathbb{D})$, for $\Omega_{\mathbb{1}_A}(a) = \{\omega,\ a \ll \mathbb{1}_A(\omega)\} = \Omega,\ A$ or $\emptyset$. Then, $\mathbb{K}(A) = \mathbb{V}(\mathbb{1}_A)$ defines an idempotent $\mathbb{D}$-probability on $\mathcal{A}$. By uniqueness of the integral (see Proposition 47), $\mathbb{V}$ is exactly the integral associated with $\mathbb{K}$. Moreover, (11) implies $\mathbb{K}(A) = \mathbb{V}(\mathbb{1}_A)$ which implies the uniqueness of $\mathbb{K}$. By Proposition 47, $\mathbb{V}$ has a density if $\mathbb{K}$ has a density and $\otimes$ is distributive with respect to infinite $\oplus$. Conversely, if $\mathbb{V}$ has a density denoted by c, then clearly $\mathbb{K}$ has c as density : $\mathbb{K}(A) = \mathbb{V}(\mathbb{1}_A) = \oplus_{\omega\in\Omega} \mathbb{1}_A(\omega) \otimes c(\omega) = \oplus_{\omega\in A} c(\omega)$. ■

If Ω is a metric space, every open set is an $\mathcal{F}_\sigma$ set. Then in a Polish space, every continuous (on converging nondecreasing sequences) linear form on $\mathcal{C}(\Omega,\mathbb{D})$ admits the representation (9) ((10) in $\mathbb{R}_{\max}$). In the other hand, by Proposition 47, the counter example of section 3 leads to a counter example for linear forms. The following continuous linear form $\mathbb{V}$ on $\mathcal{C}(\Omega = L^\infty(0,1), \mathbb{R}_{\max})$ does not have a representation of the form (10) :

$$\begin{aligned}\mathbb{V}(f) &= \sup\{a \in \mathbb{R},\ \nexists(\omega_n) \in \Omega^{\mathbb{N}},\ \{\omega,\ a < f(\omega)\} \subset \cup_n B(\omega_n, 1)\} \\ &= \inf_{(\omega_n)\in\Omega^{\mathbb{N}}} \ \sup_{\omega,\ \|\omega-\omega_n\|_\infty > 1\ \forall n} f(\omega).\end{aligned}$$

Moreover, $\mathbb{V}$ is continuous for the uniform convergence topology on $\mathcal{C}(\Omega,\mathbb{R}_{\max})$, defined by the exponential distance d, i.e. $d(x,y) = |e^x - e^y|$; $d(\mathbb{V}(f),\mathbb{V}(g)) \leq \sup_{\omega\in\Omega} d(f(\omega), g(\omega))$.

5 Application to Large Deviations

The purpose of Large Deviations is to find, for a given family of probabilities $(P_\varepsilon)_{\varepsilon>0}$ (resp. $(P_n)_{n\in\mathbb{N}}$) on $(\Omega,\mathcal{A})$, the asymptotic rate of convergence of P_ε when ε tends to 0 (resp. n tends to infinity). In practice, the limit is a Dirac measure at some point. For instance, if X_n are independent random variables with same law, the law P_n of $\frac{X_1+\cdots+X_n}{n}$ tends to the Dirac measure at the mean point $\mathbb{E}(X_1)$. For almost all sets A, $P_\varepsilon(A)$ tends to 0 exponentially fast and for particular sets A, $-\varepsilon \log P_\varepsilon(A)$ has a limit which can be expressed as the minimum of a function I over the set A. In order to formalize this theory, Varadhan has introduced [15] the following concept :

Definition 50 ([15]) *Consider a probability space $(\Omega, \mathcal{A})$, where Ω is a complete separable metric space and $\mathcal{A}$ is the set of Borel sets of Ω, and a family $(P_\varepsilon)_{\varepsilon>0}$ of probabilities on $(\Omega, \mathcal{A})$. Then (P_ε) obeys the* large deviation principle *if there exists a* rate function $I : \Omega \to [0, +\infty]$ *such that*

1. *I is lower semi-continuous (l.s.c.) and $\Omega_a = \{\omega \in \Omega, I(\omega) \le a\}$ is a compact set for any $a < +\infty$,*

2. *for each closed set $C \subset \Omega$*

$$\limsup_{\varepsilon\to 0} \varepsilon \log P_\varepsilon(C) \le -\inf_{\omega\in C} I(\omega),$$

3. *for each open set $U \subset \Omega$*

$$\liminf_{\varepsilon\to 0} \varepsilon \log P_\varepsilon(U) \ge -\inf_{\omega\in U} I(\omega).$$

Since the functions $\mathbb{K}(A) = -\inf_{\omega\in A} I(\omega)$, with I as in point 1 are particular $\mathbb{R}_{\max}$-idempotent probabilities on $(\Omega, \mathcal{A})$, we will consider the following weak form of the previous definition.

Definition 51 *Consider for any $A \in \mathcal{A}$ the quantities*

$$\mathbb{K}^\vee(A) \stackrel{def}{=} \limsup_{\varepsilon\to 0} \varepsilon \log P_\varepsilon(A),$$
$$\mathbb{K}^\wedge(A) \stackrel{def}{=} \liminf_{\varepsilon\to 0} \varepsilon \log P_\varepsilon(A).$$

We say that (P_ε) obeys the weak large deviation principle *if there exists a $\mathbb{R}_{\max}$-idempotent probability $\mathbb{K}$ on $(\Omega, \mathcal{A})$ such that*

1. *there exists a sequence (Ω_n) of compact sets such that $\mathbb{K}(\Omega_n^c) \to_{n\to+\infty} \mathbb{0} = -\infty$ (C^c denotes the complementary set of C),*

2. *for each closed set $C \subset \Omega$, $\mathbb{K}^\vee(C) \le \mathbb{K}(C)$,*

3. *for each open set $U \subset \Omega$, $\mathbb{K}^\wedge(U) \ge \mathbb{K}(U)$.*

Then (P_ε) obeys the large deviation principle iff (P_ε) obeys the weak large deviation principle with a measure $\mathbb{K}$ having an upper semi-continuous (u.s.c.) density $c = -I$ (condition 1 of both definitions are equivalent in this case). But if (P_ε) obeys the weak large deviation principle with a measure $\mathbb{K}$, the maximal extension $\mathbb{K}^*$ of the restriction of $\mathbb{K}$ to open sets is also admissible ($\mathbb{K}^*(U) = \mathbb{K}(U) \leq \mathbb{K}^\wedge(U)$, $\mathbb{K}^*(C) \geq \mathbb{K}(C) \geq \mathbb{K}^\vee(C)$). Now, the theorems of section 3 give sufficient conditions on Ω for $\mathbb{K}^*$ having a density and this density is necessarily an u.s.c. function. Conversely, if $\mathbb{K}$ has an u.s.c. density then $\mathbb{K} = \mathbb{K}^*$. In particular if Ω is a complete separable metric space, we have shown in Example 37 that $\mathbb{K}^*$ has necessarily a density and thus Definitions 50 and 51 are equivalent. We suppose now that Ω is a general topological space and search for conditions on (P_ε) in order to satisfy the weak large deviation principle. For this we construct a measure $\mathbb{K}$ which is a good candidate for the large deviation principle.

Remark 52 $\mathbb{K}^\vee$ and $\mathbb{K}^\wedge$ are nondecreasing functions on $\mathcal{A}$ and $\mathbb{K}(\emptyset) = \mathbb{0}$, $\mathbb{K}(\Omega) = \mathbb{1}$ for $\mathbb{K} = \mathbb{K}^\vee$ or $\mathbb{K}^\wedge$. Moreover $\mathbb{K}^\vee(A \cup B) = \mathbb{K}^\vee(A) \oplus \mathbb{K}^\vee(B)$ for any A and B in $\mathcal{A}$ but this is false for $\mathbb{K}^\wedge$. However, this last property is not useful, as one can construct the maximal idempotent-measure lower than a nondecreasing function $\mathbb{K}^\wedge$, but not the minimal measure greater than $\mathbb{K}^\vee$. ■

Proposition 53 *Denote by $\mathcal{U}$ the set of open sets of Ω. The maximal $\mathbb{R}_{\max}$-idempotent measure on $(\Omega, \mathcal{U})$ lower than $\mathbb{K}^\wedge$ is the following measure :*

$$\mathbb{K}(U) = \inf_{(U_n \in \mathcal{U})_n,\, \cup_n U_n = U} \sup_n \mathbb{K}^\wedge(U_n) \quad \forall U \in \mathcal{U}.$$

Proof. By additivity and continuity, any $\mathbb{R}_{\max}$-idempotent measure lower than $\mathbb{K}^\wedge$ is lower than $\mathbb{K}$. Let us prove that $\mathbb{K}$ is an idempotent measure. Firstly, $\mathbb{K}(\emptyset) \leq \mathbb{K}^\wedge(\emptyset) = \mathbb{0}$, then $\mathbb{K}(\emptyset) = \mathbb{0}$ and as $\mathbb{K}^\wedge$ is nondecreasing, $\mathbb{K}$ is also nondecreasing.

Consider a sequence (possibly finite) of open sets (U_n) and $U = \cup_n U_n$. Since $\mathbb{K}$ is monotone, $\mathbb{K}(U) \geq \sup_n \mathbb{K}(U_n)$. On the other hand

$$\begin{aligned}
\sup_n \mathbb{K}(U_n) &= \sup_n \inf_{\substack{(U_{n,m} \in \mathcal{U})_m\,, \\ \cup_m U_{n,m} = U_n}} \sup_m \mathbb{K}^\wedge(U_{n,m}) \\
&= \inf_{\substack{(U_{n,m} \in \mathcal{U})_{n,m}\,, \\ \cup_m U_{n,m} = U_n\ \forall n}} \sup_{n,m} \mathbb{K}^\wedge(U_{n,m}) \\
&\geq \inf_{\substack{(U_{n,m} \in \mathcal{U})_{n,m}\,, \\ \cup_{m,n} U_{n,m} = U}} \sup_{n,m} \mathbb{K}^\wedge(U_{n,m})
\end{aligned}$$

$$\geq \quad \mathbb{K}(U).$$

Then $\mathbb{K}(\cup_n U_n) = \sup_n \mathbb{K}(U_n)$ which implies both additivity and continuity properties. ■

Consider the maximal extension $\mathbb{K}^*$ of the measure $\mathbb{K}$ of Proposition 53 to the algebra of all subsets of Ω. We have :

$$\mathbb{K}^*(A) = \inf_{(U_n \in \mathcal{U})_n,\, \cup_n U_n \supset A} \sup_n \mathbb{K}^\wedge(U_n).$$

It is the maximal measure on $\mathcal{A}$ (or $\mathcal{P}(\Omega)$) satisfying condition 3 and it is a good candidate to satisfy the weak large deviation principle. Indeed, suppose that $\mathbb{K}'$ satisfies the weak large deviation principle, then $\mathbb{K}' \leq \mathbb{K}$ on open sets (by condition 3 and Proposition 53), thus $\mathbb{K}' \leq \mathbb{K}^*$ on $\mathcal{A}$ and $\mathbb{K}^*$ satisfies condition 2.

Remark 54 In a metric space Ω, the restriction to open sets of a measure $\mathbb{K}$ satisfying conditions 2 and 3 of the weak large deviation principle, is unique and thus equal to $\mathbb{K}^*$. Indeed, any open set U is the union of open sets U_n such that $\overline{U}_n \subset U$ ($U_n = \{\omega \in U, d(\omega, U^c) > 1/n\}$). Then, for any measure $\mathbb{K}$ and $\mathbb{K}'$ satisfying the large deviation principle, we have $\mathbb{K}(U) \geq \mathbb{K}(\overline{U}_n) \geq \mathbb{K}^\vee(\overline{U}_n) \geq \mathbb{K}^\wedge(U_n) \geq \mathbb{K}'(U_n)$ thus $\mathbb{K}(U) \geq \sup_n \mathbb{K}'(U_n) = \mathbb{K}'(U)$. By symmetry, we obtain the uniqueness.

Then, in a metric space, a measure $\mathbb{K}$ satisfying the weak large deviation principle is necessarily equal to $\mathbb{K}^*$ on open sets and $\mathbb{K}^*$ satisfies the weak large deviation principle. As a consequence, the rate function I of Definition 50 is unique and equal to the opposite of the density of $\mathbb{K}^*$. ■

We thus have the following result.

Theorem 55 *1) If P_ε obeys the weak large deviation principle, then $\mathbb{K}^\vee(C) \leq \mathbb{K}^*(C)$ for any closed subset C of Ω, that is*

$$\mathbb{K}^\vee(C) \leq \inf_{(U_n \in \mathcal{U})_n,\, C \subset \cup_n U_n} \sup_n \mathbb{K}^\wedge(U_n). \tag{12}$$

2) The following condition is sufficient in general and necessary in metric spaces for (P_ε) to obey the weak large deviation principle :

Inequality (12) holds and $\mathbb{K}^*$ satisfies condition 1 of Definition 51.

3) In a metric space, Definitions 50 and 51 are equivalent. Indeed, if $\mathbb{K}^*$ satisfies condition 1 of Definition 51, then $\mathbb{K}^*$ has necessarily as u.s.c. density the function :

$$c^*(\omega) = \inf_{U \in \mathcal{U},\, U \ni \omega} \mathbb{K}^\wedge(U).$$

Then (12) is equivalent to

$$\mathbb{K}^{\vee}(C) \leq \sup_{\omega \in C} c^*(\omega). \tag{13}$$

Moreover, if the conditions of Definition 51 hold, the rate function I is unique and equal to $-c^*$.

Proof. We only have to prove point 3. Suppose that Ω is a metric space and that $\mathbb{K}^*$ satisfies condition 1 of Definition 51. Then the non necessary complete subspace $\widetilde{\Omega} = \cup_n \Omega_n$ is such that $\mathbb{K}^*(\widetilde{\Omega}^c) \leq \inf_n \mathbb{K}^*(\Omega_n^c) = -\infty = \mathbb{0}$. Then, $\mathbb{K}^*(A) = \mathbb{K}^*(A \cap \widetilde{\Omega})$ for any subset A of Ω. From Example 39, the restriction of $\mathbb{K}^*$ to the open sets of $\widetilde{\Omega}$ has necessarily the density $c(\omega) = \inf_{U \in \mathcal{U},\, U \cap \widetilde{\Omega} \ni \omega} \mathbb{K}^*(U \cap \widetilde{\Omega}) = \inf_{U \in \mathcal{U},\, U \ni \omega} \mathbb{K}^*(U) = c^*(\omega)$ for $\omega \in \widetilde{\Omega}$, where $c^*(\omega) \stackrel{\text{def}}{=} \mathbb{K}^*(\{\omega\})$ for any $\omega \in \Omega$. Therefore $\mathbb{K}^*(A) = \mathbb{K}^*(A \cap \widetilde{\Omega}) = \sup_{\omega \in A \cap \widetilde{\Omega}} c^*(\omega) = \sup_{\omega \in A} c^*(\omega)$ for any $A \in \mathcal{U}$. Thus $\mathbb{K}^*$ has c^* as density and (12) is equivalent to (13). Moreover, $c^*(\omega) = \mathbb{K}^*(\{\omega\}) = \inf_{(U_n \in \mathcal{U})_n,\, \omega \in \cup_n U_n} \sup_n \mathbb{K}^{\wedge}(U_n) = \inf_{U \in \mathcal{U},\, U \ni \omega} \mathbb{K}^{\wedge}(U)$. ■

Thus in a metric space, the unique rate function can be calculated by using open sets or even a basis of neighborhoods only. Then, conditions 1 and 2 of Definition 50 or 51 have to be verified. If Ω is not a metric space, the same result may be obtained when open sets are replaced by $\mathcal{F}_\sigma$ open sets (see Corollary 41). It is indeed the good notion in a general normal topological space at least, since the large deviation principle has to be compared with the weak convergence of probabilities. The weak convergence of a sequence of probabilities P_n towards P is by definition equivalent to the convergence of the expectations $P_n(f)$ towards $P(f)$ for all bounded continuous functions, thus is equivalent to i) $\liminf_n P_n(U) \geq P(U)$ for all $\mathcal{F}_\sigma$ open sets which is also equivalent to ii) $\limsup_n P_n(C) \leq P(C)$ for all $\mathcal{G}_\delta$ closed sets (where a $\mathcal{G}_\delta$ set is by definition a countable intersection of open sets). Indeed, expectations of continuous functions only involves Baire sets. Then, a large deviation principle in a general normal topological space should have been defined by Definitions 50 or 51 with closed sets replaced by $\mathcal{G}_\delta$ closed sets, and open sets replaced by $\mathcal{F}_\sigma$ open sets.

Let us note that condition 1 of Definitions 50 and 51 is exactly the tightness condition for the idempotent probability $\mathbb{K}$ defined in [1] or [3] and is related to the tightness condition of classical probabilities. Although classical probabilities over Polish spaces are always tight, idempotent probabilities are in general not and this condition has to be imposed. Compactness results may be proved as in classical probability using this condition [3]. The main ingredient of this section was indeed that any tight idempotent probability on the set of $\mathcal{F}_\sigma$ open sets has a density. Thus, cases where weak and "strong" large deviation principles do not coincide may only be obtained when the tightness condition 1 is relaxed.

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