Introduction to the multi-step discretization algorithms via the controled path integral

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N° 2518
Mars 1995

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Masasuke SHIMA*

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In this report, using the formulations of controlled path integrals (Shima 1992), the author derives the one step algorithm, from which the multi-step algorithms can be obtained.

Résumé. Les algorithmes de discrétisation multipas pour les lois de contrôle à temps continu ont été étudiés par Isurugi et Shima (1985) à l'aide des algorithmes d'intégration d'équations différentielles ordinaires.

Dans ce rapport, utilisant la formulation d'intégrales de chemins contrôlés (Shima 1992), l'auteur donne l'algorithme à un pas, d'où les algorithmes multipas peuvent se déduire.

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1 Introduction

During 1980's, many computable design algorithms are obtained in the field of nonlinear control systems (Isidori, Nijmeijer and Van der Schaft, Isurugi and Shima, Yamashita and Shima). If one tries to apply these design methods to practical problems, it is clear that the feedback control laws have to be calculated by the digital computers. Thus, we encounter the necessity of discretization algorithms which have been studied by several researchers, for example, Monaco and Normand-Cyrot, Isurugi and Shima. The former have presented the one-step type discretization algorithm. The latter have studied both the one-step methods and the multi-step methods and have shown the advantages of multi-step methods and the related problems such as the stability of the whole system. The discretization of dynamic

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compensator is also studied in the papers of Isurugi and Shima, though all ones written in japanese.

Therefore the author would like to introduce the multi-step algorithms to the researchers outside Japan by the calculation based on the controlled path integral methods (Shima 1992) and show the limit of approximation via the zero-th order hold is $\mathcal{O}(T^3)$ (T is the sampling interval). This conclusion is the same as one implicitly shown by Normand-Cyrot and Isurugi.

2 Description of system Σ

The system $\sum$ to be studied is expressed as

(1) $$\sum : \{M, X, w, u, y\}.$$

M is an n-dimensional differentiable manifold and is a state space. A state of the system is a point $p \in M$. The state trajectory on M is written as $p(t)$. X is a set of vector fields on M and is sometimes a distribution. X is expressed as $X(p, u)$, where u is an input vector to the system in the forms as $u(t)$, $u(p(t))$, etc. w are in general tensor fields on M and is written as $w(p)$. In the following w is assumed to be a differential 1-form.

And, lastly, y is the output vector of the system defined by

(2) $$y^i(t) - y^i(t_0) = \int_{t_0}^{t} \langle w^i(p), X(p, u(s))\rangle_{p(s)} ds, \quad i = 1, \dots, r.$$

The given input $u(\cdot)$ determines a vector field $X(p, u(\cdot))$ on M, which yields the state trajectory $p(t)$ on M. $p(t)$ is also named as the controlled path. Then $u(\cdot)$ and $p(\cdot)$ determine $y(\cdot)$ via (2), which is named as controlled path integral.

Outputs of the system, performance index to be minimized, and Lyapunov function can be expressed in the form of "controlled path integrals".

3 Affine control system

If ($x(p, u)$ is given by

(3) $$X(p, u) = X_0(p) + X_1(p)u^1 + \cdots + X_m(p)u^m$$

with $X_0(p), X_1(p), \dots, X_m(p)$, C_∞ vector fileds on M. The system $\sum$ is called "affine control system". If we put

(4) $$u^0 \equiv 1$$

we have

$$
(5) \qquad \begin{aligned} X(p,u) &= X_0(p)u^0 + X_1(p)u^1 + \cdots + X_m(p)u^m && 5_1 \\ &= X_j(p)u^j && 5_2 \end{aligned}
$$

Using the local co-ordinates, we have

$$
(6) \qquad X_j(p) = f_j^1(x)\frac{\partial}{\partial x^1} + \cdots + f_j^n(x)\frac{\partial}{\partial x^n}.
$$

In the same way, we have

$$
(7) \qquad w(p) = w_1(x)dx^1 + \cdots + w_n(x)dx^n
$$

or

$$
(8) \qquad w^i(p) = dx^i, \quad i = 1, \ldots, n
$$

or

$$
(9) \qquad w(p) = \frac{\partial h}{\partial x^1}dx^1 + \cdots + \frac{\partial h}{\partial x^n}dx^n
$$

or

$$
(10) \qquad w(p) = c_1 dx^1 + \cdots + c_n dx^n.
$$

(6) and (10) yield

$$
(11) \qquad \begin{aligned} \dot{x} &= f_0(x) + f_1(x)u^1 + \cdots + f_m(x)u^m \\ y &= h(x) \end{aligned}
$$

which is a familiar "affine" control system.

Thus, our formulation includes the systems studied up to the present.

In addition, it is not assumed that w is an exact 1-form. Moreover, from (6) and (7), we have

$$
(12) \qquad \begin{aligned} \langle w(p), X(p,n)\rangle &= \sum_{j=0}^{m} \langle w, X_j\rangle u^j \\ &= \sum_{j=0}^{m} \left\{ \sum_{i=1}^{n} w_i(x) f_j^i(x) \right\} u^j = w_i f_j^i u^j \end{aligned}
$$

4 Expressions of the controlled path integrals (I)

In this section, we derive the necessary expressions of the integral :

$$J = \int_{t^0}^{t} \langle w, X \rangle\rangle dt. \tag{13}$$

Substituting $(5)_1$, into (13), we have

$$J = \int_{t_0}^{t} \sum_{i_1=0}^{m} \langle w, X_{i_1} \rangle_{p(s)} u^{i_1}(s) ds. \tag{14}$$

Let us define the set of suffixes:

$$I_1 := \{i_1\},\ I_N := \{i_N, i_{N-1}, \dots, i_1\},\ \begin{array}{l} N = 2, 3, \dots \\ i_j = 0, 1, , \dots, m \end{array} \tag{15}$$

For example, there are $(m+1)$ cases for I_1:

$$I_1 : \{0\}, \{1\}, \dots, \{m\}. \tag{16}$$

Using (15), we define

$$\begin{aligned} J(I_1 : t, s) &:= \int_s^t u^{i_1}(s_1) ds_1 \\ J(I_k : t, s) &:= \int_s^t u^{i_k}(s_k) J(I_{k-1} : t, s_k) ds_k,\ k = 2, 3, \dots \end{aligned} \tag{17}$$

In the same way, we define

$$\begin{aligned} w(I_1) &= w(\{i_1\}) = w(X_{i_1}) = \langle w, X_{i_1} \rangle \\ w(I_k) &= \langle d\{w(I_{k-1})\}, X_{i_k} \rangle = L_{X_{i_k}}\{w(I_{k-1})\},\ k = 2, 3, \dots \end{aligned} \tag{18}$$

where d is the exterior derivative and L_X is the Lie derivative. For example, we have

$$\begin{aligned} w(I_2) &= \langle d\{w(I_1)\}, X_{i_2} \rangle = L_{X_{i_2}}\{w(I_1)\} = L_{X_{i_2}}\{\langle w, X_{i_1} \rangle\} \\ &= \langle L_{X_{i_2}} w, X_{i_1} \rangle + \langle w, L_{X_{i_2}} X_{i_1} \rangle \end{aligned} \tag{19}$$

where the Lie derivatives $L_{X_{i_2}} w$ and $L_{X_{i_2}} X_{i_1}$ can be calculated in the following way.

If $\varphi(x)$ is a function and X is a vector field

$$X = f^1(x) \frac{\partial}{\partial x^1} + \dots + f^n(k) \frac{\partial}{\partial x^n} \tag{20}$$

expressed in the local co-ordinate

(21) $$L_x\varphi := f^1(x)\frac{\partial\varphi}{\partial x^1} + \cdots + f^n(x)\frac{\partial\varphi}{\partial x^n}$$

is the Lie derivative of $\varphi(x)$.

Furthermore, if we define a differential 1-form

(22) $$w := \frac{\partial(x)}{\partial x^1}dx^1 + \cdots + \frac{\partial\varphi(x)}{\partial x^n}dx^n =: d\varphi$$

we have

(23) $$\langle w, X\rangle = \sum_{i,j=1}^{n} \frac{\partial\varphi(x)}{\partial x^i} f^j(x)\langle dx^i, \frac{\partial}{\partial x^j}\rangle = \sum_{i=1}^{n} \frac{\partial\varphi(x)}{\partial x^i} f^i(x)$$

(24) $$[\because \langle dx^i, \frac{\partial}{\partial x^j}\rangle = \delta^i_j]$$

From (21) and (23), we obtain

(25) $$L_X\varphi = \langle w, X\rangle = \langle d\varphi, X\rangle.$$

If w is a differential 1-form:

(26) $$w = w_1(x)dx^1 + \cdots + w_n(x)dx^n$$

$L_X w$ is given by

(27) $$L_X w := \sum_{i=1}^{n}\{L_X w_i + \langle w, \frac{\partial f}{\partial x^i}\rangle\}dx^i$$

If Y is an another vector filed:

(28) $$Y = g^1(x)\frac{\partial}{\partial x^1} + \cdots + g^n(x)\frac{\partial}{\partial x^n}.$$

$L_X Y$ is given by

(29) $$L_X Y = L_f g = \sum_{i=1}^{n}\{L_f g^i - L_g f^i\}\frac{\partial}{\partial x^i}.$$

In view of these expressions, we know that the expressions of (19) are well defined. As to why (19) holds, please consult it with the books of differential geometry.

The calculations of the following special case will be used later in this report. If

$$w^i = dx^i \quad (i = 1, \dots, n), \tag{30}$$

in view of (29) and

$$w^i = \delta^i_j dx^j \tag{31}$$

we obtain

$$\begin{aligned} L_X w^i &= \sum_{j=1}^{n} \{L_X \delta^i_j + \langle w^i, \frac{\partial f}{\partial x^j} \rangle\} dx^j \\ &= \sum_{j=1}^{n} \{\langle w^i, \frac{\partial f}{\partial x^j} \rangle\} dx^j \quad [\because L_X \delta^i_j = 0] \\ &= \sum_{j=1}^{n} \frac{\partial f^i}{\partial x^j} dx^j. \end{aligned} \tag{32}$$

If we can put

$$X_{i_1} = X_{i_2} = X \tag{33}$$

in (19), we have

$$\begin{aligned} w^i(I_2) &= \langle L_X w^i, X \rangle + \langle w, L_X X \rangle \\ &= \langle \frac{\partial f^i}{\partial x^j} dx^j, f^j \frac{\partial}{\partial x^j} \rangle \quad [\because \text{(32) and } L_X X = 0] \\ &= \frac{\partial f^i}{\partial x^j} f^j \end{aligned} \tag{34}$$

which occurs for the case of the zero-th order hold function used in the sample data controlled system. If we define

$$w(I_2) := \begin{bmatrix} w^1(I_2) \\ \vdots \\ w^n(I_2) \end{bmatrix} \tag{35}$$

we have

(36)
$$w(I_2) = \begin{bmatrix} \frac{\partial f^1}{\partial x^1} & \cdots & \frac{\partial f^1}{\partial x^n} \\ \cdots & \cdots & \cdots \\ \frac{\partial f^n}{\partial x^1} & \cdots & \frac{\partial f^n}{\partial x^n} \end{bmatrix} \begin{bmatrix} f^1(x) \\ \vdots \\ f^n(x) \end{bmatrix} = \frac{\partial f}{\partial x} \cdot f.$$

Moreover, if we differentiate (32) along the trajectory $x(t)$ of the vector filed X again, we have the following equivalent expressions. Using (19)

(37)
$$\begin{aligned} w^i(I_3) &= \langle L_X^2 w, X\rangle + \langle L_X w, L_X X\rangle \\ &= \langle L_X^2 w, X\rangle \quad [\because L_X X = 0] \end{aligned}$$

On the other hand, using (34) and (19) we have

(38)
$$\begin{aligned} w^i(I_3) &= L_X\{w^i(I_2)\} \\ &= L_X\{\frac{\partial f^i}{\partial x^j} \cdot f^j\} \\ &= \sum_{j=1}^{n}\{L_X\{\frac{\partial f^i}{\partial x^j}\} \cdot f^j + \frac{\partial f^i}{\partial x^j} \cdot L_X\{f^j\}\} \end{aligned}$$

Therefore, we obtain

(39)
$$w^i(I_2) = \langle L_X^2 w, X\rangle = L_X\{\frac{\partial f^i}{\partial x^j}\} \cdot f^j + \frac{\partial f^i}{\partial x^j} \cdot L_X\{f^j\}.$$

This formula is obtained under the assumptions (30) and (33).
Expressions for more general case are given in the next section.

5 Expressions of the controlled path integrals (II)

$w(I_3)$ is calculated in the same way as (19)

(40)
$$\begin{aligned} w(I_3) &= w(\{i_3, i_2, i_1\}) \\ &= L_{X_{i_3}}\{w(I_2)\} \\ &= L_{X_{i_3}}\{\langle L_{X_{i_2}} w, X_{i_1}\rangle + \langle w, L_{X_{i_2}} X_{i_1}\rangle\} \\ &= \langle L_{X_{i_3}} L_{X_{i_2}} w, X_{i_1}\rangle + \langle L_{X_{i_2}} w, L_{X_{i_3}} X_{i_1}\rangle \\ &\quad + \langle L_{X_{i_3}} w, L_{X_{i_2}} X_{i_1}\rangle + \langle w, L_{X_{i_3}} L_{X_{i_2}} X_{i_1}\rangle. \end{aligned}$$

Using (27) and (29) repeatedly, it is clearly possible to calculate (40) in principle ; but the number of "inner products" increases with the rate of 2^k for $w(I_k)$, which yields the problem of calculation. Keeping this problem in mind, we use the expression(18).

In the next place, differentiating (19) with respect to s, we have the following relations

$$
(41) \qquad \begin{aligned} \partial_s J(I_1;t,s) &= -u^{i_1}(s) \\ \partial_s J(I_k;t,s) &= -u^{i_k}(s) J(I_{k-1};t,s) \quad k=2,3,\dots . \end{aligned}
$$

Then, we obtain the following expressions by applying the integration by parts

$$
(42) \qquad \begin{aligned} J &= \int_{t_0}^{t} \sum_{i_1=0}^{m} \langle w, X_{i_1}\rangle u^{i_1}(s) ds \\ &= \int_{t_0}^{t} \sum_{i_1=0}^{m} w(I_1)\{-\partial_s J(I_1;t,s)\} ds \\ &= \sum_{i_1=0}^{m} \{[-w(I_1)J(I_1;t,s)]_{t_0}^{t} + \int_{t_0}^{t} \sum_{i_2=0}^{m} w(I_2) u^{i_2}(s) J(I_1;t,s) ds\} \\ &= \sum_{i_1=0}^{m} w(I_1)_{p(t_0)} J(I_1;t,t_0) + \sum_{i_1=0}^{m} \sum_{i_2=0}^{m} \int_{t_0}^{t} w(I_2)\{-\partial_s J(I_2;t,s)\} ds \end{aligned}
$$

Therefore, repeating this procedure, we obtain the following expression

$$
(43) \qquad \begin{aligned} J &= \int_{t_0}^{t} \langle w, X\rangle ds \\ &= \sum_{i=0}^{m} \int_{t_0}^{t} \langle w, X_{i_1}\rangle u^{i_1}(s) ds \\ &= \sum_{k=1}^{N_1} \sum_{i_1 \cdots i_k=0}^{m} w(I_k)_{p(t_0)} . J(I_k;t,t_0) + \sum_{i_1 \cdots i_N=0}^{m} \int_{t_0}^{t} w(I_N)_{p(s)}\{-\partial_s J(I_N;t,s)\} ds \end{aligned}
$$

This expression is valid along the state trajectory $p(t)$ and not restricted in the neighborhood of $p(t_0)$. From (43), we can derive the Fliess type expression and the Volterra type expression by exchanging the orders of integration. Details are referred to Shima (1992). Moreover, under the appropriate conditions on convergence, we can obtain the functional series expansion formulas of Fliess and Volterra from (43).

Our approach is a comprehensible introduction to these expressions, though this approach may be found in the previous literatures. The author does not exclude this possibilities, since this approach is very simple and straightforward. If the reader of this report knows the previous result of this approach, it is very kind of him to inform it to the author.

6 Relative order and feedback design

Let us assume that the following conditions hold

$$
(44) \qquad \begin{aligned} & w(I_k)_p = 0,\ I_k \neq \{0^k\},\ k = 1,\dots,q-1 \\ & (w(\{1,0^{q-1}\}), w(\{2,0^{q-1}\}), \dots, w(\{m,0^{q-1}\})_p \\ & = (L_{X_1} L_{X_0}^{q-2}\{w(X_0)\}, L_{X_2} L_{X_0}^{q-2}\{w(X_0)\}, \dots, L_{X_m} L_{X_0}^{q-2}\{w(X_0)\})_p \\ & \neq (0,\dots,0) \\ & p \in U(p_0,\varepsilon),\ p(t_0) = p_0. \end{aligned}
$$

Then, we have

$$
\begin{aligned}
y(t)-y(t_0) &= \sum_{k=1}^{q-1} w(\{0^k\})_{p0}\cdot J(\{0^k\}:t,t_0) \\
&+ \sum_{j=0}^{m}\int_{t_0}^{t} w(\{j,0^{q-1}\})_{p(s)}u^j(s)J(\{0^{q-1}\}:t,s)ds.
\end{aligned} \tag{45}
$$

And, it is certain that we can control the "output" $y(t)$. Thus, in this case, y has the relative order q. Differentiating the both sides of (45) with respect to t, we have

$$
\begin{aligned}
\dot{y}(t) &= \sum_{k=1}^{q-1} w(\{0^k\})_{p0}.J(\{0^{k-1}\}:t,t_0) \\
&\quad +\sum_{j=0}^{m}\int_{t_0}^{t} w(\{j,0^{q-1}\})_{p(s)}u^j(s)J(\{0^{q-2}\}:t,s)ds \\
\ddot{y}(t) &= \sum_{k=2}^{q-1} w(\{0^k\})_{p0}.J(\{0^{k-2}\}:t,t_0) \\
&\quad +\sum_{j=0}^{m}\int_{t_0}^{t} w(\{j,0^{q-1}\})_{p(s)}u^j(s)J(\{0^{q-3}\}:t,s)ds \\
&\cdots \\
y^{(q-1)}(t) &= w(\{0^{q-1}\})_{p0}J(\{0^0\}:t,t_0) \\
&\quad +\sum_{j=0}^{m}\int_{t_0}^{t} w(\{j,0^{q-1}\})_{p(s)}u^j(s)ds
\end{aligned} \tag{46}
$$

where

$$w(\{0^0\})_{p0} = y(t_0),\ \ J(\{0^0\}:t,s)=1.$$

From (46), we have the initial conditions:

$$y(t_0)=y(t_0),\ \dot{y}(t_0)=w(\{0\})_{p0},\cdots,y^{(q-1)}(t_0)=w(\{0^{q-1}\})_{p0}. \tag{47}$$

The integrated part of (45) is equal to

$$\int_{t_0}^{t}\left\{\sum_{j=0}^{m} w(\{j,0^{q-1}\})_{p(s)}u^j(s)\right\}J(\{0^{q-1}\}:t,s)ds. \tag{48}$$

If p is "completely" known along the state trajectory, in other words, if the state is always known, we can replace $\{\cdots\}$ of (48) by $v(s)$, new input. That is, it is possible to determine $u^j(s)$ $(s=1,\ldots,m)$ such that

$$\sum_{j=0}^{m} w(\{j,0^{q-1}\})_{p(s)}u^j(s)=v(s), \tag{49}$$

which is enabled by the definition of relative order and the full knowledge of state $p(s)$. Then, we obtain

$$(50)\quad \begin{aligned} y(t)-y(t_0) &= \sum_{k=1}^{q-1} w(\{0^k\})_{p_0} J(\{0^k\}:t,t_0) + \int_{t_0}^{t} v(s)J(\{0^{q-1}\}:t,s)ds \\ y^{(\ell)}(t) &= \sum_{k=\ell}^{q-1} w(\{0^k\})_{p_0} J(\{0^{k-\ell}\}:t,t_0) + \int_{t_0}^{t} v(s)J(\{0^{q-1-\ell}\}:t,s)ds \\ & \quad \ell = 1,\dots,q-1. \end{aligned}$$

Moreover, if $y(t), \dot{y}(t), \dots$, and $y^{(q-1)}(t)$ are measurable and available for the "feedback", we may have

$$(51)\quad v(s) = \tilde{v}(s) - \alpha_1 y^{(q-1)}(s) - \alpha_2 y^{(q-2)}(s) - \cdots - \alpha_{q-1}\dot{y}(s) - \alpha_q \cdot y(s)$$

where $\tilde{v}(s)$ is a new input.

Substituting (51) into (50), we have an integral equation

$$(52)\quad \begin{aligned} y(t) = &\sum_{k=0}^{q-1} w(\{0^k\})_{p_0} J(\{0^k\}:t,t_0) \\ &+ \int_{t_0}^{t} \{\tilde{v}(s) - \alpha_1 y^{(q-1)}(s) - \cdots - \alpha_q y(s)\} \frac{1}{(q-1)!}(t-s)^{q-1} ds \end{aligned}$$

which characterizes the "input output relation" from $\tilde{v}(\cdot)$ to $y(\cdot)$. In our formulation, state feedback (49) and output feedback (51) may be different, because w may not be exact. By feedback, we have the different input output relations.

7 Vector field $\hat{f}(x)$ with state feedback and vector field $\bar{f}(x)$ with zero-th order Holder

In the following discussion, we use the expressions in the local co-ordinates, which are shown again as follows

$$(53)\quad \begin{aligned} X &= X_0 + X_1 u^1 + \cdots + X_m u^m \\ &= X_0 u^0 + X_1 u^1 + \cdots + X_m u^m \quad [\because u^0 \equiv 1] \\ &= f(x) + g_1(x)u^1 + \cdots + g_m(x)u^m \\ &= g_0(x)u^0 + g_1(x)u^1 + \cdots + g_m(x)u^m \quad [\because u^0 \equiv 1,\ g_0(x) \equiv f(x)] \\ &= g_\alpha(x)u^\alpha \end{aligned}$$

$$(54)\quad w = w_1(x)dx^1 + \cdots + w_n(x)dx^n = w_i(x)dx^i$$

(55) $$f(x) = f^i(x)\frac{\partial}{\partial x^i},\ g_\alpha(x) = g^i_\alpha(x)\frac{\partial}{\partial x^i}$$

(56) $$w(I_1) = \langle w, X_{i_1}\rangle = \langle w_i dx^i, g^j_{i_1}\frac{\partial}{\partial x^j}\rangle = w_i g^i_{i_1}$$

(57) $$\begin{aligned} w(I_2) &= w(\{i_2, i_1\}) = L_{X_{i_2}}\{w(X_{i_1})\} = L_{g_{i_2}}\{w(g_{i_1})\} \\ &= L_{g_{i_2}}\langle w, g_{i_1}\rangle = \langle L_{g_{i_2}}w.g_{i_1}\rangle + \langle w, L_{g_{i_2}}g_{i_1}\rangle \end{aligned}$$

(58) $$L_f w = (L_f\{w_i\} + \langle w, \frac{\partial f}{\partial x^i}\rangle)dx^i$$

(59) $$L_f g = (L_f g^i - L_g f^i)\frac{\partial}{\partial x^i}.$$

If a feedback control law

(60) $$u^i(x) = a^i(x) + b^i_j(x)v^j, \quad j = 1, \dots, m'$$

is given with new control input v^j, we put

(61) $$u^0(x) = u^0 \equiv 1$$

and we have a vector field

(62) $$\hat{f}(x) := g_\alpha(x)u^\alpha(x) = g_\alpha(x)\{a^\alpha(x) + b^\alpha_j(x)v^j\}.$$

In this report, we only study the case

(63) $$v^j \equiv 0.$$

Thus, we have a vector field

(64) $$\hat{f}(x) = g_\alpha(x)u^\alpha(x) = g_\alpha u^\alpha.$$

Using (64) in (56) and considering (30):

$$w^i = dx^i \tag{65}$$

we have

$$\hat{w}^i(I_1) := \langle dx^i, g_\alpha u^\alpha \rangle = g^i_\alpha u^\alpha \tag{66}$$

$$\begin{aligned} \hat{w}^i(I_2) &:= L_{\hat{f}}\{\hat{w}^i(I_1)\} \\ &= L_{\hat{f}}\{g^i_\alpha(x)\}u^\alpha(x) + g^i_\alpha(x) L_{\hat{f}}\{u^\alpha(x)\} \\ &= g^j_\beta(x)u^\beta(x)\frac{\partial g^i_\alpha(x)}{\partial x^j}u^\alpha(x) + g^i_\alpha(x)g^j_\beta(x)u^\beta(x)\frac{\partial u^\alpha(x)}{\partial x^j} \end{aligned} \tag{67}$$

On the other hand, we have

$$\begin{aligned} \hat{w}^i(I_2) &:= L_{\hat{f}}\{\hat{w}^i(I_1)\} \\ &= L_{\hat{f}}\{\langle dx^i, \hat{f}\rangle\} \\ &= \langle L_{\hat{f}} dx^i, \hat{f}\rangle + \langle dx^i, L_{\hat{f}}\hat{f}\rangle \\ &= \langle L_{\hat{f}} dx^i, \hat{f}\rangle \quad [\because L_{\hat{f}}\hat{f} = 0]. \end{aligned} \tag{68}$$

It is to be noted that $I_1 = \{i_1\} = \{0\}$ and $I_2 = \{i_2, i_1\} = \{0, 0\}$ in these expressions. In view of (32) we have

$$L_{\hat{f}} dx^i = \frac{\partial \hat{f}^i}{\partial x^j}dx^j = \left\{ \frac{\partial g^i_\alpha(x)}{\partial x^j}u^\alpha(x) + g^i_\alpha(x)\frac{\partial u^\alpha(x)}{\partial x^j} \right\} dx^j \tag{69}$$

and

$$\langle L_{\hat{f}} dx^i, \hat{f}\rangle = \left\{ \frac{\partial g^i_\alpha(x)}{\partial x^j}u^\alpha(x) + g^i_\alpha(x)\frac{\partial u^\alpha(x)}{\partial x^j} \right\} g^j_\beta(x)u^\beta(x) \tag{70}$$

which is the same expression as (67).

Moreover, we calculate $\hat{w}^i(I_3) = \hat{w}^i(\{0,0,0\})$ as follows

$$\begin{aligned} \hat{w}^i(I_3) &:= L_{\hat{f}}\{\hat{w}^i(I_2)\} \\ &= L^2_{\hat{f}}\{g^i_\alpha(x)\}u^\alpha(x) + 2L_{\hat{f}}\{g^i_\alpha(x)\}L_{\hat{f}}\{u^\alpha(x)\} + g^i_\alpha(x)L^2_{\hat{f}}\{u^\alpha(x)\}. \end{aligned} \tag{71}$$

Substituting (66), (67) and (71) into (43) with $N = 4$ and considering the fact that (65) implies

$$y^i(t) = \hat{x}^i(t) \tag{72}$$

for the vector field $\hat{f}$, we have the following expression:

$$\begin{aligned} \hat{x}^i(t) - \hat{x}^i(t_0) &= g^i_\alpha(x_0)u^\alpha(x_0)(t-t_0) \\ &+[L_{\hat{f}}\{g^i_\alpha(x_0)\}u^\alpha(x_0) + g^i_\alpha(x_0)L_{\hat{f}}\{u^\alpha(x_0)\}]\frac{1}{2!}(t-t_0)^2 \\ &+[L^2_{\hat{f}}\{g^i_\alpha(x_0)\}u^\alpha(x_0) + 2L_{\hat{f}}\{g^i_\alpha(x_0)\}L_{\hat{f}}\{u^\alpha(x_0)\} \\ &+g^i_\alpha(x_0)L^2_{\hat{f}}\{u^\alpha(x_0)\}].\frac{1}{3!}(t-t_0)^3 \\ &+\int_{t_0}^t \hat{w}(\{0^4\})_{\hat{p}(s)}\frac{1}{3!}(t-s)^3 ds \end{aligned} \tag{73}$$

If a "constant control law"

$$u^i(s) = u^i = const. \quad t_0 \le s < t \tag{74}$$

is used instead of (60), we have a vector field

$$\bar{f}(x) := g_\alpha(x)u^\alpha. \tag{75}$$

Since

$$L_{\bar{f}}u^\alpha = 0, \tag{76}$$

it is obvious that we have the expression for the vector filed $\bar{f}$:

$$\begin{aligned} \bar{x}^i(t) - \bar{x}^i(t_0) &= g^i_\alpha(x_0)u^\alpha(t-t_0) \\ &+L_{\bar{f}}\{g^i_\alpha(x_0)\}u^\alpha\frac{1}{2!}(t-t_0)^2 \\ &+[L^2_{\bar{f}}\{g^i_\alpha(x_0)\}u^\alpha\frac{1}{3!}(t-t_0)^3 \\ &+\int_{t_0}^t \bar{w}(\{0^4\})_{\bar{x}(s)}\frac{1}{3!}(t-s)^3 ds. \end{aligned} \tag{77}$$

In the calculation of (77), the following equalities are used

$$\begin{aligned} J(I_1 : t, t_0) &= \int_{t_0}^t u^{i_1} ds_1 = u^{i_1}(t-t_0) \\ J(I_2 : t, t_0) &= \int_{t_0}^t u^{i_2} ds_2 \int_{s_2}^t u^{i_1} ds_1 = u^{i_2}u^{i_1}\frac{1}{2!}(t-t_0)^2 \\ J(I_3 : t, t_0) &= u^{i_3}u^{i_2}u^{i_1}\frac{1}{3!}(t-t_0)^3 \end{aligned} \tag{78}$$

N.B. These expressions are valid only under the assumption (74), i.e., only for the "zero-th order holder". Expression (43) can be used for any functions $u^{\alpha}(\cdot)$.

8 Discretization algorithm of continuous time control law

Under the following assumptions, we study the discretization problem of continuous time control law.

Assumption 1 *The zero-th order holder is used for the sampled data (digital) control system.*

N.B. This assumption can be loosened by the so-called generalized holder, with which we have the vast open field of research. The tools for practical applications will be developped if the good and realizable results are obtained. To this approach, the expression (43) is suitable.

Assumption 2 *The state feedback control law is given by*

$$u(t) = a(x(t)) + B(x(t))v(t) \tag{79}$$

as in (60).

N.B. Only the case with $v(t) \equiv 0$ is studied in this report.

Assumption 3 *At time $t = k\Delta$ with the sampling period Δ, all the present and past state at the sampling times : $x(k\Delta)$, $x(\overline{k-1}\Delta)$, ... are exactly known and can be used for the calculations of*

$$u(t) = u_k \quad (k\Delta \le t < (k+1)\Delta). \tag{80}$$

N.B. This is an unrealistic assumption in some cases, because the time necessary for calculation is not taken into account. But, this problem is not so significant, if our multi-step algorithms are applied.

Assumption 4 *The sampling interval Δ is much smaller than the time constants of system dynamics.*

N.B. This is a very convenient and occasionally vague expression. But this problem is also studied in the multi-step approach and a reasonable estimate is given for an exacmple.

Now under these assumptions, we will try to approximate $\hat{x}(t) = \begin{pmatrix} \hat{x}^1(t) \\ \vdots \\ \hat{x}^n(t) \end{pmatrix}$ via

$$\bar{x}(t) = \begin{pmatrix} \bar{x}^1(t) \\ \vdots \\ \bar{x}^n(t) \end{pmatrix}. \tag{81}$$

$\hat{x}^i(t)$ is given by (73) and $\bar{x}^i(t)$ by (77).

We express the discretized control over $[k\Delta, (k+1)\Delta]$

$$u_k = u_{k0} + u_{k1}\Delta + u_{k2}\Delta^2 + \cdots \tag{82}$$

and omit the suffix "k" in this expression. Thus, we have

$$\begin{aligned} &u^\alpha = u_0^\alpha + u_1^\alpha\Delta + u_2^\alpha\Delta^2 + \cdots, \quad \alpha = 0, 1, \ldots, m \\ &u_0^0 = 1,\ u_1^0 = 0,\ u_2^0 = 0, \cdots \end{aligned} \tag{83}$$

Moreover, we regard

$$t_0 = k\Delta,\ t = (k+1)\Delta \tag{84}$$

In other words, we have

$$t - t_0 = \Delta \tag{85}$$

in the expressions (73) and (77).
Using the assumption 3, we have

$$\hat{x}^i(t_0) = \bar{x}^i(t_0). \tag{86}$$

Then, substracting (77) from (73), we obtain

$$\begin{aligned} \hat{x}^i(t) - \bar{x}^i(t) &= g_\alpha^i(x_0)\{u^\alpha(x_0) - u^\alpha\} \cdot \Delta \\ &+[L_f\{g_\alpha^i(x_0)\}u^\alpha(x_0) - L_f\{g_\alpha^i(x_0)\}u^\alpha + g_\alpha^i(x_0)L_f\{u^\alpha(x_0)\}]\frac{1}{2!}\Delta^2 \\ &+[L_f^2\{g_\alpha^i(x_0)\}u^\alpha(x_0) - L_f^2\{g_\alpha^i(x_0)\}u^\alpha + 2L_f\{g_\alpha^i(x_0)\}L_f\{u^\alpha(x_0)\} \\ &+g_\alpha^i(x_0)L_f^2\{u^\alpha(x_0)\}\frac{1}{3!}\Delta^3 \\ &+ \text{integral term}. \end{aligned} \tag{87}$$

Substituting (83) into (87), we have

$$g^i_\alpha(x_0)\{u^\alpha(x_0)\Delta - u^\alpha_0\Delta - u^\alpha_1\Delta^2 - u^\alpha_2\Delta^3 - \cdots\}. \tag{88}$$

So, if we put

$$u^\alpha_0 = u^\alpha(x_0) \tag{89}$$

the terms of Δ disappear from (87) and the remaining terms in (88) are

$$- g^i_\alpha(x_0)u^\alpha_1\Delta^2 - g^i_\alpha(x_0)u^\alpha_2\Delta^3 - \cdots \tag{90}$$

Combining (89) with (90), we have the terms of Δ^2 as follows

$$\begin{aligned}&\frac{1}{2!}\Delta^2\Big[g^j_\beta(x_0)u^\beta(x_0)\frac{\partial g^i_\alpha(x_0)}{\partial x^j}u^\alpha(x_0) - g^j_\beta(x_0)\\ &\{u^\beta(x_0)+u^\beta_1\Delta+u^\beta_2\Delta^2+\cdots\}\frac{\partial g^i_\alpha(x_0)}{\partial x^j}u^\alpha + g^i_\alpha(x_0)L_f\{u^\alpha(x_0)\} - g^i_\alpha(x_0)2u^\alpha_1\Big]\end{aligned} \tag{91}$$

Thus, if we put

$$\begin{aligned}u^\alpha_1 &= \frac{1}{2}L_f\{u^\alpha(x_0)\}\\ &= \frac{1}{2}g^j_\beta(x_0)u^\beta(x_0)\frac{\partial u^\alpha(x_0)}{\partial x^j}\end{aligned} \tag{92}$$

the terms of Δ^2 disappear in (91) and in (87).

The remaining terms in (91) are

$$-\frac{1}{2!}\Delta^2\left[g^j_\beta(x_0)\{u^\beta(x_0)u^\alpha_1\Delta + u^\alpha(x_0)u^\beta_1\Delta+\cdots\}\frac{\partial g^i_\alpha(x_0)}{\partial x^j}\right]. \tag{93}$$

Combining (90), (93) and (87), we have the terms of Δ^3 as follows

$$\begin{aligned}&\frac{1}{3!}\Delta^3\Big[g^k_\gamma(x_0)u^\gamma(x_0)\frac{\partial}{\partial x^k}\{g^j_\beta(x_0)u^\beta(x_0)\frac{\partial g^i_\alpha(x_0)}{\partial x^j}\}u^\alpha(x_0)\\ &-g^k_\gamma u^\gamma\frac{\partial}{\partial x^k}\{g^j_\beta(x_0)u^\beta\frac{\partial g^i_\alpha(x_0)}{\partial x^j}\}u^\alpha\\ &+2L_f\{g^i_\alpha(x_0)\}L_f\{u^\alpha(x_0)\} + g^i_\alpha(x_0)L^2_f\{u^\alpha(x_0)\}\\ &-6g^i_\alpha(x_0)u^\alpha_2 - 3g^j_\beta(x_0)\{u^\beta(x_0)u^\alpha_1 + u^\alpha(x_0)u^\beta_1\}\frac{\partial g^i_\alpha(x_0)}{\partial x^j}\Big]\end{aligned} \tag{94}$$

Substituting (89) and (92) into (94), we have the terms of Δ^3 as follows

$$\begin{aligned}\frac{1}{3!}\Delta^3 \Bigg[& g^k_\gamma(x_0)u^\gamma(x_0)g^j_\beta(x_0)\frac{\partial u^\beta(x_0)}{\partial x^k}\frac{\partial g^i_\alpha(x_0)}{\partial x^j}u^\alpha(x_0) \\ & +2L_f\{g^i_\alpha(x_0)\}L_f\{u^\alpha(x_0)\} + g^i_\alpha(x_0)\{L^2_f\{u^\alpha(x_0)\} - 6.u^\alpha_2\} \\ & -3g^j_\beta(x_0)u^\beta(x_0)\frac{1}{2}L_f\{u^\alpha(x_0)\}\frac{\partial g^i_\alpha(x_0)}{\partial x^j} \\ & -3g^j_\beta(x_0)u^\alpha(x_0)\frac{1}{2}L_f\{u^\beta(x_0)\}\frac{\partial g^i_\alpha(x_0)}{\partial x^j}\Bigg].\end{aligned} \tag{95}$$

The first term of (95) is equal to

$$L_f\{u^\beta(x_0)\}g^j_\beta(x_0)\frac{\partial g^i_\alpha(x_0)}{\partial x^j}u^\alpha(x_0). \tag{96}$$

The fourth term of (95) is equal to

$$-\frac{3}{2}L_f\{g^j_\alpha(x_0)\}L_f\{u^\alpha(x_0)\}. \tag{97}$$

Considering (96) and (97), (95) is equal to

$$\begin{aligned}\frac{1}{3!}\Delta^3 \Bigg[& \frac{1}{2}L_f\{g^i_\alpha(x_0)\}L_f\{u^\alpha(x_0)\} - \frac{1}{2}L_f\{u^\beta(x_0)\}g^j_\beta(x_0).\frac{\partial g^i_\alpha(x_0)}{\partial x^j}u^\alpha(x_0) \\ & +g^i_\alpha(x_0)\{L^2_f\{u^\alpha(x_0)\} - 6u^\alpha_2\}\Bigg]\end{aligned} \tag{98}$$

If we put

$$u^\alpha_2 = \frac{1}{3!}L^2_f\{u^\alpha(x_0)\}, \tag{99}$$

we have the remaining terms of Δ^3 as follows

$$-\frac{1}{12}\Delta^3[L_f\{u^\alpha(x_0)\}.\{L_f\{g^i_\alpha(x_0)\} - g^j_\alpha(x_0)\frac{\partial g^i_\beta(x_0)}{\partial x^j}u^\beta(x_0)\}] \tag{100}$$

$$= -\frac{1}{12}\Delta^3 L_f\{u^\alpha(x_0)\}u^\beta(x_0)\{g^j_\beta(x_0)\frac{\partial g^i_\alpha(x_0)}{\partial x^j} - g^j_\alpha(x_0)\frac{\partial g^i_\beta(x_0)}{\partial x^j}\} \tag{101}$$

$$\begin{aligned}&= -\frac{1}{12}\Delta^3 \sum_{\alpha<\beta} L_f\{u^\alpha(x_0)\}u^\beta(x_0) - L_f\{u^\beta(x_0)\}u^\alpha(x_0)\} \\ &\quad \times\{g^j_\beta(x_0)\frac{\partial g^i_\alpha(x_0)}{\partial x^j} - g^j_\alpha(x_0)\frac{\partial g^i_\beta(x_0)}{\partial x^j}\}\end{aligned} \tag{102}$$

If we express the term $u^\alpha(x_0)L_f\{u^\beta(x_0)\} - u^\beta(x_0)L_f\{u^\alpha(x_0)\}$ as

$$(103) \qquad \{u^\alpha(x_0), u^\beta(x_0)\}_{\bar{f}} := u^\alpha(x_0)L_{\bar{f}}\{u^\beta(x_0)\} - u^\beta(x_0)L_{\bar{f}}\{u^\alpha(x_0)\}$$

and use the definition (29) :

$$(104) \qquad \begin{aligned} L_{g_\alpha}g_\beta &= [g_\alpha(x_0), g_\beta(x_0)] \\ &:= \sum_{i=1}^{n}\{g^j_\alpha(x_0)\frac{\partial g^i_\beta(x_0)}{\partial x^j} - g^j_\beta(x_0)\frac{\partial g^i_\alpha(x_0)}{\partial x^j}\}(\frac{\partial}{\partial x^i})_{x_0}, \end{aligned}$$

(102) can be expressed by the remaining terms of Δ^3

$$(105) \qquad = -\frac{1}{12}\Delta^3 \sum_{\alpha<\beta}\{u^\alpha(x_0), u^\beta(x_0)\}_{\bar{f}}\{[g_\alpha(x_0), g_\beta(x_0)]\}^i.$$

Therefore, we obtain from (87)

$$(106) \qquad \hat{x}(t) - \bar{x}(t) = -\frac{1}{12}\Delta^3 \sum_{\alpha<\beta}\{u^\alpha(x_0), u\beta(x_0)\}_{\bar{f}}[g_\alpha(x_0), g_\beta(x_0)] + o(\Delta^3).$$

In other words, if we use the zero-th order holder, it is impossible in general to attain the accuracy of Δ^3. Even if we choose other u_2^α in (99), we can not make all the terms of Δ^3 disappear.

This is the conclusion of our approach which was implicitly written in Prof. Normand-Cyrot's book and was also noticed in our studies on multi-step discretization algorithms.

It is shown in (106) that both the control laws and the system structure are "responsible" to the remainder terms of Δ^3. Moreover, the expression (106) seems to be a special case of more general results on this type of discretization problems where the "generalized holder" are made use of.

9 Examples

Example 1

If the system is linear, we have

$$(107) \qquad \dot{x} = Ax + Bu = Ax + b_1u^1 + \cdots + b_mu^m$$

with control law

$$(108) \qquad u = Kx.$$

Then, we have

(109) $$g_0(x) = f(x) = Ax, \quad g_1(x) = b_1, \dots, g_m(x) = b_m$$

and

(110) $$u(x) = Kx = \begin{bmatrix} K^1 x \\ \cdots \\ K^m x \end{bmatrix}.$$

In other words,

(111) $$u^0(x) = 1, \; u^\alpha(x) = K^\alpha x \; \alpha = 1, \dots, m.$$

If $\alpha, \beta \geq 1$, we have

(112) $$[g_\alpha(x), g_\beta(x)] = [b_\alpha, b_\beta] = 0.$$

If $\alpha = \beta = 0$ we have

(113) $$[g_0(x), g_0(x)] = [Ax, Ax] = 0.$$

Thus, the remaining terms of Δ^3 : (106) is

(114) $$-\frac{1}{12}\Delta^3 \sum_{\beta=1}^{m} \{u^0(x_0), u^\beta(x_0)\}_{\hat{f}} [Ax, b_\beta]$$

with

(115)
$$\begin{aligned} \{u^0(x_0), u^\beta(x_0)\}_{\hat{f}} &= L_{\hat{f}}\{u^\beta(x_0)\} = (\hat{f}^1 \frac{\partial}{\partial x^1} + \cdots + \hat{f}^n \frac{\partial}{\partial x^n})(K_1^\beta x^1 + \cdots + K_m^\beta x^m) \\ &= K_1^\beta \hat{f}^1 + \cdots + K_m^\beta \hat{f}^m = K^\beta \hat{f} \\ &= K^\beta (A + BK)x \end{aligned}$$

and

$$[Ax, b_\beta] = -Ab_\beta.$$

Therefore, we obtain as the remaining terms of Δ^3

(116)
$$\begin{aligned} &+\frac{1}{12}\Delta^3 \sum_{\beta=1}^{m} A\{b_1 K^1 + \cdots + b_m K^m\}(A + BK)x \\ &= \frac{1}{12}\Delta^3 ABK(A + BK)x \end{aligned}$$

which is the same result as one previously obtained.

Example 2 If the system is given by

$$\begin{aligned} \dot{x}_1 &= -10x_1 - 0.01x_3u_1 + 0.1u_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= 100x_1u_1 \\ y_1 &= x_1,\ y_2 = x_2 \end{aligned} \tag{117}$$

$$\begin{aligned} u_1 &= 0.01(v_2 - x_2 - x_3)/x_1 \\ u_2 &= 10v_1 + 90x_1 + 0.001x_3(v_2 - x_2 - x_3)/x_1 \end{aligned} \tag{118}$$

which decouples the input-output relation of (117). If the control law (118) is discretized via the zero-th order holder and (89), (92), (99), there appears the coupling term inevitably.

According to the formulation of the problem, we have

$$g_0(x) = f(x) = \begin{bmatrix} -10x_1 \\ x_3 \\ 0 \end{bmatrix},\ g_1(x) = \begin{bmatrix} -0.01x_3 \\ 0 \\ 100x_1 \end{bmatrix},\ g_2(x) = \begin{bmatrix} 0.1 \\ 0 \\ 0 \end{bmatrix}. \tag{119}$$

In this case, we calculate as follows

$$\begin{aligned} &[g_0, g_1] = \begin{bmatrix} -0.1x_3 \\ -100x_1 \\ -1000x_1 \end{bmatrix},\ [g_0, g_2] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ [g_1, g_2] = \begin{bmatrix} 0 \\ 0 \\ -10 \end{bmatrix} \\ &\hat{f} = g_0 + g_1u_1(x) + g_2u_2(x) = \begin{bmatrix} -x_1 \\ x_3 \\ -x_2 - x_3 \end{bmatrix} \\ &\{u_0, u_1\} = L_{\hat{f}}u_1 = \{-x_1\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} - (x_2 + x_3)\frac{\partial}{\partial x_3}\}u_1 = \frac{-0.01x_3}{x_1} \\ &\{u_0, u_2\} = L_{\hat{f}}u_2 = -30x_1 + \frac{0.001}{x_1}(x_2^2 + 2x_2x_3) \\ &\{u_1, u_2\} = u_1(x)L_{\hat{f}}u_2 - u_2(x)L_{\hat{f}}u_1 \end{aligned} \tag{120}$$

Using (120), we have the terms of Δ^3 as follows

$$-\frac{1}{2}\Delta^3. \begin{bmatrix} -90x_1 + \frac{0.001(x_0+x_3)^2}{x_1} \\ x_3 \\ -18x_3 - 9x_2 + \frac{0.0001(x_2+x_3)^3}{x_1^2} \end{bmatrix}. \tag{121}$$

However, in deducing (106), we assumed that

$$v_1 = v_2 = 0. \tag{122}$$

Therefore, in order to estimate the effect of discretization on the cross term, it is necessary to carry out the similar analysis and calculation for the case where

$$u_1 \not\equiv 0, \ u_2 \not\equiv 0. \tag{123}$$

10 Multi-step discretization algorithm

Considering (89), (92) and (99), we obtain

$$u = u(x_0) + \frac{1}{2!}\Delta L_f u(x_0) + \frac{1}{3!}\Delta^2 L_f^2 u(x_0) + \cdots \tag{124}$$

$$\doteqdot \frac{1}{\Delta}\int_{K\Delta}^{(k+1)\Delta} u(\hat{x}(t))dt \tag{125}$$

where $\hat{x}(t)$ satisfies the equation

$$\dot{x}(t) = \hat{f}(x(t)) = g_0(x(t)) + g_1(x(t))u^1(x(t)) + \cdots + g_m(x(t))u^m(x(t)). \tag{126}$$

The expressions (124) and (125) are interesting and useful, which Prof. Isurugi told me several years ago, but is not stated in any of our papers explicitly. Especially (125) shows the close relationship among the discretization, the extrapolation, and the numerical integration, which has been studied by Prof. Isurugi and the author in a series of papers since 1985. In our studies the discretization algorithm via the integration schemes of ordinary differential equations are studied. For exemples, the Runge-Kutta scheme, the Adams-Bashforth scheme, the Predictor-Corrector scheme and so on. The possibility of one-step methods and the multi-step methods are studied with the consideration into the stability problems of the whole system and the simulation examples.

In the following, we will show the relationship between one-step method and multi-step method using (124). The conclusion obtained in section 9 is of very general character. Thus, the problem is how to realize (124). Considering the assumption 3, we know $x(k\Delta)$, $x((k-1)\Delta)$, $x((k-2)\Delta), \ldots$, exactly. Hence, we can calculate $u(x(k\Delta)), u(x(k-1)\Delta), \ldots$ exactly. We have along the trajectory of (126)

$$\begin{aligned} \bar{u}_k^i &:= u^i(x(k\Delta)) \\ \bar{u}_{k-1}^i &:= u^i(x((k-1)\Delta)) \\ &= u^i(x(k\Delta)) - \Delta L_f u^i(x(k\Delta)) + \frac{1}{2!}\Delta^2 L_f^2 u^i(x(k\Delta)) - \cdots \\ \bar{u}_{k-2}^i &:= u^i(x(k\Delta)) - 2\Delta L_f u^i(x(k\Delta)) + \frac{1}{2!}(2\Delta)^2 L_f^2 u^i(x(k\Delta)) - \cdots \end{aligned} \tag{127}$$

If we add these terms multiplied by $\alpha_0, \alpha_1 \alpha_2$ respectively, we have

(128)
$$\alpha_0\bar{u}^i_k + \alpha_1\bar{u}^i_{k-1} + \alpha_2\bar{u}_{k-2} = (\alpha_0+\alpha_1+\alpha_2)u^i(x(k\Delta)) - (\alpha_1+2\alpha_2)L_{\bar{f}}u^i(x(k\Delta))\Delta$$
$$+(\frac{1}{2}\alpha_1 + 2\alpha_2)L^2_{\bar{f}}u^i(x(k\Delta))\Delta^2 - \cdots$$

If we put

(129)
$$\alpha_0+\alpha_1+\alpha_2 = 1,\ -\alpha_1 - 2\alpha_2 = \frac{1}{2},\ \frac{1}{2}\alpha + 2\alpha_2 = \frac{1}{6},$$

we have

(130)
$$\alpha_0 = \frac{23}{12},\ \alpha_1 = -\frac{16}{12},\ \alpha_2 = \frac{5}{12}.$$

With these coefficients, we can express u of (124) as the sum of

(131)
$$u^i \doteqdot \alpha_0\bar{u}^i_k + \alpha_1\bar{u}^i_{k-1} + \alpha_2\bar{u}^i_{k-2} = \frac{23}{12}u^i(x(k\Delta))$$
$$-\frac{16}{12}u^i(x((k-1)\Delta)) + \frac{5}{12}u^i(x((k-2)\Delta)) + O(\Delta^3)$$

where Δ disappears from the coefficients, which is very useful in practice. This is the multi-step algorithm.

If α_0 is set to zero: $\alpha_0 = 0$, we have

(132)
$$\begin{aligned} \alpha_1 + \alpha_2 &= 1 \\ -\alpha_1 - 2\alpha_2 &= \frac{1}{2}. \end{aligned}$$

Then, we have

(133)
$$\alpha_1 = \frac{5}{2},\ \alpha_2 = -\frac{3}{2}$$

with the accuracy of $O(\Delta^2)$.

In this type, the computation time is taken into account by $\alpha_0 = 0$.

11 Concluding remark

The author's formulation of the system

$$\sum : \{M, X, w, u, y\} \tag{134}$$

with

$$y^i(t) - y^i(t_0) = \int_{t_0}^{t} \langle w^i(p).X(p, u(s))\rangle_{p(s)} ds \tag{135}$$

may be different from the usual one.

At least it is certain that this formulation contains the usual formulation as is explained in Section 3. In addition, this formulation is very convenient to derive the integral type expressions of input-output relations such as the Fliess and the Volterra expansion formulation. The key expression is (43) which is a result of direct and straightforward calculations and thus may be found in the previous literatures. If it as so, please inform it to the author. In the derivation of (43), Lie derivatives of functions, vector fields and covector fields are fully made use of, which is only an introductory knowledge of the differential geometry. It is the authors desire that the more fundamental knowledge will be applied to this formulation.

About the discretization problem the expression (43) can be used for the sampled data system with generalized hold functions. The analysis of this case is not done in this report. It is probable that the final form (105) is only a special case of this more general settings.

The following is so obvious and is not fully explained in section 7. Since $\hat{f}(x)$ (64) with state feedback is a vector field

$$L_{\hat{f}}\hat{f} = [\hat{f}, \hat{f}] = 0.$$

Moreover, $\bar{f}(x)$ (75) with constant control is also a vector field and the relation

$$L_{\bar{f}}\bar{f} = 0$$

holds.

Of course, these are the trivial relations. But, for the case where the generalized holder is used, the more general analysis based on (43) should be carried out. The fundamental expressions for the discretization problem are (73) and (77). (77) is derived under the assumption 1 of section 8. In addition, in (73) and (79), the expression by means of the Lie derivatives functions are used. In this respect, (67) and (73) show the relation between (67) and (68). The expressions (73) and (77) is a natural extension of the approach of Isurugi and Shima (1993) and also seems to be the same style of approach with that of Norman-Cyrot (1993).

The calculations used in deriving (89), (92), (99) and (105) are technical. Necessary explanations are often omitted as trivial. For example from (101) to (102). But in view of the simplicitly and the implications of (106) we can expect the existence of the more general and lucid reasonings.

The relations (120) and (125) are also very simple, but a little puzzling why the interpretation as (125) is valid ?

One of the advantages of multi-step algorithms is seen in (131). As to the more detailed informations and results on the multi step algorithms, readers are referred to our papers. Though all are written in Japanese to our regret.

Examples are given in section 11. Our method requires a little complicated calculations and gives a systematic way of estimating the order of approximations. The case where $v^i(t) \neq 0$ is also to be studied in the near future.

Acknowledgments This work is done during the author's stay at INRIA from April to June of 1994 with the scholarship of the Japanese Ministry of Education.

The author would like to express his sincere gratitudes for Prof. Bensoussan, Prof. Delebecque, old acquaintances and colleagues of INRIA who warmly welcomed him after the 20 years of periods. There are many things to be told about the past and the future, poverty and prosperity environments and resources, religions and nations ...

Deep thanks are due to Dr Dirk von Wissel who inspired me by the DAE problem and motivated me to introduce our works on the discretization problem to the members of INRIA, which formed the initial version of this report. Thanks are also extended to the young members of INRIA and the people of Biblioteque who supported me in the operation of computers and in the search of reference books.

I am heartily grateful to Mme Martine Verneuille for her skilful typing of the manuscript with many suffixes. Returning to Japan, I have learned to use the Word 5 and the e-mail, which is another gift from INRIA. Au revoir, au Japon.

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ISSN 0249 - 6399

RR_2518