Cover-Preserving Embeddings of Bipartite Orders into Boolean Lattices

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Abstract: We study the question which bipartite ordered sets are order preserving embeddable into two consecutive levels of a Boolean lattice. This is related to investigations on parallel computer architectures, where bipartite networks are embedded into hypercube networks. In our main Theorem we characterize these orders by the existence of a suited edge-coloring of the covering graph. We analyze the representations of cycle-free orders, crowns and glued crowns and present an infinite family of orders which are not embeddable. Their construction shows that this embeddability is not characterizable by a finite number of forbidden suborders.

Key-words: Embedding, Boolean lattice, interconnection network.

(Résumé : tsvp)

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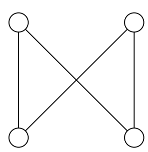
Plongement de couverture des ordres bipartis dans les treillis booléen

Résumé : Quels ordres bipartis sont plongeables dans deux niveaux consécutifs d'un treillis booléen? Cette question est liée à l'étude des architectures de machines parallèles. Dans notre théorème principal nous caractérisons ces ordres par l'existence d'une coloration appropriée d'arêtes du graphe de couverture. Nous analysons les représentations des ordres sans cycles, des couronnes et des couronnes imbriquées et présentons une famille infinie d'ordres non plongeables. Leur construction montre que la plongeabilité ne peut être caractérisée par un nombre fini de sous–ordres interdits.

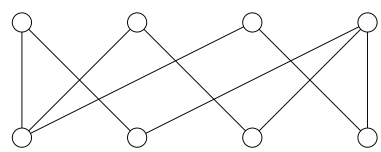
Mots-clé : Plongement, treillis booléen, réseaux d'interconnexion.

1 Introduction

In this paper we study the question which bipartite orders are order preserving embeddable into two consecutive levels of a Boolean lattice. Or to put it in another way : What are the subdiagrams of height one in Boolean lattices? We will often just speak of *representable* orders. A bipartite order, often called height one order, consists only of minimal and maximal elements. These are the orders whose comparability graph is bipartite. The order C_2 (see Figure 1a) is not representable in the sense described above, because it cannot occur as a subdiagram of any lattice. Moreover, a representable order can never contain C_2 as a subdiagram. It was asked in a paper by M. Wild [Wil92] whether this condition is also sufficient to characterize representable orders. We find that this is not the case. The order represented in Figure 1b does not contain C_2 but is yet not representable. In fact this order is the smallest order which is not representable by non–trivial reasons.



(a)



(b)

Figure 1: Not representable orders.

M. Wild used projectivities, which play a crucial role in the theory of congruences of lattices, in order to study cover preserving embeddings of orders into Boolean lattices. In the case of lattices he could give a necessary and sufficient condition. But for bipartite orders the projectivity–method does not work (at least not in a straightforward manner).

In this paper we shall characterize a representable bipartite order by the existence of an edge coloring which fulfils certain conditions. We shall speak of an *admissible coloring*. This is much in the spirit of a result of Havel and Movárek [HM72], who characterized graph embeddability by conditions of an edge coloring. Note however, that they consider the embedding of not necessarily induced subgraphs in contrast to our embedding which deals with the induced suborders of a Boolean lattice. The problem of subgraph embeddability into a hypercube has shown to be NP–complete [CKV87]. Not much seems to be known about the embedding into hypercubes of *induced* subgraphs.

In Section 3 we will show that the order of Figure 1b has no admissible coloring and is therefore not representable as induced suborder of two consecutive levels of a Boolean lattice. Furthermore we shall study the representability of cycle-free orders, crowns and glued crowns. In fact, our order of Figure 1b is an example of two crowns which are glued together in a way which makes it not representable. We will also present an infinite family of orders which are not representable. In Section 4 we discuss a different approach to construct a non-representable order. It is related to the Desargues–configuration of geometry.

An approach to study parallel computer architectures is to consider the ability that a network structure can simulate another network structure. The simulation is modelled by embeddings. A good simulation is said to exist when adjacent processors in the guest network are mapped to reasonable close processors in the host network (see [MS90],[BL92]). In our case we even require that adjacent processors are mapped to adjacent ones. In the language of interconnection networks this is called an embedding with dilation 1. In addition, we require that all processors of the same color class of the bipartite network are mapped to a fixed level in the Boolean network.

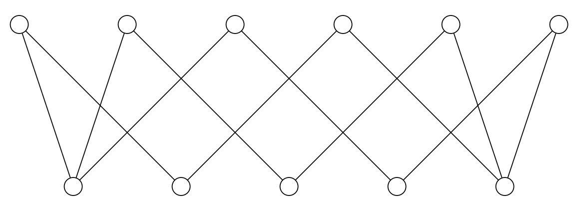


Figure 2: This order has no distance preserving representation.

Let us finally remark that there is a nice characterization of distance preserving subgraphs of hypercubes given by D.Ž. Djoković [Djo73]. In the metric case one has more structure on hand than in our case. The characterization yields also a polynomial time algorithm for deciding whether or not a graph is a distance preserving subgraph of a hypercube. In Figure 2 we give an example of a bipartite order which is representable in our sense but not in a distance preserving way.

2 Main Theorem

Definition 1 *We say that an ordered set $P = (X, \leq_P)$ is embeddable into an ordered set $Q = (Y, \leq_Q)$ if there exists a mapping $f : X \longrightarrow Y$ such that $a \leq_P b \iff f(a) \leq_Q f(b)$ for all $a, b \in X$.*

Definition 2 *A coloring of a bipartite graph $G = (X, E)$ is called* admissible *if it holds:*

(1) If two edges of the same color (respectively just one edge of some color) are/is connected by a path of other colors, then the number of edges of the path must be even.

For every path between x and y it holds:

(2) If $x \neq y$, then the path has at least one color class of odd cardinality.

(3) If the path has exactly one color class of odd cardinality, then $\{x, y\} \in E$.

Note that, since a bipartite graph has no odd cycles, condition (1) implies that on a cycle every color appears an even number of times. In particular, on a path joining the two vertices of an edge has to occur the color of this edge.

Let B_n be the lattice of all subsets of a set $\{1, \ldots, n\}$. For the covering graph of B_n there exists a natural coloring by $g(\{A, B\}) := A \Delta B$ for $\{A, B\} \in E$. The following proposition will be useful later on:

Proposition 1 *Let Γ be a path from the set A to the set B of B_n. The colors of odd cardinality of the coloring g of Γ are exactly the elements of the symmetric difference $A \Delta B$.*

Proof We prove the proposition by induction. If $A = B$, then $A\Delta B = \emptyset$ and the statement holds. Otherwise let C be the set before B on the path from A to B and let $\alpha := C\Delta B$. We have $A\Delta B = (A\Delta C)\Delta(C\Delta B) = (A\Delta C)\Delta\{\alpha\}$. A color $\beta \neq \alpha$ occurs odd times on the path from A to B if and only if it occurs odd times from A to C. But $\beta \in A\Delta B$ is equivalent to $\beta \in A\Delta C$ and the statement follows by induction for the case $\beta \neq \alpha$. The color α occurs odd times on the path from A to B if and only if it occurs even times on the path from A to C. But $\alpha \in A\Delta B$ is equivalent to $\alpha \notin A\Delta C$, and the statement follows by induction. $\square$

Theorem 1 *A bipartite ordered set $P = (X, \leq)$ is order preserving embeddable into two consecutive levels of a Boolean lattice if and only if there exists an admissible coloring of the covering graph (X, E) of P.*

Proof We assume first that P is embedded into a Boolean lattice. Hence there exists a mapping f from P into subsets of cardinality k and $k+1$ of a set Ω such that $a \leq_P b \iff f(a) \subseteq f(b)$. This induces a natural coloring by $g(\{a, b\}) := f(a)\Delta f(b)$ for $\{a, b\} \in E$. We shall show that this coloring is admissible. In order to check color condition (1), consider a path with vertices $x_0, \ldots, x_n$ and $g(\{x_0, x_1\}) = g(\{x_{n-1}, x_n\})$ and $g(\{x_0, x_1\}) \neq g(\{x_i, x_{i+1}\})$ for $i = 1, \ldots, n-2$. Assume that n (i.e. the number of edges of this path) is odd. W.l.o.g. $|f(x_0)| = k$ and $|f(x_n)| = k+1$. Let $\alpha := g(\{x_0, x_1\})$. Then $f(x_1) = f(x_0) \cup \{\alpha\}$ and $f(x_{n-1}) = f(x_n) \setminus \{\alpha\}$. By Proposition 1, $\alpha \notin f(x_1)\Delta f(x_{n-1})$, a contradiction. To prove color condition (2) let $x \neq y$ be two elements of P that are joined by a path Γ. It follows $f(x) \neq f(y)$, since f is an embedding and therefore $f(x)\Delta f(y) \neq \emptyset$. This implies, by Proposition 1, that there is at least one color class of odd cardinality on Γ. Now suppose that there exists exactly one color class of odd cardinality on a path between two elements a and b. Again with Proposition 1 we obtain $|f(a)\Delta f(b)| = 1$. Therefore there exists some α such that $f(a) = f(b) \cup \{\alpha\}$ or $f(b) = f(a) \cup \{\alpha\}$. Hence $\{a, b\} \in E$.

We now turn to the other direction of the proof and assume that there exists an admissible coloring with colors from a set Ω of the covering graph (X, E) of P. Furthermore let us first assume that the ordered set P is connected. For

$x \in \mathrm{Min}(P)$ (resp. $x \in \mathrm{Max}(P)$) we define

$$\begin{aligned} S(x) \;=\; & \{\alpha \in \Omega \mid \text{there exists a path } x = x_0, x_1, \ldots, x_{n+1} \text{ with} \\ & g(\{x_n, x_{n+1}\}) = \alpha \text{ and } g(\{x_i, x_{i+1}\}) \neq \alpha \text{ for } i = 0, \ldots, n-1, \\ & \text{where } n \text{ is odd (resp. even)}\}. \end{aligned}$$

Observe that whether n is odd or even does not depend on a specific path. This follows by color condition (1). Even multiple occurences of vertices or edges on the path does not change the property that the path has even length or not, since there exists no odd cycles. In Figure 3 we show some examples for determining $S(x)$ from a given admissible coloring. For example, no color of an edge adjacent to x is in $S(x)$ for $x \in \mathrm{Min}(P)$, whereas all colors of adjacent edges are in $S(x)$ if $x \in \mathrm{Max}(P)$. We prove the following claim:

Claim: Let $x, y \in X$ and Γ be a path from x to y. The set of colors which occur on Γ with an odd cardinality equals $S(x)\Delta S(y)$.

Proof of the claim: Consider a path starting and ending with the same color α. From condition (1) follows that this path has odd length if and only if α occurs an odd number of times on it. On the path Γ from x to y let e_1 be the first and e_2 the last α–edge.

We have $\alpha \in S(x)\Delta S(y)$ if and only if the number of edges from x to e_1 plus the number of edges from e_2 to y is

(a) odd, if x and y are on the same level,

(b) even, if they are on different levels.

In case (a), the length of Γ is even, hence the length of the path from e_1 to e_2 is odd, i.e., α occurs an odd number of times on Γ. In case (b), the length of Γ is odd and therefore the length from e_1 to e_2 is again odd and α occurs an odd number of times on Γ.

We now turn to the proof that $x \mapsto S(x)$ yields the wanted embedding. Let $x, y \in P$ with $x \leq_P y$ and α be the color of the edge $\{x, y\}$. Applying the claim above it follows that $S(x)\Delta S(y) = \{\alpha\}$. Since $y \in \mathrm{Max}(P)$ and $x \in \mathrm{Min}(P)$, we have $\alpha \in S(y) \setminus S(x)$, hence $S(x) = S(y) \setminus \{\alpha\}$, and therefore $S(x) \subset S(y)$. As a further consequence we get $|S(u)| = |S(v)|$ for $u, v \in \mathrm{Min}(P)$ or $u, v \in \mathrm{Max}(P)$ and $|S(u)| = |S(v)| - 1$ for $u \in \mathrm{Min}(P)$ and $v \in \mathrm{Max}(P)$ by considering a path from u to v.

We now show that $S(x) = S(y)$ implies $x = y$. Assume to the contrary that $x \neq y$ and let Γ be a path from x to y. By color condition (2) there exists at least one odd color class on Γ, which contradicts the fact that $S(x) \Delta S(y) = \emptyset$.

Let $x, y \in P$ and $S(x) \subset S(y)$. Then there exists $\alpha \in \Omega$ with $S(x) = S(y) \setminus \{\alpha\}$, since $|S(x)| = |S(y)| - 1$. Hence $|S(x) \Delta S(y)| = 1$ and color condition (3) yields $\{x, y\} \in E$. Thus $x <_P y$ holds, because $y <_P x$ would imply $S(y) \subset S(x)$.

Finally let us consider the case that P is not connected. The admissible coloring of the covering graph of P leads to admissible colorings for all connected components. Assume that different components use different colors and construct the embedding into a Boolean lattice for each of them. In order to obtain sets of the same size for all minimal elements (resp. for all maximal elements) we might have to fill them up with additional elements. For example, if we have an embedding for one component into level 1 and 2 of a Boolean lattice and for another component into level 2 and 3 of a Boolean lattice, then add a not yet used element x to all the sets of the first component. This way we get an embedding for the whole ordered set into the Boolean lattice B_n where n is the number of elements used in total. $\square$

3 Examples

In this section we apply Theorem 1 to describe, on the one hand, representations for embeddable orders and to argue, on the other hand, why certain orders are not embeddable into two consecutive levels of a Boolean lattice. Note that the color conditions for an admissible coloring can be formulated in a different way. Let $oc(\Gamma)$ be the number of colors which occur an odd number of times on a path Γ.

Lemma 1 *A coloring of a bipartite graph is admissible if and only if:*

(1) If two edges of the same color are connected by a path of other colors, then the number of edges of the path must be even and at least two.

For every path Γ between two elements x and y the following three conditions hold:

(A) $oc(\Gamma) = 0 \iff x = y$

(B) $oc(\Gamma) = 1 \iff x \neq y$ *and* $\{x, y\} \in E$

(C) $oc(\Gamma) \geq 2 \iff x \neq y$ *and* $\{x, y\} \notin E$

Proof Certainly conditions (1),(A),(B) and (C) imply condition (1),(2) and (3) from Definition 2. Now we show that in an admissible coloring the conditions (A),(B) and (C) are satisfied. One direction of (A) follows directly from (2), the other can be proven by means of (1) and the fact that there are no odd cycles. Use condition (A) and (3) to prove (B). Finally, (C) is a direct consequence of (A) and (B). □

Proposition 2 *Let $P = (X, \leq_P)$ be a representable order and $x \in Min(P)$ (or $x \in Max(P)$, respectively). Then $P^a := (X \cup \{a\}, \leq_P \cup \{(x,a)\})$ (or $P_a := (X \cup \{a\}, \leq_P \cup \{(a,x)\})$, respectively) are also representable.*

Proof The covering graph of P has an admissible coloring. Now assign to the only additional edge in the covering graph of P^a (or P_a) a new, not yet used, color. This yields certainly an admissible coloring for P^a (or P_a). □

Corollary 1 *Bipartite tree–like orders (i.e., bipartite cycle–free orders) are representable.*

Therefore every not representable order has to contain a cycle. The orders consisting in only such a cycle are also called *crowns*. Let C_n denote the crown with n minimal and n maximal elements (see Figure 3a). We know that in a cycle, every color has to occur an even number of times. On the other hand, two adjacent edges cannot have the same color. Furthermore, a path of three consecutive edges has to be colored with three different colors. Straightforward

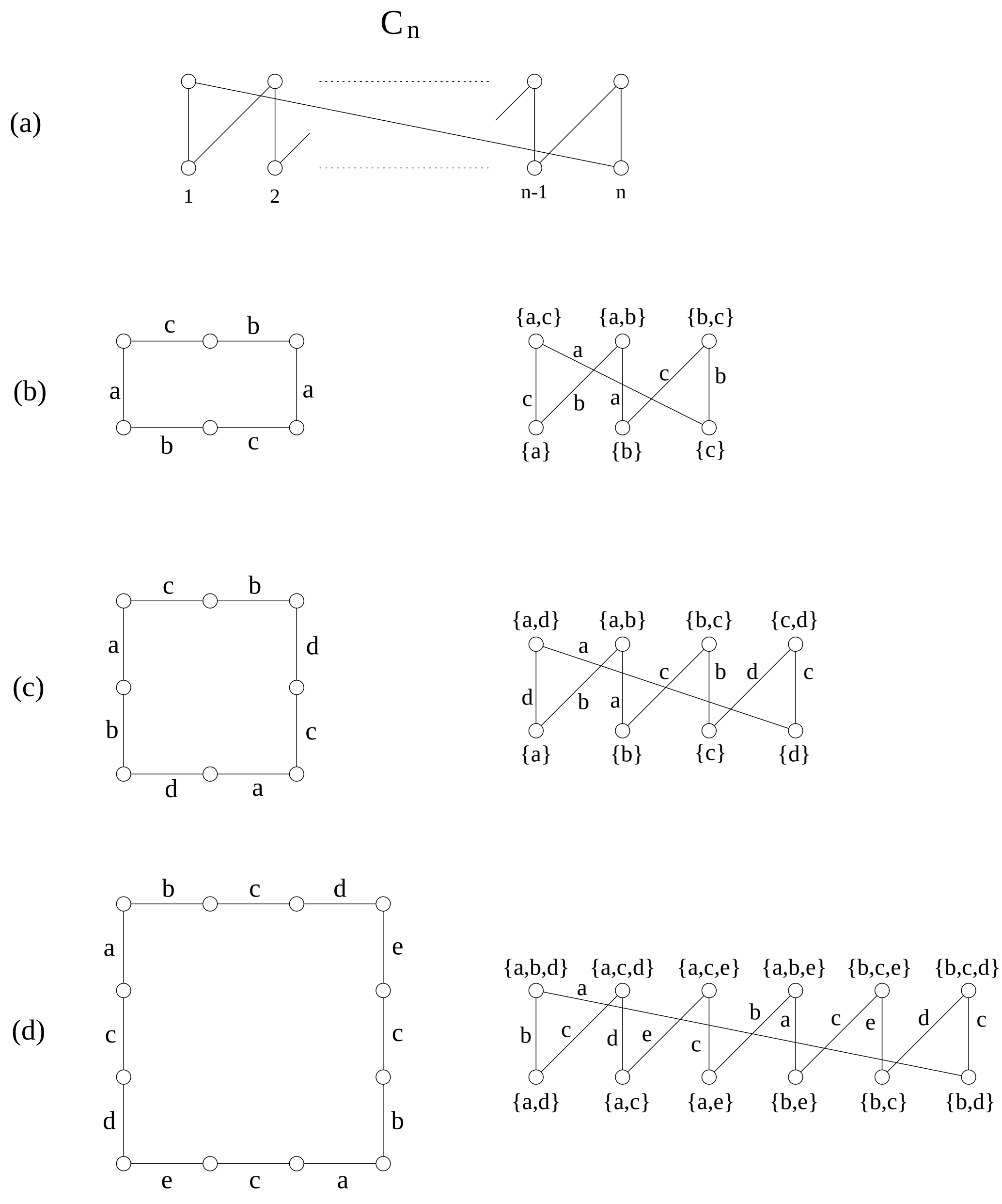


Figure 3: Embedding crowns.

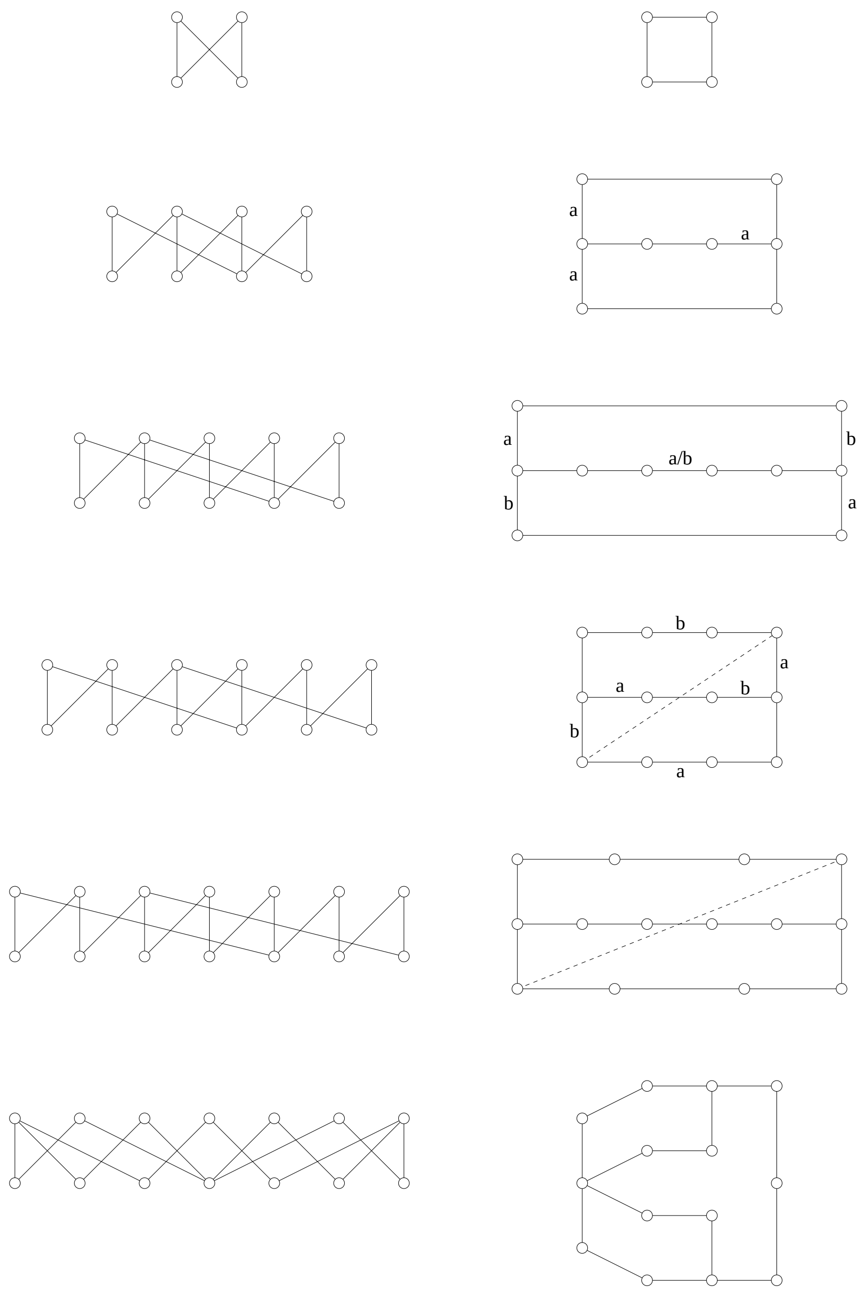


Figure 4: Some critical orders and their covering graphs.

colorings of the covering graphs of C_3 and C_4 using 3 (resp. 4) colors and corresponding embeddings are shown in Figure 3b and 3c. For both orders there is essentially only one way to color the edges. This will help later on, in finding arguments why certain orders containing several crowns are not representable. Note however that from the coloring of the covering graph of C_4 we can deduce two embeddings depending on the choice of the minimal and the maximal elements. One embedding leads to one–element and two–element sets, the other to two–element and three–element sets. In Figure 3d we show a coloring of the covering graph of C_6 using 5 colors. We remark that it is not necessary to use n colors for C_n. In fact, there are a lot of non-isomorphic colorings for a crown with at least 5 minimal elements.

Now let us turn to orders which are not representable. First, we consider orders where two crowns have been glued together once, i.e., they have some consecutive edges in common. Note that such an order contains in total three crowns. We will give arguments for the non-embeddability of the second, third and fourth order of Figure 4. The second order contains three orders isomorphic to C_3. We start with coloring one edge of the upper cycle with a. Then there is only one place on the upper cycle where a can appear a second time. This leads to only one possible edge for the second a on the lower cycle. This edge however is adjacent to the first edge colored by a. A contradiction.

For the third order note that the two upper vertical edges have to be colored different, say a and b. The outer cycle forces the coloring of two remaining vertical edges. In consequence, there is only one place for the second occurence of a on the upper cycle which is the same for b. Again we have a contradiction.

For the fourth order start with coloring two edges on the middle horizontal line with a and b as indicated. Now we have two (isomorphic) possibilities for coloring edges by a and b on the upper and the lower cycle. One is shown in the picture. We observe on a path Γ from the lower left to the upper right corner two edges colored by a and two by b. Therefore $oc(\Gamma) = 1$. The coloring can only be admissible if there would be an edge between the two corners (indicated with a dotted line). Hence the contradiction.

Surprisingly, the order obtained by adding this edge is representable. This stands in contrast to the examples which we have considered up to now: They have — in a sense — too many comparabilities to make them representable.

Also at the fifth order of Figure 4 we observe this phenomenon. The proof for the non-representability of the remaining orders of Figure 4 is left to the reader. All the orders of Figure 4 are *critical*, i.e., removing an arbitrary element would make them representable. These are all critical orders with not more than 14 elements which we have found. It would be interesting to know if there are more.

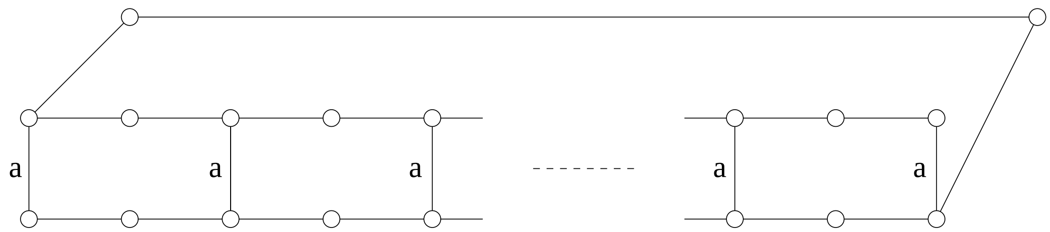


Figure 5: An infinite family of not representable orders.

A natural question to ask is whether there is a finite number of forbidden suborders which ensure representability. This turns out not to be the case. The covering graph of an infinite family of orders which are not representable is shown in Figure 5. If we color the leftmost vertical edge by a then this forces the same color for the other vertical edges. Then we do not find a place for the second a on the long cycle going along the border of the graph.

It cannot be caused by a finite number of forbidden suborders that this infinite family of orders is not representable, since the size of the long cycle grows in the same way as the size of the order grows and every suborder without this cycle is representable. So, there can be no finite family of forbidden suborders characterizing the representability of bipartite orders. It is not even clear if it can be calculated in polynomial time whether an ordered set is representable or not. Therefore we conclude this section by posing the following problem.

Problem 1 *Is it NP-complete to decide whether or not a given bipartite order is embeddable into two consecutive levels of a Boolean lattice?*

4 A different approach

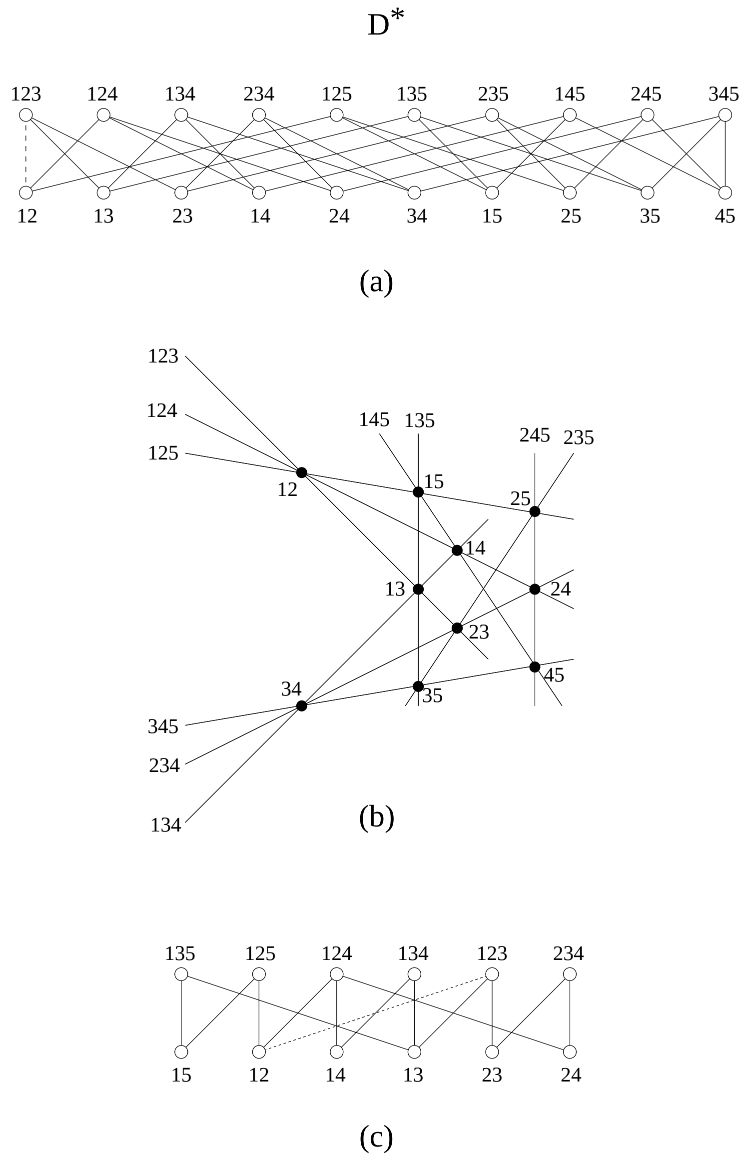


Figure 6

Let D denote the bipartite order consisting of the ten 2–sets and the ten 3-sets of B_5, the Boolean lattice consisting of all subsets of a set with five elements. Let D^* denote the order which arises from D by deleting one comparability, say "$\{1,2\} < \{1,2,3\}$" (see Figure 6a, the removed edge is indicated by a dotted line). Of course, D is trivially representable but D^* is not, which might intuitively be expected. This example, which by the way was the starting point of our investigations, is interesting in several respects. With D^* we again have an example where just one comparability is missing to make it representable. And we will prove that D^* is not representable using arguments of projective geometry.

First, observe that D has a representation by the points and lines of the Desargues configuration (see Figure 6b). A minimal element of D is below a maximal element if and only if the corresponding point is incident with the corresponding line. Let $V_3(\mathbb{R})$ denote the lattice of subspaces of the vectorspace $\mathbb{R}^3$. By the coordinatization theory of projective planes we know that D^* cannot be embedded into $V_3(\mathbb{R})$. Let $B_k^{k+1}(n)$ denote the bipartite order consisting of k–sets and $(k+1)$–sets of B_n. Now it is known that $B_k^{k+1}(n)$ can be embedded into $V_3(\mathbb{R})$. (First, $B_k^{k+1}(n)$ is embedded into $V_n(\mathbb{R})$, the lattice of subspaces of $\mathbb{R}^n$. Then by lifting and projection on suited spaces which are in general position with respect to the finite many embedded ones, one gets the wanted embedding into $V_3(\mathbb{R})$.) Thus if D^* was embeddable into $B_k^{k+1}(n)$, then D^* would be embeddable into $V_3(\mathbb{R})$, a contradiction. To put it in other words: D^* is not even embeddable into two consecutive levels of the lattice of subspaces of a vector space over $\mathbb{R}$.

D^* is not critical with respect to a representation. In fact, a suborder of D^* (see Figure 6c) is isomorphic to one of the not representable orders listed in Figure 4.

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