Weak Algebraic Monge Arrays

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INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

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N° 2501

Mars 1995

PROGRAMME 1

Rapport de recherche

ISSN 0249-6399

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Programme 1 — Architectures parallèles, bases de données, réseaux et systèmes distribués
Projet Archi

Rapport de recherche n°2501 — Mars 1995 — 10 pages

Abstract: An $n \times n$ matrix C is called a *weak Monge* matrix iff $c_{ii} + c_{rs} \leq c_{is} + c_{ri}$ for all $1 \leq i \leq r, s \leq n$. It is well known that the classical linear assignment problem is optimally solved by the identity permutation if the underlying cost-matrix fulfills the weak Monge property.

In this paper we introduce higher dimensional weak Monge arrays and prove that higher dimensional axial assignment problems can be solved efficiently if the cost-structure is a higher dimensional weak Monge array. Additionally, the concept of weak Monge arrays is related to other Monge properties and extended in an algebraic framework. Finally, the problem of testing whether or not a given array can be permuted to become a weak Monge array is solved.

Key-words: Monge array, Monge sequence

(Résumé : tsvp)

*This research has been supported by the Spezialforschungsbereich F 003 "Optimierung und Kontrolle", Projektbereich Diskrete Optimierung and was done while the first author was visiting INRIA, Rocquencourt, France.

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Tableaux de Monge faibles

Résumé : Une matrice $n \times n$ est dite de *Monge faible* ssi $c_{ii} + c_{rs} \leq c_{is} + c_{ri}$ pour tout $1 \leq i \leq r, s \leq n$. Il est bien connu que le problème classique d'affectation linéaire est résolu, de manière optimale, par la permutation identité si la matrice de coût associée satisfait la propriété de Monge faible.

Dans cet article, les tableaux de Monge faibles en dimension supérieure sont introduits; il est prouvé que les problèmes d'affectation linéaire correspondants sont aussi résolus optimalement, sous la condition de Monge faible. De plus, le concept est relié à d'autres propriétés de Monge puis étendu à un cadre algébrique plus général. Finalement, un algorithme de reconnaissance, à une permutation près, d'un tableau de Monge faible est donné.

Mots-clé : tableau de Monge, séquence de Monge

Contents

1 Introduction

Given an $n \times m$ matrix C with real entries, a nonnegative supply vector $a = (a_1, \ldots, a_n)$ and a nonnegative supply vector $b = (b_1, \ldots, b_m)$ such that $\sum_{i=1}^{n} a_i = \sum_{j=1}^{m} b_j$, the classical *Hitchcock* transportation problem (TP) is given by:

$$\min \quad \sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij}x_{ij} \tag{1}$$

$$\text{s.t.} \quad \sum_{j=1}^{m} x_{ij} = a_i \qquad \text{for all } i = 1, \ldots, n \tag{2}$$

$$\sum_{i=1}^{n} x_{ij} = b_j \qquad \text{for all } j = 1 \ldots, m \tag{3}$$

$$x_{ij} \geq 0 \qquad \text{for all } i, j. \tag{4}$$

Given a fixed sequence $\mathcal{S}$ of pairs of indices, the greedy-algorithm which maximizes all variables along of this sequence $\mathcal{S}$ in turn always produces a feasible solution of (TP). Hoffman [10] identified a necessary and sufficient condition for such a sequence $\mathcal{S}$ such that for all vectors a and b the greedy-algorithm is optimal. He named these sequences *Monge sequences* (cf. Monge [11]):

Associated with an $n \times m$ matrix C an ordering $\mathcal{S} := ((i_1, j_1), \ldots, (i_{nm}, j_{nm}))$ of the nm pairs of indices of C is called a *Monge sequence* (with respect to the matrix C) if the following condition holds:

> For every $1 \leq i, r \leq n$ and $1 \leq j, s \leq m$, whenever (i, j) precedes both (i, s) and (r, j) in $\mathcal{S}$, the corresponding matrix entries in C are such that
>
> $$c_{ij} + c_{rs} \leq c_{is} + c_{rj}.$$

A subclass of matrices C which possess a Monge sequence are the so-called *Monge matrices*, i.e. $n \times m$ matrices C which fulfill

$$c_{ij} + c_{rs} \leq c_{is} + c_{rj} \qquad \text{for all } 1 \leq i < r \leq n, 1 \leq j < s \leq m.$$

If we interprete Hoffmann's result on Monge matrices we obtain the following obvious result: the north-west corner rule always produces an optimal solution for all possible supply and demand vectors of (TP).

A special case of (TP) is the *linear assignment* problem *(AP)*, where $n = m$, $a_i = b_j = 1$ for all i, j and the variables x_{ij} are forced to be either 0 or 1. As a direct consequence of the above result for the (TP) it follows that the (AP) restricted to Monge matrices is always solved by the identity permutation. However, Derigs, Goecke and Schrader [7] proved a more general result showing that the identity permutation is always optimal whenever the underlying cost-matrix is a *weak* Monge matrix.

An $n \times m$ matrix C is called a *weak Monge* matrix if

$$c_{ii} + c_{rs} \leq c_{is} + c_{ri} \qquad \text{for all } 1 \leq i < r \leq n, 1 \leq i < s \leq m.$$

In this paper we are interested in higher dimensional Monge structures and especially in higher dimensional weak Monge arrays. Similar as in two dimensions there exist Monge sequences for higher dimensional arrays (cf. Rudolf [13]) and also Monge arrays as a natural extension to more dimensions introduced by Aggarwal and Park [2].

Moreover it is not very surprisingly that the results concerning the (TP) resp. (AP) also hold for Monge properties for higher dimensions. In detail, the following results are known for the d-dimensional axial transportation problem, (dTP), and the d-dimensional axial assignment problem, (dAP).

Theorem 1.1 *(Bein, Brucker, Park and Pathak* [3]*)*
Let C be an $n_1 \times \cdots \times n_d$ Monge array. Then the optimal solution of (dTP) *with respect to the cost array C is found by the lexicographical greedy algorithm.*

Due to the close relationship of (dTP) and (dAP) and the structure of (dAP) the subsequent result is straightforward.

Corollary 1.2 *(Bein, Brucker, Park and Pathak* [3]*)*
Let C be an $n \times \cdots \times n$ Monge array. Then the minimal value of (dAP) *with respect to the cost array C is given by $\sum_{i=1}^{n} c[i, \ldots, i]$.*

In this paper we extend weak Monge matrices to higher dimensions and introduce *weak Monge arrays*. Additionally we also extend the result of Derigs et al. [7] to arbitrary dimensions and relate weak Monge arrays to Monge arrays and arrays possessing Monge sequences. Additionally, we also embed the whole concept of weak Mongeness in an abstract algebraic setting.

Finally the recognition problem of *permuted* weak Monge arrays is also solved, i.e. we present an algorithm for detecting whether or not there exist permutations such that the permuted array fulfills the higher dimensional weak Monge property or not. Related results on the recognition problems of other Monge structures as well as a lot of applications in several fields of mathematics are compiled in Burkard et al. [5].

The paper is organized as follows: In Section 2 we present the definition of d-dimensional weak monge arrays. Section 3 is dedicated to the relationship of weak Mongeness to other Monge properties and in Section 4 the concept is extended to arbitrary algebraic weak Monge arrays. Section 5 contains the main theorem of this paper, namely that each instance of a (dAP) can be solved efficiently if the underlying cost array is weak Monge. Finally, Section 6 deals with the problem of permuting a given array in order to obtain a weak Monge array.

2 Weak Monge Arrays

In this section the notation of a weak (algebraic) Monge array is introduced. To give a better understanding of the main idea we first define 3-dimensional weak Monge arrays and later extend this definition to general d-dimensional weak algebraic Monge arrays.

Definition 2.1 *Let C be an $n_1 \times n_2 \times n_3$ array. Then C is called a (3-dimensional) weak Monge array, iff for all $1 \leq i \leq \min\{n_1, n_2, n_3\}$ and*

(i) for all $1 \leq i < r \leq n_1$, $1 \leq i < s \leq n_2$ and $1 \leq i < t \leq n_3$

- *(a) $c[i,i,i] + c[r,s,t] \leq c[i,i,t] + c[r,s,i]$,*
- *(b) $c[i,i,i] + c[r,s,t] \leq c[i,s,t] + c[r,i,i]$,*
- *(c) $c[i,i,i] + c[r,s,t] \leq c[i,s,i] + c[r,i,t]$ and*

(ii) for all $r_1 \neq r_2$, $s_1 \neq s_2$, $t_1 \neq t_2$ and $1 \leq i < r_\ell \leq n_1$, $1 \leq i < s_\ell \leq n_2$ and $1 \leq i < t_\ell \leq n_3$ for $\ell = 1, 2$ the following inequality is fulfilled:

$$c[i,i,i] + \min_{\phi,\psi} \left\{ c[r_1, s_{\phi(1)}, t_{\psi(1)}] + c[r_2, s_{\phi(2)}, t_{\psi(2)}] \right\} \leq c[i, s_1, t_1] + c[r_1, i, t_2] + c[r_2, s_2, i],$$

where ϕ and ψ are arbitrary permutations of the set $\{1, 2\}$.

Before we define the weak Monge property for higher dimensions we need further definitions and some notations. Let C be an $n_1 \times n_2 \times \cdots \times n_d$ array, $n := \min\{n_k | 1 \leq k \leq d\}$, $(i^1, \ldots, i^d)$ be a fixed feasible d-tuple of indices and $\mathcal{F} := \{(j^1_\ell, \ldots, j^d_\ell) | \ell = 1, \ldots, q\}$ be a set of q feasible d-tuples of indices. Let $J^k(\mathcal{F}) := \{j^k_\ell | (j^1_\ell, \ldots, j^d_\ell) \in \mathcal{F}\}$ be the associated multisets of indices occuring in the set $\mathcal{F}$ in dimension k.

Then $\mathcal{F}$ is a called a *covering* with respect to $(i^1, \ldots, i^d)$, iff for all $k = 1, \ldots, d$ there exists an ℓ, $1 \leq \ell \leq q$, such that $j^k_\ell \in J^k(\mathcal{F})$ and $i^k = j^k_\ell$, i.e. if each index of $(i^1, \ldots, i^d)$ is covered at least once by a tuple contained in the set $\mathcal{F}$. To each covering $\mathcal{F}$ we also define for all $1 \leq k \leq d$ by $I^k(\mathcal{F}) := J^k(\mathcal{F}) \setminus \{i^k\}$ the multisets we get by deleting the index value i^k exactly once from the multiset $J^k(\mathcal{F})$. If each multiset $J^k(\mathcal{F})$ associated with $\mathcal{F}$ has q pairwise distinct elements, $\mathcal{F}$ is said to be *simple*.

If $\mathcal{F}$ is a covering w.r.t. $(i^1, \ldots, i^d)$ and no proper subset of $\mathcal{F}$ is a covering then $\mathcal{F}$ is said to be *minimal*. And $\mathcal{F}$ is called an *upper* covering w.r.t. $(i^1, \ldots, i^d)$ whenever it is minimal and for all $k = 1, \ldots, d$ and $j^k_\ell \in J^k(\mathcal{F})$: $j^k_\ell \geq i^k$.

Now we are ready to define d-dimensional weak Monge arrays:

Definition 2.2 *An $n_1 \times \cdots \times n_d$ array C is called a d-dimensional weak Monge array, iff for all $1 \leq i \leq n$ and $2 \leq q \leq d$ and for each set $\mathcal{F}$ having q feasible d-tuples of indices which is a simple and upper covering with respect to $(i, i, \ldots, i)$ there exist permutations $\phi_1, \ldots, \phi_d$ on the set $\{1, \ldots, q-1\}$ such that*

$$c[i, i, \ldots, i] + \sum_{\ell=1}^{q-1} c[i^1_{\phi_1(\ell)}, i^2_{\phi_2(\ell)}, \ldots, i^d_{\phi_d(\ell)}] \leq \sum_{(j^1_\ell, \ldots, j^d_\ell) \in \mathcal{F}} c[j^1_\ell, \ldots, j^d_\ell] \tag{5}$$

where $i^k_\ell \in I^k(\mathcal{F})$ for all $1 \leq \ell \leq q-1$, $k = 1, \ldots, d$.

3 Relationship to Monge arrays and d-dimensional Monge sequences

This section is dedicated to relate in some sense weak Monge arrays with Monge arrays and arrays possessing a higher dimensional Monge sequence. Precisely we will show that Monge arrays form a proper subclass of weak Monge arrays and that each array possessing a Monge sequence can be permuted in such a way that it becomes a weak Monge array.

To that end we first present an equivalent characterization of Monge arrays using the notation introduced in the previous section.

Lemma 3.1 *A d-dimensional array C is a Monge array iff for all feasible tuples $(i^1, \ldots, i^d)$, $2 \leq q \leq d$ and for each set $\mathcal{F}$ with q feasible d-tuples of indices which is an upper covering with respect to $(i^1, i^2, \ldots, i^d)$ there exist permutations $\phi_1, \ldots, \phi_d$ on the set $\{1, \ldots, q-1\}$ such that*

$$c[i^1, i^2, \ldots, i^d] + \sum_{\ell=1}^{q-1} c[i^1_{\phi_1(\ell)}, i^2_{\phi_2(\ell)}, \ldots, i^d_{\phi_d(\ell)}] \leq \sum_{(j^1_\ell, \ldots, j^d_\ell) \in \mathcal{F}} c[j^1_\ell, \ldots, j^d_\ell] \tag{6}$$

where $i^k_\ell \in I^k(\mathcal{F})$ for all $1 \leq \ell \leq q-1$, $k = 1, \ldots, d$.

Proof. "$\Longleftarrow$": Let $(i^1, \ldots, i^d)$ be a feasible tuple of indices. Since for each upper covering $\mathcal{F}$ with respect to $(i^1, \ldots, i^d)$ condition (6) holds, it especially holds for all upper coverings $\mathcal{F} := \{(s^1, \ldots, s^d), (t^1, \ldots, t^d)\}$. Since $\mathcal{F}$ is a upper covering with respect to $(i^1, \ldots, i^d)$ and $q = 2$, each multiset $I^k(\mathcal{F})$ contains exactly one entry, say j^k, such that $j^k \geq i^k$, or in other words $i^k = \min\{s^k, t^k\}$ and $j^k = \max\{s^k, t^k\}$. But now condition (6) turns into

$$c[i^1, \ldots, i^d] + c[j^1, \ldots, j^d] \leq c[s^1, \ldots, s^d] + c[t^1, \ldots, t^d],$$

for $i^k = \min\{s^k, t^k\}$ and $j^k = \max\{s^k, t^k\}$. Since $\mathcal{F}$ is an arbitrary upper covering w.r.t an arbitrary tuple $(i^1, \ldots, i^d)$, we arrive exactly at the Monge property in d dimensions.

"$\Longrightarrow$": Assume that C is a Monge array. Consider an arbitrary tuple of indices, say $(i^1, \ldots, i^d)$, and let $\mathcal{F} = \{(j^1_\ell, \ldots, j^d_\ell) | 1 \leq \ell \leq q$ be an arbitrary upper covering with respect to $(i^1, \ldots, i^d)$. We show that (6) holds. Let $J^k(\mathcal{F})$ be the multisets associated with $\mathcal{F}$. Then let $B(\mathcal{F})$ be the ordered d-dimensional subarray of C we get by deleting all entries $(s^1, \ldots, s^d)$ in C where there exist at least a k such that $s^k \notin J^k(\mathcal{F})$. Note that since each ordered subarray of a Monge array is Monge, $B(\mathcal{F})$ is Monge. Let now $\tilde{B}$ be the corresponding $q \times \cdots \times q$ subarray of C we get by expanding $B(\mathcal{F})$, i.e. whenever a certain index p occurs α_p times in a multiset $J^k(\mathcal{F})$ we replace the induced $d-1$ dimensional subarray in $B(\mathcal{F})$ by exactly α_p time copies of this subarray. Since $B(\mathcal{F})$ is Monge, $\tilde{B}$ obtained in this way is also Monge (duplicating a subarray does not violate the Monge condition). Since $\mathcal{F}$ is an upper covering the lexicographically smallest tuple of indices occurring in $\tilde{B}$ is just $(i^1, \ldots, i^d)$. Due to Corollary 1.2 the minimal assignment with respect to the cost matrix $\tilde{B}$ is obtained by just summing up the entries of the main diagonal.

Therefore it directly follows that there exist permutations $\phi_1, \ldots, \phi_d$ acting on $\{1, \ldots, q-1\}$ such that

$$c[i^1, i^2, \ldots, i^d] + \sum_{\ell=1}^{q-1} c[i^1_{\phi_1(\ell)}, i^2_{\phi_2(\ell)}, \ldots, i^d_{\phi_d(\ell)}] \leq \sum_{(j^1_\ell, \ldots, j^d_\ell) \in \mathcal{F}} c[j^1_\ell, \ldots, j^d_\ell]$$

where $i^k_\ell \in I^k(\mathcal{F})$ for all $1 \leq \ell \leq q-1$, $k = 1, \ldots, d$. But this completes the proof. □

As a direct consequence of Definition 2.2 and Lemma 3.1 we obtain the following corollary:

Corollary 3.2 *Each Monge array is a weak Monge array.* □

Next the relationship between *permuted* weak Monge arrays and arrays having a d-dimensional Monge sequence is established. We say that an $n_1 \times \cdots \times n_d$ array C is a *permuted* (weak) Monge array whenever there exist d permutations ψ_k on the set $N_k := \{1, \ldots, n_k\}$ such that the permuted array $C_{\psi_1, \ldots, \psi_d} := \left(c[\psi_1(i^1), \ldots, \psi_d(i^d)]\right)$ is a (weak) Monge array.

Arrays possessing d-dimensional Monge sequences were introduced in Rudolf [13] in the following way: First $\mathcal{S}$ is called a d-dimensional sequence with respect to the array C if it is an ordering of all elements of the Cartesian product $N = N_1 \times \cdots \times N_d$. Monge sequences are defined as follows:

Definition 3.1 *$\mathcal{S}$ is called a d-dimensional Monge sequence, if and only if the subsequent condition is satisfied:*
Let $(i^1, \ldots, i^d) \in \mathcal{S}$. Then for each set $\mathcal{F}$ of q feasible d-tuples of indices which is a minimal covering with respect to $(i^1, i^2, \ldots, i^d)$ the following holds:

Whenever $(i^1, \ldots, i^d)$ is the element which occurs first in $\mathcal{S}$ among all elements contained in $M(\mathcal{F}) := \{(s^1, \ldots, s^d) | s^k \in J^k(\mathcal{F}), 1 \leq k \leq d\}$ then there exist permutations $\phi_1, \ldots, \phi_d$ on $\{1, \ldots, q-1\}$ such that

$$c[i^1, \ldots, i^d] + \sum_{\ell=1}^{q-1} c[i^1_{\phi_1(\ell)}, i^2_{\phi_2(\ell)}, \ldots, i^d_{\phi_d(\ell)}] \leq \sum_{(j^1_\ell, \ldots, j^d_\ell) \in \mathcal{F}} c[j^1_\ell, \ldots, j^d_\ell] \tag{7}$$

where $i^k_\ell \in I^k(\mathcal{F})$ for all $1 \leq \ell \leq q-1$, $k = 1, \ldots, d$.

Matrices having a Monge sequence and permuted weak monge matrices are related to each other. More precisely, if an $n \times n$ matrix C possesses a Monge sequence then C is a permuted weak Monge matrix (e.g. Burkard et al. [5]). Now we extend this result to higher dimensions.

Lemma 3.3 *Let C be an $n_1 \times \cdots \times n_d$ array. If $\mathcal{S}$ is a Monge sequence with respect to C, then C is a permuted weak Monge array.*

Proof. We show in the following that, given a d-dimensional Monge sequence $\mathcal{S}$, it is always possible to define permutations $\psi_1, \ldots, \psi_d$ such that $B := C_{\psi_1,\ldots,\psi_d}$ is weak Monge. Let $\mathcal{S} = \{(i_1^1, \ldots, i_1^d), \ldots, (i_p^1, \ldots, i_p^d)\}$ with $p = \prod_{k=1}^d n_k$. Define $\psi_k(1) = i_1^k$ for $k = 1, \ldots, d$, i.e $b[1, \ldots, 1] := c[i_1^1, \ldots, i_1^d]$. Since $\mathcal{S}$ is a Monge sequence and $(i_1^1, \ldots, i_1^d)$ is the first element in $\mathcal{S}$, condition (7) holds for each set $\mathcal{F}$ having q feasible d-tuples of indices which is a simple and minimal covering with respect to $(i_1^1, \ldots, i_1^d)$. No matter how each permutation ψ_k is completed, any such set $\mathcal{F}$ turns into a simple and upper covering with respect to $(1, \ldots, 1)$ in B. And therefore condition (5) is fulfilled for $(1, \ldots, 1)$ and the array B.
Now we are done. We just delete all $d-1$ dimensional subarrays of C which contain $(i_1^1, \ldots, i_1^d)$ and remove at the same time all the corresponding entries from the Monge sequence $\mathcal{S}$. Now an $n_1 - 1 \times \cdots \times n_d - 1$ array C' remains. Since $\mathcal{S}$ was a Monge sequence for C, the remaining sequence $\mathcal{S}'$ is also a Monge sequence for C'. Thus we can recursively define $\psi_k(i)$ for all $1 \leq k \leq d$ and $2 \leq i \leq n$ using this procedure. After n steps – note that $n = \min\{n_k | 1 \leq k \leq d\}$ – all conditions which are necessary for B to be a weak Monge array are fulfilled. Thus we can complete the permutations in an arbitrary way and the lemma is proven. $\Box$

4 Weak Algebraic Monge Arrays

Similar as Monge matrices can be extended to arbitrary algebraic Monge matrices (cf. e.g. Burkard, Klinz and Rudolf [5]), we introduce in this section weak algebraic Monge arrays.

To this end, let $(H, \oplus, \preceq)$ be a totally ordered commutative semigroup such that $\oplus$ is compatible with $\preceq$, i.e.

$$a \preceq b \quad \Longrightarrow \quad a \oplus c \preceq b \oplus c \qquad \text{for all} \;\; a, b, c \in H. \tag{8}$$

Then we call an $n_1 \times \cdots \times n_d$ array C whose elements are taken from $(H, \oplus, \prec)$ a d-dimensional *weak algebraic Monge array*, iff for all $1 \leq i \leq n$, $2 \leq q \leq d$ and for each set $\mathcal{F}$ having q feasible d-tuples of indices which is a simple and upper covering with respect to $(i, i, \ldots, i)$ there exist permutations $\phi_1, \ldots, \phi_d$ on the set $\{1, \ldots, q-1\}$ such that

$$c[i, i, \ldots, i] \bigoplus_{\ell=1}^{q-1} c[i^1_{\phi_1(\ell)}, \ldots, i^d_{\phi_d(\ell)}] \preceq \bigoplus_{(j_\ell^1, \ldots, j_\ell^d) \in \mathcal{F}} c[j_\ell^1, \ldots, j_\ell^d] \tag{9}$$

where $i_\ell^k \in I^k(\mathcal{F})$ for all $1 \leq \ell \leq q-1$, $k = 1, \ldots, d$.

By setting $(H, \oplus, \preceq)$ to $(\mathbb{R}, +, \leq)$ we arrive at weak Monge arrays introduced already in Section 2, other examples for $(H, \oplus, \preceq)$ can be found in Burkard et al. [5].

Defining d-dimensional algebraic Monge sequences in a concise way, i.e. by simply replacing $+$ with $\oplus$ and $\leq$ with $\preceq$ and assuming that all elements of C are drawn from H, it is straightforward that Lemma 3.3 also holds in the general algebraic framework.

5 The Main Result

In this section the main result of this paper is given, namely that each instance of an (algebraic) d-dimensional axial assignment problem can be solved efficiently, if the underlying cost structure fulfills the weak (algebraic) Monge property in d dimensions. This result generalizes Corollary 1.2.

Theorem 5.1 *If the $n \times \cdots \times n$ cost array C of an (algebraic) d-dimensional axial assignment problem is a d-dimensional weak Monge array, then the optimal value is obtained by*

$$\bigoplus_{i=1}^{n} c[i,i,\ldots,i].$$

Proof. Let X denote the solution obtained by setting $x[i,\ldots,i]=1$ for all $i=1,\ldots,n$ and zero otherwise and assume that X is not optimal. Let Y be those optimal solution of (dAP) with respect to C among all optimal solutions such that $i := \min\{j | y[j,\ldots,j] \neq 1\}$ is maximum. Since $Y \neq X$, $i < n-1$. Now investigate the set $\mathcal{F}_1$ of d-tuples defined as $\mathcal{F}_1 := \{(j_\ell^1, j_\ell^2, \ldots, j_\ell^d) | y[j_\ell^1, j_\ell^2, \ldots, j_\ell^d] = 1\}$. Due to the fact that Y is a d-dimensional assignment, $\mathcal{F}_1$ is a covering of $(i,\ldots,i)$. Let now $\mathcal{F} \subseteq \mathcal{F}_1$ be a minimal covering with respect to $(i,\ldots,i)$ and let $q := |\mathcal{F}|$. Again due to the structure of Y and since $y[j,\ldots,j]=1$ for all $1 \leq j < i$, the set $\mathcal{F}$ is a simple and upper covering with respect to $(i,\ldots,i)$. Applying now condition (5), there exist permutations $\phi_1,\ldots,\phi_d$ such that

$$c[i,i,\ldots,i] \bigoplus_{\ell=1}^{q-1} c[i^1_{\phi_1(\ell)},\ldots,i^d_{\phi_d(\ell)}] \preceq \bigoplus_{(j_\ell^1,\ldots,j_\ell^d)\in\mathcal{F}} c[j_\ell^1,\ldots,j_\ell^d]$$

where $i_\ell^k \in I^k(\mathcal{F})$ for all $1 \leq \ell \leq q-1$, $k=1,\ldots,d$. But now we are able to construct a new feasible solution $Y^\star$ in the following way: set $y^\star[i,i,\ldots,i]=1$, $y^\star[j_\ell^1,\ldots,j_\ell^d]=0$ for all $(j_\ell^1,\ldots,j_\ell^d)\in\mathcal{F}$, $y^\star[i^1_{\phi_1(\ell)},\ldots,i^d_{\phi_d(\ell)}]=1$ for all $\ell=1,\ldots,q-1$ and leave all other elements of Y unchanged. Due to the above inequality it is easy to see that $Y^\star$ is also an optimal assignment and since $y^\star[j,\ldots,j]=1$ for all $j=1,\ldots,i$ we have achieved a contradiction to the choice of Y w.r.t. the maximality of i. Thus the theorem is proven. □

6 Recognizing Permuted Algebraic Weak Monge Arrays

Before we investigate the problem of recognizing permuted weak (algebraic) Monge arrays we briefly mention the results and algorithms derived for the two-dimensional case. Given an $n \times n$ matrix C we ask for permutations ϕ and ψ such that $C_{\phi,\psi}$ is a permuted weak (algebraic) Monge matrix.

First of all it is clear that there exists a trivial $O(n^5)$ algorithm to solve this problem (cf. Faigle [9]). A more efficient algorithm with running time $O(n^4)$ which can be improved to $O(n^3 \log n)$ time is due to Cechlárová and Szabó [6]. Another approach is due to Burkard et al. [5]. Although it does not improve the time complexity, it is much simpler and also works in an abstaract algebraic setting. Their algorithm is based on the fact that matrices having a Monge sequence are permuted weak Monge. Therefore their algorithm can be seen as a minor adaption of the algorithm of Alon, Cosares, Hochbaum and Shamir [1] for the detection and construction of Monge sequences. It works as follows:

Start with an empty graph consisting of n^2 nodes corresponding to all pairs of indices. Investigate all pairs (i,j) and (r,s) with $i \neq r$ and $j \neq s$ and add an edge from (i,j) to (r,s) whenever $c_{ij} \oplus c_{rs} \succ c_{is} \oplus c_{jr}$. Set $k = 1$. As long as there exists an isolated node (i,j) in G, set $\phi(k) = i$ and $\psi(k) = j$, increase k by one and update G as follows: Delete the node (i,j), all nodes (i,s) and (r,j) and all incident edges to those nodes. If G is empty, $C_{\phi,\psi}$ is a weak (algebraic) Monge matrix, otherwise C is no permuted weak (algebraic) Monge matrix.

Similar as in two dimensions this idea also works in higher dimensions. Starting from the algorithm for constructing and detecting d-dimensional (weak) Monge sequences given in [13] we are able to modify this algorithm in the same way as before to recognize permuted weak (algebraic) Monge arrays.

For the ease of illustration let us assume w.l.o.g. that we only treat $n \times n \times \cdots \times n$ arrays (an extension to $n_1 \times n_2 \times \cdots \times n_d$ arrays is straightforward).

Again we start by building a directed graph G having n^d nodes corresponding to all possible d-tuples of indices. The arcs of G are constructed in the following way. For each d-tuple of indices $(i^1, \ldots, i^d)$ we investigate all minimal and simple coverings $\mathcal{F}$ w.r.t. $(i^1, \ldots, i^d)$ and look at all *active* inequalities, i.e. we identify each set of d permutations $\phi_1, \ldots, \phi_d$ such that

$$c[i^1, i^2, \ldots, i^d] + \sum_{\ell=1}^{q-1} c[i^1_{\phi_1(\ell)}, i^2_{\phi_2(\ell)}, \ldots, i^d_{\phi_d(\ell)}] < \sum_{(j^1_\ell, \ldots, j^d_\ell) \in \mathcal{F}} c[j^1_\ell, \ldots, j^d_\ell] \tag{10}$$

where $i^k_\ell \in I^k(\mathcal{F})$ for all $1 \leq \ell \leq q-1$, $k = 1, \ldots, d$. Then we draw for each active inequality directed arcs from each d-tuple of indices occurring on the left side of inequality (10) to each d-tuple occurring on the right and associate with each of these arcs a common unique label.

After having constructed G we are able to fix d permutations $\psi_1, \ldots, \psi_d$ such that $C_{\psi_1, \ldots, \psi_d}$ is a weak Monge array step by step (if they exist at all): In each step j, $j = 1, \ldots, n$, we are looking for a node $v \in G$ with indegree equal to zero. If no such node v exists, we can stop, C is no permuted weak Monge array. Otherwise — assuming that $(i^1, \ldots, i^d)$ is the corresponding d-tuple to node v — fix $\psi_k(j) = i^k$ for $k = 1, \ldots, d$. The remaining part in this step concerns the update of the graph G. Besides node $(i^1, \ldots, i^d)$ we also delete all nodes $(j^1, \ldots, j^d)$ with $j^k = i^k$ for at least one k and all incident arcs to these nodes and all arcs having the same label of at least one deleted arc. Note that this is eqivalent to cancelling all active inequalities of type (10) in which at least one index i^k is involved, or in other words we delete all $d-1$ dimensional subarrays of C which contain the entry $(i^1, \ldots, i^d)$.

Finally, after n steps, all permutations are completely determined.

Theorem 6.1 *The algorithm above either determines permutations $\psi_1, \dots, \psi_d$ such that $C_{\psi_1,\dots,\psi_d}$ is a weak Monge array or proves that no such permutations exist.*
Its running time is $O(d^2(d-1)!^d n^{d^2})$.

Proof. The correctness of the algorithm is straightforward: Whenever the algorithm stops while G is non-empty, then each node $v \in G$ has an indegree greater than zero. This is equivalent that all d-tuples of indices occur at least once on the right side of an active inequality of type (10) and therefore no d-tuple satisfies condition (5).

If the algorithm stops with d permutations $\psi_1, \dots, \psi_d$, we have to show that $B := C_{\psi_1,\dots,\psi_d}$ is a weak Monge array. Assume the contrary, i.e. that B is no weak Monge array, then there exists a d-tuple of indices $(i, \dots, i)$ and a simple and upper covering $\mathcal{F}$ w.r.t. $(i, \dots, i)$ such that for all permutations $\phi_1, \dots, \phi_d$

$$b[i, i, \dots, i] + \sum_{\ell=1}^{q-1} b[i^1_{\phi_1(\ell)}, i^2_{\phi_2(\ell)}, \dots, i^d_{\phi_d(\ell)}] > \sum_{(j^1_\ell,\dots,j^d_\ell)\in\mathcal{F}} b[j^1_\ell, \dots, j^d_\ell] \tag{11}$$

where $i^k_\ell \in I^k(\mathcal{F})$ for all $1 \leq \ell \leq q-1$, $k = 1, \dots, d$. Note that this means that the corresponding node $(\psi_1(i), \dots, \psi_d(i))$ in the graph G in Step i has indegree greater than zero, a contradiction to its choice.

Next we prove the complexity bound of the algorithm. First observe that the directed graph G described above can be constructed in $O(d^2(d-1)!^d n^{d^2})$ time. For each fixed d-tuple $(i^1_1, i^2_1, \dots, i^d_1)$ we have $O(n^{d(d-1)})$ possible simple and minimal coverings. For each such covering there are $O((d-1)!^d)$ possible sets of permutations $\phi_1, \dots, \phi_d$ and therefore the same maximal number of active inequalities. And since each active inequality generates at most d^2 arcs, G can be constructed in overall $O(d^2(d-1)!^d n^{d^2})$ time. Since during the deletion steps only arcs and nodes constructed in the initialization step are deleted, we end up with the same overall time complexity. □

Since the above algorithm for recognizing permuted weak Monge arrays is mainly based on condition (11), we can simply replace the operation $+$ by $\oplus$ and $>$ by $\succ$ in (11) and finally arrive at an algorithm for recognizing permuted weak algebraic Monge arrays having the same time complexity.

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Éditeur
INRIA, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)
ISSN 0249- 6399