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A Simple Approach for Multiple Singular Points

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Abstract: In this paper, we present a simple approach for determining multiple limit points and multiple bifurcation points.

Key-words: bifurcation, limit point, singular points, multiple points

(Résumé : tsvp)

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Une Approche Simple pour la Détermination de Points Singuliers Multiples

Résumé : Dans cet article, nous présentons une approche simple du problème de la détermination de points limite multiples et de points de bifurcation multiples

Mots-clé : bifurcation, point limite, points singuliers, points multiples

1 Introduction

We consider a nonlinear functional equation of the form

$$G(u,\lambda)=0, \tag{1}$$

where u is an element of some Banach space X with the norm $\parallel . \parallel$, λ is a real parameter, and G is an operator mapping $X \times R$ into some Banach space Y. It is assumed that a solution (u_0, λ_0) is known at which the Frechet derivative of G with respect to u, denoted by $D_uG_0 = D_uG_0(u_0, \lambda_0)$, is singular. Of course, this is necessary for branching to occur at (u_0, λ_0).

Definition 1 *$(u_0, \lambda_0) \in X \times R$ is called a multiple singular point if*

1. *$G(u_0, \lambda_0) = 0$*

2. *D_uG_0 is singular, 0 is an eigenvalue of D_uG_0 with geometric and algebraic multiplicity m. There exist m linearly independent function $\phi_1, \phi_2, ..., \phi_m$, $\parallel \phi_1 \parallel = 1, i = 1, 2, ..., m$ such that*

$$\begin{array}{l} \ker(D_uG_0) = \ \mathit{span}\ \{\phi_1, \phi_2, ..., \phi_m\}, \\ \mathit{Range}\ (D_uG_0)\ \mathit{is\ closed,\ codim\ Range}\ (D_uG_0) = m. \end{array} \tag{2}$$

Assumption (2) requires D_uG_0 to be a Fredholm operator of index zero. The adjoint or dual operator $D_uG_0^* : Y^* \to X^*$ will have, say

$$\ker(D_uG_0^*) = \ \text{span}\ \{\psi_1^*, \psi_2^*, ..., \psi_m^*\} \tag{3}$$

and then Range (D_uG_0) is characterized by

$$\text{Range}\ (D_uG_0) = \{y \in Y; < \psi_i^*, y > 0, i = 1, 2, ..., m\}$$

We also choose $\psi_i^* \in Y^*, i = 1, 2, ..., m$, to satisfy

$$< \psi_i^*, \phi_j >= \delta_{ij}, \quad i, j = 1, 2, ..., m$$

where $< .,. >$ denotes pair product between X and Y^*.

Definition 2 *An m-multiple singular point (u_0, λ_0) is called an m-multiple limit point (LP) if*

$$D_\lambda G_0 \equiv D_\lambda G(u_0, \lambda_0) \notin \textit{ Range } (D_u G_0)$$

and is called an m-multiple regular bifurcation point (RBP) if

$$D_\lambda G_0 \in \textit{ Range } (D_u G_0). \tag{4}$$

By virtue of (4) there exist a unique solution ϕ_0 of the following equation

$$D_u G_0 \phi_0 + D_\lambda G_0 = 0, \; < \psi_i^*, \phi_0 >= 0, \; i = 1, 2, ..., m. \tag{5}$$

For simplicity, in the sequel we introduce the following matrices and vectors :

$$\begin{array}{ll} A(\xi) = (A_{ij}(\xi))_{i,j=1,2,...,m}, & B = (b_{ij})_{i,j=1,2,...,m}, \\ C = (c_1, c_2, ..., c_m)^T, & D = (d_1, d_2, ...d_m)^T, \end{array} \tag{6}$$

where

$$A_{ij}(\xi) = \sum_{k=1}^{m} a_{ijk} \xi^k, \quad \xi = (\xi^1, \xi^2, ..., \xi^m)^T \in R^m, \tag{7}$$

$$\begin{array}{l} a_{ijk} = a_{ikj} =< \psi_i^*, D_{uu} G_0 \phi_j \phi_k >, \\ b_{ij} =< \psi_i^*, D_u D G_0(\phi_0, 1) \phi_j >=< \psi_i^*, D_{uu} G_0 \phi_0 \phi_j + D_{u\lambda} G_0 \phi_j >, \\ c_i =< \psi_i^*, D^2 G_0(\phi_0, 1)^2 >=< \psi_i^*, D_{uu} G_0 \phi_0 \phi_0 + 2 D_{u\lambda} G_0 \phi_0 + D_{\lambda\lambda} G_0 >, \\ d_i =< \psi_i^*, D_\lambda G_0 > . \end{array}$$

2 The Bifurcation Equations and the Solution Branch

Assume that $G(.,.)$ is smooth enough, and there exists a solution arc $(u(\varepsilon), \lambda(\varepsilon))$ on the branch depending smoothly on some parameter ε, i.e.

$$G(u(\varepsilon), \lambda(\varepsilon)) = 0, \quad |\varepsilon| \leq \varepsilon_0 \tag{8}$$

with $(u(0), \lambda(0)) = (u_0, \lambda_0)$. Differentiating (8) twice with respect to ε and evaluating at $\varepsilon = 0$ we get

$$D_u G_0 \dot{u}(0) + D_\lambda G_0 \dot{\lambda}(0) = 0, \tag{9}$$

$$\begin{aligned} D_u G_0 \ddot{u}(0) + D_\lambda G_0 \ddot{\lambda}(0) = -(D_{uu} G_0 \dot{u}(0) \dot{u}(0) \\ +2 D_{u\lambda} G_0 \dot{u}(0) \dot{\lambda}(0) + D_{\lambda\lambda} G_0 \dot{\lambda}(0) \dot{\lambda}(0)). \end{aligned} \tag{10}$$

The general solution of (9) can be written as

$$\dot{u}(0) = \sum_{i=0}^{m} \xi^i \phi_i, \tag{11}$$

where $\{\phi_1, \phi_2, ..., \phi_m\}$ is a basis of $\ker(D_u G_0)$ and ϕ_0 is defined by (5). It is easy to verify that

$$\xi^0 = \dot{\lambda}(0) \begin{cases} = 0 \text{ for } LP \\ \neq 0 \text{ for } RBP. \end{cases}$$

Using (11) and (7), we find that a necessary condition for the existence of a solution $\ddot{u}(0)$ of (10) is

$$\sum_{j,k=1}^{m} a_{ijk} \xi^j \xi^k + \eta d_i = 0, \; i = 1, 2, ..., m \text{ for } LP, \tag{12}$$

$$\sum_{j,k=1}^{m} a_{ijk} \xi^j \xi^k + 2 \sum_{j=1}^{m} b_{ij} \xi^j \xi^0 + c_i \xi^0 \xi^0 = 0, i = 1, 2, ..., m \text{ for } RBP \tag{13}$$

where $\eta = \ddot{\lambda}(0)$. (12) is called the Limit Point Bifurcation Equation and (13) is called the Algebraic Bifurcation Equation.

On the other hand, in view of smoothness of G there are eigenvalues $\rho_i(\varepsilon)$ and eigenfunctions $\phi_i(\varepsilon)$ of $D_u G(u(\varepsilon), \lambda(\varepsilon))$:

$$D_u G(u(\varepsilon), \lambda(\varepsilon)) \phi_i(\varepsilon) = \rho_i(\varepsilon) \phi_i(\varepsilon), \quad i = 1, 2, ..., m, \tag{14}$$

with $\rho_i(0) = 0, \phi_i(0) = \phi_i, i = 1, 2, ..., m$. Differentiating (14) with respect to ε and evaluating at $\varepsilon = 0$ we obtain

$$\begin{aligned} D_{uu} G \dot{u}(\varepsilon) \phi_i(\varepsilon) + D_{u\lambda} G \dot{\lambda}(\varepsilon) \phi_i(\varepsilon) + D_u G \dot{\phi}_i(\varepsilon) \\ = \dot{\rho}_i(\varepsilon) \phi_i(\varepsilon) + \rho_i(\varepsilon) \dot{\phi}_i(\varepsilon), \quad i = 1, 2, ..., m, \end{aligned} \tag{15}$$

$$(16) \quad D_u G_0 \dot{\phi}_0(0) = \begin{cases} -D_{uu} G_0 \dot{u}(0)\phi_j + \dot{\rho}_j(0)\phi_j & \text{for LP} \\ -D_{uu} G_0 \dot{u}(0)\phi_j - D_{u\lambda} G_0 \phi_j \xi^0 + \dot{\rho}_j(0)\phi_j & \text{for RBP} \end{cases}$$

Applying (11) and (7), we derive from (16) that

$$(17) \qquad -\sum_{k=1}^{m} a_{ijk}\xi^k + \dot{\rho}_j(0)\delta_{ij} = 0, \quad i,j = 1,2,...,m \quad \text{for LP}$$

$$(18) \qquad -\sum_{k=1}^{m} a_{ijk}\xi^k - b_{ij}\xi^0 + \dot{\rho}_j(0)\delta_{ij} = 0, \quad i,j = 1,2,...,m \quad \text{for RBP.}$$

Equations (12), (13) and (17), (18) can be written as :
for the case of the limit point bifurcation equation

$$(19) \qquad A(\xi)\xi + \eta D = 0, \quad \xi^T \xi = 1$$

$$(20) \qquad A(\xi) = \text{ diag } (\dot{\rho}_1(0), \dot{\rho}_2(0), ..., \dot{\rho}_m(0)).$$

For the case of the algebraic bifurcation equation

$$(21) \qquad A(\xi)\xi + B\xi\xi^0 + C\xi^0\xi^0 = 0, \quad \xi^T\xi = 1$$

$$(22) \qquad A(\xi) + B\xi^0 = \text{ diag } (\dot{\rho}_1(0), \dot{\rho}_2(0), ..., \dot{\rho}_m(0)).$$

In a neighborhood of $\varepsilon = 0$, (11) can be written as

$$(23) \qquad \dot{u}(\varepsilon) = \sum_{i=0}^{m} \xi^i(\varepsilon)\phi_i(\varepsilon),$$

$$(24) \qquad \eta(\varepsilon) = \ddot{\lambda}(\varepsilon)$$

with

$$(25) \qquad \xi^i(0) = \xi^i, \ \eta(0) = \eta, \ i = 1,2,...,m.$$

Now we return to (10).

For the case of LP, let $v_0 = \ddot{u}(0)$; v_0 satisfies

$$(26) \qquad \begin{array}{l} D_u G_0 v_0 + \eta D_\lambda G_0 = -D_{uu} G_0 \dot{u}(0)\dot{u}(0), \\ < \psi_i^*, v_0 >= 0, \ i = 1,2,...,m. \end{array}$$

For the case of RBP, let $v_0 = \ddot{u}(0) - \ddot{\lambda}(0)\phi_0; v_0$ satisfies

$$\begin{aligned} & D_u G_0 v_0 = -(D_{uu} G_0 \dot{u}(0)\dot{v}(0) + 2D_{u\lambda} G_0 \dot{u}(0)\xi^0 + D_{\lambda\lambda} G_0 \xi^0 \xi^0), \\ & < \psi_i^*, v_0 >= 0, \; i = 1, 2, ..., m. \end{aligned} \tag{27}$$

It is well known that not all roots of the limit point bifurcation equation and the algebraic bifurcation equation can be guaranteed to generate solution arcs, but isolated roots of the limit point bifurcation equation and algebraic bifurcation equation do generate a solution arc through $(u_0 \lambda_0)$. On the other hand, the root (ξ_0, η_0) of the limit point bifurcation equation is isolated if and only if its Jacobian matrix

$$J_l(\xi_0) = \begin{pmatrix} 2A(\xi_0) & D \\ \xi_0^T & \eta \end{pmatrix} \tag{28}$$

evaluated at the root is nonsingular, and the root (ξ_0, ξ_0^0) of the algebraic bifurcation equation is isolated if and only if its Jacobian matrix

$$J_b(\xi_0, \xi_0^0) = \frac{1}{2} \begin{pmatrix} A(\xi_0) + \xi_0^0 B & B\xi_0 + \xi_0^0 C \\ \xi_0^T & \xi_0^0 \end{pmatrix} \tag{29}$$

evaluated at the root is nonsingular.

Precisely, we have the following theorem ([DK80]).

Theorem 1 *Assume that $G(u, \lambda)$ is a C^k mapping from $X \times R$ into Y, $D_u G_0$ satisfies (2) and $\{\psi_i^*, i = 1, 2, ..., m\}$ satisfies $< \psi_i^*, \phi_j >= \delta_{ij}$. Let (ξ_0, η_0) or (ξ_0, ξ_0^0) be an isolated root of the limit point bifurcation equation or the algebraic bifurcation equation respectively, and let v_0 be the unique solution of (26) or (27). Then there exist $\varepsilon_0 > 0$ and unique functions $(\Phi(\varepsilon), v(\varepsilon), \eta(\varepsilon))$ or $((\Phi(\varepsilon), v(\varepsilon), \xi^0(\varepsilon))$ such that :*
for the limit point bifurcation equation, $\forall |\varepsilon| \leq \varepsilon_0$

$$\begin{aligned} & G(u_0 + \varepsilon\Phi(\varepsilon) + \frac{1}{2}\varepsilon^2 v(\varepsilon), \lambda_0 + \frac{1}{2}\varepsilon^2 \eta(\varepsilon)) = 0, \\ & \Phi(\varepsilon) = \sum_{j=1}^{m} \xi^j(\varepsilon)\phi_j, < \psi_i^*, v(\varepsilon) > 0, i = 1, 2, ..., m, \end{aligned} \tag{30}$$

for the algebraic bifurcation equation, $\forall |\varepsilon| \leq \varepsilon_0$

$$(31) \qquad \begin{aligned} &G(u_0 + \varepsilon\Phi(\varepsilon) + \frac{1}{2}\varepsilon^2 v(\varepsilon), \lambda_0 + \frac{1}{2}\varepsilon^2 \xi^0(\varepsilon)) = 0, \\ &\Phi(\varepsilon) = \sum_{j=1}^{m} \xi^j(\varepsilon)\phi_j, \quad < \psi_i^*, v(\varepsilon) > 0, \quad i = 1, 2, ..., m. \end{aligned}$$

Where the $\xi(\varepsilon), \eta(\varepsilon), \xi^0(\varepsilon), v(\varepsilon)$ *are* C^{k-2} *functions of* ε *with*

$$\xi(0) = \xi_0, \ \eta(0) = \eta_0, \ \xi^0(0) = \xi_0^0, \ v(0) = v_0$$

3 A Simple Approach for the Limit Point and the Regular Bifurcation Point and Isolated Root

Firstly we discusse the relation between isolated roots of the bifurcation equation and the character of the eigenvalues $\rho_i(\varepsilon), i = 1, 2, ..., m$.

Theorem 2 *$(\xi_0, \eta_0 \neq 0)$ is an isolated root of the limit point bifurcation equation if and only if $\dot{\rho}_i(0) \neq 0$, $i = 1, 2, ..., m$, where $\dot{\rho}_i(0)$ defined by (20).*

Proof : It follows from (20) that

$$(32) \qquad \det[A(\xi_0)] = \Pi_{i=1}^m |\dot{\rho}_i(0)|$$

on the other hand, the Jacobian matrix $J_l(\xi_0)$ defined by (28) is nonsingular if and only if

$$A(\xi_0) \text{ is nonsingular}$$

or

$$\eta_0 = 0; \ker(A(\xi_0)) = \text{ span } \{\xi_0\}, \ D \notin \text{ Range } (A(\xi_0)).$$

Hence $(\xi_0, \eta_0 \neq 0)$ is an isolated root of the limit point bifurcation equation if and only if $A(\xi_0)$ is nonsingular, i.e. $\dot{\rho}_i(0) \neq 0, i = 1, 2, ..., m$.

■

Theorem 3 *$(\xi_0, \xi_0^0 \neq 0)$ is an isolated root of the algebraic bifurcation equation if and only if $\dot{\rho}_i(0) \neq 0, i = 1, 2, ..., m$, where $\dot{\rho}_i(0)$ is defined by (22).*

Proof : It follows from (22) that

$$\det[A(\xi_0) + \xi_0^0 B] = \Pi_{i=1}^m |\dot{\rho}_i(0)|. \tag{33}$$

We derive that the Jacobian $J_b(\xi_0, \xi_0^0)$ defined by (29) is nonsingular if and only if

$$A(\xi_0) + \xi_0^0 B \text{ is nonsingular}$$

or

$$\xi_0^0 = 0, \ker(A(\xi_0)) = \text{ span } \{\xi_0\},\ B\xi_0 \notin \text{ Range } (A(\xi_0)).$$

Therefore $(\xi_0, \xi_0^0 \neq 0)$ is an isolated root of the algebraic bifurcation equation if and only if $A(\xi_0) + \xi_0^0 B$ is nonsingular. (33) yields $\dot{\rho}_i(0) \neq 0, i = 1, 2, ..., m$.

■

By theorems 2 and 3 we obtain

Theorem 4 *Along any solution branch generated by an isolated root of the bifurcation equation the real eigenvalues $\rho_i(\varepsilon), i = 1, 2, ..., m$ change sign at (u_0, λ_0). Furthermore, if m is odd, then $\det(D_u G(u(\varepsilon), \lambda(\varepsilon)))$ changes sign.*

Proof : It is easy to derive that a real $\rho_i(\varepsilon)$ changes sign at (u_0, λ_0) because of $\rho_i(0) = 0$ and theorems 2 and 3. If there exist complex eigenvalues they must appear in conjugate pairs, for example, $\alpha(\varepsilon) \pm \beta(\varepsilon)i$. This follows from theorems 2 and 3 and that $\dot{\rho}_i(0) \neq 0, i.e. \dot{\alpha}(0)\dot{\beta}(0) \neq 0$

1. If a pair of complex eigenvalues are still complex when they pass through (u_0, λ_0) then $|\alpha + i\beta||\alpha - i\beta| = \alpha^2 + \beta^2$, it is not change sign.

2. If a pair of complex eigenvalues become two real eigenvalues $\alpha_1(\varepsilon), \alpha_2(\varepsilon)$ when they pass through (u_0, λ_0) then $\beta(0) = 0$ and $\alpha(0) \neq 0$. Hence $\alpha_1(\varepsilon)$ and $\alpha_2(\varepsilon)$ have the same sign, and the sign of $\det(D_u G(u(\varepsilon), \lambda(\varepsilon)))$ does not change.

3. If a real eigenvalue becomes a pair of complex eigenvalues pass through (u_0, λ_0) it is also impossible for the sign of $\det(D_u G(u(\varepsilon), \lambda(\varepsilon)))$ to change.

Because m is odd, the number of eigenvalues which are still real when they pass through (u_0, λ_0) is odd, hence their product will change sign at $\varepsilon = 0$. Therefore, $\det(D_u G(u(\varepsilon), \lambda(\varepsilon)))$ changes sign at $\varepsilon = 0$.

■

4 The case of a Simple Singular Point

We consider the case $m = 1$. For a simple limit point

$$\begin{array}{c} a \equiv a_{111} =< \psi, D_{uu} G_0 \phi\phi > \\ A(\xi) = a\xi,\ \xi = \xi^1,\ \dot{u}(0) = \xi\phi. \end{array} \tag{34}$$

Nonsingular of A implies $a \not\equiv 0$. Then (26) becomes

$$D_u G_0 v_0 + \eta D_\lambda G_0 = D_{uu} G_0 \phi\phi\xi, \tag{35}$$

and $d =< \psi^*, D_\lambda G_0 > \neq 0$. Hence

$$\eta d = -a\xi,$$

i.e.

$$\eta = \ddot{\lambda}(0) = -a\xi/d \neq 0.$$

This means that (u, λ_0) is nondegenerated limit point. So the isolated root of the limit point bifurcation equation is equivalent to the nondegeneration condition for the simple limit point.

For the simple bifurcation point

$$
\begin{array}{lll}
a & \equiv a_{111} & =<\psi^*, D_{uu}G_0\phi\phi>, \\
b & \equiv b_{11} & =<\psi^*, D_uDG_0(\phi_0,1)\phi>, \\
c & \equiv c_1 & =<\psi^*, D^2G_0(\phi_0,1)(\phi_0,1)>.
\end{array}
\tag{36}
$$

The algebraic bifurcation equation becomes

$$
\begin{array}{l}
a\xi^2 + 2b\xi\xi^0 + c\xi^0\xi^0 = 0,\ \xi^2 + (\xi^0)^2 = 1, \\
A(\xi) = a\xi,\ B = b.
\end{array}
\tag{37}
$$

Then $(\xi_0, \xi_0^0 \neq 0)$ is an isolated root of the algebraic bifurcation equation if and only if

$$a\xi_0 + b\xi_0^0 \neq 0. \tag{38}$$

Setting $\chi = \dfrac{\xi_0}{\xi_0^0}$, then χ satisfies (38) and (37), i.e.

$$a\chi + b \neq 0 \tag{39}$$

$$a\chi^2 + 2b\chi + c = 0 \tag{40}$$

If $a = 0$, then $b \neq 0$, and $\chi = -\dfrac{c}{2b}$ is an isolated root. If $a \neq 0$, combining (39) and (40), $\chi \neq -\dfrac{b}{a}$ and we have

$$\chi = -\frac{b}{a} \pm \frac{\sqrt{b^2 - ac}}{a},\ b^2 - ac \neq 0. \tag{41}$$

Equations (41) show that the algebraic bifurcation equation (37) does not have a multiple root.

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