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Abstract: This study deals with the theorical basis of the optimal control methods in primitive variable formulation and penalty function of the Navier-Stokes equations.

Key-words: Navier-Stokes equations, optimal control

(Résumé : tsvp)

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Aspects Mathématiques du Contrôle Optimal dans les Équations de Navier-Stokes

Résumé : Ce travail concerne la base théorique des méthodes de contrôle optimal pour les équations de Navier-Stokes formulées en variables primitives ou pénalisées

Mots-clé : équations de Navier-Stokes, contrôle optimal

1 Introduction

Let $\Omega \subset R^n$ $n = 2$ or 3 be a bounded domain with a Lipschitz continuous boundary $\partial\Omega = \Gamma = \Gamma_1 \cup \Gamma_2$.The stationary Navier-Stokes equations are

$$\begin{cases} -\lambda\Delta \mathbf{u} + (\mathbf{u}\nabla)\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \text{div } \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u}|_{\Gamma_1} = 0, \left(\lambda \dfrac{\partial \mathbf{u}}{\partial n} - p\mathbf{n} \right)_{\Gamma_2} = \mathbf{g} & \text{on } \Gamma, \end{cases} \tag{1}$$

where $\mathbf{u}$ is the velocity vector of the fluid, p is the pressure, $\lambda^{-1} = Re$ is the Reynolds number.

Let Sobolev space be $X = (H^1(\Omega))^n$ with norm $\| \mathbf{u} \|_1^2 = \sum_{i=1}^n \| u_i \|_1^2$ $\forall \mathbf{u} \in X$ and seminorm $|\mathbf{u}|_1^2 = \sum_{i=1}^n |u_i|_1^2, \forall \mathbf{u} \in X$, $X_0 = (H_0^1(\Omega))^n$, $V = \{\mathbf{u}; \mathbf{u} \in X, \mathbf{u}|_{\Gamma_1} = 0\}$ and $V_0 = \{\mathbf{u}; \mathbf{u} \in V, \text{ div } \mathbf{u} = 0\}$.

We introduce the following linear, bilinear and trilinear functionals :

$$a_0(\mathbf{u}, \mathbf{v}) = \lambda(\nabla \mathbf{u}, \nabla \mathbf{v}) = \lambda \sum_{i,j=1}^n \int_\Omega \frac{\partial u^i}{\partial x^j} \frac{\partial v^i}{\partial x^j} dx,$$

$$G(\mathbf{u}, \mathbf{v}) = (\text{div } \mathbf{u}, \text{ div } \mathbf{v}),\ a_\varepsilon(\mathbf{u}, \mathbf{v}) = a_0(\mathbf{u}, \mathbf{v}) + \varepsilon^{-1} G(\mathbf{u}, \mathbf{v}),$$

$$a_1(\mathbf{u}; \mathbf{v}, \mathbf{w}) = ((\mathbf{u}\nabla)\mathbf{v}, \mathbf{w}) = \sum_{i,j=1}^n \int_\Omega u^j \frac{\partial v^i}{\partial x^j} w^j dx,$$

$$< \mathbf{F}, \mathbf{v} > = < \mathbf{f}, \mathbf{v} > + < \mathbf{g}, \mathbf{v} >_{\Gamma_2},$$

where we assume $\mathbf{f} \in V^*$ (the dual space to V) , $\mathbf{g} \in (H^{-\frac{1}{2}}(\Gamma_2))^n$ and $H^{\frac{1}{2}}(\Gamma_2) = \{\gamma_0 g : g \in H^1(\Omega)\}$ is the image of the trace mapping γ_0 on $H^1(\Omega)$

$\forall \mu \in H^{\frac{1}{2}}(\Gamma_2)$, we define

$$\| \mu \|_{\frac{1}{2},\Gamma_2} = \inf_{g \in H^1(\Omega)} \{\| g \|_1, \mu = \gamma_0 g|_{\Gamma_2}\}.$$

$H^{-\frac{1}{2}}(\Gamma_2)$ is the dual space to $H^{\frac{1}{2}}(\Gamma_2)$, so $\forall \mu^* \in H^{-\frac{1}{2}}(\Gamma_2)$,

$$\| \mu^* \|_{-\frac{1}{2},\Gamma_2} = \sup_{\mu \in H^{\frac{1}{2}}(\Gamma_2)} | < \mu^*, \mu >_{\Gamma_2} | / \| \mu \|_{\frac{1}{2},\Gamma_2}$$

where $< .,. >_{\Gamma_2}$ denotes the duality between $H^{\frac{1}{2}}(\Gamma_2)$ and $H^{-\frac{1}{2}}(\Gamma_2)$. It is not difficult to prove that $\forall \mathbf{f} \in V^*, \mathbf{g} \in (H^{-\frac{1}{2}}(\Gamma_2))^n$ $< \mathbf{f}, \mathbf{v} > + < \mathbf{g}, \mathbf{v} >_{\Gamma_2}$, is a continuous linear functional on the space V. Therefore there is a $\mathbf{F} \in V^*$ such that $< \mathbf{F}, \mathbf{v} >=< \mathbf{f}, \mathbf{v} > + < \mathbf{g}, \mathbf{v} >_{\Gamma_2}$ $\forall \mathbf{v} \in V$ and $\| \mathbf{F} \|_* \leq \| \mathbf{f} \|_* + \| \mathbf{g} \|_{-\frac{1}{2},\Gamma_2}$, where $\| \,.\, \|_*$ is the norm of V^*.

In the velocity-pressure formulation, the variational form of (1) is

$$\text{to find } \mathbf{u} \in V_0 \text{ such that } a_0(\mathbf{u}, \mathbf{v}) + a_1(\mathbf{u}; \mathbf{u}, \mathbf{v}) =< \mathbf{F}, \mathbf{v} > \quad \forall \mathbf{v} \in V_0. \tag{2}$$

In the penalty function formulation, the variational form of (1) is

$$\text{to find } \mathbf{u}_\varepsilon \in V \text{ such that } a_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{v}) + a_1(\mathbf{u}_\varepsilon; \mathbf{u}_\varepsilon, \mathbf{v}) =< \mathbf{F}, \mathbf{v} > \quad \forall \mathbf{v} \in V, \tag{3}$$

where $\varepsilon p_\varepsilon + \text{ div } \mathbf{u}_\varepsilon = \mathbf{0}, \quad \varepsilon$ is the penalty parameter.

In both case, the variational form of (1) can be written as

$$\text{to find } \mathbf{u} \in H \text{ such that } A(\mathbf{u}, \mathbf{v}) + a_1(\mathbf{u}; \mathbf{u}, \mathbf{v}) =< \mathbf{F}, \mathbf{v} > \quad \forall \mathbf{v} \in H \tag{4}$$

In the case of (2), $H = V_0$ and $A(\mathbf{u}, \mathbf{v}) = a_0(\mathbf{u}, \mathbf{v})$; in the case of (3) $H = V$ and $A(\mathbf{u}, \mathbf{v}) = a_\varepsilon(\mathbf{u}, \mathbf{v})$.

2 Optimal Control Formulation

We define the functional $J(\mathbf{v})$ by

$$J(\mathbf{v}) = A(\mathbf{v} - \xi, \mathbf{v} - \xi)/2, \tag{5}$$

where ξ is a solution of the Stokes equation

$$\xi \in H,\ A(\xi, \eta) =< \mathbf{F}, \eta > -a_1(\mathbf{v}; \mathbf{v}, \eta), \quad \forall \eta \in H. \tag{6}$$

The optimal control problem associated with (4) is

$$\text{find } \mathbf{u} \in H \text{ such that } J(\mathbf{u}) = \min_{\mathbf{v} \in H} J(\mathbf{v}). \tag{7}$$

It is obvious that (6), (7) have the structure of an optimal control problem where $\mathbf{v}$ is the control vector and ξ is the state vector. (6) is a state equation, while the functional (5) is a cost function.

It is clear that if $\mathbf{u}$ satisfies (4), then $\mathbf{u}$ is also the solution of (5), (6), (7). Conversely, if $\mathbf{u}$ is a solution of (5) (6), (7) such that $J(\mathbf{u}) = 0$ then $\mathbf{u}$ also satisfies (4). The trilinear form $a_1(\mathbf{u}; \mathbf{v}, \mathbf{w})$ is continuous on H. So we can introduce the norm of $a_1(\mathbf{u}; \mathbf{v}, \mathbf{w})$ as

$$N = \sup_{u,v,w \in H} \frac{|a_1(\mathbf{u}; \mathbf{v}, \mathbf{w})|}{|\mathbf{u}|_1 |\mathbf{v}|_1 |\mathbf{w}|_1},$$

and we have

$$|a_1(\mathbf{u}; \mathbf{v}, \mathbf{w})| \leq N |\mathbf{u}|_1 |\mathbf{v}|_1 |\mathbf{w}|_1. \tag{8}$$

When meas $\Gamma_2 = 0, a_1(\mathbf{u}; \mathbf{v}, \mathbf{w})$ is antisymmetric with respect to $\mathbf{v}, \mathbf{w}$. i.e.

$$a_1(\mathbf{u}; \mathbf{v}, \mathbf{w}) = -a_1(\mathbf{u}; \mathbf{w}, \mathbf{v}) \quad \forall \mathbf{u} \in V_0, \mathbf{v}, \mathbf{w} \in X. \tag{9}$$

(8) shows that, $\forall \mathbf{u} \in H, \exists \mathbf{h}(\mathbf{u}) \in H^*$ (the dual space to H), such that

$$< \mathbf{h}(\mathbf{u}), \mathbf{v} >=< \mathbf{F}, \mathbf{v} > -a_1(\mathbf{u}; \mathbf{u}, \mathbf{v}). \tag{10}$$

So

$$\| \mathbf{h}(\mathbf{u}) \|_* = \sup_{\mathbf{v} \in H} | < \mathbf{h}(\mathbf{u}), \mathbf{v} > | / |\mathbf{v}|_1 \leq \| \mathbf{F} \|_* + N |\mathbf{u}|_1^2. \tag{11}$$

Especially, if $\mathbf{F} \in (L^{\frac{4}{3}}(\Omega))^n$, then $\forall \mathbf{u} \in H, \mathbf{h}(\mathbf{u}) \in (L^{\frac{4}{3}}(\Omega))^n$,

$$\| \mathbf{h}(\mathbf{u}) \|_{0,\frac{4}{3}} = \sup_{v \in L^4(\Omega)} \frac{| < \mathbf{F}, \mathbf{v} > -a_1(\mathbf{u}; \mathbf{u}, \mathbf{v})|}{\| \mathbf{v} \|_{0,4}} \leq \| \mathbf{F} \|_{0,\frac{4}{3}} + C |\mathbf{u}|_1^2 \tag{12}$$

in view of the Sobolev embedding theorem $L^4(\Omega) \hookrightarrow H^1(\Omega)$, we have

Lemma 1 *The mapping defined by (10)* $\mathbf{u} \to \mathbf{h}(\mathbf{u})$ *is a continuous operator from* H *into* H^**. When* $\mathbf{F} \in (L^{\frac{4}{3}}(\Omega))^n$ *it is also continuous from* H *into* $(L^{\frac{4}{3}}(\Omega))^n$*. In the meantime the following estimates hold*

$$\begin{aligned} \| \mathbf{h}(\mathbf{u_1}) - \mathbf{h}(\mathbf{u_2}) \|_* &\leq N(|\mathbf{u}_1|_1 + |\mathbf{u}_2|_1)|\mathbf{u}_1 - \mathbf{u}_2|_1, \quad \forall \mathbf{u}_1, \mathbf{u}_2 \in H, \\ \| \mathbf{h}(\mathbf{u_1}) - \mathbf{h}(\mathbf{u_2}) \|_{0,\frac{4}{3}} &\leq C(|\mathbf{u}_1|_1 + |\mathbf{u}_2|_1)|\mathbf{u}_1 - \mathbf{u}_2|_1, \quad \forall \mathbf{u}_1, \mathbf{u}_2 \in H. \end{aligned}$$

The bilinear functionnal $A(\mathbf{u},\mathbf{v})$ is continuous and coercive on $H\times H$

$$|A(\mathbf{u},\mathbf{v})|\le M|\mathbf{u}|_1|\mathbf{v}|_1,\quad |A(\mathbf{u},\mathbf{v})|\ge\gamma|u|_1^2.$$

The introduction of (10) into (6) leads to

$$\text{find } \xi\in H \text{ such that } A(\xi,\eta)=<\mathbf{h}(\mathbf{v}),\eta>\quad\forall\eta\in H. \tag{13}$$

There exists a unique solution $\xi=T\mathbf{v}$ for (13) according to Lax-Milgram's theorem, so the operator $\mathbf{v}\to\xi=T\mathbf{v}$ defined by (13) is a mapping from H into H, and

$$|T\mathbf{v}|_1\le\nu^{-1}\parallel\mathbf{h}(\mathbf{v})\parallel_*\le\nu^{-1}(\parallel F\parallel_*+N|\mathbf{v}|_1^2)\quad\forall\mathbf{v}\in H. \tag{14}$$

In addition, when $\mathbf{F}\in(L^{\frac{4}{3}}(\Omega))^n$ and meas $\Gamma_2=0$, then $T\mathbf{v}=\xi\in(H^{2,\frac{4}{3}}(\Omega))^n$ and

$$\parallel T\mathbf{v}\parallel_{2,\frac{4}{3}}=C\parallel\mathbf{h}(\mathbf{v})\parallel_{0,\frac{4}{3}}\le C(\parallel\mathbf{F}\parallel_{0,\frac{4}{3}}+|\mathbf{v}|_1^2)\quad\forall\mathbf{v}\in H \tag{15}$$

In other words, T is a mapping from H into $(H^{2,\frac{4}{3}}(\Omega))^n$. Using lemma 1 and Lax-Milgram's theorem we can obtain

Lemma 2 *Suppose Ω is a bounded domain of R^n with Lipschitz boundary Γ. Then the mapping T defined by (13) is continuous from H into H. If the boundary is of class $C^2,\mathbf{F}\in(L^{\frac{4}{3}}(\Omega))^n$, and meas $\Gamma_2=0$, then T is also continuous from H into $(H^{2,\frac{4}{3}}(\Omega))^n$ and the following hold*

$$\begin{array}{lll}|T\mathbf{v}_1-T\mathbf{v}_2|_1 & \le N\nu^{-1}(|\mathbf{v_1}|_1+|\mathbf{v}_2|_1)|\mathbf{v_1}-\mathbf{v}_2|_1 & \forall\mathbf{v_1},\mathbf{v}_2\in H\\ \parallel T\mathbf{v}_1-T\mathbf{v}_2\parallel_{2,\frac{4}{3}} & \le C(|\mathbf{v_1}|_1+|\mathbf{v}_2|_1)|\mathbf{v_1}-\mathbf{v}_2|_1 & \forall\mathbf{v_1},\mathbf{v}_2\in H.\end{array}$$

From lemmas 1 and 2 we immediately obtain

Lemma 3 *The operator T is compact from H into H.*

Lemma 4 *Suppose Ω is a bounded domain of R^n with Lipschitz boundary $\Gamma=\Gamma_1\cup\Gamma_2,H=V_0$. Then*

1. *the* $\mathbf{u} \to a_1(\mathbf{u};\mathbf{u},\mathbf{v})$ *is weakly continuous in* H*, i.e.*

$$\begin{aligned} &\mathbf{u}_m \to \mathbf{u} \text{ (weakly) in } H \\ &\Longrightarrow \lim_{m\to\infty} a_1(\mathbf{u}_m;\mathbf{u}_m,\mathbf{v}) = a_1(\mathbf{u};\mathbf{u},\mathbf{v}) \quad \forall \mathbf{v} \in H. \end{aligned} \tag{16}$$

2. *the operator* T *is weakly continuous from* H *into* H *i.e.*

$$\mathbf{u}_m \to \mathbf{u} \text{ (weakly) in } H \Longrightarrow T\mathbf{u}_m \to T\mathbf{u} \text{ (weakly) in } H. \tag{17}$$

<u>Proof</u>

1. Since the embedding operator $H \hookrightarrow (L^2(\Omega))^n$ is compact and an arbitrary linear compact operator from a reflexive Banach space into a Banach space is certainly a strongly continuous operator, so $\mathbf{u}_m \to \mathbf{u}$ (strongly) in $(L^2(\Omega))^n$.

   Now, $\forall \mathbf{u} \in H, \forall \mathbf{v}, \mathbf{w} \in X$,

$$a_1(\mathbf{u};\mathbf{v},\mathbf{w}) + a_1(\mathbf{u};\mathbf{w},\mathbf{v}) = \oint_\Gamma (\mathbf{v}\mathbf{w})(\mathbf{u}\mathbf{n})ds - \int_\Omega \mathbf{v}\mathbf{w} \text{ div } \mathbf{u} dx = \int_{\Gamma_2} (\mathbf{v}\mathbf{w})(\mathbf{u}\mathbf{n})ds$$

   Let $\mathbf{w} \in (\mathcal{D}(\Omega))^n$, then

$$\begin{aligned} &|a_1(\mathbf{u}_m;\mathbf{u}_m,\mathbf{w}) - a_1(\mathbf{u};\mathbf{u},\mathbf{w})| \\ &= |a_1(\mathbf{u}-\mathbf{u}_m;\mathbf{w},\mathbf{u}) + a_1(\mathbf{u}_m;\mathbf{w},\mathbf{u}-\mathbf{u}_m) + \int_{\Gamma_2} ((\mathbf{u}_m - \mathbf{u})\mathbf{w})(\mathbf{u}_m\mathbf{n})ds \\ &- \int_{\Gamma_2} (\mathbf{u}\mathbf{w})((\mathbf{u}_m - \mathbf{u})\mathbf{n})ds| \\ &\leq N \parallel \mathbf{u} - \mathbf{u}_m \parallel_{0,2} |\mathbf{w}|_{1,4}(\parallel \mathbf{u} \parallel_{0,4} + \parallel \mathbf{u}_m \parallel_{0,4}) \\ &+ C \parallel \mathbf{u}_m - \mathbf{u} \parallel_{0,\frac{8}{3},\Gamma} |\mathbf{w}|_{0,\frac{8}{3},\Gamma}(|\mathbf{u}|_{0,4,\Gamma} + |\mathbf{u}_m|_{0,4,\Gamma}). \end{aligned}$$

   It is well known that as $q < 2(n-1)/(n-2)$, $H^1(\Omega) \hookrightarrow L^q(\Gamma)$ is compact, and $H^1(\Omega) \hookrightarrow L^4(\Omega)$. From this we have $\mathbf{u}_m \to \mathbf{u}$ (strongly) in $(L^{\frac{8}{3}}(\Gamma))^n$ and

$$\begin{aligned} &|a_1(\mathbf{u}_m;\mathbf{u}_m,\mathbf{w}) - a_1(\mathbf{u};\mathbf{u},\mathbf{w})| \\ &\leq CN|\mathbf{w}|_{1,4}(|\mathbf{u}|_1 + |\mathbf{u}_m|_1) \parallel \mathbf{u} - \mathbf{u}_m \parallel_{0,2} + C|\mathbf{w}|_1(|\mathbf{u}|_1 + |\mathbf{u}_m|_1) \parallel \mathbf{u}_m - \mathbf{u} \parallel_{0,\frac{8}{3}\Gamma}. \end{aligned}$$

Therefore

$$\lim_{m\to\infty} a_1(\mathbf{u}_m;\mathbf{u}_m,\mathbf{w}) = a_1(\mathbf{u};\mathbf{u},\mathbf{w})$$

in view of the boundedness of $\{\mathbf{u_m}\}$ in H. Thus (16) holds by virtue of the density of $(\mathcal{D}(\Omega))^n$ in H.

2. It is easy to obtain (17).

Lemma 5 *Under the conditions of lemmas 3 and 4 and* $\mathbf{F} \in (L^{\frac{4}{3}}(\Omega))^n$*, the operator* T *is strongly continuous from* H *into* H*, i.e.*

$$\mathbf{u}_m \to \mathbf{u} \text{ (weakly) in } H \Longrightarrow T\mathbf{u}_m \to T\mathbf{u} \text{ (strongly) in } H. \tag{18}$$

Proof If (18) is not true. Then there exists a subsequence $\{\mathbf{u}_{mp}\}$ such that

$$|T\mathbf{u} - T\mathbf{u}_{mp}|_1 > \varepsilon \quad \forall\varepsilon > 0. \tag{19}$$

But we have $\mathbf{u}_{mp} \to \mathbf{u}$ in H and hence $\{\mathbf{u}_{mp}\}$ is uniformly bounded in H. As T is compact, there exists a subsequence $\{\mathbf{u}_{mq}\}$ of $\{\mathbf{u}_{mp}\}$ with $T\mathbf{u}_{mq} \to \mathbf{w}$ in H. On the other hand, $\mathbf{u}_{mq} \to \mathbf{u} \Longrightarrow T\mathbf{u}_{mq} \to Tu$ by virtue of (17). As the weak limit is unique we conclude that $\mathbf{w} = T\mathbf{u}$, and thus $T\mathbf{u}_{mq} \to T\mathbf{u}$ in H. This contradicts (19), and so we conclude that $T\mathbf{u}_m \to T\mathbf{u}$ in H, i.e. T is strongly continuous.

Lemma 6 *The operator* T *is Gateaux differentiable everywhere in* H*, i.e.*

$$A(T'(\mathbf{u})\mathbf{w},\mathbf{v}) = a_1(\mathbf{u};\mathbf{w},\mathbf{v}) - a_1(\mathbf{w};\mathbf{u},\mathbf{v}) \quad \forall\mathbf{v},\mathbf{w} \in H. \tag{20}$$

and $T'(\mathbf{u})$ *is Lipschitz continuous in* H*,*

$$\| T'(\mathbf{u}_1) - T'(\mathbf{u}_2) \| \leq 2N\nu^{-1}|\mathbf{u}_1 - \mathbf{u}_2|_1 \quad \forall\mathbf{u}_1,\mathbf{u}_2 \in H \tag{21}$$

Furthermore, if the conditions of lemma 3 are satisfied, $\forall\mathbf{u} \in H$*,* $T'(\mathbf{u})$ *is compact for* $H \to H$*.*

Proof It is not difficult to obtain (20).

To prove (21) we observe that

$$A((T'(\mathbf{u}_1) - T'(\mathbf{u}_2))\mathbf{w}, \mathbf{v}) = a_1(\mathbf{u}_2 - \mathbf{u}_1; \mathbf{w}, \mathbf{v}) + a_1(\mathbf{w}; \mathbf{u}_2 - \mathbf{u}_1, \mathbf{v}).$$

Hence

$$\| T'(\mathbf{u}_1) - T'(\mathbf{u}_2) \| \leq \nu^{-1} \sup_{v,w\in H} \frac{|a_1(\mathbf{u}_2 - \mathbf{u}_1; \mathbf{w}, \mathbf{v}) + a_1(\mathbf{w}; \mathbf{u}_2 - \mathbf{u}_1, \mathbf{v})|}{|\mathbf{w}|_1 |\mathbf{v}|_1}.$$

Employing (8), we obtain (21).

By virtue of (12) and the regular theorem for Stokes problems, $T'(\mathbf{u})\mathbf{w} \in (H^{2,\frac{4}{3}}(\Omega))^n \hookrightarrow H$. Hence $T'(\mathbf{u})$ is compact.

3 The Existence Theorem and the Minimizing Sequence

The variational problem (6) is equivalent to the operator equation

$$\mathbf{u} = T\mathbf{u}. \tag{22}$$

That $\mathbf{u}$ is a solution of (6) if and only if $\mathbf{u}$ is a fixed point of T.

Theorem 1 *Suppose $\mathbf{F} \in H^*$ and*

$$4N\nu^{-2} \| \mathbf{F} \|_* < \mathbf{1}. \tag{23}$$

Let $K = \{\mathbf{u}; \mathbf{u} \in H, |\mathbf{u}|_1 \leq \mathbf{2}\nu^{-1} \| \mathbf{F} \|_*\}$. Then there exists a unique fixed point of T in the set K.

Proof First, we prove that T is a mapping $K \to K$. If $\mathbf{v} \in K$, then

$$|T\mathbf{v}|_1 \leq \nu^{-1}(\| \mathbf{F} \|_* + N|\mathbf{v}|_1^2) \leq \nu^{-1}(\| \mathbf{F} \|_* + 4N\nu^{-2} \| \mathbf{F} \|_*^2) \leq 2\nu^{-1} \| \mathbf{F} \|_*$$

by virtue of (14) and (23). So $T : K \to K$.

Secondly, T is a contraction mapping in K. If $\mathbf{v_1}, \mathbf{v_2} \in K$, then in view of (23)

$$|T\mathbf{v_1} - T\mathbf{v_2}|_1 \leq N\nu^{-1}(|\mathbf{v_1}|_1 + |\mathbf{v_2}|_1)|\mathbf{v_1} - \mathbf{v_2}| < |\mathbf{v_1} - \mathbf{v_2}|_1$$

So, T is a contraction mapping $K \to K$. We conclude that the mapping T has a unique fixed point in K.

It is well know that if $A(\mathbf{u}, \mathbf{v}) = a_0(\mathbf{u}, \mathbf{v}), H = V_0$ and meas $\Gamma_2 = 0$, then problem (22) has at least one solution $\mathbf{u}^*$ which satisfies

$$|\mathbf{u}^*|_{\mathbf{1}} \leq \nu^{-\mathbf{1}} \parallel \mathbf{F} \parallel_* . \tag{24}$$

In addition, if $4N\nu^{-2} \parallel \mathbf{F} \parallel_* < \mathbf{1}$, then problem (22) has a unique solution which satisfies (24).

In another case, we have

Theorem 2 *Suppose* $A(\mathbf{u}, \mathbf{v}) = a_\varepsilon(\mathbf{u}, \mathbf{v}), \mathbf{F} \in H^*, H = V$, *meas* $\Gamma_2 = 0$, *and the constants* α, β *are such that*

$$\nu > \alpha > 0, \ \textit{and} \ \nu - C\beta/2 \geq \alpha > 0$$

where C *is a Sobolev embedding constant*

$$\parallel \mathbf{u} \parallel_{0,4} \leq C|\mathbf{u}|_1 \quad \forall \mathbf{u} \in X, n \geq 4.$$

Then the penalty variational problem (3) for the Navier-Stokes equation has a unique solution in set $K_1 = \{\mathbf{u}; \mathbf{u} \in V, \parallel \textit{div } \mathbf{u} \parallel_0 \leq \beta, |\mathbf{u}|_1 \leq \alpha^{-1} \parallel \mathbf{F} \parallel_*\}$ *if*

1.

$$\alpha^{-1}N \parallel \mathbf{F} \parallel_* < \mathbf{1}, \tag{25}$$

2. *the penalty parameter* ε *is small enough such that*

$$0.5(\varepsilon/\alpha)^{\frac{1}{2}} \parallel \mathbf{F} \parallel_* \leq \beta. \tag{26}$$

<u>Proof</u> It is easy to prove that

$$|a_1(\mathbf{u};\mathbf{v},\mathbf{v})| \leq 0.5C|\mathbf{v}|_1^2 \parallel \text{ div } \mathbf{u} \parallel_0 \quad \forall \mathbf{u},\mathbf{v} \in H \tag{27}$$

Let $C_u(\mathbf{w},\mathbf{v}) = a_\varepsilon(\mathbf{w},\mathbf{v}) + a_1(\mathbf{u};\mathbf{w},\mathbf{v})$. Then $\forall \mathbf{u} \in K_1$

1. $C_u(.,.)$ is a continuous bilinear form on $H \times H$

2. $C_u(.,.)$ is H - elliptic, i.e.

$$C_u(\mathbf{v},\mathbf{v}) \geq \alpha|\mathbf{v}|_1^2 \quad \forall \mathbf{v} \in H. \tag{28}$$

In fact, by (27) and (24), $\forall \mathbf{u} \in K_1$

$$C_u(\mathbf{v},\mathbf{v}) = \nu|\mathbf{v}|_1^2 + \varepsilon^{-1} \parallel \text{ div } \mathbf{v} \parallel_0^2 + a_1(\mathbf{u};\mathbf{v},\mathbf{v}) = (\nu - C\beta/2)|\mathbf{v}|_1^2 \geq \alpha|\mathbf{v}|_1^2 \quad \forall \mathbf{v} \in H.$$

Hence the existence of the unique solution to

$$C_u(\mathbf{w},\mathbf{v}) = <\mathbf{F},\mathbf{v}> \quad \forall \mathbf{v} \in H \tag{29}$$

is guaranteed by the Lax-Milgram theorem. That is, a mapping $P : \mathbf{u} \to \mathbf{w} = P\mathbf{u}$ is well defined by (29)

$$|P\mathbf{u}|_\mathbf{1} = |\mathbf{w}|_\mathbf{1} \leq \alpha^{-\mathbf{1}} \parallel \mathbf{F} \parallel_* .$$

In addition, by virtue of (29) and (28) we have

$$\varepsilon^{-1} \parallel \text{ div } \mathbf{w} \parallel_0^2 + \alpha|\mathbf{w}|_1^2 \leq \parallel \mathbf{F} \parallel_* |\mathbf{w}|_1.$$

Using $ab \leq \sigma a^2 + b^2/(4\sigma)$ (where $\sigma > 0$ is an arbitrary constant) we have

$$\varepsilon^{-1} \parallel \text{ div } \mathbf{w} \parallel_0^2 + (\alpha - \sigma)|w|_1^2 \leq \parallel \mathbf{F} \parallel_*^2 /(4\sigma).$$

Setting $\alpha = \sigma$ and using (26) we obtain

$$\| \operatorname{div} \mathbf{w} \|_0 \leq \beta.$$

Consequently P is the mapping $K_1 \to K_1$.

Furthermore, P is a contraction mapping. In fact, if $\mathbf{w}_1 = P\mathbf{u}_1$ and $\mathbf{w}_2 = P\mathbf{u}_2$, we have

$$a_\varepsilon(\mathbf{w}_1 - \mathbf{w}_2, \mathbf{v}) + a_1(\mathbf{u}_1 - \mathbf{u}_2; \mathbf{w}_1, \mathbf{v}) + a_1(\mathbf{u}_2; \mathbf{w}_1 - \mathbf{w}_2, \mathbf{v}) = 0.$$

Setting $\mathbf{v} = \mathbf{w_1} - \mathbf{w_2}$ we obtain

$$C_{u_2}(\mathbf{w}_1 - \mathbf{w}_2, \mathbf{w}_1 - \mathbf{w}_2) = a_1(\mathbf{u}_2 - \mathbf{u}_1; \mathbf{w}_1, \mathbf{w}_1 - \mathbf{w}_2).$$

It follows that

$$|\mathbf{w}_1 - \mathbf{w}_2|_1 \leq N\alpha^{-2} \| \mathbf{F} \|_* |\mathbf{u}_1 - \mathbf{u}_2|_1$$

by virtue of $\mathbf{w}_1 \in K_1$. Using (25) we conclude thet P is a contraction mapping. So $\mathbf{w} = P\mathbf{w}$ as the unique fixed point in K_1.

Assume that $(\mathbf{u}_\varepsilon, p_\varepsilon)$ and $(\mathbf{u}, p)$ are the solutions of (3) and (2) respectively. Then we have [Kai]

Theorem 3 *Under hypothesis (23) the following estimate holds :*

$$|\mathbf{u} - \mathbf{u}_\varepsilon|_1 + \| p - p_\varepsilon \|_0 \leq c_1 \varepsilon,$$

where c_1 is a constant.

Lemma 7 *Assume that $\mathbf{u}^*$ is a solution of (22) and condition (23) is satisfied. Then we have*

$$\mathbf{w} - T'(\mathbf{u}^*)\mathbf{w} = 0 \Longrightarrow \mathbf{w} = 0.$$

Proof In fact, $A(\mathbf{w}-T'(\mathbf{u}^*)\mathbf{w},\mathbf{v}) = A(\mathbf{w},\mathbf{v})+a_1(\mathbf{u}^*;\mathbf{w},\mathbf{v})+a_1(\mathbf{w};\mathbf{u}^*,\mathbf{v}), \forall \mathbf{v} \in H$. Let $C(\mathbf{u}^*;\mathbf{w},\mathbf{v}) = A(\mathbf{w}-T'(\mathbf{u}^*)\mathbf{w},\mathbf{v})$. Employing (8), (23) and that $A(.,.)$ is continuous and coercive on $H \times H$, we obtain

$$C(\mathbf{u}^*;\mathbf{w},\mathbf{w}) \geq \nu(1-4N\nu^{-2} \parallel \mathbf{F} \parallel_*)|\mathbf{w}|_1^2 > |\mathbf{w}|_1^2 \quad \forall \mathbf{w} \in H;$$

i.e. coerciveness of $C(\mathbf{u}^*;.,.)$. $C(\mathbf{u}^*;.,.)$ is also continuous on $H \times H$. Hence $C(\mathbf{u}^*;\mathbf{w},\mathbf{v}) = 0, \forall \mathbf{v} \in H \Longrightarrow \mathbf{w} = 0$.

Let $J(\mathbf{u})$ be a functional defined by (5) :

$$J(\mathbf{u}) = A(\mathbf{u}-T\mathbf{u},\mathbf{u}-T\mathbf{u})/2 = A(\mathbf{u}-\xi,\mathbf{u}-\xi)/2.$$

It is easy to check that $J(\mathbf{u})$ is Gateaux differentiable everywhere in H, and

$$\begin{aligned} &< \text{Grad } J(\mathbf{u}),\mathbf{w} >= A(T\mathbf{u}-\mathbf{u},\mathbf{w}-T'(\mathbf{u})\mathbf{w}) \\ &= A(T\mathbf{u}-\mathbf{u},\mathbf{w}) + a_1(\mathbf{u};\mathbf{w},T\mathbf{u}-\mathbf{u}) + a_1(\mathbf{w};\mathbf{u},T\mathbf{u}-\mathbf{u}), \forall \mathbf{u},\mathbf{w} \in H. \end{aligned} \tag{30}$$

So we conclude that if $\mathbf{u}^*$ is a solution of (22), i.e. a fixed point of T, then $\mathbf{u}^*$ is a stationary point of J

$$\text{Grad } J(\mathbf{u}^*) = 0. \tag{31}$$

If $\mathbf{u}^*$ is a stationary point of J, then we have, by (30)

$$A(T\mathbf{u}^*-\mathbf{u}^*,\mathbf{w}-T'(\mathbf{u}^*)\mathbf{w}) = 0 \quad \forall \mathbf{w} \in H. \tag{32}$$

From lemmas 6 and 7, we conclude that $I - T'(\mathbf{u}^*)$ is an isomorphism of H. So (32) yields $T\mathbf{u}^* - \mathbf{u}^* = \mathbf{0}$. Hence we obtain

Theorem 4 *Under condition (23), a solution of (22) is a stationary point of J. Conversely, a stationary point of J is a solution of (22).*

Theorem 5 *Suppose Ω is a bounded domain of R^n with Lipschitz boundary Γ and one of the two hypotheses :*

1.

$$A(.,.) = a_0(.,.),\ H = V_0, \mathbf{F} \in V_0^*, \tag{33}$$

2.

$$A(.,.) = a_\varepsilon(.,.),\ H = V, \mathbf{F} \in (L^{\frac{4}{3}}(\Omega))^n, \Gamma \textit{ is of class } C^2 \tag{34}$$

holds. Then $J(\mathbf{u})$ is weakly lower semicontinuous on H and achieves its infimum at some point in H.

<u>Proof</u> Let $\mathbf{u}_m \to \mathbf{u}$ (weakly) in H.

1. When (33) is valid, then by lemma 4, $T\mathbf{u}_m \to T\mathbf{u}$ (weakly) in H.

2. When (34) is valid, then there exists a subsequence (still denoted by $\{T\mathbf{u}_m\}$) such that, by lemma 3,

$$T\mathbf{u}_m \to T\mathbf{u} \text{ (strongly) in} H.$$

So we have $\mathbf{z}_m = \mathbf{u}_m - T\mathbf{u}_m \to \mathbf{z} = \mathbf{u} - T\mathbf{u}$ (weakly) in H. In addition,

$$A(\mathbf{z}_m - \mathbf{z}; \mathbf{z}_m - \mathbf{z}) \geq 0 \Longrightarrow A(\mathbf{z}_m, \mathbf{z}_m) \geq 2A(\mathbf{z}_m, \mathbf{z}) - A(\mathbf{z}, \mathbf{z}).$$

Hence we get

$$\liminf J(\mathbf{u}_m) \geq J(\mathbf{u}). \tag{35}$$

Similary we can prove that (35) is also true for the whole sequence $\{\mathbf{u}_m\}$. So we conclude that J is weakly lower semicontinuous on H.

On the other hand, $J(\mathbf{u}) \geq 0, \forall \mathbf{u} \in H$. Let $\alpha = \inf_{\mathbf{u}\in H} J(\mathbf{u}), \{\mathbf{u}_m\}$ be a minimizing sequence

$$\lim_{m\to\infty} J(\mathbf{u}_m) = \alpha.$$

From this ,

$$A(\mathbf{z}_m, \mathbf{z}_m) \leq M.$$

Hence $\{\mathbf{z}_m\}$ is uniformaly bounded. Furthermore, it follows from equation (6) that

$$A(\mathbf{u}_m, \mathbf{v}) + a_1(\mathbf{u}_m; \mathbf{u}_m, \mathbf{v}) =< \mathbf{F}, \mathbf{v} > -A(\mathbf{z}_m, \mathbf{v}). \tag{36}$$

So

$$|\mathbf{u}_m|_1 \leq \nu^{-1}(\| \mathbf{F} \|_* + \nu |\mathbf{z}_m|_1) \leq C_2.$$

In the case of (34), setting $\mathbf{v} = \mathbf{u}_m$ in (36)

$$(\nu - 0.5C \parallel \text{ div } \mathbf{u}_m \parallel_0)|\mathbf{u}_m|_1 \leq \| \mathbf{F} \|_* + \nu|\mathbf{z}_m|_1 \leq \nu C_2$$

by virtue of(27). From the proof of theorem 2 we know that if ε is small enough then $\nu - 0.5C \parallel \text{div } \mathbf{u}_m \parallel_0 > \nu - 0.5C\beta \geq \alpha$. So $\{\mathbf{u}_m\}$ is uniformly bounded. Therefore we can extract a subsequence $\{\mathbf{u}_{mp}\}$ of $\{\mathbf{u}_m\}$ such that

$$\mathbf{u}_{mp} \rightarrow \mathbf{u}_0 \text{ (weakly) in } \; H$$

$$\lim_{m_p \rightarrow \infty} J(\mathbf{u}_{mp}) = \inf_{\mathbf{v} \in H} J(\mathbf{v}) = \alpha.$$

Since $J(\mathbf{u})$ is weakly lower semicontinuous on H

$$\alpha = \lim_{m_p \rightarrow \infty} J(\mathbf{u}_{mp}) \geq J(\mathbf{u_0}).$$

But by the definition of α we must have $J(\mathbf{u}_0) \geq \alpha$, i.e $J(\mathbf{u}_0) = \alpha$.

Theorem 6 *Suppose the conditions of lemma 2 are satisfied and the minimizing sequence* $\{\mathbf{u}_m\}$ *of* J *is such that*

$$\lim_{m \rightarrow \infty} J(\mathbf{u}_m) = 0. \tag{37}$$

Then $\{\mathbf{u}_m\}$ *converges strongly to the solution* $\mathbf{u}$ *of (4) :*

$$\mathbf{u}_m \rightarrow \mathbf{u}_0 \; in \; H. \tag{38}$$

<u>Proof</u> The proof of theorem 5 showed that there exists a subsequence $\mathbf{u}_{mp}$ of $\mathbf{u}_m$ such that $\mathbf{u}_{mp} \to \mathbf{u}_0, J(\mathbf{u}_0) = \inf J(\mathbf{v}) = 0$. According to lemma 5 there also exists a subsequence (still denoted by $\mathbf{u}_{mp}$) of $\mathbf{u}_{mp}$ such that

$$T\mathbf{u}_{mp} \to T\mathbf{u}_0 \text{ (strongly) in } H. \tag{39}$$

In addition, (37) shows that

$$\lim_{m\to\infty} A(\mathbf{z}_m, \mathbf{z}_m) = 0, \mathbf{z}_m = \mathbf{u}_m - T\mathbf{u}_m, i.e.$$

$$\lim_{m\to\infty} |\mathbf{z}_m|_1 = 0. \tag{40}$$

From (39), (40) we conclude that

$$\mathbf{u}_{mp} \to \mathbf{u}_0 \text{ (strongly) in } H. \tag{41}$$

Using (33) we have

$$A(\mathbf{u}_0, \mathbf{v}) + a_1(\mathbf{u}_{mp}; \mathbf{u}_{mp}, \mathbf{v}) =< \mathbf{F}, \mathbf{v} > -A(\mathbf{z}_{mp}, \mathbf{v}), \forall \mathbf{v} \in H.$$

Letting $mp \to \infty$ we obtain

$$A(\mathbf{u}_0, \mathbf{v}) + a_1(\mathbf{u}_0; \mathbf{u}_0, \mathbf{v}) =< \mathbf{F}, \mathbf{v} > \quad \forall \mathbf{v} \in H \tag{42}$$

by virtue of (40) and (41), i.e. $\mathbf{u_0}$ is a solution of (4).

It remains to prove (38). Assume that it is not true. Then there exists a subsequence $\{\mathbf{u}_{mq}\}$ of $\{\mathbf{u}_m\}$ such that

$$|\mathbf{u}_{mq} - \mathbf{u}_0|_1 > \varepsilon \quad \forall \varepsilon > 0. \tag{43}$$

But $\{\mathbf{u}_{mq}\}$ is also a minimizing sequence of J which satisfies (37). According to the previous discussion we obtain

$$\mathbf{u}_{mv} \to \mathbf{u}_0 \text{ (strongly) in} H,$$

where $\{\mathbf{u}_{mv}\}$ is a subsequence of $\{\mathbf{u}_{mq}\}$. This contradicts assumption (43), and we have (38).

4 The Construction of a Minimizing Sequence

Now, let us make a minimizing sequence.

Step 0 We take $\mathbf{u_0}$ as the solution of the corresponding Stokes equation :

$$A(\mathbf{u}_0, \mathbf{v}) =< \mathbf{F}, \mathbf{v} >, \quad \forall \mathbf{v} \in H.$$

Then compute $\mathbf{g}_0 \in H$ by $A(\mathbf{g}_0, \mathbf{v}) =<$ Grad $J(\mathbf{u}_0), \mathbf{v} > \ \forall \mathbf{v} \in H$, and set $\mathbf{z}_0 = \mathbf{g}_0$.

Assuming $\mathbf{u_m}, \mathbf{g_m}, \mathbf{z_m}$, are known for $m = 0, 1, 2...$, compute $\mathbf{u_{m+1}}, \mathbf{g_{m+1}}, \mathbf{z_{m+1}}$ as follows

Step 1 Descent. Compute

$$\lambda_m = \arg\min_{\lambda \in R} J(\mathbf{u}_m - \lambda \mathbf{z}_m), \mathbf{u}_{m+1} = \mathbf{u}_m - \lambda_m \mathbf{z}_m \tag{44}$$

Step 2 Calculation of the New Descent Direction.
Find $\mathbf{g}_{m+1} \in H$ such that

$$A(\mathbf{g}_{m+1}, \mathbf{v}) =< \text{ Grad } J(\mathbf{u}_{m+1}), \mathbf{v} > \quad \forall v \in H$$
$$\rho_{m+1} = A(\mathbf{g}_{m+1}, \mathbf{g}_{m+1} - \mathbf{g}_m)/A(\mathbf{g}_m, \mathbf{g}_m),$$
$$\mathbf{z}_{m+1} = \mathbf{g}_{m+1} + \rho_{m+1}\mathbf{z}_m.$$

Step 3 Test.
If $\| \mathbf{u}_{m+1} - \mathbf{u}_m \|_1 \leq \varepsilon$ or $|J(\mathbf{u}_{m+1})| \leq \varepsilon$ then stop else go to step 1 to continue.
Let $t \in R$. It follows that

$$A(T(\mathbf{u} + t\mathbf{w}), \mathbf{v}) =< \mathbf{F}, \mathbf{v} > -a_1(\mathbf{u} + t\mathbf{w}; \mathbf{u} + t\mathbf{w}, \mathbf{v}),$$

and

$$T(\mathbf{u} + t\mathbf{w}) = T\mathbf{u} + T'(\mathbf{u})\mathbf{w}t + T'(\mathbf{w})\mathbf{w}t^2/2.$$

Since $J(\mathbf{u})$ is defined by (5), (6). So $J(\mathbf{u}+t\mathbf{v})$ is a polynomial of degree 4 of $t : \forall t \in R, \mathbf{v} \in H$,

$$2J(\mathbf{u}+t\mathbf{v}) = 2f(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3 + \alpha_4 t^4, \tag{45}$$

where

$$\begin{array}{ll}
\alpha_0 = A(\mathbf{u}+T\mathbf{u}, \mathbf{u}-T\mathbf{u}), & \alpha_1 = 2A(\mathbf{u}-T\mathbf{u}, \mathbf{v}-T'(\mathbf{u})\mathbf{v}) \\
\alpha_2 = A(\mathbf{v}-T'(\mathbf{u})\mathbf{v}, \mathbf{v}-T'(\mathbf{u})\mathbf{v}) - A(\mathbf{u}-T\mathbf{u}, T'(\mathbf{v})\mathbf{v}) & \\
\alpha_3 = A(T'(\mathbf{u})\mathbf{v}-\mathbf{v}, T'(\mathbf{v})\mathbf{v}), & \alpha_4 = A(T'(\mathbf{v})\mathbf{v}, T'(\mathbf{v})\mathbf{v})/4.
\end{array}$$

It is clear that $\xi_0 = T\mathbf{u},\ \xi_1 = T'(\mathbf{u})\mathbf{v},\ \xi_2 = T'(\mathbf{v})\mathbf{v}/2$ are the solutions of the following variational problems respectively

$$\begin{array}{l}
A(\xi_0, \eta) =< \mathbf{F}, \eta > -a_1(\mathbf{u}; \mathbf{u}; \eta), \\
A(\xi_1, \eta) = -a_1(\mathbf{u}; \mathbf{v}, \eta) - a_1(\mathbf{v}; \mathbf{u}, \eta), \quad \forall \eta \in H \\
A(\xi_2, \eta) = -a_1(\mathbf{v}; \mathbf{v}, \eta).
\end{array}$$

Theorem 7 $\forall \mathbf{u}, \mathbf{v} \in H$, *there exists a solution of the single variable minimizing problem*

$$t^* = \arg\min_{t \in R} J(\mathbf{u}+t\mathbf{v}),$$

and t^* is a zero point of polynomial $f'(t) = \dfrac{df}{dt}$ where $f(t)$ is defined by (45).

Proof It is known from (45) that

1. If $\boldsymbol{\xi_2} \neq \mathbf{0}$, one has $\alpha_4 = (\xi_2, \xi_2) > 0$, owing to the coerciveness of $A(.,.)$, so $f(t)$ is a polynomial of degree 4. Certainly, $f(t)$ achieves its infimum in the finite interval.

2. If $\boldsymbol{\xi_2} = \mathbf{0}$, one has $\alpha_4 = 0$. However,

$$\begin{array}{l}
\alpha_3 = -A(\mathbf{v}-\xi_1, \xi_2) = 0 \\
\alpha_2 = A(\mathbf{v}-\xi_1, \mathbf{v}-\xi_1) - 2A(\mathbf{u}-\xi_0, \xi_2) = A(\mathbf{v}-\xi_1, \mathbf{v}-\xi_1).
\end{array}$$

Likewise, when $\mathbf{v} - \xi_1 \neq 0, \alpha_2 > 0, f(t)$ is a polynomial of degree 2, and achieves its infimum in finite interval.

3. If $\mathbf{v} - \xi_1 = 0, \xi_2 = 0$, one has $\alpha_4 = \alpha_3 = \alpha_2 = 0$ In this case

$$\alpha_1 = 2A(\mathbf{u} - \xi_0, \mathbf{v} - \xi_1) = 0$$

so $f(t) = \alpha_0$. Then $f(t)$ can achieve its infimum. According to 1,2 and 3, $f'(t) = \dfrac{\alpha_1}{2} + \alpha_2 t + \dfrac{3}{2}\alpha_3 t^2 + 2\alpha_4 t^3$ is a polynomial of odd degree, hence there exists at least a zero point of $f'(t)$ at which $f(t)$ achieves its infimum.

In practice, this theorem is very important. Due to it, there are many improvements in computational efficiency and accuracy for the conjugate gradient method. For this reason to solve the minimization problem (44) one only has to find a root of the equation of degree 3. In order to evalute J it is necessary to compute $a_1(.;.,.)$, and $A(.,.)$ and to solve the Stokes problem once.

References

[BE79] M. Bercoviex and M. Engelman. A finite element for the numerical solution of viscous incompressible flows. *J. Comp. Phys*, (30):181–201, 1979.

[Kai] Li Kaitai. Mixed boundary value problem for Navier-Stokes equations. *Scientific and Technical Report.* 92-306. Xi'an Jiaotong University.

[KAYD82] Li Kaitai, Huang Aixiang, Ma Yichen, and Li Du. Optimal control finite element approximation for penaly variational problem of Navier-Stokes equations. (16 - 1):85–88, 1982.

[Ode] J. T. Oden. RIP-Methods for Stokesian Flows. *Finite Element in Fluids*, (4). Jhon Willey and Sons.

[Red80] J. N. Reddy. On the mathematical theory of the penalty finite elements for Navier-Stokes Equations. *Proceedings of the 3rd International Confernce on Finite Element in Flow Problems*, (2), 1980.

[Tem77] R. Temam. Navier-stokes equations. 1977. North-Holand - Amsterdam.

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