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Laurent Baratchart, Juliette Leblond, Jonathan R. Partington, N. Torkhani

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

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N° 2488
Février 1995

PROGRAMME 5

Rapport de recherche

ISSN 0249-6399

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Laurent Baratchart*, Juliette Leblond*,
Jonathan R. Partington **, N. Torkhani*

Programme 5 — Traitement du signal, automatique et productique
Projet MIAOU

Rapport de recherche n° 2488 — Février 1995 — 31 pages

Abstract: We consider the problem of identifying a scalar BIBO-stable transfer function from a series of experimental pointwise measurements at a set of frequencies lying within the bandwidth. To this end, we propose an algorithm which consists in building a sequence of maps from data to BIBO-stable models which uniformly converges to the seeked transfer function on the bandwidth when the number of measurements goes to infinity and the noise level goes to zero, while asymptotically meeting some gauge constraint outside. Error bounds are derived for this approximation scheme which is also illustrated by numerical experiments.

Key-words: Robust identification, Hardy spaces, dual extremal problems, Nehari extension.

(Résumé : tsvp)

*INRIA, BP 93, 2004 route des Lucioles, 06902 Sophia–Antipolis Cedex, FRANCE, {baratcha}{leblond}{torkhani}@sophia.inria.fr

**School of Mathematics, University of Leeds, Leeds LS2 9JT, U.K.,{pmt6jrp}@gps1.leeds.ac.uk

Identification robuste dans l'algèbre du disque depuis des données limitées à une bande de fréquences

Résumé : Nous considérons le problème de l'identification d'une fonction de transfert scalaire EBSB-stable depuis un échantillon de mesures ponctuelles expérimentales sur un ensemble de fréquences appartenant à la bande passante. A cet effet, nous proposons un algorithme consistant à bâtir une suite d'applications depuis les données dans les modèles EBSB-stables qui converge uniformément vers la fonction de transfert cherchée sur la bande de fréquences lorsque le nombre de mesures tend vers l'infini et le niveau de bruit tend vers zéro, tout en satisfaisant asymptotiquement une certaine contrainte de gabarit à l'extérieur. Nous calculons des bornes sur l'erreur pour ce schéma d'approximation que nous illustrons aussi par des simulations numériques.

Mots-clé : Identification robuste, espaces de Hardy, problèmes extrêmaux duaux, extension de Nehari.

1 Introduction

This paper is concerned with the problem of harmonic identification, that is, of recovering a (SISO) and BIBO-stable transfer function from a family of experimental pointwise values on the imaginary axis. In [10], a setting to approach this issue was proposed, in which the error in measurements is handled in a deterministic fashion and the identification procedure consists of a map from finite sets of data to (stable) transfer-functions that converges uniformly to the "true" transfer function when the noise goes uniformly to zero and the number of data goes to infinity.
In the present work, we shall consider the (realistic) case where the experiments are only available in some range of frequencies corresponding to the bandwidth of the system. In this case, none of the algorithm that were proposed [8, 10, 14, 15] converges, and we shall see that the setting itself has to be modified: we shall adapt to the new situation by requiring the map from data to models to converge uniformly in the bandwidth while meeting some norm-constraints at remaining frequencies.
Our working space will be the unit disc rather than the half-plane, the two frameworks being equivalent by means of a Möbius transform. Since the transfer function of a BIBO-stable system is continuous on the imaginary axis including at infinity, a model for us has to be found in the disc algebra.

2 A set-up for band-limited identification

Let H_∞ be the familiar Hardy space of bounded analytic functions in the disc, and $A(\mathbb{D})$ (the disc algebra) be the subspace of such functions that are continuous on the closed disc. On a couple of occasions, we shall also use the symbol H_∞ to mean the Hardy space of the right half-plane $\Pi_+ = \{s \in \mathbb{C};\ \mathrm{Re}s > 0\}$ but the context will always keep the meaning clear. The algebra $A(\Pi_+)$ of the right half-plane will then consist of those functions in H_∞ of this half-plane that extend continuously to the imaginary axis including at infinity. The symbol $C(X)$ stands for the space of complex continuous functions on X endowed with the sup norm. Spaces X used in this paper will be arcs on the unit circle or intervals on the imaginary axis.

In the problem of robust H_∞ identification of functions in the disc algebra as stated in the above-mentioned references, one is given experimental data as complex numbers $(a_k)_{k=1}^N = (f(z_k) + \eta_k)_{k=1}^N$, where f is an unknown function in the disc algebra $A(\mathbb{D})$, and $z_1, \ldots, z_N$ are points on the unit circle $\mathbb{T}$ while $(\eta_1, \ldots, \eta_N)$ is some unknown but bounded noise sequence (possibly deterministic). From this, one wishes to construct an approximation $f_N = T_N(a_1, \ldots, a_N)$ such that in the limit, as the noise level tends to zero and the number of observations tends to infinity, one has convergence in the H_∞ norm, that is,

$$\lim_{\substack{N\to\infty \\ \epsilon\to 0}} \sup_{\|\eta\|_\infty \leq \epsilon} \|f_N - f\|_\infty = 0 \qquad \text{for all } f \in A(\mathbb{D}). \tag{1}$$

To approach this problem, a two-stage algorithm has been found useful [8, 10, 14, 15]: to proceed, one first computes a trigonometric polynomial p_N which interpolates the given data (but is not in $A(\mathbb{D})$), and one applies then the (nonlinear) Nehari extension [21] to obtain the best approximation to p_N by a function f_N bounded and analytic in the disc (it will in fact be rational).
When the points (z_k) are equally spaced on the circle, p_N can be obtained using the classical Jackson or de la Vallée-Poussin trigonometric polynomials [14, 22]). When the points are not equally spaced, the problem becomes computationally harder but one can design a transformation from the given points into equally spaced ones and proceed as before (see e.g. [17]), or else rely on a more general principle of linear programming [16].
In the last reference, the overall error of the identification procedure can be expressed as a sum of two terms, one corresponding to the noise and the other to the maximum gap between the interpolation points. One such theoretical bound is $4\epsilon + 5\,\mathrm{dist}\,(f, P_p)$, where $\epsilon \geq \|\eta\|_\infty$ and P_p is the space of polynomials of degree p and the maximum gap is less than $1/p$. Thus, the error goes to zero as $\epsilon \to 0$ provided the maximum gap between the measurements points (z_k) goes to zero.
However, in practical applications, one may not be able to measure f at all points on the circle. For example, in the identification of continuous-time, linear, time-invariant and BIBO-stable control systems by frequency response measurements, which can be reduced to the above problem by means of the Möbius transformation $s = (1+z)/(1-z)$ and $G(s) = f(z)$ where G is the

transfer-function, one is not able to measure $G(i\omega)$ for arbitrarily high values of ω. Moreover one is not normally concerned about modelling G arbitrarily well at high frequencies. In this situation, no algorithm can guarantee uniform convergence over the whole circle $\mathbb{T}$ without further *a priori* knowledge on f [16]. It is nevertheless natural to ask whether the unknown function f can be recovered in a robust fashion at least in the range of frequencies where measurements are available, through a model which is still under control at the remaining frequencies.

Let us go over to the half-plane for a while, and discuss a bit further the situation where measurements are only available in the bandwidth, say Ω. In this connexion, some work on band-limited identification has been published by Bai and Raman [1], in which they essentially approximate separately the real and imaginary parts of the transfer function by polynomials over the frequency interval Ω, plugging in some arbitrary polynomial weight of sufficiently high degree to become the denominator of the approximant so as to end up with a stable and proper model. In doing so, they are not concerned about controlling the behaviour off the set Ω and, since their scheme is (real) linear, it is a routine matter to check, by the same arguments as in [15], that their sequence of estimates is unbounded outside Ω for almost every noise in l^∞ (*i.e.* for every noise sequence in a set of second category in the sense of Baire). In fact, we claim that *any* H^∞ band-limited identification scheme *must* incorporate some constraints that impinge on the behaviour of the transfer-function outside Ω. This can be inferred from two facts:
(i) in the space $C(\Omega)$, the subspace $A(\Pi_+)|_\Omega$ obtained by restricting $A(\Pi_+)$ to Ω is dense;
(ii) if $G \in C(\Omega)$ does not belong to $A(\Pi_+)|_\Omega$, any sequence of functions in $A(\Pi_+)$ (or even in H_∞) that converges to G on Ω is unbounded in H_∞.
Fact (i) is an easy consequence of Runge's theorem while fact (ii) follows from the weak–* compactness of balls in H_∞, and we refer the reader to [2] for a proof which is phrased on the disc rather than the half-plane (and also works in $L^p(\Omega)$ for $1 \leq p \leq \infty$). Altogether, (i) and (ii) indicate that no matter the data, we can always construct an excellent model on Ω at the cost of nearly destabilizing it at the remaining frequencies, a problem which is familiar to identification practitioners. At this point, it is perhaps interesting to draw a

parallel with the seemingly different process of stochastic parametric identification: there, the constraints on the model are often imposed in terms of bounded rational degree, and the analogue of the above mentioned phenomenon would be that allowing the degree to grow too large destabilizes the model because it starts fitting the noise.

In this paper, we choose to constrain the behaviour of the model to lie within some tolerance of a prescribed function at non-measured frequencies. Thus, back to the disc, we propose the following modified set-up. We suppose that $0 < a < \pi$ and consider $I = \{e^{i\theta} : a \leq \theta \leq 2\pi - a\}$ which is a proper closed symmetric subarc of the unit circle. We define J to be the closure of the complement of I, i.e. $J = \{e^{i\theta} : -a \leq \theta \leq a\}$. Also, we define the norm

$$\|g\|_{I,\infty} = \max\{|g(e^{i\theta})| : e^{i\theta} \in I\}, \tag{2}$$

for g in $L^\infty(I)$, and similarly for J.
We provide ourselves with measurements $a_k = f(z_k)+\eta_k$, with $k = -N, \cdots, N$, where the z_k all lie within I with $z_{-N} = e^{-ia}$ and $z_N = e^{ia}$. We shall assume that the function f satisfies an *a priori* estimate of the form

$$|f(z) - h(z)| \leq r(z), \quad \text{for all } z \in J, \tag{3}$$

for some functions h and r belonging to $C(J)$, with r a non-negative gauge function that vanishes at the endpoints of J.
Our aim is to find an approximate model f_N of f on I converging robustly on I, namely:

$$\lim_{\substack{N\to\infty \\ \epsilon\to 0}} \sup_{\|\eta\|_\infty \leq \epsilon} \|f_N - f\|_{I,\infty} = 0.$$

Moreover, we also require this approximation procedure asymptoticaly meets the gauge constraint on J:

$$\lim_{\substack{N\to\infty \\ \epsilon\to 0}} \sup_{\|\eta\|_\infty \leq \epsilon} \left\{ \sup_{z\in J} |f_N(z) - h(z)| - r(z) \right\} \leq 0.$$

However, from our incomplete set of data, we cannot construe the model f_N to converge robustly to f on the whole circle; on J, we will only get that f_N

converges weakly–* to f:

$$\lim_{N\to\infty}\int_J f_N u\,d\theta = \int_J f u\,d\theta \quad \text{for all} \quad u \in L^1(J).$$

Note also that this scheme is not untuned in the terminology of [10], and this is natural since we emphasized the necessity of constraining the model on J in one way or another: here, we need a pointwise bound of the form (3) on J.

A few comments on the role of h and r are perhaps in order. Firstly, h is meant to be a reference behaviour for the transfer function f on J and this transfer function is assumed to be continuous, so that h has to coincide with f at the endpoints $e^{\pm ia}$ of the interval. However, our guess may well be false within the interval J, that is, r need not be small except at the endpoints of J. Clearly, *any* choice for h will generate infinitely many functions r meeting (3) and what we need to do in practice is to guess one of them, so it seems more secure to choose r to be large on J. On the opposite, it is necessary to have a good guess on the behaviour of f outside the bandwidth, that is, to be able to make r small if one wants to get accurate modelling at infinity. Indeed, the approximation f_N to f that we are about to construct is such that $|f_N - h| \to r$ uniformly on J as $N \to \infty$ and $\epsilon \to 0$. Thus, if $|f - h|$ is significantly smaller than r, the values of f and f_N will not be close to each other on J and the weak–* convergence of f_N to f will cause f_N to oscillate on J with an amplitude which depends on the size of r. Still, the model f_N asymptotically meets the gauge constraint (3) which is the main feature of our approach and warrants applications where one is not so much concerned with the behaviour at high frequencies except for its boundedness.

Secondly, it appears that the existence of h is somewhat theoretical as we cannot consistently assign numerical values to it since $h(z_N) = f(z_N)$ and $f(z_N)$ is known up to η_N only. There is no way around this implicit definition of h, to the effect that the true value we have been using for it is revealed only in the limit when the noise goes to zero. However, this will not prevent us from designing a convergent identification scheme in the sense indicated above. In order to account for this uncertainty and still be able to handle (3) in a constructive manner, it is convenient to use a relative version of h obtained by normalizing the boudary values to be zero. Specifically, given a complex number c, we let $e_c(e^{i\theta})$ be the function defined on J by

$$e_c(e^{i\theta}) = \frac{1}{2i\sin a}(c(e^{i\theta} - e^{-ia}) - \bar{c}(e^{i\theta} - e^{ia})); \tag{4}$$

thus, e_c is linear in $z = e^{i\theta}$ and satisfies $e_c(e^{ia}) = c$ and $e_c(e^{-ia}) = \bar{c}$. We let now

$$h_0 = h - e_{f(e^{ia})}, \tag{5}$$

so that $h_0(e^{\pm ia}) = 0$, and all we shall need beyond the values a_k to make our procedure effective is to specify numerically h_0 and r. This means, in fact, that the function h can be assigned numerically up to some precision less than $2\epsilon/\sin a$ since the latter is a bound for $|e_{f(e^{ia})} - e_{a_N}|$ on J.
There is nothing so special about the function e_c defined in (4) except that $e_c(e^{ia}) = c$, $e_c(e^{-ia}) = \bar{c}$, and e_c goes uniformly to zero on J with c; any function with the same properties could be used in its place and this choice was mainly for simplicity and definiteness. When nothing is known on the shape of f except being proper and stable, a particularly simple choice is $h = e_{f(e^{ia})}$ whence $h_0 = 0$; if one wants a strictly proper model, one may use quadratic interpolants rather than linear ones for h to interpolate the value 0 at 1. We then need to choose r large enough so that (3) is satisfied. Of course, there is no way to ensure beforehand that it is the case, and this is revealed *a posteriori* only if the convergence gets ruined which means that r is too small somewhere on J.

In this paper, we describe an identification procedure meeting the above requirements (section 3) and derive error bounds in the case of equally spaced points on I with $h_0 = 0$ (section 4); the procedure rests on results demonstrated in [2] that we will recall without proof. We then report on two numerical experiments (section 5): one from academic data previously considered in the literature, and one from real data measured on a hyperfrequency filter by the French CNES.

We shall make the standing assumption, required for system-theoretical reasons though not for mathematical ones, that either the unknown function f and the analytic model f_N we are seeking are real-symmetric, namely that $f(\bar{z}) = \overline{f(z)}$ and the same for f_N: thus we need only take measurements in

$a \le \theta \le \pi$ and obtain the others by complex conjugation. The reference function h is also assumed to verify this hypothesis on the (symmetric) arc J.

3 An algorithm for approximate modelling

Suppose, for some unknown function $f \in A(\mathbb{D})$, that we are given the values $(a_k) = (f(z_k) + \eta_k)_{k=-N}^{N}$, where z_k belongs to I and (η_k) is a (deterministic) noise sequence, assumed to be ϵ-small in the l^∞ norm. We also assume that $z_0 = \pi$ and that $z_{-k} = \bar{z}_k$, $a_{-k} = \bar{a}_k$, and $\eta_{-k} = \bar{\eta}_k$ for $1 \le k \le N$, which is the real-symmetric assumption made above.

Although we are seeking models in $A(\mathbb{D})$ only, we shall need to make excursions into H_∞. If $g \in H_\infty$ and

$$\sup_{z \in \mathbb{D}} |g(z)| = \|g\|_\infty,$$

recall (see e.g.[11, chap.3]) that the radial limit

$$\lim_{r \to 1} g(re^{i\theta})$$

exists for almost every θ (even nontangential limits exist) and this serves as a definition for $g(e^{i\theta})$. In this way, $g(e^{i\theta})$ becomes a member of $L_\infty(\mathbb{T})$, with norm $\|g\|_\infty$, whose Fourier coefficients of negative index do vanish and whose restriction to any subset of positive measure on $\mathbb{T}$ is nonzero if g is nonzero.

Given functions $\alpha \in L^\infty(I)$, $\beta \in L^\infty(J)$, we denote by $\alpha \vee \beta$ the $L^\infty(\mathbb{T})$ function which is equal to α on I and to β on the interior $\mathring{J}$ of J; when $\alpha > 0$ and $\beta > 0$, we also denote by $w_{\alpha,\beta}$ the outer function:

$$w_{\alpha,\beta} = \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log(\alpha \vee \beta)\, d\theta \right\}. \tag{6}$$

This function is characterized by the following properties (see e.g. [11, ch.5]): $w_{\alpha,\beta}(0) > 0$, $w_{\alpha,\beta}$ and $w_{\alpha,\beta}^{-1}$ are both in H_∞, and $|w_{\alpha,\beta}| = \alpha \vee \beta$, that is

$$|w_{\alpha,\beta}(z)| = \begin{cases} \alpha(z) \text{ a.e. on} I, \\ \beta(z) \text{ a.e. on } \mathring{J}\,. \end{cases} \tag{7}$$

Moreover, observe that $w_{\alpha,\beta} = w_{\alpha,1}\, w_{1,\beta}$ so that $w_{\alpha,\beta}^{-1} = w_{1/\alpha,1/\beta}$.

We begin with a result asserting that robust band–limited identification, as defined in the introduction, is theoretically possible at least when r satisfies a Lipschitz condition. In addition, the arguments in the proof will turn out to be explicit providing us with a constructive way of solving the problem.

Theorem 3.1 *Assume the sequence (z_k) is dense in I. Let $h \in C(J)$ and let r be a non negative Lipschitz-continuous function on J of exponent μ, $0 < \mu \leq 1$, which satisfies $r(e^{ia}) = r(e^{-ia}) = 0$. There exists a sequence of maps $T_N \ : \ \mathbb{C}^{N+1} \to A(\mathbb{D})$ such that*

$$\lim_{\substack{N\to\infty \\ \epsilon\to 0}} \sup_{\|\eta\|_\infty \leq \epsilon} \|T_N(a_0, \cdots, a_N) - f\|_{I,\infty} = 0 \,, \tag{8}$$

and

$$\lim_{\substack{N\to\infty \\ \epsilon\to 0}} \sup_{\|\eta\|_\infty \leq \epsilon} \left\{ \sup_{z\in J} |T_N(a_0, \cdots, a_N)(z) - h(z)| - r(z) \right\} \leq 0, \tag{9}$$

for all $f \in A(\mathbb{D})$ such that $f \vee h \in C(\mathbb{T})$ and $|f(z) - h(z)| \leq r(z)$ for every $z \in J$.

Before proving Theorem 3.1, we need to establish a few facts concerning a bounded (dual) extremal problem, which plays here the same role as the Nehari extension does in robust identification over the whole circle. These results will extend some of those established in [2].
For every pair of functions $\kappa, \rho \in C(J)$ with $\rho > 0$, we define

$$\mathcal{B}_{\rho,\kappa} = \{\gamma \in H_\infty \,, \ |\kappa - \gamma| \leq \rho \ \text{ a.e. on } J\} \,.$$

Proposition 1 *Let ψ be in $L^\infty(I)$, h and ρ be in $C(J)$ with $\rho > 0$, and consider the following minimization problem:*

$$\|\psi - g_0\|_{I,\infty} = \min_{g\in\mathcal{B}_{\rho,h}} \|\psi - g\|_{I,\infty} = \beta_\infty. \tag{10}$$

(i) Problem (10) admits a solution $g_0 \in \mathcal{B}_{\rho,h}$; when $\psi \vee h \in H_\infty + C(\mathbb{T})$, the solution g_0 is unique.
We assume now that ψ is not already the trace on I of a function in $\mathcal{B}_{\rho,h}$ so that $\beta_\infty > 0$.
(ii) When $\psi \vee h \in H_\infty + C(\mathbb{T})$, we have that:

$$\begin{cases} |\psi - g_0| = \beta_\infty \ \text{ a.e. on } I \,, \\ |h - g_0| = \rho \ \text{ a.e. on } J \,. \end{cases}$$

(iii) The function g_0 is a solution to problem (10) *if and only if*

$$v_0 = g_0 \, w_{1/\beta_\infty,1/\rho} \tag{11}$$

is a solution to the implicit Nehari problem:

$$\min_{v \in H_\infty} \|(\psi \vee h)\, w_{1/\beta_\infty,1/\rho} - v\|_\infty = \|(\psi \vee h)\, w_{1/\beta_\infty,1/\rho} - v_0\|_\infty = 1\,. \tag{12}$$

Proof: the case where $\rho = M > 0$ is constant on J is contained in Theorems 3 and 4 and also section 4 of [2]. What we need here is to consider instead of the constant M an arbitrary positive function $\rho \in C(J)$.
The first step is to make sure that $\mathcal{B}_{\rho,h}$ is non empty. For this, put $m = \min_J \rho > 0$. Since any $g \in H_\infty$ such that $\|g - h\|_{J,\infty} \leq m$ belongs to $\mathcal{B}_{\rho,h}$, the conclusion follows from the density of $A(\mathbb{D})_{|J}$ in $C(J)$ already pointed out (but for the half-plane) as fact (i) in section 2.
Next, setting

$$\gamma = g \, w_{1,1/\rho} \tag{13}$$

and taking (7) into account, we get

$$\begin{aligned}\min_{g \in \mathcal{B}_{\rho,h}} \|\psi - g\|_{I,\infty} &= \min_{\gamma \in \mathcal{B}_{1,hw_{1,1/\rho}}} \|\psi - \gamma \, w_{1,1/\rho}^{-1}\|_{I,\infty} \\ &= \min_{\gamma \in \mathcal{B}_{1,hw_{1,1/\rho}}} \|\psi \, w_{1,1/\rho} - \gamma\|_{I,\infty} = \beta_\infty\,. \end{aligned} \tag{14}$$

We are now back to the case of a constant bound on J so that the cited results of [2] apply. This yields γ_0 realizing the infimum above hence also $g_0 = \gamma_0 \, w_{1,1/\rho}^{-1}$ as asserted in (i). If $\psi \vee h \in H_\infty + C(\mathbb{T})$, so does $(\psi \vee h)\, w_{1,1/\rho}$ since $H_\infty + C(\mathbb{T})$ is an algebra (see e.g. [6, IX,thm.2.2]); again from [2], we get uniqueness of γ_0, hence of g_0, thereby proving (i).
We turn to the proof of (ii) and we observe, since $\beta_\infty > 0$ by assumption, that [2, thm.4] implies:

$$\begin{cases} |\psi \, w_{1,1/\rho} - \gamma_0| = \beta_\infty \text{ a.e. on } I\,, \\ |h \, w_{1,1/\rho} - \gamma_0| = 1 \text{ a.e. on } J\,. \end{cases}$$

Now, (ii) follows at once from (7) and (13).

As regards (iii), we get from [2, thm.3] that $\gamma_0 w_{1/\beta_\infty,1}$ is the solution to (12), and from section 4 of the cited paper that the value of this problem is indeed 1. Now, (11) follows immediately from (13). ■

Notice that β_∞ is *defined* by (10), so that the weight $w_{1/\beta_\infty,1/\rho}$ depends on ρ, ψ and h through β_∞. Hence, problem (12) is an implicit one and the right value for β_∞ is the one that makes the infimum in (12) equal to 1. That such a value is unique will follow from Lemma 3.2 below.

We are now in position to establish our main result.

Proof of Theorem 3.1: we introduce an approximation h_N to h by the formula

$$h_N = h_0 + e_{a_N},$$

where h_0 and e_{a_N} are defined by (5) and (4), so that $h_N(e^{\pm ia}) = a_{\pm N}$. Here, we should keep in mind as in the introduction that h_0 is supposed to be known so that h_N, unlike h, is also known from the data.

The first step will be to construct a trigonometric polynomial p_N satisfying

$$\lim_{\substack{N\to\infty \\ \epsilon\to 0}} \sup_{\|\eta\|_\infty \le \epsilon} \|p_N - f \vee h\|_\infty = 0. \qquad (15)$$

To this end, we rely on the methods detailed in [14, 16, 20] which are simple and robustly convergent to recover continuous functions on the whole of $\mathbb{T}$ from densely located (noisy) interpolation data. Specifically, let $\xi_N = \sup_{k=0,\dots,N-1} |z_{k+1} - z_k|$. Then, choose $N' = N'(N)$ large enough and N' points z_{N+k} belonging to $J \cap \{\mathrm{Im} z \ \geq 0\}$ such that $\sup_{k=1,\dots,N'-1} |z_{N+k+1} - z_{N+k}| \le \xi_N$. This can be ensured by taking $N' \ge \frac{a}{\pi - a} N$. Take now the associated N' pointwise values $h_N(z_{N+k})$ of h_N and obtain N' others by complex conjugation.

We thus obtain $2\,(N+N')+1$ pointwise data

$$\{f(z_k)\,,\ 0 \le |k| \le N\,,\ h_N(z_p)\,,\ N+1 \le |p| \le N+N'\}$$

that are distributed on the whole of $\mathbb{T}$ with a maximum gap less than ξ_N. Note that $\|h - h_N\|_{J,\infty} \le 2\epsilon/\sin a$ by the hypothesis that $|a_N| \le \epsilon$, hence the measurements of h_N can be regarded as noisy measurements of h (with noise at most $2\epsilon/\sin a$) and it is this which will guarantee robust convergence to $f \vee h$.

If $\xi_N = O(\frac{1}{N+N'})$, p_N may be chosen so as to be a Jackson polynomial of degree $2\,(N+N')$ which interpolates exactly the prescribed data or a de la Vallée Poussin polynomial which only interpolates approximatively these data. An alternative procedure is to build p_N as a trigonometric polynomial of least deviation from the data, of degree $d(N) < 2\,(N+N')+1$: such a p_N will do whenever $d(N)\xi_N$ converges to 0 as N goes to ∞. In any case, we thus obtain a trigonometric polynomial

$$p_N(z) = \sum_{k=-d}^{d} p_N^k z^k$$

satisfying (15).

However, p_N cannot serve as a model because it does not belong to $A(\mathbb{D})$ in general, that is, the p_N^k's need not be zero for negative k. If $p_N \in A(\mathbb{D})$ for some N and some a_k's, we simply set $T_N(a_0, \cdots, a_N) = p_N$ which meets all our requirements. We now assume throughout the proof that $p_N \notin A(\mathbb{D})$ and we notice in this case that p_N cannot be the trace of any H_∞ function on I: if g were such a function, $z^d(p_N - g) \in H_\infty$ would vanish on I hence should vanish identically, yielding $p_N = g$ so that p_N would be in $A(\mathbb{D})$.

Let

$$\sigma_N(z) = r(z) + \varepsilon_N\,,\ \ \forall z \in J\,, \tag{16}$$

for a sequence (ε_N) of positive numbers to be determined later. This defines a μ-Lipschitz-continuous positive function σ_N on J.

The next stage is to get a function $f_N \in \mathcal{B}_{\sigma_N,p_N}$ solution to the following bounded extremal problem:

$$\min\{\|p_N - g\|_{I,\infty}\,,\ g \in \mathcal{B}_{\sigma_N,p_N}\} = \|p_N - f_N\|_{I,\infty} = \beta_\infty(N)\,. \tag{17}$$

For simplicity, we will write in the sequel $\beta_\infty = \beta_\infty(N)$. It follows from Proposition 1 that $\beta_\infty > 0$ and that f_N does exist, is unique and satisfies:

$$|p_N - f_N| = \begin{cases} \beta_\infty \text{ a.e. on } I\,, \\ \sigma_N \text{ a.e. on } J\,. \end{cases} \tag{18}$$

Again from Proposition 1, it follows that problem (17) is equivalent to that of finding v_N which solves the Nehari problem:

$$\min_{v\in H_\infty} \|p_N w_{1/\beta_\infty,1/\sigma_N} - v\|_\infty = \|p_N w_{1/\beta_\infty,1/\sigma_N} - v_N\|_\infty = 1\,, \tag{19}$$

where f_N and v_N are related by

$$v_N = f_N \, w_{1/\beta_\infty, 1/\sigma_N}.$$

This provides us with $f_N \in H_\infty$ and the problem is now, for each N, to choose ε_N ensuring that $f_N \in A(\mathbb{D})$. Observe, indeed, that for arbitrary values of ε_N, the outer function $w_{1/\beta_\infty, 1/\sigma_N}$ is discontinuous at $e^{\pm ia}$ and that neither v_N nor *a fortiori* f_N need to be continuous on $\mathbb{T}$. The following lemma will allow us to obtain this continuity from an appropriate choice of ε_N.

Lemma 3.1 *Under the hypotheses of Theorem* 3.1 *and still assuming* $p_N \notin H_\infty$*, the following assertions hold.*
(i) For every fixed N*, the quantity* β_∞ *defined by* (17) *and* (16) *is continuous and decreasing with respect to* ε_N *and the implicit equation:*

$$\varepsilon_N = \beta_\infty \tag{20}$$

admits a solution.
(ii) For every N *and the choice* (20) *of* ε_N*, the outer function* $w_{1/\beta_\infty, 1/\sigma_N}$ *is Lipschitz-continuous on* $\mathbb{T}$*, of exponent* μ*.*
(iii) If for every N *we choose* ε_N *as given by* (20)*, then:*

$$\lim_{\substack{N\to\infty \\ \epsilon\to 0}} \beta_\infty = 0\,. \tag{21}$$

Proof: in order to establish (i), observe from the convexity of the set $\mathcal{B}_{\sigma_N, p_N}$ and of the norm function $\| \, \|_{I,\infty}$ that β_∞ is a decreasing convex function of ε_N hence is continuous.
Now, $p_{N|_I} \in C(I)$ which is contained in the $L^\infty(I)$ closure of $H_{\infty|_I}$ (this is once again fact (i) of section 2 translated to the disk), so (17) and (16) imply that $\beta_\infty \to 0$ as $\varepsilon_N \to \infty$. Thus, for ε_N large enough, $\beta_\infty < \varepsilon_N$.
Let then $\varepsilon_N \to 0$. Assume that $\beta_\infty < \varepsilon_N$ so that in particular $\beta_\infty \to 0$. In view of (17), and since f_N remains bounded on J, this implies $p_N \in H_\infty$ (see [2, prop.3], this is fact (ii) of the introduction, again on the disk) which is a contradiction. Hence, $\beta_\infty \geq \varepsilon_N$ eventually, which proves (i) by the mean value theorem.

We turn to the proof of (ii). Since the gauge function r is assumed to be μ-Lipschitz on J, so is σ_N from its definition (16), and also $1/\sigma_N$ as $\sigma_N \geq \varepsilon_N > 0$. Hence, writing $w_N = w_{1/\beta_\infty, 1/\sigma_N}$ for simplicity,

$$|w_N| = \begin{cases} 1/\beta_\infty = 1/\varepsilon_N = 1/\sigma_N(e^{\pm} ia) \text{ on } I\,, \\ 1/\sigma_N \text{ on } J\,, \end{cases}$$

is μ-Lipschitz on $\mathbb{T}$ and it remains for us to show that w_N is also μ-Lipschitz. This follows from the fact that an outer function whose log-modulus is μ-Lipschitz is itself μ-Lipschitz, but we could not readily locate this result in the literature. So, we give the argument in the case at hand. By (6),we get for each $z \in \mathbb{D}$:

$$w_N(z) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\log|w_N(e^{i\theta})|\,d\theta\right\},$$

or else:

$$\begin{aligned} w_N(z) = & \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\,\frac{e^{i\theta}+z}{e^{i\theta}-z}\log|w_N(e^{i\theta})|\,d\theta\right\} \\ & \times\exp\left\{\frac{i}{2\pi}\int_0^{2\pi} \mathrm{Im}\,\frac{e^{i\theta}+z}{e^{i\theta}-z}\log|w_N(e^{i\theta})|\,d\theta\right\} \\ = & \ \exp\{u(z)\}\ \{\exp\, i\tilde{u}(z)\}\,, \end{aligned}$$

where

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\,\frac{e^{i\theta}+z}{e^{i\theta}-z}\log|w_N(e^{i\theta})|\,d\theta = \log|w_N(z)|$$

by the Poisson formula as applied to the bounded harmonic function $\log|w_N(z)|$, and

$$\tilde{u}(z) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Im}\,\frac{e^{i\theta}+z}{e^{i\theta}-z}\log|w_N(e^{i\theta})|\,d\theta$$

is the conjugate function. Since

$$\frac{1}{\sup_J r + \beta_\infty} \leq |w_N(e^{i\theta})| \leq \frac{1}{\beta_\infty} \text{ on } \mathbb{T}\,,$$

$\log|w_N(e^{i\theta})|$ is bounded and μ-Lipschitz on $\mathbb{T}$. It now follows from a sharp result on conjugate functions, see [6, III,thm.1.3], that

$$\tilde{u}(e^{i\theta}) = lim_{r\rightarrow 1}\tilde{u}(re^{i\theta})$$

exists for every θ and is again μ-Lipschitz on $\mathbb{T}$. By Fatou's theorem (see e.g.[11, chap.3]), the relation

$$\lim_{r\to 1} u(re^{i\theta}) = \log |w_N(e^{i\theta})|$$

is valid at every point of continuity of $\log |w_N(e^{i\theta})|$, that is, everywhere on $\mathbb{T}$. Hence

$$w_N(e^{i\theta}) = \lim_{r\to 1} \exp\left\{u(re^{i\theta})\right\} \left\{\exp\, i\tilde{u}(re^{i\theta})\right\}$$

exists for every θ and defines a μ-Lipschitz function on $\mathbb{T}$ as desired. This achieves the proof of (ii).

To prove part (iii), choose $\varepsilon_N = \beta_\infty$ for each N, as this is possible by (i), and assume that (21) is false. Then, (15) implies that for N large enough and ϵ small enough we will get

$$\|p_N - h\|_{J,\infty} \leq \varepsilon_N \,,$$

and, since $|f - h| \leq r$ on J by hypothesis, it turns out that

$$|f - p_N| \leq \sigma_N \text{ on } J\,.$$

Hence, for such N and ϵ, $f \in \mathcal{B}_{\sigma_N, p_N}$ and necessarily

$$\beta_\infty \leq \|p_N - f\|_{I,\infty} \,,$$

which, still from (15), tends to 0 as $N \to \infty$, $\epsilon \to 0$, a contradiction. This proves (iii) and the lemma. ■

To complete the proof of Theorem 3.1, choose $\varepsilon_N = \beta_\infty$. It follows from part (ii) of Lemma 3.1 that $p_N w_{1/\beta_\infty, 1/\sigma_N}$ is μ-Lipschitz hence *a fortiori* Dini continuous on $\mathbb{T}$, and the Carleson-Jacobs theorem [6, IV,thm.2.1] implies that the solution v_N to (19) belongs to $A(\mathbb{D})$. Again from part (ii) of Lemma 3.1,

$$w_{\beta_\infty, \sigma_N} = \left(w_{1/\beta_\infty, 1/\sigma_N}\right)^{-1}$$

is continuous (since it is μ-Lipschitz) so that

$$f_N = v_N w_{\beta_\infty, \sigma_N}$$

lies in $A(\mathbb{D})$.

We finally verify that $T_N(a_0, \cdots, a_N) = f_N$ does the job. Indeed, on I, we have the inequality $|f - f_N| \leq |f - p_N| + |p_N - f_N|$, and the last term is equal to β_∞ by (18); thus, (15) and (21) give (8).
Moreover, on J, we get $|h - f_N| \leq |h - p_N| + |p_N - f_N|$ and, since $\sigma_N = r + \beta_\infty$, we obtain (9) from (15), (18), and (21). This ends the proof of Theorem 3.1.
■

Having established Theorem 3.1, we must tie one loose end to make the proof constructive, namely how does one find in practice β_∞ in order to solve the Nehari problem (19) and to select ε_N according to (20). This can be done by a dichotomy procedure which rests on Lemma 3.2 below.
For every $\varepsilon > 0$, define the map Δ_ε:

$$\begin{array}{rccl} \Delta_\varepsilon : &]\,0,\,\infty[& \longrightarrow &]\,0\,,\infty\,[\\ & \delta & \longmapsto & \min\limits_{v \in H_\infty} \|p_N w_{1/\delta, 1/(\varepsilon+r)} - v\|_\infty \end{array}$$

Lemma 3.2 *If $p_N \notin H_\infty$, then for every $\varepsilon > 0$, the map Δ_ε is defined from $]0, \infty[$ onto $]0, \infty[$, is continuous, and monotonically decreasing.*

Proof of lemma 3.2: let $\varepsilon > 0$. Then for every real $\delta > 0$, the function $p_N w_{1/\delta, 1/(\varepsilon+r)} \in H_\infty + C(\mathbb{T})$. Hence, by [6, IV, thm.1.3,thm.1.7], there is a unique function $v_\delta \in H_\infty$ such that:

$$\Delta_\varepsilon(\delta) = \|p_N w_{1/\delta, 1/(\varepsilon+r)} - v_\delta\|_\infty \,. \tag{22}$$

Let $\delta_1\,,\,\delta_2 > 0$, $\delta_1 \neq \delta_2$. Then, from the definition of Δ_ε, we get:

$$\Delta_\varepsilon(\delta_1) < \|p_N w_{1/\delta_1, 1/(\varepsilon+r)} - w_{\delta_2/\delta_1, 1}\, v_{\delta_2}\|_\infty = \|w_{\delta_2/\delta_1, 1}\,(p_N w_{1/\delta_2, 1/(\varepsilon+r)} - v_{\delta_2})\|_\infty.$$

That the inequality above is strict follows from the uniqueness of v_δ and the fact that $w_{\delta_2/\delta_1, 1}\, v_{\delta_2} \neq v_{\delta_1}$: indeed,

$$|p_N w_{1/\delta_1, 1/(\varepsilon+r)} - v_{\delta_1}| = \Delta_\varepsilon(\delta_1)$$

and

$$|p_N w_{1/\delta_2, 1/(\varepsilon+r)} - v_{\delta_2}| = \Delta_\varepsilon(\delta_2)$$

are constant a. e. on $\mathbb{T}$, while

$$|p_N w_{1/\delta_1,1/(\varepsilon+r)} - w_{\delta_2/\delta_1,1}\, v_{\delta_2}| = \Delta_\varepsilon(\delta_2)|w_{\delta_2/\delta_1,1}|$$

assumes different values on I and J. Therefore:

$$\Delta_\varepsilon(\delta_1) < \max\left(\frac{\delta_2}{\delta_1}\|p_N w_{1/\delta_2,1/(\varepsilon+r)} - v_{\delta_2}\|_{I,\infty}, \|p_N w_{1/\delta_2,1/(\varepsilon+r)} - v_{\delta_2}\|_{J,\infty}\right)$$

$$= \Delta_\varepsilon(\delta_2)\max\left(\frac{\delta_2}{\delta_1}, 1\right).$$

Taking $\delta_2 < \delta_1$ shows that Δ_ε decreasing and then $\delta_1 < \delta_2$ that it is continuous.

As a continuous and positive decreasing map, Δ_ε has a limit at ∞. Given $\xi > 0$, there exists a function $g \in H_\infty$ such that $\|p_N w_{1,1/(\varepsilon+r)} - g\|_{J,\infty} < \xi$ because $p_N w_{1,1/(\varepsilon+r)} \in H_\infty + C(\mathbb{T})$ and $H_{\infty|_J}$ is dense in $C(J)$ (this follows at once from fact (i) in section 2). For every $n > 0$, we have:

$$\Delta_\varepsilon(n) \leq \|p_N w_{1/n,1/(\varepsilon+r)} - g w_{1/n,1}\|_\infty ,$$

which implies that for n large enough:

$$\Delta_\varepsilon(n) \leq \max\left(\frac{1}{n}\|p_N w_{1,1/(\varepsilon+r)} - g\|_{I,\infty}, \|p_N w_{1,1/(\varepsilon+r)} - g\|_{J,\infty}\right) < \xi .$$

As ξ is arbitrarily small, we necessarily get

$$\lim_{\delta\to\infty} \Delta_\varepsilon(\delta) = 0.$$

To analyse the behaviour of Δ_ε when $\delta \to 0$, we write

$$\Delta_\varepsilon(\delta) = \max\left(\frac{1}{\delta}\,\|p_N w_{1,1/(\varepsilon+r)} - v_\delta\, w_{\delta,1}\|_{I,\infty}\ ,\ \|p_N w_{1,1/(\varepsilon+r)} - v_\delta\, w_{\delta,1}\|_{J,\infty}\right) .$$

We claim that if the first argument of the max remains bounded as $\delta \to 0$, then the second does not. Indeed, $v_\delta\, w_{\delta,1}$ would otherwise be a family of H_∞ functions converging to $p_N w_{1/\delta,1/(\varepsilon+r)}$ in $L^\infty(I)$ as δ tends to 0, but remaining bounded on J; in view of $p_N \notin H_\infty$, this would contradict [2, prop.3] (fact (ii) of section 2 rephrased on the disk). Thus, we get

$$\lim_{\delta \to 0} \Delta_\varepsilon(\delta) = \infty.$$

This shows that Δ_ε is onto $(0, \infty)$. ∎

By Lemma 3.2, we can associate to every $\varepsilon > 0$ a unique $\beta_\infty(\varepsilon) > 0$ such that $\Delta_\varepsilon(\beta_\infty(\varepsilon)) = 1$, and $\beta_\infty(\varepsilon)$ may be computed by a dichotomy procedure in view of the monotonicity of Δ_ε.
Given p_N, which in turn defines Δ_ε, what we want to find is now the unique value $\varepsilon = \varepsilon_N$ for which $\beta_\infty(\varepsilon) = \varepsilon$ so that both (19) and (20) are satisfied. In view of the monotonicity asserted in part (i) of Lemma 3.1, this can again be solved by dichotomy.
This process, which is somehow similar in spirit to the γ-iteration used in H_∞-control, settles our constructive approach to Theorem 3.1. However, it requires solving a series of Nehari problems the solution of which can be numerically estimated only when the function to be approximated is continuous. Indeed, in this case, it can be represented arbitrarily well in $L^\infty(\mathbb{T})$ by a rational function (using for instance the Jackson polynomials previously introduced to compute p_N) whose Hankel operator has finite rank and thus posseses a finite singular-values decomposition allowing to solve the associated Nehari problem in various fashions (see e.g. [3, 5]).
Now, the typical Nehari problem we must solve here is associated to a function of the form

$$p_N w_{1/\delta, 1/(\varepsilon + r)} \tag{23}$$

for some positive numbers ε and δ, and such a function is clearly discontinuous at $e^{\pm ia}$ in general. However, (23) is continuous at any other point on $\mathbb{T}$, because it is even μ-Lipschitz there; indeed, an outer function whose log-modulus is μ-Lipschitz in the neighborhood of some point is itself μ-Lipschitz at this point: this is the local version of (ii) of Lemma 3.1 and it is proved in the same manner except that we must appeal, this time, to a local version of the regularity theorem for conjugate functions (see the proof of [6, III, cor. 1.4]). To circumvent the discontinuity problem at $e^{\pm ia}$, we introduce another Nehari problem, equivalent to (19). Let p be the first order trigonometric polynomial which coincides with p_N at $e^{\pm ia}$, so that $(p_N - p)w_{1/\delta,1/(\varepsilon+r)}$ is continuous on $\mathbb{T}$. The Nehari problem

$$\min_{g\in H_\infty} \|(p_N - p)w_{1/\delta,1/(\varepsilon+r)} - g\|_\infty \tag{24}$$

is clearly equivalent to (19) under the transformation

$$v = g + pw_{1/\delta,1/(\varepsilon+r)},$$

and consequently assumes the same value. The dichotomy procedures described before may now be performed numerically by solving (24) iteratively, and this what was done in the examples presented in section 5.

4 Error estimates

In this section we analyse the worst-case error bounds in an approximate model obtained by the methods of the previous section.
We shall make the hypothesis that the points z_k are κ^{th} roots of unity, for some even integer $\kappa > 2N$, and hence equally spaced in I and that $a = \pi - 2\pi N/\kappa)$. We take here h to be the function $e_{f(e^{ia})}$ defined by (4), so that the function h_0 defined in (5) will simply be zero. The approximation h_N to h becomes then e_{a_N}, formed by linear interpolation through the measured values at the endpoints of J. This choice is mainly for definiteness and is not essential although it leads to simpler computations.
We write ω_f for the modulus of continuity of f, that is

$$\omega_f(\alpha) = \sup_{|\theta-\phi|\le\alpha} |f(e^{i\theta}) - f(e^{i\phi})|. \tag{25}$$

To construct the trigonometric polynomial p_N we use the noisy values $(a_k)_{k=-N}^{N}$ of f on I together with the values of h_N on J to produce the discrete de la Vallée Poussin polynomial $V_{s,\kappa}$ with $\kappa \geq 4s+1$, as in [14]. Because $|f(z_k)-a_k| = |\eta_k| \leq \epsilon$ while $\|h_N - h\|_{J,\infty} < 2\epsilon/\sin a$, this is equivalent to using measurements of $\hat{f} = f \vee h$ with an error of at most $2\epsilon/\sin a$ and hence

$$\|\hat{f} - p_N\|_\infty \leq (4 + 2/s)(\,\mathrm{dist}\,(\hat{f}, P_s) + 2\epsilon/\sin a),$$

where P_s is the space of trigonometric polynomials of degree at most s (see [14, thm.3.1]). Now $\operatorname{dist}(\hat{f}, P_s) \leq \frac{3}{2}\omega_{\hat{f}}\left(\frac{\pi}{s+1}\right)$ by Jackson's theorem [19], which, given the definition of $\hat{f}$ implies that

$$\operatorname{dist}(\hat{f}, P_s) \leq \frac{3}{2} \max_{0\leq\lambda\leq 1}\left[\omega_f\left(\frac{\lambda\pi}{s+1}\right) + (1-\lambda)\frac{\pi}{s+1}\frac{\|f\|_{I,\infty}}{\sin a}\right].$$

One could improve upon this bound by considering a smoother extension $\hat{f}$ to f. One way to do this would be to choose h to be a function cubic (in θ) which matches f and its derivatives at the points $z_{\pm N}$ (this, of course, assumes one is able to estimate these derivatives). An approximation h_N to h could in this case be a cubic polynomial matching the noisy values $a_{\pm(N-1)}$ and $a_{\pm N}$, as in [16].
Recall now that the final model f_N is the solution to the extremal problem (17). Moreover by the proof of Lemma 3.1, we see that $\beta_\infty(N) \leq \|\hat{f} - p_N\|_\infty$. This gives us the following estimates for the error in the identified model f_N:

$$\|f - f_N\|_{I,\infty} \leq \|f - p_N\|_{I,\infty} + \|p_N - f_N\|_{I,\infty} \leq 2\|\hat{f} - p_N\|_\infty,$$

and on J,

$$|h - f_N| \leq |h - p_N| + |p_N - f_N| \leq \|h - p_N\|_{J,\infty} + (r + \beta_\infty(N)) \leq r + 2\|\hat{f} - p_N\|_\infty.$$

It is of perhaps more interest to have a bound for $|f - f_N|$ on J, and this follows immediately from the triangle inequality as well, giving on J

$$|f - f_N| \leq 2r + 2\|\hat{f} - p_N\|_\infty.$$

Observe that the bounds for $|f - f_N|$ on I and $|h - f_N|$ on J are explicit and satisfy conditions (8) and (9) of theorem 3.1 (where T_N is taken to be f_N).

5 Numerical examples

As a first example, we consider the function $f(z) = 3(z^2+1)/(z^2+2z+5)$, which has been studied (using information on the whole circle) in [8, 14, 15]. We consider the arc I defined by $a = 1$ and we choose $N = 400$. The values

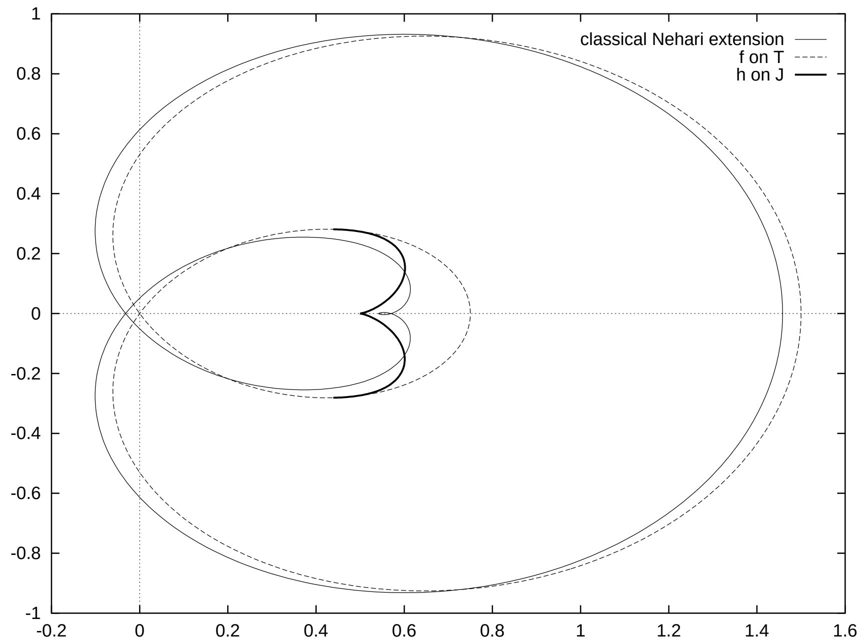

Figure 1: example 1, $d(p_N, H^\infty) = 0.04183$.

of h on J are shown in bold in figure 1, where they differ considerably from the values of f. The trigonometric polynomial p_N, of degree 800, has been built using discrete de la Vallée Poussin polynomials, and a classical Nehari extension of p_N is performed on the same figure; the error, which is equal to the $L^\infty(\mathbb{T})$ distance from p_N to H_∞, has value 0.04183 and is circular, as expected in this case. The result f_N of the constrained extension approach developped in section 3 is shown in figure 2 where the gauge function r has been chosen to be the modulus of the difference $h - f$ on J, which is the appropriate choice in order to recover f on I; indeed, the $L^\infty(I)$ approximation error, $\beta_\infty = \|p_N - f_N\|_{I,\infty}$, is equal to 0.00007145.

Our second example consists in real data measured on a hyperfrequencies filter of the CNES (the French National Center for Spacial Research). The band-

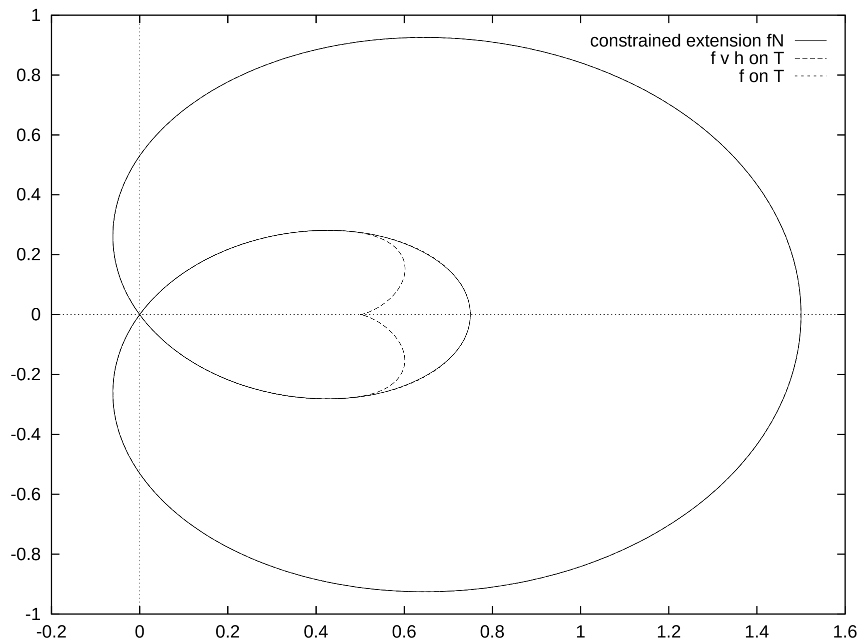

Figure 2: example 1, $r = |f - h|$ on J, $\beta_\infty = 0.00007145$.

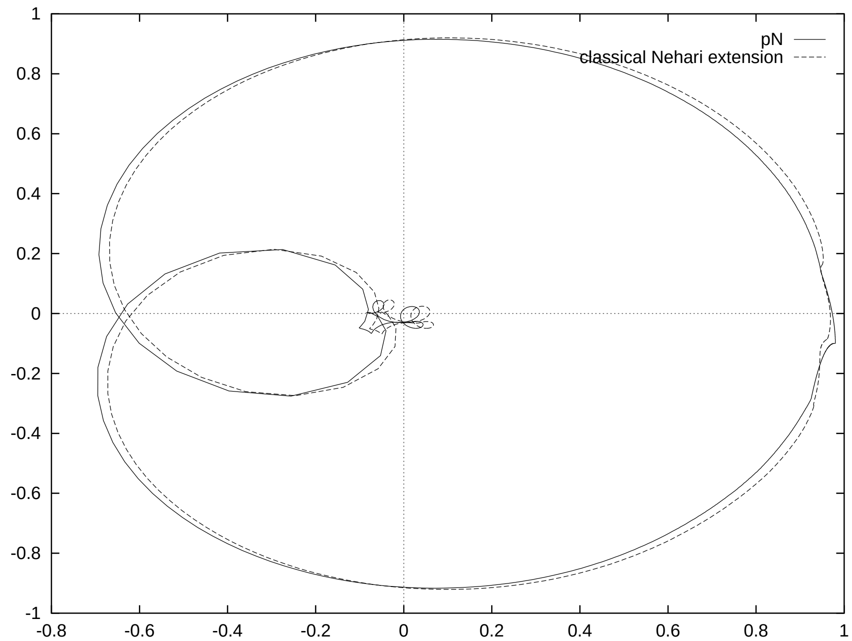

Figure 3: example 2, $d(p_N, H^\infty) = 0.0236$.

width I is now defined by $a = \pi/2$ and we are given 801 noisy pointwise values (a_k), so that $N = 400$. We first complete these data outside the bandwidth by a rough estimate h_N and we construct the trigonometric polynomial p_N as in the first example. Figure 3 shows the result of the classical Nehari extension to p_N, which gives rise to an error of value 0.0236 in $L^\infty(\mathbb{T})$. We then compute the solution f_N to the constrained approximation problem associated to p_N for different gauge functions r (as we cannot compute the best one) until an acceptable trade off is found between β_∞ and r; these gauge functions are plotted in figure 4. If no satisfactory compromise can be found, one can change the reference behaviour h_N on J, using the previous computations in order to make a more accurate choice. The corresponding results are shown in figures 5, 6, and 7. As expected, β_∞ globally decreases with respect to the size of r on J: r_0 does not allow enough elbow room for f_N on J while r_2 allows too much, giving rise

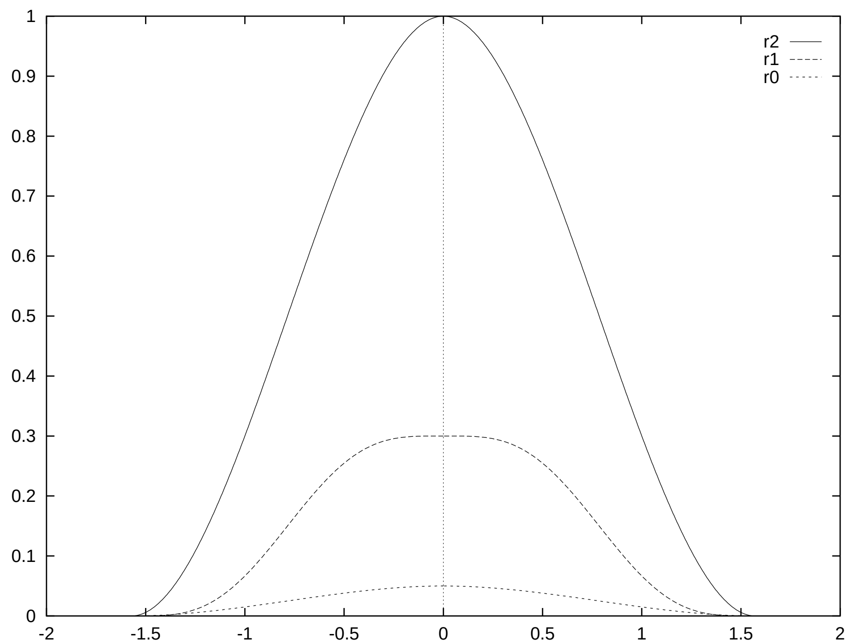

Figure 4: the gauge functions r_0, r_1, and r_2, on J.

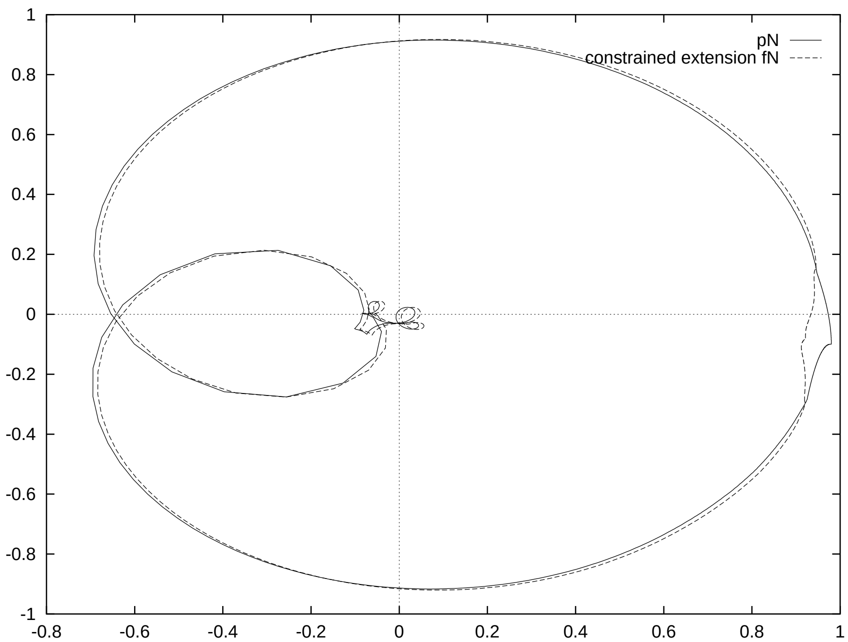

Figure 5: example 2, gauge function r_0, $\beta_\infty = 0.0122$

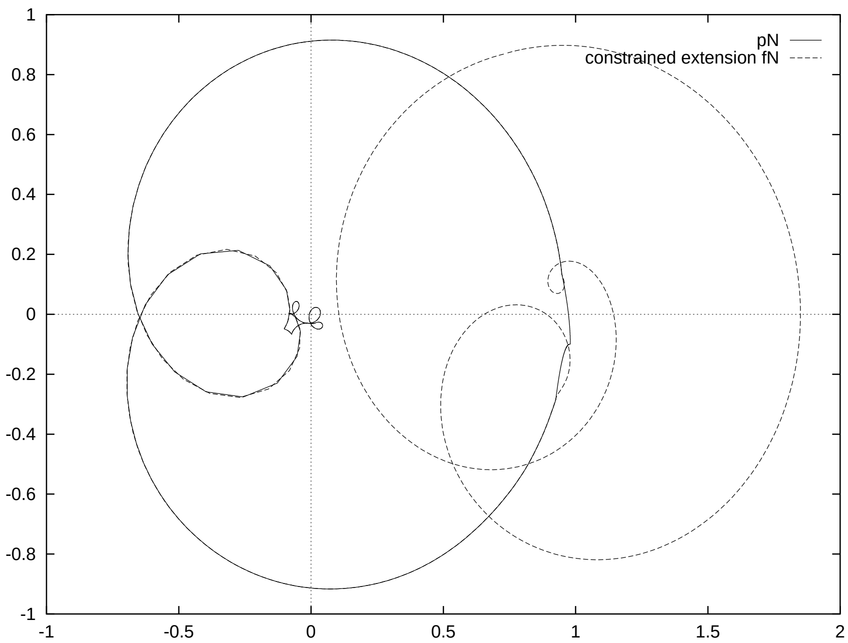

Figure 6: example 2, gauge function r_2, $\beta_\infty = 0.000347$

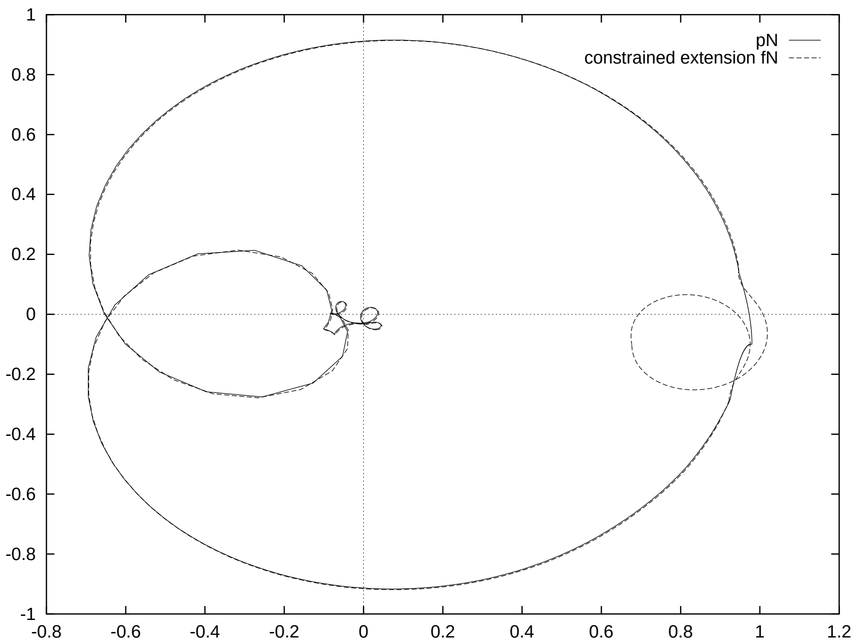

Figure 7: example 2, gauge function r_1, $\beta_\infty = 0.00328$

to a wild and winding behaviour of f_N on J which illustrates the oscillating phenomenon described in section 2; r_1 seems a reasonable compromise.

6 Conclusion

In this paper, we introduced a framework for robust band-limited identification, which extends the existing framework for robust identification on the whole axis (or circle) that was introduced in [10]. We also developed a constructive algorithm to perform such a band-limited identification, which recovers the transfer-function on the bandwidth in a robust fashion while meeting gauge constraints at the remaining frequencies. The procedure is very similar in spirit to the two-stages algorithms proposed in [8, 10, 14, 15], but appeals to a bounded extremal problem which may be seen as a generalization of the classical Nehari problem. We also derived error bounds in a standard case and presented examples on real data.

There are at least two further questions which, in our opinion, deserve some more study. The first arises from the observation that the identification procedure can be applied to any sequence of data $a_0, a_1 ...$; the question is: what is the limit behaviour of $T_N(a_1, ... a_N)$ if the data do not converge (pointwise in l_∞) to some interpolation sequence $f(a_0), f(a_1), ...$ with $f \in A(\mathbb{D})$? The second question stems from the fact that our identification scheme converges uniformly to f on I but only weak–* to f on J. This is enough to recover f uniformly on compact subsets of the half-plane (or of the disk) by the Poisson formula, but not to recover f itself. Now, still assuming (3), what additional hypotheses would be needed on f in order to design an algorithm producing some stronger type of convergence? We think both questions are important in connection with the practical value of such schemes.

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Unité de recherche INRIA Lorraine, Technopôle de Nancy-Brabois, Campus scientifique,
615 rue du Jardin Botanique, BP 101, 54600 VILLERS LÈS NANCY
Unité de recherche INRIA Rennes, Irisa, Campus universitaire de Beaulieu, 35042 RENNES Cedex
Unité de recherche INRIA Rhône-Alpes, 46 avenue Félix Viallet, 38031 GRENOBLE Cedex 1
Unité de recherche INRIA Rocquencourt, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex
Unité de recherche INRIA Sophia-Antipolis, 2004 route des Lucioles, BP 93, 06902 SOPHIA-ANTIPOLIS Cedex

Éditeur

INRIA, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)

ISSN 0249-6399