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NUMERICAL ANALYSIS OF A MIXED ELLIPTIC PROBLEM WITH FLUX AND CONVECTIVE BOUNDARY CONDITIONS TO OBTAIN A DISCRETE SOLUTION OF NON-CONSTANT SIGN

ANALYSE NUMERIQUE D'UN PROBLEME ELLIPTIQUE MIXTE AVEC DES CONDITIONS AU BORD DU FLUX ET CONVECTIVE POUR OBTENIR UNE SOLUTION DISCRETE DE SIGNE NON-CONSTANTE

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Abstract

We consider a material $\Omega \subset \mathbb{R}^n$ which occupies a convex polygonal bounded domain, with regular boundary $\Gamma=\Gamma_1 \cup \Gamma_2$ (with $\overset{o}{\Gamma}_1 \cap \overset{o}{\Gamma}_2 = \emptyset$) with $\mathrm{meas}(\Gamma_1)=|\Gamma_1|>0$ and $|\Gamma_2|>0$. We assume, without loss of generality, that the melting temperature is 0°C. We consider the following steady-state heat conduction problem in Ω :

$$\Delta u = 0 \text{ in } \Omega \ , \quad -\frac{\partial u}{\partial n}\Big|_{\Gamma_1} = \alpha\,(u-B) \ , \quad -\frac{\partial u}{\partial n}\Big|_{\Gamma_2} = q \ ,$$

where α, q, $B=\mathrm{Const.}>0$. In a previous paper (Tabacman–Tarzia, J. Diff. Eq., 77 (1989), 16 – 37) sufficient and/or necessary conditions on data α, q, B, Ω, Γ_1, Γ_2 to obtain a temperature u of non-constant sign in Ω (that is, a multidimensional steady-state, two-phase, Stefan problem) were studied.

We consider a regular triangulation by finite element method of the domain Ω with Lagrange triangles of the type 1, being $h>0$ the parameter of the discretization. We obtain sufficient (and/or necessary) conditions on data α, q, B, Ω, Γ_1, Γ_2 , as functions of the parameter h, to obtain a change of phase (steady-state, two-phase, discretized Stefan problem) in the corresponding discretized domain, that is a discrete temperature u_h of non-constant sign in Ω. We continue and use the results obtained in Tarzia, "Numerical analysis for the heat flux in a mixed elliptic problem to obtain a discrete steady-state two-phase Stefan problem", To appear on SIAM J. Numer. Anal. (See Rapport de Recherche INRIA N° 1593 (1992)).

Résumé

On considère un matériel $\Omega \subset \mathbb{R}^n$, un domaine polygonal borné et convexe avec une frontière réguliere $\Gamma=\Gamma_1 \cup \Gamma_2$ (avec $\overset{o}{\Gamma}_1 \cap \overset{o}{\Gamma}_2 = \emptyset$) avec $\mathrm{mes}(\Gamma_1) = |\Gamma_1| > 0$ et $|\Gamma_2| > 0$. On suppose, sans perte de généralité, que la température de changement de phase est 0°C. On considère le suivant cas stationnaire de conduction de la chaleur dans Ω :

$$\Delta u = 0 \ \text{ dans } \ \Omega \ , \quad -\frac{\partial u}{\partial n}\Big|_{\Gamma_1} = \alpha\,(u-B) \ , \quad -\frac{\partial u}{\partial n}\Big|_{\Gamma_2} = q \ ,$$

où α, q, $B=\mathrm{Const.}>0$. Dans le travail (Tabacman–Tarzia, J. Diff. Eq., 77(1989), 16 – 37) on a étudié des conditions suffisantes (et/ou nécessaires) sur les données α, q, B, Ω, Γ_1, Γ_2 pour obtenir une température u de signe non-constante dans Ω (c'est-à-dire, un cas stationnaire du problème de Stefan à deux phases).

On considère une triangulation régulier de Ω avec des triangles de Lagrange de type 1 avec $h>0$ le paramètre de la discretisation. On obtient des conditions suffisantes (et/ou nécessaires) sur les données α, q, B, Ω, Γ_1, Γ_2 , comme fonctions du paramètre h, pour obtenir un changement de phase (cas stationnaire discret d'un problème de Stefan à deux phases) dans le domaine discrète correspondante, c'est-à-dire une température discrète de signe non-constante dans Ω. On continue la recherche faite dans Tarzia, "Numerical analysis for the heat flux in a mixed elliptic problem to obtain a discrete steady-state two-phase Stefan problem", To appear on SIAM J. Numer. Anal. (Voir Rapport de Recherche INRIA N° 1593 (1992)).

Key words : Steady-state Stefan problem, finite element method, mixed elliptic problem, numerical analysis, variational inequalities, error bounds.

Mots Clés : Problème de Stefan stationnaire, méthode d'éléments finis, problème elliptique mixte, analyse numérique, inéquations variationnelles, estimation de l'erreur.

AMS Subject Classification : 35R35, 35J85, 65N15, 65N30.

I. INTRODUCTION.

We consider a material $\Omega \subset \mathbb{R}^n$ which occupies a convex polygonal bounded domain, with regular boundary $\Gamma = \Gamma_1 \cup \Gamma_2$ (with $\mathring{\Gamma}_1 \cap \mathring{\Gamma}_2 = \emptyset$) with $\mathrm{meas}(\Gamma_1) = |\Gamma_1| > 0$ and $|\Gamma_2| > 0$. We assume, without loss of generality, that the melting temperature is 0°C. We consider the following steady-state heat conduction problem in Ω [11, 12] :

$$(1) \qquad \left| \begin{array}{l} \Delta u = 0 \text{ in } \Omega, \\ \\ -\dfrac{\partial u}{\partial n}\Big|_{\Gamma_1} = \alpha\,(u - B) \text{ on } \Gamma_1 \;, \\ \\ -\dfrac{\partial u}{\partial n}\Big|_{\Gamma_2} = q \;\text{ on } \Gamma_2 \;, \end{array} \right.$$

where α, q, $B = \text{Const.} > 0$. Its variational formulation is given by [5, 11]

$$(2) \qquad \left| \begin{array}{l} a_\alpha(u, v) = L_{\alpha q B}(v) \;, \;\; \forall\, v \in V, \\ \\ u \in V \;, \end{array} \right.$$

where

$$(3) \qquad \left| \begin{array}{ll} V = H^1(\Omega) \;, & V_0 = \{\, v \in V \,/\, v|_{\Gamma_1} = 0 \,\}, \\ \\ a_\alpha(u,v) = a(u,v) + \alpha \displaystyle\int_{\Gamma_1} u\, v\, d\gamma \;, & a(u,v) = \displaystyle\int_\Omega \nabla u \,.\, \nabla v\, dx, \\ \\ L_{\alpha q B}(v) = L_q(v) + \alpha\, B \displaystyle\int_{\Gamma_1} v\, d\gamma \;, & L_q(v) = -\, q \displaystyle\int_{\Gamma_2} v\, d\gamma \,. \end{array} \right.$$

Problem (2) is equivalent to the following minimization problem

$$(4) \qquad \left| \begin{array}{l} G(u) \le G(v), \qquad \forall\, v \in V, \\ \\ u \in V \;, \end{array} \right.$$

where

$$(5) \qquad G(v) = \tfrac{1}{2}\, a_\alpha(v, v) - L_{\alpha q B}(v) \,.$$

The unique solution $u = u(\alpha) = u(\alpha, q, B)$ of problem (2) is given by

$$(6) \qquad u(\alpha, q, B) = B - q\, U(\alpha) \;\text{ in } \Omega \,,$$

where $U = U(\alpha)$ is the unique solution of the variational equality

$$(7)\qquad \left|\begin{array}{l} a_\alpha(U(\alpha),v) = \displaystyle\int_{\Gamma_2} v\, d\gamma, \quad \forall\, v \in V, \\ U(\alpha) \in V\,. \end{array}\right.$$

We suppose that Ω and Γ have the necessary regularity so that $U(\alpha) \in C^o(\overline{\Omega})$ as in [6] ([10, 12] show us three examples in which this condition is satisfied). A evolution Stefan problem with mixed *boundary conditions is considered in* [7,8].

In [10] the following results are obtained

THEOREM 1.– (i) Si $(\alpha,q) \in S^2(B)$ then we obtain a steady-state, two-phase, Stefan problem, with

$$(8)\qquad \left|\begin{array}{l} S^2(B) = \{\, (\alpha,q) \in (\mathbb{R}^+)^2 \,/\, q_m(\alpha,B) < q < q_M(\alpha,B) \,\}, \\ q_m(\alpha,B) = \dfrac{B\,|\,\Gamma_2\,|}{A(\alpha)} \quad , \qquad q_M(\alpha,B) = \dfrac{B\,\alpha\,|\,\Gamma_1\,|}{|\,\Gamma_2\,|} \quad , \end{array}\right.$$

where $A = A(\alpha)$ is a decreasing function in variable α which verify the following properties :

$$(9)\qquad \left|\begin{array}{ll} A(\alpha) > \dfrac{|\Gamma_2|^2}{|\Gamma_1|}\dfrac{1}{\alpha} \;, & \displaystyle\lim_{\alpha\to+\infty} A(\alpha) = C > 0 \;, \\ \displaystyle\lim_{\alpha \to +\infty} \alpha\, A'(\alpha) = 0 \;, & (\alpha\, A(\alpha)\,)' = \dfrac{1}{q^2}\; a(u_{\alpha qB}\,,\, u_{\alpha qB}) \;, \\ A(\alpha) = \displaystyle\int_{\Gamma_2} U(\alpha)\, d\gamma = a_\alpha(U(\alpha)\,,\, U(\alpha))\;, & \end{array}\right.$$

where " $'$ " represents the derivative with respect to α .

On the other hand, the constant $C > 0$ is given by

$$(10)\qquad C \doteq \int_{\Gamma_2} u_3\, d\gamma = a(u_3, u_3) > 0 \;,$$

where u_3 is the unique solution of the variational equality

$$(11)\qquad \left|\begin{array}{l} a(u_3, v) = \displaystyle\int_{\Gamma_2} v\, d\gamma\,, \qquad \forall\, v \in V_o\,, \\ u_3 \in V_o\,. \end{array}\right.$$

(ii) If we have the <u>continuous particular case</u> , defined by the condition

(12) $$\frac{1}{q^2}\, a(u(\alpha,q,B),\, u(\alpha,q,B)) = \text{Const.}\ (= C\ >\ 0)$$

then we deduce

(13) $$A(\alpha) = C + \frac{|\Gamma_2|^2}{|\Gamma_1|}\,\frac{1}{\alpha}\ .$$

Now, we consider τ_h, a regular triangulation of the polygonal domain Ω with Lagrange triangles of type 1, constituted by affine-equivalent finite element of class C^0, where $h > 0$ is a parameter which goes to zero. We can take h equal to the longest side of the triangles $T \in \tau_h$ and we can approximate V by [1, 3] :

(14) $$V_h = \left\{ v_h \in C^0(\bar{\Omega}) \;/\; v_h\,|_T \in P_1(T)\,,\ \forall T \in \tau_h \right\} \subset V\,,$$

where P_1 is the set of the polynomials of degree less than or equal to 1. Let Π_h be the corresponding linear interpolation operator.

We consider the following finite dimensional approximate variational problem, corresponding to the continuous variational problem (2), given by :

(15) $$\left|\begin{array}{ll} a_\alpha(u_h, v_h) = L_{\alpha q B}(v_h) & ,\ \forall\, v_h \in V_h\,, \\ u_h = u_h(\alpha,q,B) \in V_h\ , & \end{array}\right.$$

and then we obtain the following results :

THEOREM 2.– (i) There exists a unique solution $u_h(\alpha,q,B) \in V_h$ of the discretized problem (15). Moreover, $u_h(\alpha,q,B)$ is given by

(16) $$u_h(\alpha,q,B) = B - q\, U_h(\alpha)\ ,$$

where $U_h(\alpha)$ is the unique solution of the following variational equality :

(17) $$\left|\begin{array}{ll} a_\alpha(U_h(\alpha), v_h) = \displaystyle\int_{\Gamma_2} v_h\, d\gamma\ , & \forall\, v_h \in V_h\,, \\ U_h(\alpha) \in V_h\ . & \end{array}\right.$$

(ii) Function $U_h(\alpha)$ verifies the following property :

$$(18) \qquad \int_{\Gamma_1} U_h(\alpha)\, d\gamma = \frac{|\Gamma_2|}{\alpha} , \qquad U_h(\alpha) > 0 \text{ in } \Omega .$$

(iii) Function $u_h(\alpha, q, B)$ verifies the following properties :

$$(19) \qquad \left| \begin{array}{l} u_h(\alpha,q,B) \leq B \text{ in } \bar{\Omega} , \quad u_h(\alpha,q,B) \leq u_h(q,B) \text{ in } \bar{\Omega} , \\ \\ u_h(\alpha,q,B) \to u_h(q,B) \text{ in } V \text{ when } \alpha \to +\infty , \\ \\ \underset{\Gamma_2}{\mathrm{Min}}\, u_h(\alpha,q,B) \leq u_h(\alpha,q,B) \leq \underset{\Gamma_1}{\mathrm{Max}}\, u_h(\alpha,q,B) \text{ in } \bar{\Omega} , \end{array} \right.$$

where $u_h(q, B) = B - q\, u_{3h}$, being u_{3h} the unique solution of the following discrete variational equality

$$(20) \qquad \left| \begin{array}{l} a(u_{3h}, v_h) = \int_{\Gamma_2} v_h \, d\gamma , \quad \forall\, v_h \in V_{oh} , \\ \\ u_3 \in V_{oh} , \end{array} \right.$$

where

$$(21) \qquad V_{oh} = \left\{ v_h \in C^0(\bar{\Omega}) \;/\; v_h \mid_T \in P_1(T) ,\ \forall T \in \tau_h ,\ v_h \mid_{\Gamma_1} = 0 \right\} ,$$

(iv) We have the equality

$$(22) \qquad a(u_h(\alpha,q,B), u_h(q,B)) = a(u_h(q,B), u_h(q,B)) \ .$$

(v) We have the following monotony property :

$$\alpha_1 \leq \alpha_2 ,\ q_2 \leq q_1 \Rightarrow u_h(\alpha_1,q_1,B) \leq u_h(\alpha_2,q_2,B) \text{ in } \bar{\Omega},\ \forall\, B > 0 .$$

Proof.– We use the discrete variational equalities (15), (16) and (17), and we conclude the proof by following a method similar to the one developed for the continuous case [10].

We define, for each $h > 0$, the following real functions

$$(23) \qquad \left| \begin{array}{l} A_h(\alpha) = \int_{\Gamma_2} U_h(\alpha)\, d\gamma = a_\alpha(U_h(\alpha), U_h(\alpha)) > 0 \ , \ \forall\, \alpha > 0 , \\ \\ q_{m_h}(\alpha, q) = \dfrac{B\, |\Gamma_2|}{A_h(\alpha)} \quad , \ \forall\, \alpha\, , q > 0 . \end{array} \right.$$

The goal of this paper is to consider the discrete equivalent of the inequalities (8) which define the set $S^2(B)$. We shall obtain sufficient (and/or necessary) conditions on data α, q, B, h, Ω, Γ_1 and Γ_2 to get that the discrete solution $u_h(\alpha, q, B)$ is of non-constant sign in Ω (that is, a steady-state, two-phase, discretized Stefan problem). We shall obtain error bounds for $A_h(\alpha) - A(\alpha)$ and for $q_{m_h}(\alpha, B) - q_m(\alpha, B)$, as funtions of the parameter h. We also will analyse the discrete particular case corresponding to the continuous particular case (12).

In other words, we shall obtain for the solution of the mixed elliptic discretized problem (15), defined as $u_h(\alpha, q, B)$, analogous conditions to the ones obtained for the corresponding continuous problem [10], defined by $u(\alpha, q, B)$. For the corresponding numerical analysis, we use ideas developed in [13].

II. CONDITIONS FOR THE EXISTENCE OF A DISCRETE SOLUTION OF NON-CONSTANT SIGN.

For each $q > 0$, $\alpha > 0$, $B > 0$, we consider the functions $u(\alpha, q, B) \in V$ and $u_h(\alpha, q, B) \in V_h$, respectively, as the unique solution of the variational equalities (2) (continuous problem) and (15) (discrete problem).

Therefore, we obtain the following properties :

LEMMA 3.– We have :

(i) Function $A_h(\alpha)$ is also given by the expression :

(24) $$A_h(\alpha) = a_\alpha(U(\alpha), U_h(\alpha)).$$

(ii) We have

(25) $$A(\alpha) - A_h(\alpha) = a_\alpha(U_h(\alpha) - U(\alpha), U_h(\alpha) - U(\alpha)) \geq 0 .$$

(iii) Functions q_m and q_{m_h} are related by the following inequality :

(26) $$q_m(\alpha, B) \leq q_{m_h}(\alpha, B) .$$

(iv) We have the following integrals expressions :

(27) $$\int_{\Gamma_1} u_h(\alpha, q, B)\, d\gamma = \frac{|\Gamma_2|}{\alpha} [q_M(\alpha, B) - q] , \quad \forall h > 0,$$

(28) $$\int_{\Gamma_2} u_h(\alpha, q, B)\, d\gamma = A_h(\alpha) [q_{m_h}(\alpha, B) - q] , \quad \forall h > 0.$$

Proof.– (i) It follows from the variational equalities which define $U(\alpha) \in V$ and $U_h(\alpha) \in V_h \subset V$.
(ii) We have

$$0 \leq a_\alpha(U_h(\alpha) - U(\alpha), U_h(\alpha) - U(\alpha)) = a_\alpha(U_h(\alpha), U_h(\alpha)) + a_\alpha(U(\alpha), U(\alpha)) -$$

$$- 2\, a_\alpha(U_h(\alpha), U(\alpha)) = A_h(\alpha) + A(\alpha) - 2\, A_h(\alpha) = A(\alpha) - A_h(\alpha)\,.$$

(iii) (26) follows from the fact that $A(\alpha) \geq A_h(\alpha)$.
(iv) Taking into account (18) and (22) we obtain the expressions (27) and (28) respectivelly.

REMARK 1.– From (27) and (28) we obtain the following equivalences :

$$\int_{\Gamma_2} u_h(\alpha, q, B)\, d\gamma < 0 \Leftrightarrow q > q_{m_h}(\alpha, B)\,, \tag{29}$$

$$\int_{\Gamma_1} u_h(\alpha, q, B)\, d\gamma > 0 \Leftrightarrow q < q_M(\alpha, B)\,. \tag{30}$$

For each $h > 0$, we define the real function g_h: $(\mathbb{R}^+)^3 \to \mathbb{R}$ in the following way

$$g_h(\alpha, q, B) = G_{\alpha q B}(u_h(\alpha, q, B)) = -\tfrac{1}{2}\, L_{\alpha q B}(u_h(\alpha, q, B)) = \tag{31}$$

$$= -\tfrac{1}{2}\, a_\alpha(u_h(\alpha, q, B)\,,\, u_h(\alpha, q, B)) < 0 \;,\quad \forall\, \alpha, q, B > 0.$$

REMARK 2.– Owing to (31) we deduce

$$g_h(\alpha, q, B) = -\frac{A_h(\alpha)}{2}\, q^2 + B\, q|\Gamma_2| - \frac{\alpha\, B^2}{2}\, |\Gamma_1| < 0 \;, \forall\, \alpha, q, B > 0\,. \tag{32}$$

COROLLARY 4.– From (32) we have

$$A_h(\alpha) > \frac{2\, B\, |\Gamma_2|}{q} - \frac{\alpha\, B^2\, |\Gamma_1|}{q^2} \;, \forall\, \alpha, q, B > 0\,, \tag{33}$$

and therefore, we obtain

$$A_h(\alpha) > \frac{|\Gamma_2|^2}{\alpha\, |\Gamma_1|} \;, \forall\, \alpha > 0\,. \tag{34}$$

THEOREM 5. – (i) We have the following inequalities

$$q_m(\alpha, B) \leq q_{m_h}(\alpha, B) < q_M(\alpha, B), \;\; \forall\, \alpha, B > 0\,, \tag{35}$$

therefore the set $S_h^2(B)$ is non-empty, where

$$S_h^2(B) = \{ (\alpha, q) \in (\mathbb{R}^+)^2 \ / \ q_{m_h}(\alpha, B) < q < q_M(\alpha, B) \} \neq \emptyset. \tag{36}$$

(ii) If $(\alpha, q) \in S_h^2(B)$, for a given $B > 0$, then the function $u_h(\alpha, q, B)$ is of non-constant sign in Ω, that is, we have a steady-state, two-phase, Stefan problem.

Proof.– (i) The second inequality in (35) is obtained from (34).

(ii) It follows from (29) and (30).

LEMMA 6.– (i) We have the following estimates :

$$\left| \begin{array}{l} u_{3h} \leq U_h(\alpha) \text{ in } \bar{\Omega} , \\ C_h \leq A_h(\alpha) , \\ q_{m_h}(\alpha, B) \leq q_{o_h}(B) , \end{array} \right. \tag{37}$$

where

$$C_h = \int_{\Gamma_2} u_{3h} \, d\gamma , \qquad q_{o_h}(B) = \frac{B \, |\Gamma_2|}{C_h} . \tag{38}$$

(ii) The function $A_h = A_h(\alpha)$ is a decreasing function in α and verifies

$$\lim_{\alpha \to +\infty} A_h(\alpha) = C_h. \tag{39}$$

(iii) The function $q_{m_h} = q_{m_h}(\alpha, B)$ is an increasing function in α and verifies the following properties:

$$q_{m_h}(0^+, B) = 0 , \qquad q_{m_h}(+\infty, B) = q_{o_h}(B) , \qquad \forall \, B > 0 . \tag{40}$$

THEOREM 7. – If $q > q_{o_h}(B)$ then $u_h(\alpha, q, B)$ is a function of non-constant sign in Ω when

$$\alpha > \alpha_o(q, B) = \frac{q \, |\Gamma_2|}{B \, |\Gamma_1|} . \tag{41}$$

Proof.– If $q > q_{o_h}(B)$ then the corresponding discrete problem for $u_h(q, B)$ (i.e. $\alpha = +\infty$ for $u_h(\alpha, q, B)$) is a two-phase one [13] and therefore $u_h(q, B) < 0$ in some part of Γ_2 , that is $u_h(\alpha, q, B) < 0$ on some part of Γ_2.

On the other hand, from (27) we get the following equivalence

$$\int_{\Gamma_1} u_h(\alpha, q, B) \, d\gamma > 0 \Leftrightarrow \alpha > \alpha_o(q, B) , \tag{42}$$

then the proof is completed.

THEOREM 8.– (i) The function $g_h = g_h(\alpha, q, B)$ satisfies the following properties :

$$\frac{\partial g}{\partial q}(\alpha, q, B) = \int_{\Gamma_2} u_h(\alpha, q, B)\, d\gamma \,, \tag{43}$$

$$\frac{\partial g}{\partial B}(\alpha, q, B) = -\,\alpha \int_{\Gamma_1} u_h(\alpha, q, B)\, d\gamma \,, \tag{44}$$

$$\frac{\partial g}{\partial \alpha}(\alpha, q, B) = \frac{1}{2} \int_{\Gamma_1} [u_h^2(\alpha, q, B) - B\, u_h(\alpha, q, B)]\, d\gamma \,. \tag{45}$$

(ii) The function $A_h = A_h(\alpha)$ satisfies the following properties :

$$A_h'(\alpha) = \frac{dA_h}{d\alpha}(\alpha) = \frac{B^2|\Gamma_1|}{q^2} - \frac{2\,B\,|\Gamma_2|}{\alpha\, q} - \frac{1}{q^2}\int_{\Gamma_1} u_h^2(\alpha, q, B)\, d\gamma \,, \tag{46}$$

$$A_h'(\alpha) \geq -\frac{2\,B\,|\Gamma_2|}{\alpha\, q} \quad , \forall\, h > 0 \,, \tag{47}$$

$$\frac{d}{d\alpha}[\alpha\, A_h(\alpha)] = \frac{1}{q^2}\, a(u_h(\alpha, q, B), u_h(\alpha, q, B)) \,, \tag{48}$$

$$\lim_{\alpha \to +\infty} \alpha\, A_h'(\alpha) = 0 \qquad , \forall\, h > 0 \,, \tag{49}$$

$$\lim_{\alpha \to +\infty} \frac{\partial q_{m_h}}{\partial \alpha}(\alpha, B) = 0 \qquad , \forall\, B > 0 \,, \tag{50}$$

Proof.– (i) We use (31) and the definition of the derivative as the limit of the incremental quotient. (ii) (46) follows from (32) and (43). (47) is obtained from (19) and (46). (48) follows from (31) and (46). (49) is a consequence of (19), (47) and (48). At the end, (50) follows from (23) and (49).

III. A PARTICULAR DISCRETE CASE.

We define a <u>particular discrete case</u> (PDC), similarly to the continuous case (see (12)) [10], as the problem in variables h, α, q, B which verifies the condition

$$\frac{1}{q^2}\, a(u_h(\alpha, q, B), u_h(\alpha, q, B)) = \text{Const.}\,. \tag{51}$$

Necessarily the constant must be Const. $= C_h > 0$ taking $\alpha \to +\infty$ in (51). On the other hand, taking into account (48), we have the equivalence :

$$(\text{PDC}) \Leftrightarrow \frac{1}{q^2}\, a(u_h(\alpha, q, B), u_h(\alpha, q, B)) = C_h \Leftrightarrow \frac{d}{d\alpha}[\alpha\, A_h(\alpha)] = C_h \,. \tag{52}$$

Since (16) and (51), we deduce

(53) $$C_h = a(U_h(\alpha), U_h(\alpha)) = A_h(\alpha) - \alpha \int_{\Gamma_1} U_h^2(\alpha)\, d\gamma\ ,$$

that is

(54) $$(PDC) \Leftrightarrow A_h(\alpha) = C_h + \alpha \int_{\Gamma_1} U_h^2(\alpha)\, d\gamma\ .$$

THEOREM 9.– The following propositions are all equivalents to the particular discrete case (52) :

(55) $$u_h(q,B) - u_h(\alpha,q,B) = \frac{q\ |\Gamma_2|}{\alpha\ |\Gamma_1|} \quad \text{in } \Omega,$$

(56) $$u_h(\alpha,q,B)|_{\Gamma_1} = B - \frac{q\ |\Gamma_2|}{\alpha\ |\Gamma_1|} \quad \text{on } \Gamma_1\ ,$$

(57) $$U_h(\alpha,q,B)|_{\Gamma_1} = \frac{|\Gamma_2|}{\alpha\ |\Gamma_1|} \quad \text{on } \Gamma_1\ ,$$

(58) $$A_h(\alpha) = C_h + \frac{|\Gamma_2|^2}{\alpha\ |\Gamma_1|}\ ,$$

(59) $$\frac{d}{d\alpha}[\alpha\ A_h(\alpha)] = C_h\ .$$

Proof.– Firstly we remark if we have $u_h(q,B) - u_h(\alpha,q,B) = \text{Const.}$ in Ω in condition (55), then the constant must necessarily be given by Const.$=q\ |\Gamma_2|/\alpha\ |\Gamma_1|$ because we integre equality (55) on Γ_1 and we use (27) and the fact that $u_h(q,B)|_{\Gamma_1} = B$. Similarly, the propositions (55) $\Rightarrow$ (56) and (56)$\Rightarrow$(57) are true.

(57) $\Rightarrow$ (58). It follows from (54).

(58) $\Rightarrow$ (59). We obtain (59) by deriving the expression (58).

(59) $\Rightarrow$ (55). We deduce the following equivalence :

$$u_h(q,B) - u_h(\alpha,q,B) = \text{Const. in } \Omega \Leftrightarrow a(u_h(q,B) - u_h(\alpha,q,B), u_h(q,B) - u_h(\alpha,q,B)) = 0$$

$$\Leftrightarrow a(u_h(\alpha,q,B), u_h(\alpha,q,B)) = a(u_h(q,B), u_h(q,B)) \Leftrightarrow (52).$$

REMARK 3.– For the particular discrete case we have obtained an analytical expression for the function $A_h(\alpha)$, given by (58), and therefore the description of the set $S_h^2(B)$ is completed.

IV. ERROR BOUNDS AS FUNCTIONS OF THE PARAMETER h.

If we take into account the following interpolation result [1, 3]

(60) $$\| v - \Pi_h v \|_V \leq D_o\ h^{r-1}\ ||v||_{r,\Omega}\ , \forall\ v \in H^r(\Omega)\ (\ D_o > 0\ )\ ,$$

and we suposse the regularity property $U(\alpha) \in H^r(\Omega)$ with $r > 1$ [2, 4, 6, 9] (see in [10, 12] three examples in which the function $U(\alpha)$ is C^∞) then we deduce the following aproximation results in function of the discretization parameter h.

THEOREM 10.– (i) We have

(61)
$$A(\alpha) - A_h(\alpha) = a_\alpha(U_h(\alpha) - U(\alpha), U_h(\alpha) - U(\alpha)) =$$
$$= \frac{1}{q^2} a_\alpha(u(\alpha,q,B) - u_h(\alpha,q,B), u(\alpha,q,B) - u_h(\alpha,q,B)) \geq 0.$$

(ii) We have

(62) $$a_\alpha(U(\alpha) - U_h(\alpha), U_h(\alpha)) = 0 ,$$

(63) $$a_\alpha(U(\alpha) - U_h(\alpha), v_h) = 0 , \forall\, v_h \in V_h .$$

(iii) We have that $U_h(\alpha) = P^\alpha_{V_h}(U(\alpha))$ is the projection of $U(\alpha)$ over the space V_h with respect to the norm associated to the scalar product a_α given by

(64) $$\| v \|_\alpha = \sqrt{a_\alpha(v,v)} , \qquad \forall\, v \in V .$$

(iv) *We have the following abstract estimate :*

(65) $$\left\| U(\alpha) - U_h(\alpha) \right\|_V \leq M_\alpha \operatorname*{Inf}_{v_h \in V_h} \left\| U(\alpha) - v_h \right\|_V$$

with

(66)
$$\left| \begin{array}{l} M_\alpha = \dfrac{\| a_\alpha \|}{\lambda_\alpha} \leq \dfrac{1 + \alpha \,\| \gamma_o \|^2}{\lambda_1 \operatorname{Inf}(1,\alpha)} , \\[2ex] \lambda_\alpha = \lambda_1 \operatorname{Inf}(1,\alpha) , \end{array} \right.$$

where $\lambda_1 > 0$ is the coercive constant of the bilinear form a_1 (a_1 is the particular case of a_α when α=1), that is

(67) $$a_1(v,v) \geq \lambda_1 \| v \|^2_V , \forall\, v \in V,$$

being $\gamma_o : V \to L^2(\Gamma)$ the trace operator.

(v) We have the following abstract estimate :

(68) $$0 < A(\alpha) - A_h(\alpha) \leq \operatorname*{Inf}_{v_h \in V_h} a_\alpha(v_h - U(\alpha), v_h - U(\alpha)).$$

(vi) If we have the regularity property $U(\alpha) \in H^r(\Omega)$ with $r > 1$, then we deduce the following estimates:

$$\| U(\alpha) - U_h(\alpha) \|_V \le C_1(\alpha)\ h^{r-1}\ , \tag{69}$$

$$0 < A(\alpha) - A_h(\alpha) \le C_2(\alpha, r)\, h^{2(r-1)}\ , \tag{70}$$

$$0 < q_{m_h}(\alpha, B) - q_m(\alpha, B) \le \frac{C_2(\alpha, r)}{A(\alpha)}\, q_{m_h}(\alpha, B)\, h^{2(r-1)}\ , \tag{71}$$

where

$$C_1(\alpha) = D_o\, M_\alpha\, \|U(\alpha)\|_{r,\Omega}\ ,\qquad C_2(\alpha, r) = D_o^2\, \|a_\alpha\|\, \|U(\alpha)\|_{r,\Omega}\ . \tag{72}$$

Proof.– (i) and (ii) follow from the definitions of $U(\alpha)$, $U_h(\alpha)$, $A(\alpha)$ and $A_h(\alpha)$. From (63) we deduce (iii).

(iv) The bilinear form a_α is coercive with a coercive constant λ_α, that is

$$a_\alpha(v, v) \ge \lambda_\alpha \| v \|_V^2\ ,\qquad \forall\, v \in V. \tag{73}$$

For $v_h \in V_h$ we have

$$\lambda_\alpha \| U(\alpha) - U_h(\alpha) \|_V^2 \le a_\alpha(U(\alpha) - U_h(\alpha), U(\alpha) - U_h(\alpha)) = a_\alpha(U_\alpha - U_h(\alpha), U(\alpha)) =$$

$$= a_\alpha(U_\alpha - U_h(\alpha), U(\alpha) - v_h) \le \| a_\alpha \| \| U(\alpha) - U_h(\alpha) \|_V \| U(\alpha) - v_h \|_V\ , \tag{74}$$

which implies (65). On the other hand, we have

$$\begin{aligned} |a_\alpha(u, v)| &\le |a(u, v)| + \alpha \int_{\Gamma_1} |u|\, |v|\, d\gamma \le \| u \|_V \| v \|_V + \alpha \| u \|_{L^2(\Gamma_1)} \| v \|_{L^2(\Gamma_1)} \le \\ &\le (1 + \alpha \| \gamma_o \|^2) \| u \|_V \| v \|_V\ , \end{aligned} \tag{75}$$

that is (66).

(v) We have

$$\begin{aligned} A(\alpha) - A_h(\alpha) &= a_\alpha(U(\alpha) - U_h(\alpha), U(\alpha) - U_h(\alpha)) = \| U(\alpha) - U_h(\alpha) \|_\alpha^2 \le \\ &\le \| U(\alpha) - v_h \|_\alpha^2 = a_\alpha(U(\alpha) - v_h, U(\alpha) - v_h)\ ,\ \forall\, v_h \in V_h\ , \end{aligned} \tag{76}$$

that is (68).

(vi) Taking into account the interpolation result (60) and the fact that $\Pi_h(U(\alpha)) \in V_h$, then from (65) and (68) we deduce (69) and (70) respectively. On the other hand, we have

(77) $$q_{m_h}(\alpha, B) - q_m(\alpha, B) = \frac{B\,|\Gamma_2|}{A_h(\alpha)} \frac{A(\alpha) - A_h(\alpha)}{A(\alpha)} \le \frac{C_2(\alpha, r)}{A(\alpha)} q_{m_h}(\alpha, B)\, h^{2(r-1)} ,$$

that is (71).

REMARK 4.– If $U(\alpha) \in V$ (no necessarily $U(\alpha) \in H^r(\Omega)$ with $r > 1$) then when $h \to 0$ we have [1, 3]:

(78) $$0 < A(\alpha) - A_h(\alpha) \le \| U(\alpha) - \Pi_h(U(\alpha)) \|_\alpha^2 \to 0 \ , \quad \text{when} \quad h \to 0 .$$

We define the function

(79) $$Z(\alpha, h, r) = 1 - \frac{C_2(\alpha, r)\, h^{2(r-1)}}{A(\alpha)} < 1 \ , \ \forall\, \alpha, h > 0.$$

and we have the following equivalence (for any $0 < \epsilon < 1$) :

(80) $$\epsilon < Z(\alpha, h, r) < 1 \Leftrightarrow h < h_r(\epsilon, \alpha),$$

where

(81) $$h_r(\epsilon, \alpha) = \sqrt{\frac{(1-\epsilon)\, A(\alpha)}{D_o\, C_2(\alpha, r)\ \ ||U(\alpha)||_{r,\Omega}^2}} \ , \qquad 0 < \epsilon < 1, \alpha > 0.$$

THEOREM 11.– If we let $h, B > 0$, and $0 < \epsilon < 1$ (ϵ is a parameter to be chosen arbitrarily), then we have as functions of the parameter h, the following estimates

(82) $$q_{m_h}(\alpha, B) \le \tfrac{1}{\epsilon} q_m(\alpha, B) \ , \forall\, h \le h_r(\epsilon, \alpha) ,$$

(83) $$0 < q_{m_h}(\alpha, B) - q_m(\alpha, B) \le \frac{B\,|\Gamma_2|\, C_2(\alpha, r)}{\epsilon\, A^2(\alpha)} h^{2(r-1)} \ , \forall\, h \le h_r(\epsilon, \alpha).$$

Proof.– From (71) we deduce

(84) $$Z(\alpha, h, r)\, q_{m_h}(\alpha, B) \le q_m(\alpha, B)$$

and therefore (82), because the equivalence (80). From (71) and (82) we obtain (83).

COROLLARY 12. – We have the following limit

(85) $$\lim_{h \to 0^+} q_{m_h}(\alpha, B) = q_m(\alpha, B) \ , \ \forall\, \alpha, B > 0.$$

LEMMA 13.– If the continuous and discrete cases are particular cases then we have

(86) $$0 < A(\alpha) - A_h(\alpha) = C - C_h \leq C_0^2 \ | u_3 |^2_{r,\Omega} \ h^{2(r-1)} ,$$

(87) $$q_{m_h}(\alpha, B) < \frac{B \alpha |\Gamma_1|}{|\Gamma_2|} = q_M(\alpha, B) ,$$

(88) $$q_{m_h}(\alpha, B) - q_m(\alpha, B) \leq \frac{B C_0^2 |\Gamma_1|}{|\Gamma_2|} ||u_3||^2_{r,\Omega} \frac{\alpha}{A(\alpha)} h^{2(r-1)} .$$

Proof.– If the continuous and discrete cases are particular cases then $A(\alpha)$ and $A_h(\alpha)$ are given explicitely by (13) and (58) respectively, and we obtain (86) taking into account [13]. On the other hand, we have

(89) $$q_{m_h}(\alpha, B) = \frac{B |\Gamma_2|}{C_h + \frac{|\Gamma_2|^2}{\alpha |\Gamma_1|}} < \frac{B \alpha |\Gamma_1|}{|\Gamma_2|} = q_M(\alpha, B) ,$$

that is (87). Moreover, from (71) and (87) we obtain (88).

REMARK 5.– If $U(\alpha) \in H^2(\Omega) \cap C^0(\bar{\Omega})$ then the convergence of $A_h(\alpha)$ to $A(\alpha)$ and $q_{m_h}(\alpha, B)$ to $q_m(\alpha, B)$ is of order h^2 when $h \to 0$.

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