# Dualities between Nets and Automata Induced by Schizophrenic Objects

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# Dualities between Nets and Automata Induced by Schizophrenic Objects

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**Abstract:** The so-called synthesis problem for nets, which consists in deciding whether a given graph is isomorphic to the case graph of some net, and then constructing the net, has been solved in the litterature for various types of nets, ranging from elementary nets to Petri nets. The common principle for the synthesis is the idea of regions in graphs, representing possible extensions of places in nets. When the synthesis problem has a solution, the set of regions viewed as properties of states provides a set-theoretic representation of the transition system. We show that such correspondences between nets and transition systems can be described as dualities induced by schizophrenic objects living in the corresponding categories, leading further the analogy with classical representation theorems. This gives us a means to describe uniformly previously known translations between nets and automata, and to compare on that basis the related categories of nets.

**Key-words:** Synthesis Problem for Nets, Regions, Duality, Galois Connection, schizophrenic object

*(Résumé : tsvp)*

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# Dualités entre réseaux et automates induites par des objets schizophrènes

**Résumé :** Le problème de synthèse de réseaux, qui consiste à décider pour un automate donné s'il est isomorphe à un graphe de marquage d'un réseau, est résolu pour diverses classes de réseaux allant des réseaux élémentaires aux réseaux de Petri. Le principe général est de caractériser les sous ensembles d'états de l'automate, appelés *régions*, qui représentent les extensions des places du réseau à synthétiser. Lorsque le problème de synthèse admet une solution, les régions vues comme propriétés des états donnent une représentation ensembliste de l'automate en ce sens que tout état est caractérisé par les propriétés qu'il vérifie, tout événement est caractérisé par les propriétés qu'il modifie et enfin le fait qu'un événement soit autorisé en un état se déduit de la représentation de l'événement et de l'état comme ensembles de propriétés. L'analogie avec des théorèmes de représentation tels que le théorème de Birkhoff ou le théorème de Stone pour les algèbres de Boole, peut être faite une fois qu'on a observé, ce qui est le but de cette étude, que de telles correspondances entre réseaux et automates sont des dualités concrètes induites par des objets schizophrènes. Ceci nous donne une présentation uniforme de correspondances bien connues entre réseaux et automates et nous permet, sur cette base, de comparer les diverses catégories de réseaux mises en jeu.

**Mots-clé :** Synthèse de réseaux, régions, dualité, connexion galoisienne, objet schizophrène

# 1 Introduction

The so-called synthesis problem for nets, which consists in deciding whether a given graph is isomorphic to the case graph of some net, and then constructing the net, has been solved in the litterature for various types of nets, ranging from elementary nets to Petri nets. The common principle for the synthesis is the idea of regions in graphs, representing possible extensions of places in nets. When the synthesis problem has a solution, the set of regions viewed as properties of states provides a set-theoretic representation of the transition system where transition systems and nets can be viewed respectively as the *extensional* versus the *intensional* description of discrete event systems. Namely, a state of a transition system can be represented by the set of properties it satisfies, and an event which is given in extension by a set of transitions between states may also be described intensionally by the set of properties it modifies. Now the fact that an event is enabled in a state can be deduced from the representations of the event and the state.

In order to illustrate these points let us recall the synthesis problem for elementary net systems. In the formalism of elementary net systems, properties are figured out by places, and an event $a$ is encoded by a pair of disjoint sets of places $<{}^\bullet a, a^\bullet>$. The properties in ${}^\bullet a$ (the *preconditions* of $a$) are necessary conditions for event $a$ to proceed and they no longer hold after this event has occurred. Symmetrically, properties in $a^\bullet$ (the *postconditions* of $a$) never hold in states where the event $a$ is enabled, and always hold after the execution of $a$. The transition system associated with an elementary net, the so-called *state graph* of the net, thus consists of those transitions $M \xrightarrow{a} M'$ where $M$ and $M'$ are sets of places (*markings*) such that $M \setminus M' = {}^\bullet a$ and $M' \setminus M = a^\bullet$. This can be rephrased as

$$M \xrightarrow{a} M' \quad \textbf{iff} \quad {}^\bullet a \subseteq M \;\;\wedge\;\; a^\bullet \cap M = \emptyset \;\;\wedge\;\; M' = (M \setminus {}^\bullet a) \cup a^\bullet$$

i.e. an event is enabled at every marking that contains all its preconditions and none of its postconditions, and the new marking is obtained by withdrawing preconditions and adding postconditions. An *elementary net system* is an elementary net together with a distinguished initial marking $M_0$. [1] The synthesis problem consists in deciding whether a given finite automaton $\mathcal{A} = (Q, A, T, q_0)$ [2] is isomorphic to the case graph of some elementary net system, i.e. to the restriction of the state graph of the underlying net accessible from its initial marking.

An automaton which is isomorphic to the state graph of some elementary net system is called an *elementary transition system*. Let $\mathcal{A} = (Q, A, T, q_0)$ be an elementary transition system and $NS = (P, A, {}^\bullet(), ()^\bullet, M_0)$ be an elementary net system whose state graph is isomorphic to $\mathcal{A}$, where $P$ is the set of places, the mappings ${}^\bullet(), ()^\bullet : A \to 2^P$ indicate respectively the preconditions and postconditions of each event $a \in A$, and $M_0 \subseteq P$ is the initial marking. Since each state of the automaton $\mathcal{A}$ may be identified with a corresponding marking of the elementary net system $NS$, one can define a binary relation $\models \subseteq Q \times P$ between states of $\mathcal{A}$ and places of $NS$, where $q \models x$ (read "the state $q$ satisfies the property/place $x$") when the marking associated with the state $q$ contains the place $x$. The elementary net $\mathcal{N} = (P, A, {}^\bullet(), ()^\bullet)$ provides a *set-theoretic representation* of the transition system $TS = (Q, A, T)$, in the sense that we have a pair of mappings $[\![\cdot]\!]_Q : Q \to 2^P$ and $[\![\cdot]\!]_A : A \to 2^P \times 2^P$ defined as $[\![q]\!]_Q = \{x \in P |\ q \models x\}$ (hence $[\![q]\!]_Q$ is the marking associated with $q$) and $[\![a]\!]_A = <{}^\bullet a, a^\bullet>$ such that *(i)* the representation is faithful, i.e. both mappings are injective and *(ii)* the transition relation in $TS$ is characterized by

$$q \xrightarrow{a} q' \in T \quad \textbf{iff} \quad [\![q]\!]_Q \setminus [\![q']\!]_Q = {}^\bullet a \ \ \wedge \ \ [\![q']\!]_Q \setminus [\![q]\!]_Q = a^\bullet$$

In order to construct such a representation for a given transition system $TS$ one should guess the adequate set of places (tokens of the representation). For that purpose we proceed the other way round: we suppose that such a net $\mathcal{N} = (P, A, {}^\bullet(), ()^\bullet)$ exists and we represent each place $x \in P$ by the set $[\![x]\!]_P = \{q \in Q |\ q \models x\}$ of states satisfying this property in $TS$. This set $X = [\![x]\!]_P$,

[1] An elementary net is *simple* : $\forall x, y \in P \cup A$ $({}^\bullet x = {}^\bullet y$ and $x^\bullet = y^\bullet) \Rightarrow x = y$ –where ${}^\bullet x = \{a \in A |\ x \in a^\bullet\}$ and $x^\bullet = \{a \in A |\ x \in {}^\bullet a\}$ for $x \in P$– it is *pure*: $\forall a \in A\ {}^\bullet a \cap a^\bullet = \emptyset$, and it has no *isolated* event: $\forall a \in A\ {}^\bullet a \neq \emptyset$ or $a^\bullet \neq \emptyset$

[2] where $Q$ is a set of states, $A$ a set of events, $T \subset Q \times A \times Q$ a transition relation, $q_0 \in Q$ is the initial state and the sets $Q$, $A$, and $T$ are finite

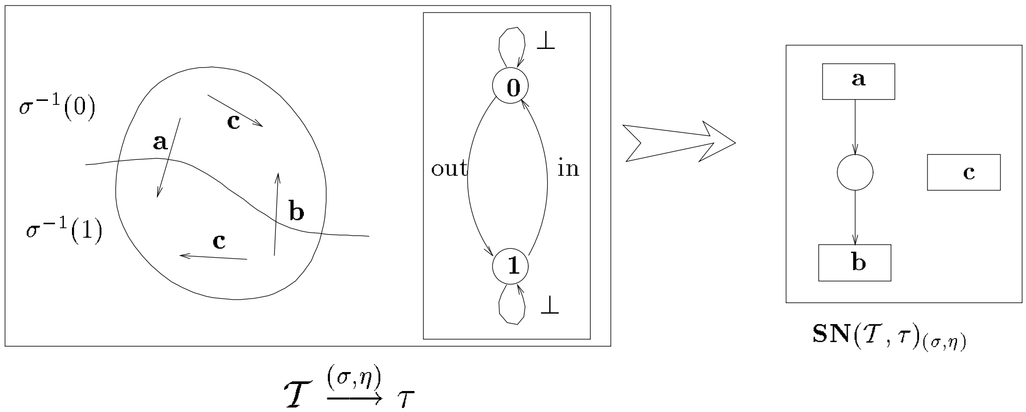


Figure 1: regions as morphisms

called the *extension* of $x$, satisfies the property

$$\begin{array}{rcl} \mathbf{Region}(X) & \equiv & \textit{for every event } a \in A \textit{ one has} \\ \textit{either} \quad q \xrightarrow{a} q' & \Rightarrow & (q \in X \ \text{ and } \ q' \notin X) \\ \textit{or} \quad q \xrightarrow{a} q' & \Rightarrow & (q \notin X \ \text{ and } \ q' \in X) \\ \textit{or} \quad q \xrightarrow{a} q' & \Rightarrow & (q \in X \ \text{ iff } \ q' \in X) \end{array}$$

The three cases above correspond respectively to $x \in {}^\bullet a$, $x \in a^\bullet$, and $x \notin {}^\bullet a \cup a^\bullet$ in $\mathcal{N}$. We call a *region* any subset $X \subseteq Q$ satisfying $\mathbf{Region}(X)$; thus the extension of a place is a region of the state graph. Notice that a set $X \subseteq Q$ is a region if and only if its characteristic function $\sigma = \chi_X : Q \to \{0,1\}$ admits a (unique) companion map $\eta : A \to \{-1,0,1\}$ such that $\sigma(q') = \sigma(q) + \eta(a)$ for every transition $q \xrightarrow{a} q'$ in $T$. From now on we shall identify regions with these pairs of mappings; which are in turn exactly the morphisms of transitions systems [3] from $TS = (Q, A, T)$ to the *classifying* transition system $2 = (\{0,1\}, A_2, T_2)$, where $A_2 = \{-1,0,1\}$ and $T_2 = \{0 \xrightarrow{0} 0,\ 0 \xrightarrow{1} 1,\ 1 \xrightarrow{-1} 0,\ 1 \xrightarrow{0} 1\}$ (see Fig. 1). A region $X \equiv (\sigma, \eta)$ induces an *atomic* elementary net $\mathcal{N}_X = (\{X\}, A, {}^\bullet(), ()^\bullet)$ with flow relations set according to the mapping $\eta$, namely

$$X \in {}^\bullet a \ \textbf{ iff } \ \eta(a) = -1 \quad \textit{and} \quad X \in a^\bullet \ \textbf{ iff } \ \eta(a) = 1$$

[3] A morphism of transition systems $(\sigma,\eta) : (Q_1, A_1, T_1) \to (Q_2, A_2, T_2)$ is a pair of mappings $\sigma : Q_1 \to Q_2$ and $\eta : A_1 \to A_2$ such that $q \xrightarrow{a} q' \in T_1 \Rightarrow \sigma q \xrightarrow{\eta a} \sigma q' \in T_2$

If $X = [\![x]\!]_P$ is the extension of a place $x \in P$ of a net $\mathcal{N} = (P, A, {}^\bullet(), ()^\bullet)$; then $\mathcal{N}_X$ is the atomic subset of $\mathcal{N}$ induced by the place $x$. Let $TS^* = \sum_{X \in \mathbf{R}(TS)} \mathcal{N}_X$ be the elementary net obtained by gluing these atomic nets $\mathcal{N}_X$ for $X$ ranging over the set $\mathbf{R}(TS)$ of regions of $TS$. $TS^*$ is the elementary net *synthetized* from the transition system $TS$. Every region $X \subset Q$ of $TS$ imposes an *elementary synchronic constraint* on the events occurring in $TS$. For instance, the region depicted in Fig. 1 expresses the constraint that each new occurrence of an event $b$ should be preceded by a new occurrence of an event $a$. If we add an initial state $q_0 \in Q$ to $TS$, then the elementary net $\mathcal{N}$ (and thus also each of its atomic components $\mathcal{N}_X$) comes equipped with an initial marking $M_0 \subseteq 2^{\mathbf{R}(TS)}$, containing exactly those regions which the initial state belongs to: $X \in M_0$ if and only if $q_0 \in X$. $(TS^*, M_0)$ is the elementary net system *synthetized* from the automaton $A = (TS, q_0)$. Suppose for instance that the initial state does not belong to the region represented in Fig. 1; then the unique place of the induced atomic net is not initially marked, and the language of this net is the shuffle of $(a \cdot b)^*$ with $c^*$. Since the regions of $TS$ are the morphisms $(\sigma, \eta) : TS \to \mathbf{2}$, the language of $\mathcal{A}$ is included in every language of the atomic net system induced by each of its regions and thus it is included in their intersection i.e. in the language of the synthetized elementary net system. Adding up synchronic constraints imposed by the regions amounts in fact to intersecting behaviours. The elementary transition systems are precisely those automata whose behaviour may be totally captured in terms of elementary synchronic constraints (see Theo. 1 below). This situation is reminiscent to the representation theory for boolean algebras where an element appears as the least upper bound of the atomic elements below this element.

If $\mathcal{A} = (Q, A, T, q_0)$ is an elementary transition system and $\mathcal{N} = (P, A, {}^\bullet(), ()^\bullet)$ is an elementary net system whose state graph is isomorphic to $\mathcal{A}$, then *(i)* if $q_1$ and $q_2$ are two distinct states of $Q$ viewed as markings of $\mathcal{N}$, then there exists a place $x \in P$ which belongs to exactly one of them, hence there exists a region (the extension of the place $x$) that distinguishes between $q_1$ and $q_2$; *(ii)* if $q$ is a state viewed as a marking $M = [\![q]\!]_Q$ and $a$ is an event disabled at $q$ then either ${}^\bullet a \not\subseteq M$ or $a^\bullet \cap M \neq \emptyset$, hence there exists a region $X \equiv (\sigma, \eta)$ (the extension of a place $x \in {}^\bullet a \setminus M$ in the first case or the complement of the extension of a place $x' \in a^\bullet \cap M$ in the second case) such that $\sigma(s) = 0$ and

$\eta(a) = -1$. Ehrenfeucht and Rozenberg proved the following characterization of elementary transition systems in [ER90].

**Theorem 1** *An automaton is an elementary transition system if and only if it has neither loop ($q \xrightarrow{a} q' \Rightarrow q \neq q'$) nor multiple transitions ($[q \xrightarrow{a} q' \wedge q \xrightarrow{a'} q']$ $\Rightarrow a = a'$), it is reduced ($\forall a \in A\ \exists\ q \xrightarrow{a} q'$) and accessible ($\forall q \in Q\ q_0 \xrightarrow{*} q$ where $\rightarrow = \cup_{a \in A} \xrightarrow{a}$), and it satisfies the following two* **separation axioms***:* (i) *every pair of distinct states is separated by some region:*

$$\forall q, q' \in Q \quad R_q = R_{q'} \Rightarrow q = q'$$

*where $R_q$ stands for the set of regions containing state $q$; and* (ii) *for every state and for every event disabled in this state, there exists a region which separates this event from this state:*

$$\forall q \in Q \quad \forall a \in A \quad {}^\bullet a \subseteq R_q \Rightarrow \exists q' \quad q \xrightarrow{a} q'$$

*where ${}^\bullet a$ is the set of regions $X \equiv (\sigma, \eta)$ such that $\eta(a) = -1$ (i.e. the set of preconditions of $a$ in the synthetized net). When these conditions are met, the automaton is isomorphic to the state graph of the synthetized elementary net system.*

A similar representation theorem exists for general (pure) Petri nets, although that representation is no longer set-theoretic. A pure Petri net is a structure $\mathcal{N} = (P, A, W)$, where $P$ and $A$ are disjoint sets of *places* and *events* respectively, and $W : P \times A \rightarrow \mathbb{Z}$ is the *weight* function. The set of *input* (resp. *output*) places of an event $a$ is the set ${}^\bullet a$ (resp. $a^\bullet$) of places $x$ such that $W(x, a) < 0$ (resp. $W(x, a) > 0$). A *marking* of $\mathcal{N}$ is a map $m : P \rightarrow \mathbb{N}$. [4] Let $\mathcal{N} = (P, A, W)$ be a net and $M$ a marking of $\mathcal{N}$; an event $a \in A$ is *enabled* at $M$ if $\forall x \in P \quad M(x) + W(x, a) \geq 0$. An event $a$ enabled at marking $M$ can fire; in doing so, it produces a new marking $M'$, defined by $\forall x \in P \quad M'(x) = M(x) + W(x, a)$. This firing step is denoted $M[a > M'$. The synthesis problem for (pure) Petri nets consists in deciding whether a finite automaton is isomorphic to the state graph of some pure Petri net. In order

[4] Nets are usually equipped with a flow relation $F \subseteq (P \times A) \cup (A \times P)$ and a weight function $W : F \rightarrow \mathbb{N}^+$. We can adopt here a more compact notation because we consider exclusively *pure* nets, i.e. nets such that $\forall x, y \in P \cup A \quad (x, y) \in F \Rightarrow (y, x) \notin F$

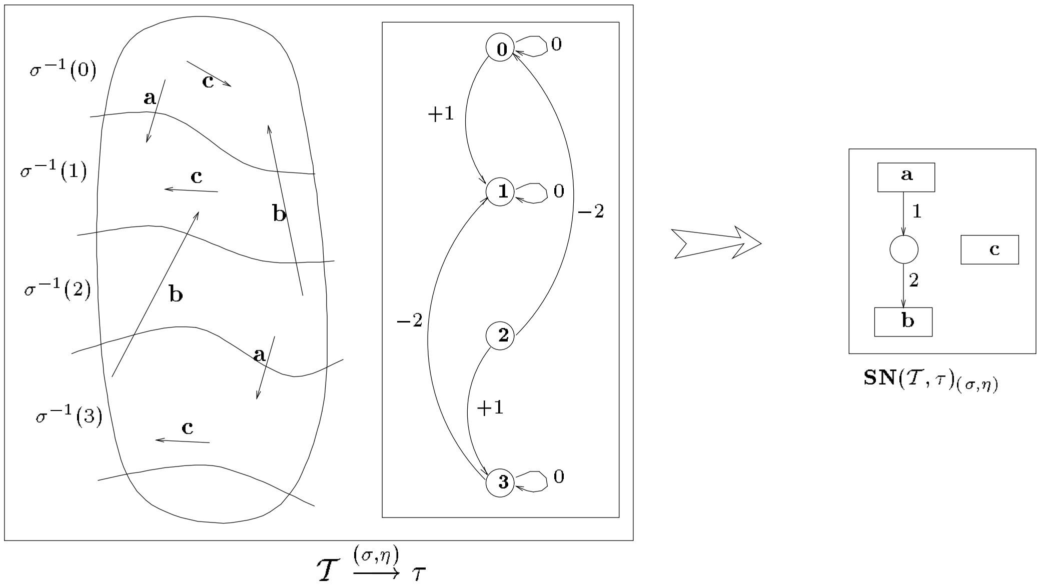


Figure 2: generalized regions as morphisms

to solve this problem, we proceed as in the elementary case by looking at the extension of a potential place of a pure Petri net in the associated state graph. That is we consider the projection of the state graph on the contents of a given place. The range of values of this place $x \in P$ is the set $\mathbb{N}$ of non negative integers, and each event modifies this value uniformly according to the flow relations $W(x,e)$. Therefore a *generalized region* in a transition system $TS = (Q, A, T)$ is a pair of mappings $\sigma : Q \to \mathbb{N}$ and $\eta : A \to \mathbb{Z}$ such that $\sigma(q') = \sigma(q) + \eta(a)$ for every transition $q \xrightarrow{a} q'$ in $T$. (Variant forms of generalized regions were introduced by Droste and Shortt [DS93], Mukund [Muk93], and Bernardinello, De Michelis and Petruni [BDP93].) Generalized regions can be identified with morphisms of transition systems $(\sigma,\eta) : TS \to \mathbb{N}$, where $\mathbb{N} = (Q_{\mathbb{N}}, A_{\mathbb{N}}, T_{\mathbb{N}})$ is the transition system given by $Q_{\mathbb{N}} = \mathbb{N}$, $A_{\mathbb{N}} = \mathbb{Z}$ and $n \xrightarrow{m} n' \in T_{\mathbb{N}}$ if and only if $n + m = n'$, see Fig. 2. A generalized region induces an atomic Petri net $\mathcal{N}_X = (\{X\}, A, W)$ whose weight function is set according to the mapping $\eta$, namely $W(X,a) = \eta(a)$. Such a region imposes a synchronic constraint on the behaviour of the automaton which may be more specific than the elementary synchronic constraints. Bernardinello, De Michelis and Petruni proved in [BDP93] that the generalized regions correspond exactly to the *synchronic distances* which were introduced by C.A. Petri as a tool to measure the relative degree of freedom between sets of events in a concurrent system. An analogue of Theo. 1 can be proved in this extended context.

Thus in both contexts (elementary net systems and pure Petri nets) regions can be viewed as morphisms onto a classifying transition system describing a type of synchronic constraints. Regions used for the synthesis of trace nets [BD93] can be presented as well as morphisms. Examples of "hybrid nets" can be produced by simply combining synchronic constraints of different nature, e.g. continuous and discrete or deterministic and stochastic. The presentation of regions as morphisms enables one to draw an analogy with classical representation theorems like Birkhoff or Stone representation theorems for boolean algebras. The purpose of this study is to lead this analogy further by establishing that the above correspondences between nets and transition systems can be described as concrete dualities induced by schizophrenic objects living in the corresponding categories. In the next section we adapt a work by Porst and Tholen [PT91] to describe dual adjunctions arising from schizophrenic objects in a context suitable for our purpose. We apply this theory in Section 3 where we derive dual adjunctions between a category of transition systems and various categories of nets. In Section 4 we enrich these constructions by adding information on the state space so as to obtain Galois connections.

# 2 Schizophrenic Objects and Dual Adjunctions

**Definition 2** *A* Set-category *(or category over* **Set***) is a pair* $<\mathcal{C}, U>$ *where* $\mathcal{C}$ *is a category and* $U : \mathcal{C} \to \mathbf{Set}$ *is a functor called the underlying functor. It is a* concrete category *if* $U$ *is faithful.*

In most cases the underlying functor will be left implicit, and we shall use the uniform notation $|C|$ and $|f|$ for respectively the underlying set of an object $C$ and the underlying mapping of an arrow $f$ of any Set-category $\mathcal{C}$.

If $\mathcal{C}$ is a Set-category, we recall that the *initial lift* of a *structured source* $\{C_i; f_i : X \to |C_i|\}$, where the $C_i$'s are objects of $\mathcal{C}$ and the $f_i$'s are mappings from a set $X$ to the underlying sets of the $C_i$'s, is a corresponding family of arrows $\tilde{f}_i : C \to C_i$ in $\mathcal{C}$ such that $|\tilde{f}_i| = f_i$ (and therefore $|C| = X$) and which is initial in the following sense: whenever one has an object $C'$ and arrows $g_i : C' \to C_i$ in $\mathcal{C}$ such that for all indices $|g_i| = f_i \circ f$ for some mapping $f : |C'| \to X$, then there exists a unique arrow $\tilde{f} : C' \to C$ such that $|\tilde{f}| = f$ and $g_i = \tilde{f}_i \circ \tilde{f}$. The following definition is an adaptation from [PT91].

**Definition 3 (Schizophrenic Object)** *A schizophrenic object between two Set-categories $\mathcal{A}$ and $\mathcal{B}$ is a pair of objects $<K_{\mathcal{A}}, K_{\mathcal{B}}> \in |\mathcal{A}| \times |\mathcal{B}|$ having the same underlying set $K = |K_{\mathcal{A}}| = |K_{\mathcal{B}}|$ and such that*

1. *for all object $A$ in $\mathcal{A}$, the family $\{K_{\mathcal{B}}; ev_A(a) : \mathcal{A}(A, K_{\mathcal{A}}) \to K\}_{a \in |A|}$ of evaluation mappings $ev_A(a)(f) = |f|(a)$ has an initial lift $\{\epsilon_A(a) : A^* \to K_{\mathcal{B}}\}_{a \in |A|}$, and symmetrically*

2. *for all object $B$ in $\mathcal{B}$, the family $\{K_{\mathcal{A}}; ev_B(b) : \mathcal{B}(B, K_{\mathcal{B}}) \to K\}_{b \in |B|}$ has an initial lift $\{\epsilon_B(b) : B^* \to K_{\mathcal{A}}\}_{b \in |B|}$.*

$A^*$, called the *dual* of $A$, is therefore an object of the category $\mathcal{B}$ whose underlying set is the set of $\mathcal{A}$-morphisms from $A$ to the *classifying object* $K_{\mathcal{A}}$. If $K = \{0, 1\}$ and if $\mathcal{A}$ is concrete, then the elements of the underlying set of the dual of $A$ can be identified with subsets of the underlying set of $A$: $|A^*| \subseteq 2^{|A|}$. Of course, as an initial lift the dual of an object is only defined up to (a unique) isomorphism. However, once those lifts are (arbitrarily) chosen, we obtain a functorial correspondence, more precisely:

**Lemma 4** *Let $<K_{\mathcal{A}}, K_{\mathcal{B}}>$ be a schizophrenic object between two Set-categories $\mathcal{A}$ and $\mathcal{B}$. For every morphism $f : A_1 \to A_2$ in $\mathcal{A}$, the mapping "composing with $f$" given by $f^\bullet : \mathcal{A}(A_2, K_{\mathcal{A}}) \to \mathcal{A}(A_1, K_{\mathcal{A}})$: $g \mapsto g \circ f$, is the underlying mapping of an arrow $f^* : A_2^* \to A_1^*$ in $\mathcal{B}$ such that the functoriality laws: $(1_A)^* = 1_{A^*}$ and $(f \circ g)^* = g^* \circ f^*$ are satisfied.*

*Proof:* For $g : A_2 \to K_{\mathcal{A}}$ in $\mathcal{A}$, and $a \in |A_1|$ one has $|\epsilon_{A_2}(|f|(a))|(g) = ev_{A_2}(|f|(a))|(g)$ $= |g|(|f|(a)) = |g \circ f|(a) = ev_{A_1}(a)(g \circ f) = (ev_{A_1}(a) \circ f^\bullet)(g)$ i.e. $|\epsilon_{A_2}(|f|(a))| = ev_{A_1}(a)$ $\circ f^\bullet$. By initiality of $\{\epsilon_{A_1}(a)\}_{a \in |A_1|}$ we deduce a unique $f^* : A_2^* \to A_1^*$ such that *(i)* $\epsilon_{A_2}(|f|(a)) = \epsilon_{A_1}(a) \circ f^*$ and *(ii)* $|f^*| = f^\bullet$. Thanks to this characterization of $f^*$, the functoriality laws immediately follow. ∎

**Lemma 5** *Let* $<K_\mathcal{A}, K_\mathcal{B}>$ *be a schizophrenic object between two Set-categories* $\mathcal{A}$ *and* $\mathcal{B}$. *The initial lift* $\{\epsilon_A(a) : A^* \to K_\mathcal{B}\}_{a \in A}$ *of the evaluation mappings, viewed as a mapping* $\epsilon_A : |A| \to \mathcal{B}(A^*, K_\mathcal{B})$ *is the underlying mapping of an arrow* $Ev_A : A \to A^{**}$.

*Proof :* For $f \in |A^*|$ i.e. $f : A \to K_\mathcal{A}$ in $\mathcal{A}$, and $a \in |A|$ one has

$$ev_{A^*}(f)(\epsilon_A(a)) = |\epsilon_A(a)|(f) = ev_A(a)(f) = |f|(a)$$

i.e. $|f| = ev_{A^*}(f) \circ \epsilon_A$, by initiality of $\{\epsilon_{A^*}(f)\}_{f \in |A^*|}$, we deduce a unique morphism $Ev_A : A \to A^{**}$ such that *(i)* $\epsilon_{A^*}(f) \circ Ev_A = f$ and *(ii)* $|Ev_A| = \epsilon_A$. ∎

**Definition 6 (Span)** *Let* $\mathcal{K} = <K_\mathcal{A}, K_\mathcal{B}>$ *be a schizophrenic object between two Set-categories* $\mathcal{A}$ *and* $\mathcal{B}$. *A* $\mathcal{K}$*-span* $\varphi \in \mathbf{Span}_\mathcal{K}(A, B)$ *from* $A \in |\mathcal{A}|$ *to* $B \in |\mathcal{B}|$ *consists of families of morphisms* $\{\varphi_a : B \to K_\mathcal{B}\}_{a \in |A|}$ *and* $\{\varphi^b : A \to K_\mathcal{A}\}_{b \in |B|}$ *in* $\mathcal{B}$ *and* $\mathcal{A}$ *respectively, such that*

$$\forall a \in |A| \;\; \forall b \in |B| \quad |\varphi_a|(b) = |\varphi^b|(a)$$

If $\mathcal{A}$ and $\mathcal{B}$ are concrete categories, then spans coincide with bimorphisms, i.e. those mappings $\varphi : |A| \times |B| \to K$ such that

1. $\forall a \in |A| \;\; \varphi(a, -) : |B| \to K$ is the underlying mapping of a morphism $\varphi_a$ from $B$ to $K_\mathcal{B}$, and
2. $\forall b \in |B| \;\; \varphi(-, b) : |A| \to K$ is the underlying mapping of a morphism $\varphi^b$ from $A$ to $K_\mathcal{A}$.

For concrete categories, a span is a $K$-valued relation between the underlying sets of $A$ and $B$, and it can be represented by a matrix with values in $K$ whose rows and columns are respectively indexed by the sets $|A|$ and $|B|$. Of course $\mathbf{Span}_\mathcal{K}(A, B)$= $\mathbf{Span}_\mathcal{K}(B, A)$ corresponding to the transposition of matrices.

**Lemma 7** *Let* $<K_{\mathcal{A}}, K_{\mathcal{B}}>$ *be a schizophrenic object between two Set-categories* $\mathcal{A}$ *and* $\mathcal{B}$. *There is a bijective correspondence between the hom-set* $\mathcal{A}(A, B^*)$ *and the set of* $\mathcal{K}$*-spans* $\mathbf{Span}_{\mathcal{K}}(A, B)$ *given by the following identities, where* $f \in \mathcal{A}(A, B^*)$ *and* $\varphi \in \mathbf{Span}_{\mathcal{K}}(A, B)$.

$$\forall a \in |A| \;\; \forall b \in |B| \quad \varphi_a = |f|(a) \;\; \textit{and} \;\; \varphi^b = \epsilon_B(b) \circ f \tag{1}$$

*Proof :* What is meant in the statement of this lemma is the following: for every morphism $f : A \to B^*$ there exists a unique span $\varphi \in \mathbf{Span}_{\mathcal{K}}(A, B)$ such that the pair $<f, \varphi>$ satisfies the identities (1), and conversely for every span $\varphi \in \mathbf{Span}_{\mathcal{K}}(A, B)$ there exists a unique morphism $f : A \to B^*$ such that the pair$<f, \varphi>$ satisfies the identities (1). From which the bijective correspondence $\mathcal{A}(A, B^*) \cong \mathbf{Span}_{\mathcal{K}}(A, B)$ follows from the fact that both sets are in bijective correspondence with the set of pairs $<f, \varphi>$ which satisfy the identities (1). For the one hand the identities (1) clearly determine $\varphi$ in terms of the morphism $f$, and it is indeed a span because

$$|\varphi_a|(b) = ||f|(a)|(b) = ev_B(b)(|f|(a)) = |\epsilon_B(b) \circ f|(a) = |\varphi^b|(a)$$

For the converse direction, assume $\varphi$ is a span. Then $ev_B(b)(\varphi_a) = |\varphi_a|(b) = |\varphi^b|(a)$ i.e. $|\varphi^b| = ev_B(b) \circ \varphi_{(-)}$ for every $b \in |B|$, and the initiality of $\{\epsilon_B(b)\}_{b \in |B|}$ precisely ensure the existence and unicity of a morphism $f : A \to B^*$ verifying the identities (1). ∎

**Proposition 8 (Dual Adjunction Induced by a Schizophrenic Object)**

*Let* $<K_{\mathcal{A}}, K_{\mathcal{B}}>$ *be a schizophrenic object between two Set-categories* $\mathcal{A}$ *and* $\mathcal{B}$. *There is a bijective correspondence* $\mathcal{A}(A, B^*) \cong \mathcal{B}(B, A^*)$ *given by the following identities, where* $f : A \to B^*$ *and* $g : B \to A^*$,

$$g = f^* \circ Ev_B \quad \textit{and} \quad f = g^* \circ Ev_A \tag{2}$$

*i.e. the functors* $(-)^*$ *are adjoint to the right with the evaluations as units.*

*Proof:* By Lem. 7 we have $\mathcal{A}(A, B^*) \cong \mathbf{Span}_{\mathcal{K}}(A, B) = \mathbf{Span}_{\mathcal{K}}(B, A) \cong \mathcal{B}(B, A^*)$ given by the following identities where $f : A \to B^*$ and $g : B \to A^*$.

$$\forall a \in |A| \;\; \forall b \in |B| \quad \epsilon_B(b) \circ f = |g|(b) \;\; \text{and} \;\; |f|(a) = \epsilon_A(a) \circ g \tag{3}$$

In order to establish the proposition it suffices to prove that given a morphism $g : B \to A^*$, the morphism $f = g^* \circ Ev_A$ satisfies the identities (3). For that purpose we recall that (by Lem. 4) $g^*$ is the unique morphism $g^* : A^{**} \to B^*$ such that *(i)* $\epsilon_{A^*}(|g|(b)) = \epsilon_B(b) \circ g^*$ for every $b \in |B|$, and *(ii)* $|g^*| = g^\bullet$. We recall also that (by Lem.5) $Ev_A$ is the unique morphism $Ev_A : A \to A^{**}$ such that *(i)* $\epsilon_{A^*}(f) \circ Ev_A = f$ for every $f \in |A^*|$, and *(ii)* $|Ev_A| = \epsilon_A$. Now we can proceed to the verification that $f = g^* \circ Ev_A$ satisfies (3)

1. $\epsilon_B(b) \circ g^* \circ Ev_A = \epsilon_{A^*}(|g|(b)) \circ Ev_A = |g|(b)$, and
2. $|g^* \circ Ev_A|(a) = (g^\bullet \circ \epsilon_A)(a) = \epsilon_A(a) \circ g$.

∎

We can turn the set of $\mathcal{K}$-spans into a category whose objects are triples $(A, \varphi, B)$ where $\varphi \in \mathbf{Span}_{\mathcal{K}}(A, B)$ (equivalently $(A, f, B)$ where $f \in \mathcal{A}(A, B^*)$ or $(A, f^\sharp, B)$ where $f^\sharp \in \mathcal{B}(B, A^*)$) and whose morphisms are pairs of *reindexing* morphisms $\alpha \in \mathcal{A}(A_1, A_2)$, and $\beta \in \mathcal{B}(B_2, B_1)$ such that

- $\forall a \in A_1 \;\; \forall b \in B_2 \quad \varphi_1^{|\beta|b} = \varphi_2^b \circ \alpha \;\;$ and $\;\; (\varphi_2)_{|\alpha|a} = (\varphi_1)_a \circ \beta$,
- or equivalently $\beta^* \circ f_1 = f_2 \circ \alpha$,
- or equivalently $\alpha^* \circ f_2^\sharp = f_1^\sharp \circ \beta$.

For concrete categories this condition on morphisms reduces to :

$$\forall a \in A_1 \;\; \forall b \in B_2 \quad \varphi_1(a, \beta b) = \varphi_2(\alpha a, b) \tag{4}$$

Now the projection $\pi_1 : \mathbf{Span}_{\mathcal{K}} \to \mathcal{A}$ has a right-inverse left-adjoint $\rho_1 : \mathcal{A} \to \mathbf{Span}_{\mathcal{K}}$ which takes an object $A$ of $\mathcal{A}$ to the unit $Ev_A : A \to A^{**}$ and takes the arrow $f : A \to A'$ to the pair $(f, f^*)$. Symmetrically, the second projection $\pi_2 : \mathbf{Span}_{\mathcal{K}} \to \mathcal{B}^{op}$ has a right-inverse right-adjoint $\rho_2 : \mathcal{B}^{op} \to \mathbf{Span}_{\mathcal{K}}$ which takes an object $B$ of $\mathcal{B}$ to the unit $Ev_B : B \to B^{**}$ and takes the arrow $g : B' \to B$ to the pair $(g^*, g)$. The original dual adjunction $()^* \dashv (()^*)^{op} : \mathcal{A} \to \mathcal{B}^{op}$ can be recovered as the composite of the adjunctions $\rho_1 \dashv \pi_1$ (a co-reflection) and $\pi_2 \dashv \rho_2$ (a reflection). $\mathcal{A}$ and $\mathcal{B}^{op}$ are co-reflective (respectively reflective) full subcategories of $\mathbf{Span}_{\mathcal{K}}$. The *kernel* of $\mathbf{Span}_{\mathcal{K}}$ is the full subcategory consisting of those spans $\varphi \in \mathbf{Span}_{\mathcal{K}}(A, B)$ such that $B \cong A^*$ and $A \cong B^*$, it is equivalent to the respective full subcategories of $\mathcal{A}$ and $\mathcal{B}^{op}$ consisting of those objects for which units are isomorphisms. This yields a duality between the respective subcategories of $\mathcal{A}$ and $\mathcal{B}$. An interesting case of dual adjunction (not necessarily induced by a schizophrenic object) is when fixed points of the units coincide with elements of the image of the dual category, more precisely we recall the following from [Isb72].

**Definition 9 (Galois Connections)** *Let $\mathcal{A}(A, B^*) \cong \mathcal{B}(B, A^*)$ be a dual adjunction with units $\{E_A : A \to A^{**}\}_{A \in |\mathcal{A}|}$ and $\{E_B : B \to B^{**}\}_{B \in |\mathcal{B}|}$. We further let $\mathcal{B}^*$ (the image of $\mathcal{B}$) denote the full subcategory of $\mathcal{A}$ consisting of those objects $B^*$ for $B \in |\mathcal{B}|$. The image of $\mathcal{A}$ is defined similarly. Then the dual adjunction is a* Galois Connection *whenever one of the following equivalent conditions is satisfied :*

1. *It restricts to a duality between the images: $\mathcal{B}^* \overset{op}{\cong} \mathcal{A}^*$,*
2. *the arrows $(E_A)^*$ are isomorphisms for $A \in |\mathcal{A}|$,*
3. *their left-inverses $E_{A^*}$ are isomorphisms,*
4. *the arrows $(E_B)^*$ are isomorphisms for $B \in |\mathcal{B}|$,*
5. *their left-inverses $E_{B^*}$ are isomorphisms,*
6. *the maps $\{E_A : A \to A^{**}\}_{A \in |\mathcal{A}|}$ constitute a reflection of $\mathcal{A}$ into $\mathcal{B}^*$,*
7. *the maps $\{E_B : B \to B^{**}\}_{B \in |\mathcal{B}|}$ constitute a reflection of $\mathcal{B}$ into $\mathcal{A}^*$.*

# 3 Dual Adjunction between Nets and Transition Systems

## 3.1 Transition Systems

A transition system is a triple $(Q, A, T)$ that consists of a set of states $Q$, a set of events $A$, and a transition relation $T \subseteq Q \times A \times Q$. We write $q \xrightarrow{a} q'$ as an abbreviation for $(q, a, q') \in T$, but also to denote the corresponding element of $T$. A morphism of transition systems $(\sigma, \eta) : (Q_1, A_1, T_1) \to (Q_2, A_2, T_2)$ is a pair of mappings $\sigma : Q_1 \to Q_2$ and $\eta : A_1 \to A_2$ relating states and events of both systems, such that

$$q \xrightarrow{a} q' \quad \Rightarrow \quad \sigma(q) \xrightarrow{\eta(a)} \sigma(q') \tag{5}$$

We can regard transition systems as structured sets in at least three different ways depending on which of the respective sets of states, events or transitions is considered the underlying set of the transition system. In the present case we will consider that the underlying set of a transition system is the set of its transitions: $|(Q, A, T)| = T$. Therefore the underlying mapping of a morphism $|(\sigma, \eta)|$ is the mapping that takes the transition $q \xrightarrow{a} q'$ to the transition $\sigma(q) \xrightarrow{\eta(a)} \sigma(q')$. As the set of transitions plays a central rôle, it is technically convenient that it can be referred to without mentionning the sets of states and events. We shall therefore use the following alternative definition of transition systems which is almost a restatement of the above definition (the resulting categories are equivalent).

**Definition 10 (Transition Systems)**
*A transition system is a structure* $A \xleftarrow{\lambda} T \overset{\partial^0}{\underset{\partial^1}{\rightrightarrows}} Q$ *, where* $A$*,* $T$*, and* $Q$ *are sets of events, transitions and states respectively, and* $\lambda$*,* $\partial^0$*, and* $\partial^1$ *are the labelling, source, and target mappings respectively. Those mappings are required to be jointly monic i.e.*

$$[\lambda(t) = \lambda(t') \;\wedge\; \partial^0(t) = \partial^0(t') \;\wedge\; \partial^1(t) = \partial^1(t')] \Rightarrow t = t' \tag{6}$$

*A morphism between two transition systems consist of three mappings* $\sigma : Q_1 \to Q_2$*,* $\eta : A_1 \to A_2$*, and* $f : T_1 \to T_2$ *such that the following diagrams commute.*

$$\begin{array}{ccccc} A_1 & \xleftarrow{\lambda_1} & T_1 & \overset{{\partial^0}_1}{\underset{{\partial^1}_1}{\rightrightarrows}} & Q_1 \\ \eta \downarrow & & f \downarrow & & \downarrow \sigma \\ A_2 & \xleftarrow{\lambda_2} & T_2 & \overset{{\partial^0}_2}{\underset{{\partial^1}_2}{\rightrightarrows}} & Q_2 \end{array} \tag{7}$$

Transition systems and their morphisms constitute a Set-category denoted **Trans** where $|A \xleftarrow{\lambda} T \overset{\partial^0}{\underset{\partial^1}{\rightrightarrows}} Q| = T$ and $|(\sigma, f, \eta)| = f$. Condition (6) ensures that the set of transitions may be viewed as an actual set of transitions between elements of $Q$ labelled in $A : T \subseteq Q \times A \times Q$. We shall denote $q \xrightarrow{a} q'$ the element $t \in T$ such that $\partial^0(t) = q$, $\lambda(t) = a$, and $\partial^1(t) = q'$. Because of that condition the component $f$ on transitions is unambiguously determined by the components $\sigma$ and $\eta$ on states and events. Definition of morphisms may therefore be restated as follows: they are those pairs of mappings $(\sigma, \eta)$ for which a mapping $f : T_1 \to T_2$ can be defined such that the diagrams (7) commute. Which is a reformulation of condition (5). Conversely the mappings $\sigma$ and $\eta$ are unambiguously determined by the component $f$ if the initial transition system is reduced.

**Definition 11 (Reduced Transition Systems)**
*A transition system* $A \xleftarrow{\lambda} T \underset{\partial^1}{\overset{\partial^0}{\rightrightarrows}} Q$ *is said to be reduced when* $\lambda$ *is onto and* $(\partial^0, \partial^1)$ *are jointly onto, i.e.* $A = Im(\lambda)$ *and* $Q = Im(\partial^0) \cup Im(\partial^1)$.

In a reduced transition system each event is the label of at least one transition, and each state is at least the source or the target of one transition. The full subcategory **Trans**° of **Trans** consisting of the reduced transition systems is therefore a concrete category.

**Proposition 12** *The categories* **Trans** *and* **Trans**° *have initial lifts of jointly monic structured sources.*

*Proof:*
*(i) The category* **Trans** *has initial lifts of jointly monic structured sources.*
Let $(Q_i, A_i, T_i)$ be a family of transitions systems, and a corresponding jointly monic family of mappings $f_i : T \to T_i$, i.e. $[(\forall i \;\; f_i(t) = f_i(t')) \Rightarrow t = t']$. We equip the set $T$ with a structure of transition system as follows. Let $\lambda : T \to \prod_i A_i$ and $\partial^0, \partial^1 : T \to \prod_i Q_i$ be the mappings such that $\lambda(t)_i = a_i$, $\partial^0(t)_i = q_i$ and $\partial^1(t)_i = q_i'$ where $f_i(t) = q_i \xrightarrow{a_i} q_i'$. The set of mappings $\lambda$, $\partial^0$ and $\partial^1$ is jointly monic because the family of mappings $\{f_i\}_i$ is jointly monic. The pair of projections $Q = \prod_i Q_i \xrightarrow{\pi_i} Q_i$ and $A = \prod_i A_i \xrightarrow{\pi_i} A_i$ is a morphism of transition systems from $(Q, A, T)$ to $(Q_i, A_i, T_i)$ with underlying mapping $f_i$. Now let $(Q', A', T') \xrightarrow{(\sigma_i, \eta_i)} (Q_i, A_i, T_i)$ be a family of transition system morphisms, whose

underlying mappings coincide with the compositions $f_i \circ f$ for some mapping $f : T' \to T$.

$$
\begin{array}{ccccc}
A' & \xleftarrow{\lambda'} & T' & \overset{\partial^{0\prime}}{\underset{\partial^{1\prime}}{\rightrightarrows}} & Q' \\
\eta \downarrow & & f \downarrow & & \downarrow \sigma \\
A & \xleftarrow{\lambda} & T & \overset{\partial^{0}}{\underset{\partial^{1}}{\rightrightarrows}} & Q \\
\pi_i \downarrow & & f_i \downarrow & & \downarrow \pi_i \\
A_i & \xleftarrow{\lambda_i} & T_i & \overset{\partial^{0}{}_i}{\underset{\partial^{1}{}_i}{\rightrightarrows}} & Q_i
\end{array}
$$

By the universal property of products we deduce a unique morphism of transition system $(\sigma, \eta) : (Q', A', T') \to (Q, A, T)$ such that $|(\sigma, \eta)| = f$ and $\forall i \ (\sigma_i, \eta_i) = (\pi_i, \pi_i) \circ (\sigma, \eta)$.

*(ii) The category* **Trans°** *has initial lifts of jointly monic structured sources.* The construction is as above except that now $Q \subseteq \prod_i Q_i$ and $A \subseteq \prod_i A_i$ are given by $Q = Im(\partial^0) \cup Im(\partial^1)$ and $A = Im(\lambda)$. If $\eta : A \to \prod_i A_i$ and $\sigma : Q \to \prod_i Q_i$ are the mappings induced by the product structure, i.e. $\eta(a)_i = \eta_i(a)$ and $\sigma(q)_i = \sigma_i(q)$, then we only have to prove that $Im(\eta) \subseteq Im(\lambda)$ and $Im(\sigma) \subseteq Im(\partial^0) \cup Im(\partial^1)$. We check the first condition, the second case is similar. Let $a = \eta(a')$ be an element of $Im(\eta)$, since $(Q', A', T')$ is reduced we deduce a transition $t' \in T'$ such that $\lambda'(t') = a'$, then $t = f(t') \in T$ satisfies $\lambda(t) = a$. ∎

*Note:* Following the lines of the part *(ii)* in the above proof, the reader may wish to verify that the family of morphisms

$$
\begin{array}{ccccc}
Im(\lambda) & \xleftarrow{\lambda} & T & \overset{\partial^{0}}{\underset{\partial^{1}}{\rightrightarrows}} & Im(\partial^0) \cup Im(\partial^1) \\
\cap & & \| & & \cap \\
A & \xleftarrow{\lambda} & T & \overset{\partial^{0}}{\underset{\partial^{1}}{\rightrightarrows}} & Q
\end{array}
$$

where $A \xleftarrow{\lambda} T \overset{\partial^0}{\underset{\partial^1}{\rightrightarrows}} Q$ range over all objects of **Trans**, constitute a concrete coreflection of **Trans** into **Trans**$^\circ$, where concreteness means that both the coreflector and its adjoint commute with the underlying functors. We shall denote $\mathcal{T}^\circ$ the coreflection (i.e. reduction) of a transition system $\mathcal{T}$.

## 3.2 Nets

If $I$ is a set of incidence values, the category **Nets**$(I)$ whose objects are triples $(P, A, F)$ consisting of a set of places $P$, a set of events $A$, and an incidence matrix $P \times A \to I$ and whose morphisms $(\beta, \eta) : (P_1, A_1, F_1) \longrightarrow (P_2, A_2, F_2)$ are pairs of mappings $\beta : P_1 \to P_2$, $\eta : A_2 \to A_1$ such that

$$\forall x \in P_1 \;\; \forall a \in A_2 \quad F_2(\beta x, a) = F_1(x, \eta a) \tag{8}$$

is a Set-category where $|(P, A, F)| = P$ and $|(\beta, \eta)| = \beta$. The full subcategory **Nets**$^\circ(I)$ of **Nets**$(I)$ consisting of the *event-simple nets* which are those objects verifying the following condition

$$(\forall x \in P \quad F(x, a) = F(x, b)) \Rightarrow a = b \tag{9}$$

is a concrete category, since the mapping $\eta$ on events is unambiguously determined by the mapping $\beta$ on places.

**Proposition 13** *The categories* **Nets**$(I)$*, and* **Nets**$^\circ(I)$ *have initial lifts.*

*Proof:*
*(i) The category* **Nets**$(I)$ *has initial lifts.*
Let $(P_i, A_i, F_i)$ be a family of nets, together with a corresponding family of mappings $\beta_i : P \to P_i$. The initial structure of net on the set $P$ is defined as follows. The set of events is the coproduct $A = \coprod_i A_i$ with the associated injections $in_i : A_i \to A$. The incidence matrix $F : P \times A \to I$ is the "coproduct" of the matrices $F_i$, in the sense that

$$\forall x \in P \;\; \forall a_i \in A_i \quad F_i(\beta_i x, a_i) = F(x, in_i(a_i))$$

Therefore $(\beta_i, in_i)$ is a morphism of nets from $(P, A, F)$ to $(P_i, A_i, F_i)$. We now verify that this family of morphisms satisfies the required universal property. Let

$\{(\beta'_i,\eta'_i):(P',A',F')\to(P_i,A_i,F_i)\}$ be another family of morphisms for which there exists a mapping $\beta:P'\to P$ such that $\forall i\ \beta'_i=\beta_i\circ\beta$. We have to prove that there exists a unique morphism $(\beta,\eta):(P',A',F')\to(P,A,F)$ such that $(\beta'_i,\eta'_i)=(\beta_i,in_i)\circ(\beta,\eta)$. If a solution exists, it is necessarily unique since, due to the coproduct structure of the family $\{in_i:A_i\to A\}_i$, the mapping $\eta$ is fully characterized by the equations $\eta'_i=\eta\circ in_i$. We just have to check that the pair of mappings $(\beta,\eta)$ is indeed a morphism of nets. Let $a\in A$ and $x'\in P'$, there exists a (unique) pair $<i,a_i>$ with $a_i\in A_i$ and $a=in_i(a_i)$. Now $F(x',\eta(a))=F(x',\eta'_i(a_i))=F_i(\beta'_i(x'),a_i)=F_i(\beta_i\circ\beta(x'),a_i)=F(\beta(x'),in_i(a_i))=F(\beta(x'),a)$

*(ii) The category* $\mathbf{Nets}^\circ(I)$ *has initial lifts.*

Let $(P_i,A_i,F_i)$ be a family of nets in $\mathbf{Nets}^\circ(I)$, together with a corresponding family of mappings $\beta_i\ :P\to P_i$. The initial structure of net on the set $P$ is defined in two stages. First as above we define the net $(P,A,F)$ where $A=\coprod_i A_i$ with injections $in_i\ :A_i\to A$ and $\forall x\in P\ \forall a_i\in A_i\ F_i(\beta_i x,a_i)=F(x,in_i(a_i))$. It is generally not an object of $\mathbf{Nets}^\circ(I)$, we thus define the following equivalence relation on $A$

$$a\equiv b\quad\textbf{iff}\quad\forall x\in P\ \ F(x,a)=F(x,b)$$

We let $A_\equiv$ be the set of equivalence classes, with projection $\pi:A\to A_\equiv$, and we let $F_\equiv:P\times A_\equiv\to I$ be characterized by

$$\forall x\in P\ \ \forall a\in A\quad F_\equiv(x,\pi(a))=F(x,a)$$

Hence $(P,A_\equiv,F_\equiv)$ is an object of $\mathbf{Nets}^\circ(I)$ and the pairs $(\beta_i,\pi\circ in_i)$ are morphisms of nets as composite of two morphisms $(P,A_\equiv,F_\equiv)\xrightarrow{(1,\pi)}(P,A,F)\xrightarrow{(\beta_i,in_i)}(P_i,A_i,F_i)$. We now verify that this family of morphisms satisfies the required universal property in the category $\mathbf{Nets}^\circ(I)$. Let $\{(\beta'_i,\eta'_i):(P',A',F')\to(P_i,A_i,F_i)\}$ be another family of morphisms in $\mathbf{Nets}^\circ(I)$ for which there exists a mapping $\beta:P'\to P$ such that $\forall i\ \beta'_i=\beta_i\circ\beta$. By initiality of the lift $\{(P,A,F)\xrightarrow{(\beta_i,in_i)}(P_i,A_i,F_i)\}$, we know there exists a unique mapping $\eta:A\to A'$ such that $(\beta,\eta)$ is a morphism of nets and $\forall i\ (\beta'_i,\eta'_i)=(\beta_i,in_i)\circ(\beta,\eta)$. Let $s:A_\equiv\to A$ be an arbitrary section of $\pi$, i.e. $\pi\circ s=1_{A_\equiv}$. $(1,s):(P,A,F)\to(P,A_\equiv,F_\equiv)$ is a morphism of nets, so is $(\beta,\eta\circ s)$ from $(P',A',F')$ to $(P,A_\equiv,F_\equiv)$. In order to check $\eta'_i=(\eta\circ s)\circ(\pi\circ in_i)$ we use the fact that $(P',A',F')$ is an object of $\mathbf{Nets}^\circ(I)$ and notice that $\forall a_i\in A_i$ and $\forall x\in P'$ one has

$$
\begin{array}{rcll}
F'(x, \eta'_i(a_i)) & = & F_i(\beta'_i(x), a_i) & \textbf{by} \;\; (\beta'_i, \eta'_i) : (P', A', F') \rightarrow (P_i, A_i, F_i) \\
& = & F_i(\beta_i \circ \beta(x), a_i) & \text{since} \;\; \beta'_i = \beta_i \circ \beta \\
& = & F(\beta(x), in_i(a_i)) & \textbf{by} \;\; (\beta_i, in_i) : (P, A, F) \rightarrow (P_i, A_i, F_i) \\
& = & F_{\equiv}(\beta(x), \pi \circ in_i(a_i)) & \textbf{by} \;\; (1, \pi) : (P, A_{\equiv}, F_{\equiv}) \rightarrow (P, A, F) \\
& = & F(\beta(x), s \circ \pi \circ in_i(a_i)) & \textbf{by} \;\; (1, s) : (P, A, F) \rightarrow (P, A_{\equiv}, F_{\equiv}) \\
& = & F'(x, \eta \circ s \circ \pi \circ in_i(a_i)) & \textbf{by} \;\; (\beta, \eta) : (P', A', F') \rightarrow (P, A, F)
\end{array}
$$

We now assume there exists another mapping $\overline{\eta} : A_{\equiv} \rightarrow A'$ sharing with $\eta \circ s$ the two following conditions : *(i)* $(\beta, \overline{\eta})$ is a morphism of nets from $(P', A', F')$ to $(P, A_{\equiv}, F_{\equiv})$ and *(ii)* $\eta'_i = \overline{\eta} \circ (\pi \circ in_i)$. Then $\overline{\eta} = \eta \circ s$ because for every element $a \in A_{\equiv}$ there exists a pair $<i, a_i>$ where $a_i \in A_i$ and $a = \pi \circ in_i(a_i)$ and thus $\overline{\eta}(a) = \overline{\eta} \circ \pi \circ in_i(a_i) = \eta \circ s \circ \pi \circ in_i(a_i) = \eta \circ s(a)$. ∎

*Note :* The reader may wish to verify that the family of morphisms $\{(P, A_{\equiv}, F_{\equiv}) \stackrel{(1,\pi)}{\longrightarrow} (P, A, F)\}$ where $(P, A, F)$ range over all objects of $\mathbf{Nets}(I)$, constitutes a concrete coreflection of $\mathbf{Nets}(I)$ into $\mathbf{Nets}^{\circ}(I)$. We shall denote $\mathcal{N}^{\circ}$ the coreflection of a net $\mathcal{N}$, i.e. the event-simple net associated with $\mathcal{N}$.

## 3.3 Type of Nets

**Definition 14 (Types of Nets)** *If $I$ is a set of incidence values, a type $\tau \in$ $\mathbf{Types}(I)$ for nets in $\mathbf{Nets}(I)$ is a finite and reduced transition system with labels in $I$.*

A net is a purely syntactic object, the purpose of a type is to define the behaviour of such a net. More precisely

**Definition 15 (State Graphs)** *The state graph of a net $\mathcal{N} = (P, A, F) \in$ $\mathbf{Nets}(I)$ relative to a type $\tau = (S, I, \tau) \in \mathbf{Types}(I)$ is the transition system $\mathbf{SG}(\mathcal{N}, \tau) = (S^P, A, T)$ whose transition relation is given by:*

$$
q \stackrel{a}{\rightarrow} q' \in T \quad \textbf{iff} \quad \forall x \in P \;\; q(x) \stackrel{F(x,a)}{\longrightarrow} q'(x) \in \tau \tag{10}
$$

If $x \in P$ is a place of the net $\mathcal{N}$, we let $\mathcal{N}_x = (\{x\}, A, F_x)$ where $F_x$ is the restriction of $F$ to the set $A \times \{x\}$. Then $\mathbf{SG}(\mathcal{N}, \tau)$ is the synchronized product à la Arnold-Nivat ([AN82]) of the transition systems $\{\mathbf{SG}(\mathcal{N}_x, \tau)\}_{x \in P}$ synchronized on their common actions, i.e. the vectors of synchronization are the elements of the diagonal i.e. of the form $(a, a, \ldots, a, \ldots, a)$. Equation (10) in particular says that for every place $x \in P$ of $\mathcal{N}$, the mapping $ev_{\mathcal{N}}(x)$ that takes the transition $q \xrightarrow{a} q'$ to the transition $q(x) \xrightarrow{F(x,a)} q'(x)$ is a morphism of transition systems from the state graph of $\mathcal{N}$ to the type $\tau$. We call this morphism the *extension* of the place $x$.

**Definition 16 (Regions)** *The set of regions of a transition system $\mathcal{T}$ relative to a type $\tau = (S, I, \tau) \in$ **Types**$(I)$ is the set of morphisms of transition systems from $\mathcal{T}$ to $\tau$. The synthesized net* $\mathbf{SN}(\mathcal{T}, \tau)$=$(P, A, F)$*, is the net where $P$=*$\mathbf{Trans}(\mathcal{T}, \tau)$ *is the set of regions and the incidence matrix $F : P \times A \to I$ is given by $F(<\sigma, \eta>, a) = \eta(a)$.*

## 3.4 Schizophrenic Objects between Nets and Transition Systems

We recall the following from [Joh82]. Suppose we have a dual adjunction $\mathcal{A}(A, B^*) \cong \mathcal{B}(B, A^*)$ between two Set-categories whose underlying functors have left adjoints (let $F_{\mathcal{A}}$ and $F_{\mathcal{B}}$). Then the underlying functors are representable: $|A| = \mathbf{Set}(1, |A|) \cong \mathcal{A}(F_{\mathcal{A}}(1), A)$ more precisely $|-| \cong \mathcal{A}(F_{\mathcal{A}}(1), -)$ and similarly $|-| \cong \mathcal{B}(F_{\mathcal{B}}(1), -)$. Hence the objects $K_{\mathcal{A}} = (F_{\mathcal{B}}(1))^*$ and $K_{\mathcal{B}} = (F_{\mathcal{A}}(1))^*$ satisfy the following properties.

1. They have isomorphic underlying sets:
   $|K_{\mathcal{A}}| \cong \mathcal{A}(F_{\mathcal{A}}(1), (F_{\mathcal{B}}(1))^*) \cong \mathcal{B}(F_{\mathcal{B}}(1), (F_{\mathcal{A}}(1))^*) \cong |K_{\mathcal{B}}|$.

2. They "classify" the elements of the dual of an object:
   $|A^*| \cong \mathcal{B}(F_{\mathcal{B}}(1), A^*) \cong \mathcal{A}(A, (F_{\mathcal{B}}(1))^*) = \mathcal{A}(A, K_{\mathcal{A}})$.

Therefore the pair $\mathcal{K} = <K_{\mathcal{A}}, K_{\mathcal{B}}>$ is a good candidate as a schizophrenic object inducing the given adjunction. The underlying functors for the categories of (reduced) transition systems and (event-simple) nets clearly have left adjoints. The object $F_{\mathrm{Trans}}(1)$ is the (reduced) transition system with exactly one transition: a morphism from $F_{\mathrm{Trans}}(1)$ to a transition system $\mathcal{T} = (Q, A, T)$ is nothing else than a transition of $\mathcal{T}$ i.e. an element of $T = |\mathcal{T}|$. The object $F_{\mathrm{Nets(I)}}(1)$ is the (event-simple) net with exactly one place and one event per incidence value, i.e. $F_{\mathrm{Nets(I)}} = (P, A, F)$ where $P = \{\bullet\}$, $A = I$ and $F(\bullet, x) = x$: a morphism from $F_{\mathrm{Nets(I)}}$ to a net $N = (P, A, F)$ is nothing else than a place of $\mathcal{N}$ i.e. an element of $P = |\mathcal{N}|$ together with its flow relations. Now if we want the correspondence that takes a net to its state graph and conversely a transition system to its synthesized net, be a dual adjunction induced by a schizophrenic object, we are led to consider the pair $< \mathbf{SN}(F_{\mathrm{Trans}}(1), \tau), \mathbf{SG}(F_{\mathrm{Nets(I)}}(1), \tau) >$. Now the state graph of the net $F_{\mathrm{Nets(I)}}(1)$ relative to a type $\tau$ is the type $\tau$ itself.

**Observation 17 (Shizophrenic Object associated with a Type)** *Let $\tau = (S, I, \tau) \in \mathbf{Types}(I)$ be a type, then the pair $K_\tau = < K_{Nets(I)^\circ,\tau}, K_{Trans^\circ,\tau} >$ where $K_{Nets(I)^\circ,\tau} = \mathbf{SN}(F_{\mathrm{Trans}}(1), \tau)$ and $K_{Trans^\circ,\tau} = \tau$ is a schizophrenic object between the categories of reduced transition systems and event-simple nets.*

*Proof :* The only point we have to verify is that the family of mappings

$$\{ev_{\mathcal{N}}(x) : \mathbf{Nets}^\circ(I)(\mathcal{N}, K_{Nets(I)^\circ}) \to K\}_{x \in P}$$

is jointly monic (where $P = |\mathcal{N}|$ is the set of places of the net $\mathcal{N}$). Now the equality $ev_{\mathcal{N}}(x)(f) = ev_{\mathcal{N}}(x)(g)$ i.e. $|f|(x) = |g|(x)$, for every $x \in |\mathcal{N}|$ says that both morphisms have the same underlying mapping, since the category of event-simple nets is concrete, we deduce that $f = g$. ∎

Figure 3 represents the schizophrenic object for elementary systems.

### 3.4.1 Dual of a Net

**Proposition 18** *There is a bijective correspondence $\mathbf{Nets}^\circ(I)(\mathcal{N}, K_{Nets(I)^\circ,\tau}) \cong |\mathbf{SG}(\mathcal{N}, \tau)|$ relating morphisms from a net $\mathcal{N}$ to the "classifying net" $K_{Nets(I)^\circ,\tau}$ to transitions of the state graph of $\mathcal{N}$.*

*Proof:* $K_{Nets(I)^\circ,\tau} = \mathbf{SN}(F_{Trans^\circ}(1), \tau)$ is the net with one event (we call "enabled" this event), whose places are morphisms from $F_{Trans^\circ}(1)$ to $\tau$, i.e. transitions $s \xrightarrow{i} s' \in \tau$. The flow relation $F : P \times \{\text{enabled}\} \to I$ is given by $F(s \xrightarrow{i} s', \text{enabled}) = i$. Moreover we recall that the state graph of a net $\mathcal{N} = (P, A, F) \in \mathbf{Nets}(I)$ relative to type $\tau = (S, I, \tau)$ is the transition system $\mathbf{SG}(\mathcal{N}, \tau) = (S^P, A, T)$ whose transition relation is given by:

$$q \xrightarrow{a} q' \in T \quad \textbf{iff} \quad \forall x \in P \;\; q(x) \xrightarrow{F(x,a)} q'(x) \in \tau$$

Therefore, if $(\beta, \eta) : \mathcal{N} \to K_{Nets(I)^\circ,\tau}$ is a morphism of nets, and if we let

$$a = \eta(\text{enabled}) \tag{11}$$

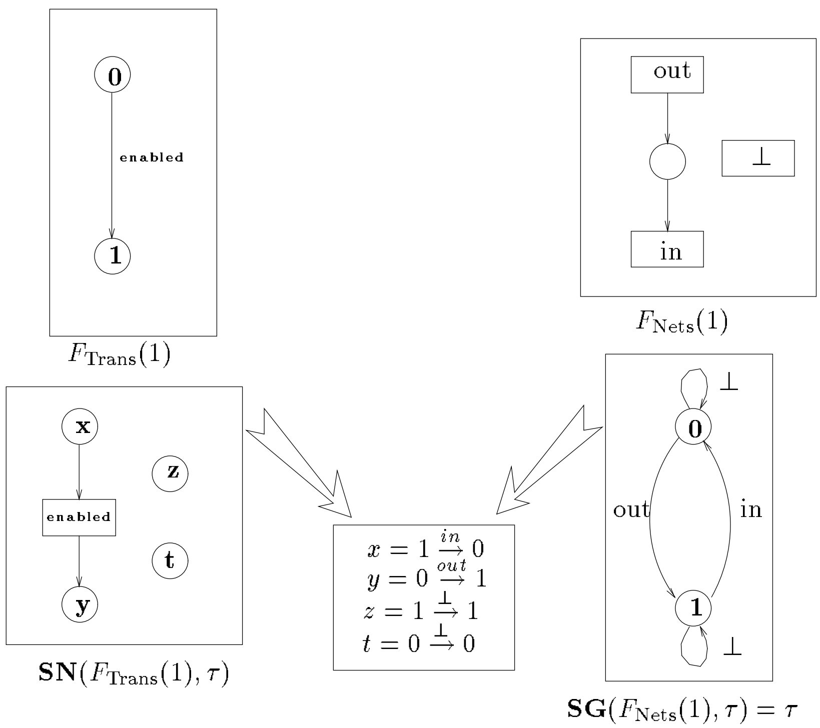


Figure 3: schizophrenic object for elementary systems

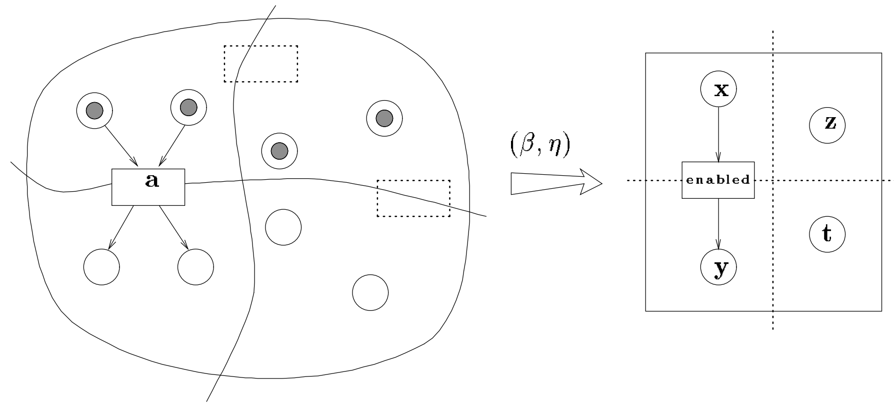


Figure 4: transition of the state graph of a net as morphism

be the event in $\mathcal{N}$ "pointed out" by this morphism, we have $\forall x \in P$, $\beta(x)$ $= s \xrightarrow{i} s'$ with $F(x,a) = i$ because of condition (8) on morphisms of nets. We deduce the transition $q \xrightarrow{a} q'$ in $|\mathbf{SG}(\mathcal{N},\tau)|$ where for all places $x \in P$

$$q(x) = \partial^0(\beta(x)) \quad \textbf{and} \quad q'(x) = \partial^1(\beta(x)) \tag{12}$$

Conversely the equations (11) and (12) determine a morphism of nets $(\beta,\eta) : \mathcal{N}$ $\rightarrow K_{Nets(I)^\circ,\tau}$ from any transition $q \xrightarrow{a} q'$ in $|\mathbf{SG}(\mathcal{N},\tau)|$. ∎

For instance, (see Fig. 4) each morphism of elementary nets $(\beta,\eta) : \mathcal{N} \rightarrow K_{\text{Nets}}$ induces a transition $M \xrightarrow{a} M'$ where $a = \eta(\mathit{enabled})$ $M = \beta^{-1}(\{x,z\})$, and $M' =$ $\beta^{-1}(\{y,z\})$. Conversely the morphism $(\beta,\eta)$ may be retrieved from the transition $M \xrightarrow{a} M'$ by $\eta(\mathit{enabled}) = a$ and

$$\beta(u) = \begin{cases} x & \textbf{if} \quad u \in M \setminus M' \\ y & \textbf{if} \quad u \in M' \setminus M \\ z & \textbf{if} \quad u \in M \cap M' \\ t & \textbf{if} \quad u \notin M \cup M' \end{cases}$$

**Proposition 19** *Let $\tau = (S, I, \tau) \in \mathbf{Types}(I)$ be a type, the dual of an event-simple net $\mathcal{N}$ w.r.t. the schizophrenic object $K_\tau$ is isomorphic to its state graph (relative to type $\tau$):*

$$\mathcal{N}^*_\tau \cong \mathbf{SG}(\mathcal{N},\tau)$$

*Proof:* By Prop. (18), we have $|\mathcal{N}_\tau^*| = \mathbf{Nets}^\circ(I)(\mathcal{N}, K_{Nets(I)^\circ,\tau}) \cong |\mathbf{SG}(\mathcal{N},\tau)|$. $\mathcal{N}_\tau^*$ is therefore isomorphic to the initial lift of

$$\{ev_{\mathcal{N}}(x) : |\mathbf{SG}(\mathcal{N},\tau)| \to \tau\}_{x\in P}$$

where $ev_{\mathcal{N}}(x)(q \xrightarrow{a} q') = q(x) \xrightarrow{F(x,a)} q'(x)$. By Prop. (12) $|\mathcal{N}_\tau^*|$ is given by

$$\begin{array}{ccccc} \mathcal{N}_\tau^* = \Big\{ Im(\lambda) & \xleftarrow{\lambda} & |\mathbf{SG}(\mathcal{N},\tau)| & \overset{\partial^0}{\underset{\partial^1}{\rightrightarrows}} & Im(\partial^0)\cup Im(\partial^1) \Big\} \\ \cap & & \| & & \cap \\ I^P & \xleftarrow{\lambda} & |\mathbf{SG}(\mathcal{N},\tau)| & \overset{\partial^0}{\underset{\partial^1}{\rightrightarrows}} & S^P \end{array}$$

where $\lambda(q \xrightarrow{a} q')(x) = F(x,a)$, $\partial^0(q \xrightarrow{a} q')(x) = q(x)$, and $\partial^1(q \xrightarrow{a} q')(x) = q'(x)$. Since the net $\mathcal{N}$ is event-simple $Im(\lambda) \cong A$ and $\mathcal{N}_\tau^*$ is isomorphic to the state graph of $\mathcal{N}$. ∎

### 3.4.2 Dual of a Transition System

**Proposition 20** *Let $\tau = (S, I, \tau) \in \mathbf{Types}(I)$ be a type, the dual $\mathcal{T}_\tau^*$ of a reduced transition system $\mathcal{T} = (Q, A, T)$ w.r.t. the schizophrenic object $K_\tau$ is isomorphic to the net $(P, A_\equiv, F_\equiv)$ where $P = \mathbf{Trans}^\circ(\mathcal{T}, K_{\mathrm{Trans},\tau})$ is the set of regions viewed as morphisms. If $F : P \times A \to I$ is given by $F(<\sigma,\eta>, a) = \eta(a)$, then $A_\equiv$ is the quotient of $A$ by the equivalence relation*

$$a \equiv b \quad \textbf{iff} \quad \forall x \in P \;\; F(x,a) = F(x,b)$$

*and $F_\equiv : P \times A_\equiv \to I$ is the resulting flow relation.*

*Proof:* By Prop. (13), $\mathcal{T}_\tau^* = (P, E, F)^\circ$ is the event-simple net associated with the net whose places are the regions of $\mathcal{T}$ (i.e. $P = \mathbf{Trans}^\circ(\mathcal{T},\tau)$), with set of events $E = \coprod_{t\in T}\{\text{enabled}\} \cong T$, and whose flow relation $F : \mathbf{Trans}^\circ(\mathcal{T},\tau) \times T \to I$ is given "by evaluation on events" (i.e. $F(<\sigma,\eta>, q \xrightarrow{a} q') = \eta(a)$). Hence $\mathcal{T}_\tau^* = (P, T_\equiv, F_\equiv)$ where $\equiv$ is the equivalence relation given by

$$(q_1 \xrightarrow{a_1} q_1') \equiv (q_2 \xrightarrow{a_2} q_2') \quad \textbf{iff} \quad \forall(\sigma,\eta) : \mathcal{T} \to \tau \;\; \eta(a_1) = \eta(a_2)$$

Since there exists at least one transition labelled by each event, ($\mathcal{T}$ is reduced), the alphabet $E_{\equiv}$ is a quotient of the alphabet of $\mathcal{T}$ where two events are identified if and only if they are indistinguishable by all regions of $\mathcal{T}$. ∎

# 4 Galois connection between Net Systems and Transition Systems

The dual adjunction between the category $\mathbf{Nets}^{\circ}(I)$ of reduced nets and the category $\mathbf{Trans}^{\circ}$ of reduced transition systems obtained above is not a Galois connection, for the arrows $E_{\mathcal{T}^*_\tau} : \mathcal{T}^*_\tau \to \mathcal{T}^{***}_{\tau\tau\tau}$ are generally not isomorphisms. Given a reduced transition system $\mathcal{T} = (Q, A, T)$, some states of its double dual $\mathcal{T}^{**}_{\tau\tau}$ may in fact be disconnected from the subset of states $\{ev_{\mathcal{T}^*_\tau}(q) \mid q \in Q\}$. In that case, a single region of $\mathcal{T}$, seen as a place of $\mathcal{T}^*_\tau$, gives rise to several regions of $\mathcal{T}^{**}_{\tau\tau}$. These regions have an identical restriction on the connected components which contain states $ev_{\mathcal{T}^*_\tau}(q)$, but they are independent on the other connected components. A possible way to avoid an infinite expansion of the set of regions by iterated double dual operations is to add an explicit information about significant markings in nets, permitting to ignore the markings of the net $\mathcal{T}^*_\tau$ which are not connected to any state in $\{ev_{\mathcal{T}^*_\tau}(q) \mid q \in Q\}$. This leads us to the following definition of Net Systems.

**Definition 21 (Net Systems)** *A net system of type $\tau = (S, I, \tau)$ is a pair $(\mathcal{N}, \mathcal{Q})$ where $\mathcal{N} = (P, A, F) \in |\mathbf{Nets}^{\circ}(I)|$ and $\mathcal{Q} \subseteq S^P$ is an invariant set of markings of $\mathcal{N}$, i.e. a subset of states of $\mathbf{SG}(\mathcal{N}, \tau)$ closed under direct transitions. A morphism of net systems $(\beta, \eta) : (\mathcal{N}_1, \mathcal{Q}_1) \to (\mathcal{N}_2, \mathcal{Q}_2)$ is a morphism $(\beta, \eta) : \mathcal{N}_1 \to \mathcal{N}_2$ satisfying the condition $\forall q_2 \in \mathcal{Q}_2\ q_2 \circ \beta \in \mathcal{Q}_1$. $\mathbf{Netsys}(\tau)$ is the category of net systems of type $\tau$.*

It is easily checked that $\mathbf{Netsys}(\tau)$ is a concrete category, whose free object on one generator $F_{Netsys(\tau)}(1)$ is equal to $((\{\bullet\}, I, F), S)$ with $F(\bullet, i) = i$. For any net system $(\mathcal{N}, \mathcal{Q})$ of type $\tau$, let $\mathbf{SG}(\mathcal{N}, \tau) = (\mathcal{Q}, A, T)$ be the state graph of $(\mathcal{N}, \mathcal{Q})$, with set of transitions given by:

$$T = \{q \xrightarrow{a} q' \mid q \in \mathcal{Q} \ \wedge\ \forall x \in P \ \ q(x) \xrightarrow{F(x,a)} q'(x) \in \tau\}$$

Thus $\mathbf{SG}(F_{Netsys(\tau)}(1)) \cong \tau$. Given a reduced transition system $\mathcal{T} = (Q, A, T)$, let the net system $\mathbf{SNS}(\mathcal{T}, \tau)$ synthesized from $\mathcal{T}$ be defined as $(\mathbf{SN}(\mathcal{T}, \tau)^{\circ}, \mathcal{Q})$ for the smallest invariant set of markings $\mathcal{Q}$ of $\mathbf{SN}(\mathcal{T}, \tau)^{\circ}$ including the subset $\{ev_1(q) \mid q \in Q\}$, where $ev_1(q)(\sigma, \eta) = \sigma(q)$. We are interested in imposing constraints on the type of nets $\tau$ ensuring that $\mathbf{SNS}(F_{Trans^{\circ}}(1), \tau)$ equals $(\mathbf{SN}(F_{Trans^{\circ}}(1), \tau), \{ev_1(0), ev_1(1)\})$, where $ev_1(0)(s \xrightarrow{i} s') = s$ and $ev_1(1)(s \xrightarrow{i} s') = s'$ for every transition $s \xrightarrow{i} s' \in \tau$ (a transition $s \xrightarrow{i} s'$ is identified here with the associated morphism $(\sigma, \eta)$ from the transition system $F_{Trans^{\circ}}(1) = (\{0, 1\}, \{\text{enabled}\}, \{0 \xrightarrow{\text{enabled}} 1\})$ to the transition system $\tau$). Suitable constraints, met by the types $\tau$ of elementary nets, Petri nets and trace nets, are embodied in the following definition of restricted types for net systems.

**Definition 22 (Restricted types for net systems)**
*A restricted type for net systems is a type of nets given by a deterministic transition system $\tau = (S, I, \tau)$ in which there exists at least one pair $(s, i) \in S \times I$ such that $s' \xrightarrow{i} s$ for some $s'$ but $s \xrightarrow{i} s"$ for no $s"$.*

In fact, determinism guarantees that $ev_1(1)$ is the unique marking of the synthetized net $\mathbf{SN}(F_{Trans^{\circ}}(1), \tau)$ which may be reached in one step from $ev_1(0)$, and the remaining condition entails that it is a sink state since the place $s' \xrightarrow{i} s$ has reached the value $s$ which is not compatible with a new firing of the (unique) transition of the net: $F((s' \xrightarrow{i} s)\ ,\ \text{enabled}) = i$ and $s \xrightarrow{i} s"$ for no $s"$ in $\tau$. We assume from now on that $\tau$ is a restricted type of nets. An analogue of Obs. **??**, depending on this assumption, may be stated as follows.

**Proposition 23** *The pair $< K_{Netsys(\tau)}, K_{Trans^{\circ},\tau} >$ where $K_{Netsys(\tau)} = \mathbf{SNS}(F_{Trans^{\circ}}(1), \tau)$ and $K_{Trans^{\circ},\tau} = \tau$ is a schizophrenic object between the categories of nets systems and reduced transition systems, with underlying set $K = \tau$.*

*Proof :* The main point is to establish that the family of evaluation mappings $ev_{\mathcal{T}}(t) : \mathbf{Trans}^{\circ}(\mathcal{T}, \tau) \to K$, where $t$ ranges over $|\mathcal{T}|$, admits an initial lift in $\mathbf{Netsys}(\tau)$. Let $P = \mathbf{Trans}^{\circ}(\mathcal{T}, \tau)$, $T = |\mathcal{T}|$, and $\beta_t = ev_{\mathcal{T}}(t)$ for $t \in T$. Thus $\beta_t$ is a family of mappings from $P$ to the set of places $K = \tau$ of the net system $\mathbf{SNS}(F_{Trans^{\circ}}(1), \tau) = ((\tau, \{\text{enabled}\}, F), \{ev_1(0), ev_1(1)\})$, where $F((s \xrightarrow{i} s'), \text{enabled}) = i$. Proceeding in the same way as in the proof of Prop. (13), one obtains an initial lift $(\beta_t, \pi \circ in_t) : (P, A_{\equiv}, F_{\equiv}) \to \mathbf{SN}(F_{Trans^{\circ}}(1), \tau)$ in the category $\mathbf{Nets}^{\circ}(I)$.

Let $\mathcal{Q}$ be the smallest invariant set of markings of the net $(P, A_\equiv, F_\equiv)$ including $\{ev_1(0) \circ \beta_t, ev_1(1) \circ \beta_t\}$ for every $t \in T$. We will show that $\{(\beta_t, \pi \circ in_t) : ((P, A_\equiv, F_\equiv), \mathcal{Q}) \to ((\tau, \{\text{enabled}\}, F), \{ev_1(0), ev_1(1)\})\}_{t \in T}$ is the initial lift of the family $\{\beta_t\}_{t\in T}$ in the category $\mathbf{Netsys}(\tau)$. Let us consider a net system $((P', A', F'), \mathcal{Q}')$ and a $t$-indexed family of morphisms $(\beta'_t, \eta'_t)$ from that net system to $\mathbf{SNS}(\ F_{Trans^\circ}\ (1), \tau)$ such that $\beta'_t = \beta_t \circ \beta$ for every $t \in T$ for some fixed $\beta : P' \to P$. From the proof of Prop. (13), there exists a unique morphism of nets $(\beta, \eta \circ s) : (P', A', F') \to (P, A_\equiv, F_\equiv)$ such that $\forall t \in T \ \ (\beta'_t, \eta'_t) = (\beta_t, \pi \circ in_t) \circ(\beta, \eta \circ s)$. It is enough to show that $(\beta, \eta \circ s)$ is a morphism of net systems from $((P', A', F'), \mathcal{Q}')$ to $((P, A_\equiv, F_\equiv), \mathcal{Q})$, i.e. that $q \circ \beta \in \mathcal{Q}'$ for every $q \in \mathcal{Q}$. Suppose $q = ev_1(0) \circ \beta_t$ or $q = ev_1(1) \circ \beta_t$ for some $t \in T$. Since $(\beta'_t, \eta'_t)$ is a morphism of net systems, both $ev_1(0) \circ \beta'_t$ and $ev_1(1) \circ \beta'_t$ belong to the set $\mathcal{Q}'$ and the desired conclusion follows from the relation $\beta'_t = \beta_t \circ \beta$. Consider now a sequence of markings $q_0, q_1 \dots q_{n-1}, q_n$ in $\mathcal{Q}$ such that $q_0 = ev_1(0) \circ \beta_t$ or $q_0 = ev_1(1) \circ \beta_t$ for some $t \in T$ and there exists a transition from $q_{j-1}$ to $q_j$ in $\mathbf{SG}((P, A_\equiv, F_\equiv), \mathcal{Q})$ for every $j \le n$. By induction on the length of sequences, one may assume that $q_{n-1} \circ \beta \ \in \ \mathcal{Q}'$. Now $q_{n-1} \stackrel{\pi \circ in_t(\text{enabled})}{\longrightarrow} q_n$ for some $t \in T$, and since $(\beta, \eta \circ s)$ is a morphism in the category $\mathbf{Nets}^\circ(I)$, there must exist a corresponding transition $q_{n-1} \circ \beta \stackrel{\eta \circ s \circ \pi \circ in_t(\text{enabled})}{\longrightarrow} q_n \circ \beta$ in the state graph of $(P', A', F')$, whence $q_n \circ \beta \in \mathcal{Q}'$. ∎

Prop. 23 establishes a dual adjunction between the categories $\mathbf{Trans}^\circ$ and $\mathbf{Netsys}(\tau)$. The proof of the proposition shows that the dual $\mathcal{T}^*_\tau$ of a reduced transition system $\mathcal{T}$ is isomorphic to the net system $\mathbf{SNS}(\mathcal{T}, \tau)$ synthesized from $\mathcal{T}$ (since $t = q \stackrel{a}{\to} q'$ entails $ev_1(q) = ev_1(0) \circ ev_\mathcal{T}(t)$ and $ev_1(q') = ev_1(1) \circ ev_\mathcal{T}(t)$). Similarly, the dual $(\mathcal{N}, \mathcal{Q})^*$ of a net system is isomorphic to its state graph $\mathbf{SG}(\mathcal{N}, \mathcal{Q})$ (since $q(x) = \partial^0(\beta(x))$ and $q'(x) = \partial^1(\beta(x))$ entail $q = ev_1(0) \circ \beta$ and $q' = ev_1(1) \circ \beta$ in equation 12).

**Theorem 24** *The dual adjunction between net systems and transition systems induced by the shizophrenic object* $< K_{Netsys(\tau)}, K_{Trans^\circ, \tau} >$ *is a Galois connection.*

*Proof :* Let $\mathcal{T} = (Q, A, T) \in |\mathbf{Trans}^\circ|$ and $\mathcal{T}^*_\tau = (\mathcal{N}, \mathcal{Q})$ with $\mathcal{N} = (P, A_\equiv, F_\equiv)$. Thus, $P = \mathbf{Trans}^\circ(\mathcal{T}, \tau)$, $a = b$ iff $\forall(\sigma, \eta) \in P \ \ \eta(a) = \eta(b)$, $(F_\equiv)(a_\equiv) = \eta(a)$, and $\mathcal{Q}$ is the smallest invariant set of markings of $\mathcal{N}$ containing $ev_1(q)$ for every

$q \in Q$. In order to prove the relation $(\mathcal{N}, \mathcal{Q}) \cong \mathbf{SNS} \circ \mathbf{SG}(\mathcal{N}, \mathcal{Q})$, the main point is to establish that $ev_{(\mathcal{N},\mathcal{Q})} : P \to \mathbf{Trans}^{\circ}(\mathbf{SG}(\mathcal{N}, \mathcal{Q}), \tau)$ is a bijective mapping. We show first that it is injective. Observe that $q \xrightarrow{a} q'$ in $\mathcal{T}$ entails $ev_1(q) \xrightarrow{a_{\equiv}} ev_1(q')$ in $\mathbf{SG}(\mathcal{N}, \mathcal{Q})$. Assume $ev_{(\mathcal{N},\mathcal{Q})}(\sigma, \eta) = ev_{(\mathcal{N},\mathcal{Q})}(\sigma', \eta')$. Evaluating both maps at $ev_1(q) \xrightarrow{a_{\equiv}} ev_1(q')$ yields a pair of identical transitions in $\tau = (S, I, \tau)$, namely $\sigma q \xrightarrow{\eta a} \sigma q'$ and $\sigma' q \xrightarrow{\eta' a} \sigma' q'$. Since $\mathcal{T}$ is reduced, the assumption entails $(\sigma, \eta) = (\sigma', \eta')$. We show now that $ev_{(\mathcal{N},\mathcal{Q})}$ is surjective. Observe that $(ev_1, \cdot_{\equiv})$ is a morphism from $\mathcal{T}$ to $\mathbf{SG}(\mathcal{N}, \mathcal{Q})$. Thus, for every region $(\Sigma, H) \in \mathbf{Trans}^{\circ}(\mathbf{SG}(\mathcal{N}, \mathcal{Q}), \tau)$, the composite morphism $(\Sigma \circ ev_1, H(\cdot_{\equiv}))$ is a region of $\mathcal{T}$ i.e a pair $(\sigma, \eta) \in P$. Moreover, $(\Sigma, H)$ and $ev_{(\mathcal{N},\mathcal{Q})}(\sigma, \eta) = (\Sigma', H')$ take the same values $\Sigma(ev_1(q)) = \Sigma'(ev_1(q))$ and $H(a_{\equiv}) = H(a'_{\equiv})$ for $q \in Q$ and $a \in A$. We will prove that $(\Sigma, H) = (\Sigma', H')$. For that purpose, let us consider a sequence of markings $s_0, s_1, \ldots, s_{n-1}, s_n$ in $\mathcal{Q}$ such that $s_0 = ev_1(q_0)$ and $s_{j-1} \xrightarrow{(a_j)_{\equiv}} s_j$ in $\mathbf{SG}(\mathcal{N}, \mathcal{Q})$ for every $j \leq n$, where $q_0 \in Q$ and $a_j \in A$. By induction on the length of sequences, one may assume $\Sigma(s_{n-1}) = \Sigma'(s_{n-1})$. Now $\Sigma(s_{n-1}) \xrightarrow{H((a_n)_{\equiv})} \Sigma(s_n)$ and $\Sigma'(s_{n-1}) \xrightarrow{H'((a_n)_{\equiv})} \Sigma'(s_n)$ in $\tau$, and because $\tau$ is deterministic, this entails $\Sigma(s_n) = \Sigma'(s_n)$. Hence $ev_{(\mathcal{N},\mathcal{Q})}$ is a bijective mapping, and the two nets $\mathcal{N}$ and $\mathbf{SN}(\mathbf{SG}(\mathcal{N}, \mathcal{Q}), \tau)$ are isomorphic. Since $\mathcal{Q}$ is an invariant set of markings of $\mathbf{SN}(\mathbf{SG}(\mathcal{N}, \mathcal{Q}), \tau)$, the set $\{ev_1(s) \mid s \in \mathcal{Q})\}$ is an invariant set of markings of $\mathbf{SN}(\mathbf{SG}(\mathcal{N}, \mathcal{Q}), \tau)$, whence finally $(\mathcal{N}, \mathcal{Q}) \cong \mathbf{SNS} \circ \mathbf{SG}(\mathcal{N}, \mathcal{Q})$ as required. ∎

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