Long time asymptotics for some dynamical noise free non-linear filtering problems

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Abstract: A good way to study the accuracy of optimal non-linear filtering is to look at the concentration of the conditional law on arbitrary neighborhood of the current (unknown) state as the time goes to infinity. In this paper we consider a restricted class of systems without noise in their state equation and under some more assumptions we show some results concerning the concentration of the conditional law, and also how estimating the current state could be different from estimating the initial condition. Our assumptions could seem quite restrictive, however they concern only the "observability" of the system, without any reference to compactness or ergodicity as in previous works.

Key-words: Non-linear filtering, long time asymptotics, measure concentration

(Résumé : tsvp)

Asymtotiques en temps long pour des problèmes de filtrage non-linéaires sans bruit de dynamique

Résumé : Une bonne approche pour étudier l'efficacité du filtrage non-linéaire optimal consiste à regarder la concentration de la loi conditionnelle sur un voisinage arbitraire de la position courante (inconnue) lorsque le temps croît vers l'infini. Dans cet article nous considérons une classe reduite de systèmes, sans bruit sur l'équation d'état, et sous certaines hypothèses supplémentaires nous montrons des résultats concernant la concentration de la loi conditionnelle, et aussi en quoi estimer l'état courant diffère d'estimer la condition initiale. Nos hypothèses peuvent sembler assez restrictives, cependant elles se rapportent uniquement à l'"observabilité" du système, sans aucune référence aux notions de compacité ou d'ergodicité comme dans les travaux précédents.

Mots-clé : Filtrage non-linéaire, asymptotique en temps long, concentration de mesure

Contents

1 Introduction

The optimal filtering of non linear systems, in particular in the framework of stochastic differential equations of Itô type, is now a well studied problem: the conditional law of the current state, given the past observations, is solution of the Kushner–Stratonovitch equation. For numerical purpose, the Zakai equation, which gives an unnormalized conditional law, is more suitable See for instance [11] for a review on non linear filtering. Nevertheless, a crucial question is still open, concerning the "accuracy" of the filtering: that is, what are the conditions to be be imposed to the system in order to make the filtering error goes to zero as the time grows to infinity. Some results have been proved in [5] and [6] by Kunita when the state process is ergodic. Viewing this problem as concerning the asymptotic stability of the Kushner–Stratonovitch equation (which can be considered as a stochastic dynamical system on the infinite dimensional space of probability measures on the state space), it becomes clear that it is closely related to the sensitivity of the filter to the prior distribution. That is, given an erroneous initial distribution, does the filter give results close to the ones obtained with the right initialization, as the time goes to infinity ? Using again ergodicity arguments, Ocone and Pardoux [10] showed some results concerning this topic.

But it seems natural that without any ergodicity, only some good fitting between the dynamics and the observations should be sufficient (related to some "observability" notion) to make the filtering error asymptotically zero for large time. That is the aim of the present paper, in the restricted frame of noise–free dynamics, that is with identically zero diffusion coefficient on the state equation. This is a much simpler but non trivial problem, used with success in some applications (such as passive target tracking), and for which well adapted numerical algorithms have been developed (see [1] for details). In fact we can adopt here a direct approach: as we can write an explicit formula for the conditional unnormalized density, under some assumptions insuring a uniform good "observability", we can directly estimate the long time behaviour of the conditional law. More precisely we will show that for any ball centered at the current state (and also moving with it), the integral of the conditional density on the ball converges in probability to 1 as the time goes to infinity.

Before studying the filtering problem we present a new approach of an estimation problem, keeping in mind that the method developed will be useful in the sequel.

2 Estimation

We consider the following estimation problem:

$$dY(t) = S\left(t, \theta_0\right) dt + dB_t \tag{1}$$

where S is a measurable function from $\mathbb{R} \times \mathbb{R}^n$ into $\mathbb{R}^d$, $\theta_0 \in \Theta \subset \mathbb{R}^n$ is a random vector and B denotes standard d-dimensional Brownian motion. For all $\theta \in \Theta$ we take the notation

$$g(t,\theta) = S(t,\theta) - S(t,\theta_0)$$

Let p_0 be the density of the law of θ_0, that is, from a bayesian point of view, the *a priori* law. We assume that θ_0 and B are independent. This section is devoted to the study of the asymptotic behaviour of the conditional law of θ_0, given the observations $\{Y_s,\, s \leq t\}$, as $t \to +\infty$. This is a well studied problem, see [4] for instance, but the standard approach is difficult to use for related filtering problems that we will study in the next sections, so we present here some methods that will be useful in the sequel. Assume that the following hypothesis are fulfilled, insuring the existence of a conditional density given by a Bayes formula. Let $\{\mathcal{F}_t^Y,\, t \geq 0\}$ be the filtration of the process Y, and for all $t \geq 0$:

$$\forall \theta \in \Theta,\ \int_0^t \|S(s,\theta)\|\, ds < \infty, \tag{2}$$

$$P\left(\int_0^t \|S(s,\theta_0)\|^2\, ds < \infty\right) = 1 \tag{3}$$

and

$$P\left(\int_0^t \left\|E\left[S(s,\theta_0)\,|\,\mathcal{F}_s^Y\right]\right\|^2\, ds < \infty\right) = 1. \tag{4}$$

Then by [9], theorem 7.23 p. 289, the Bayes formula gives for all bounded continuous ψ:

$$E\left[\psi(\theta_0)\,|\,\mathcal{F}_t^Y\right] = G.\int_\Theta \psi(\theta)\, \exp\left[-\frac{1}{2}\int_0^t S^2(s,\theta)\, ds + \int_0^t S(s,\theta)\, dY_s\right] p_0(\theta)\, d\theta \tag{5}$$

where G is a $\mathcal{F}_t^Y$ measurable nomalizing factor. By multiplying G^{-1} and the integral term by $\exp\left[-\frac{1}{2}\int_0^t S^2(s,\theta_0)\, ds - \int_0^t S(s,\theta_0)\, dB_s\right]$ we get the following form for the unnormalized *a posteriori* density:

$$f_t(\theta) = \exp\left[\int_0^t g(s,\theta) dB_s - \frac{1}{2}\int_0^t g^2(s,\theta) ds\right] p_0(\theta) \tag{6}$$

In the whole section we suppose that the following hypothesis are fulfilled:

(H1) There are two functions v and V from $\mathbb{R}_+$ into $\mathbb{R}_+$ and a positive constant C such that $1 \leq \frac{V(t)}{v(t)} \leq C$ and:

$$\forall (\theta_1,\theta_2) \in \Theta^2 \quad v(t)\|\theta_1-\theta_2\|^2 \leq \int_0^t \|S(s,\theta_1) - S(s,\theta_2)\|^2 ds \leq V(t)\|\theta_1-\theta_2\|^2$$

Notice that it is reasonable to assume that $\int_0^t \|S(s,\theta)\|^2 ds < \infty$ for some θ, and then this assumption is much more stronger than (2)-(4).

Proposition 2.1 *There exists $A_1 > 0$ such that for all $a > 0$,*

$$E\left[\sup_{\|\theta-\theta_0\| \leq a} \left|\int_0^t g(s,\theta) dB_s\right|\right] \leq A_1 a \sqrt{V(t)} \tag{7}$$

Proof We note

$$\Theta_a = \{\theta \in \Theta / \|\theta - \theta_0\| \leq a\} \tag{8}$$

As θ_0 and B are independent, we can consider that $P = P_{\theta_0} \otimes P_B$ on some probability space $\Omega_{\theta_0} \times \Omega_B$. We denote by E_{θ_0} (resp. E_B) the expectation over Ω_{θ_0} (resp Ω_B). The integral in the supremum is a Gaussian process, so [8] theorem 11.17 page 321 gives the estimate:

$$E\left[\sup_{\|\theta-\theta_0\|\leq a}\left|\int_0^t g(s,\theta)dB_s\right|\right] = E_{\theta_0}E_B\left[\sup_{\|\theta-\theta_0\|\leq a}\left|\int_0^t g(s,\theta)dB_s\right|\right] \tag{9}$$

$$\leq 24E_{\theta_0}\left[\int_0^\infty \left(\log N\left(\Theta_a, d_{U_t};\zeta\right)\right)^{\frac{1}{2}} d\zeta\right] \tag{10}$$

where

$$U_t(\theta_1,\theta_2) = \int_0^t \left(S(s,\theta_1) - S(s,\theta_2)\right) dB_s \tag{11}$$

with d_{U_t} the pseudo-metric given by U_t,

$$d_{U_t}(\theta_1,\theta_2) = \|U_t(\theta_1,\theta_2)\|_{L^2(\Omega)} \tag{12}$$

$$= \left[\int_0^t |S(s,\theta_1) - S(s,\theta_2)|^2 ds\right]^{\frac{1}{2}} \tag{13}$$

and $N\left(\Theta_a, d_{U_t};\zeta\right)$ is the smallest number of balls of radius ζ, for the pseudo-metric d_{U_t}, needed to cover Θ_a. The hypothesis above give

$$d_{U_t}(\theta_1,\theta_2) \leq (V(t))^{\frac{1}{2}}\|\theta_1-\theta_2\| \leq K(V(t))^{\frac{1}{2}}\|\theta_1-\theta_2\|_\infty \tag{14}$$

where K comes from the equivalence of the norms in $\mathbb{R}^d$. So we get:

$$N\left(\Theta_a, d_{U_t};\zeta\right) \leq N\left(\Theta_a, K(V(t))^{\frac{1}{2}}\|.\|_\infty;\zeta\right) \tag{15}$$

$$= N\left(\Theta_a, \|.\|_\infty; \frac{\zeta}{K(V(t))^{\frac{1}{2}}}\right) \tag{16}$$

$$\leq \left(\frac{2aK(V(t))^{\frac{1}{2}}}{\zeta} + 1\right)^d \tag{17}$$

From which follows:

$$\int_0^\infty \left(\log N\left(\Theta_a, d_{U_t};\zeta\right)\right)^{\frac{1}{2}} d\zeta \tag{18}$$

$$\leq \int_0^\infty \left(\log N\left(\Theta_a, \|.\|_\infty; \frac{\zeta}{K(V(t))^{\frac{1}{2}}}\right)\right)^{\frac{1}{2}} d\zeta \tag{19}$$

$$\leq \int_0^{2aK(V(t))^{\frac{1}{2}}} d^{\frac{1}{2}} \left(\log\left(\frac{2aK(V(t))^{\frac{1}{2}}}{\zeta}+1\right)\right)^{\frac{1}{2}} d\zeta \tag{20}$$

$$= d^{\frac{1}{2}} 2aK(V(t))^{\frac{1}{2}} \int_0^1 \left(\log\left(1+\frac{1}{u}\right)\right)^{\frac{1}{2}} du \tag{21}$$

with the change of variables:

$$u = \frac{\zeta}{2aK(V(t))^{\frac{1}{2}}} \tag{22}$$

As $0 \leq \left(\log\left(1+\frac{1}{u}\right)\right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{u}}$ the last integral is convergent, and the proposition is proved. □

Proposition 2.2 *There exists $A_2 > 0$ such that for all $a > 0$:*

$$E\left[\sup_{\|\theta-\theta_0\|\geq a} \left|\frac{\int_0^t g(s,\theta)dB_s}{\|\theta-\theta_0\|^2}\right|\right] \leq A_2 a^{-1}\sqrt{V(t)} \tag{23}$$

Proof Let n_0 be greatest relative integer such that $2^{n_0} \leq a$. Triangular inequality gives

$$E\left[\sup_{\|\theta-\theta_0\|\geq a} \frac{\left|\int_0^t g(s,\theta)dB_s\right|}{\|\theta-\theta_0\|^2}\right] \leq \sum_{n\geq n_0} \frac{1}{2^{2n}} E\left[\sup_{2^n\leq\|\theta-\theta_0\|\leq 2^{n+1}} \left|\int_0^t g(s,\theta)dB_s\right|\right] \tag{24}$$

$$\leq \sum_{n\geq n_0} \frac{1}{2^{2n}} E\left[\sup_{\|\theta-\theta_0\|\leq 2^{n+1}} \left|\int_0^t g(s,\theta)dB_s\right|\right] \tag{25}$$

$$\leq A_1(V(t))^{\frac{1}{2}} \sum_{n\geq n_0} \frac{2^{n+1}}{2^{2n}} \quad \text{(proposition 2.1)} \tag{26}$$

$$\leq A_1(V(t))^{\frac{1}{2}} 2^{-n_0} \sum_{p\geq 0} \frac{2^{p+1}}{2^{2p}} \tag{27}$$

$$\leq A_1(V(t))^{\frac{1}{2}} \frac{2}{a} \sum_{p\geq 0} \frac{2^{p+1}}{2^{2p}} \tag{28}$$

Because $2^{n_0} \leq a \leq 2^{n_0+1}$. The convergence of the series concludes the proof. □

The L^1 convergence implies the convergence in probability, so we have immediately:

Corollary 2.3 *Assume that* $\lim_{t\to+\infty} V(t) = +\infty$. *Then for all* $\delta > 0$

$$\frac{1}{aV(t)^{\frac{1}{2}+\delta}} \sup_{\|\theta-\theta_0\|\leq a} \left|\int_0^t g(s,\theta)dB_s\right| \xrightarrow{P} 0 \quad as \quad t\to\infty \tag{29}$$

and

$$\frac{a}{V(t)^{\frac{1}{2}+\delta}} \sup_{\|\theta-\theta_0\|\geq a} \left| \frac{\int_0^t g(s,\theta)dB_s}{\|\theta-\theta_0\|^2} \right| \xrightarrow{P} 0 \quad as \quad t \to \infty \tag{30}$$

uniformly a.

In order to show the concentration of the *a posteriori* law, we will need the

Lemma 2.4 $\forall a > 0, \forall \eta > 0, \forall \epsilon > 0, \exists T_1 > 0$ *such that* $\forall t > T_1$

$$P\left(\inf_{\|\theta-\theta_0\|\leq a} \int_0^t g(s,\theta)dB_s - \frac{1}{2}\int_0^t g^2(s,\theta)ds \geq V(t)\left(-\frac{1}{2}a^2 - \epsilon\right)\right) \geq 1-\eta \tag{31}$$

Proof It is sufficient to see that:

$$P\left(\frac{1}{V(t)} \sup_{\|\theta-\theta_0\|\leq a} \left|\int_0^t g(s,\theta)dB_s\right| \leq \epsilon\right) \tag{32}$$

$$\leq P\left(\frac{1}{V(t)} \inf_{\|\theta-\theta_0\|\leq a} \int_0^t g(s,\theta)dB_s \geq -\epsilon\right) \tag{33}$$

$$\leq P\left(\inf_{\|\theta-\theta_0\|\leq a} \int_0^t g(s,\theta)dB_s - \frac{1}{2}\int_0^t g^2(s,\theta)ds \geq V(t)\left(-\epsilon - \frac{1}{2}a^2\right)\right) \tag{34}$$

and to use the convergence in probability of the corollary. □

In a very symmetric way we get the

Lemma 2.5 $\forall a > 0, \forall \eta > 0, \forall 0 < \epsilon < \frac{1}{2C}, \exists T_2 > 0$ *such that* $\forall t > T_2$

$$P\left(\sup_{\|\theta-\theta_0\|\geq a} \frac{\int_0^t g(s,\theta)dB_s - \frac{1}{2}\int_0^t g^2(s,\theta)ds}{\|\theta-\theta_0\|^2} \leq v(t)\left(C\epsilon - \frac{1}{2}\right)\right) \geq 1-\eta \tag{35}$$

Proof Here again it is sufficient to apply the corollary after having seen that for t large enough:

$$1-\eta \leq P\left(\frac{1}{V(t)} \sup_{\|\theta-\theta_0\|\geq a} \left| \frac{\int_0^t g(s,\theta)dB_s}{\|\theta-\theta_0\|^2} \right| \leq \epsilon\right) \tag{36}$$

$$\leq P\left(\sup_{\|\theta-\theta_0\|\geq a} \frac{\int_0^t g(s,\theta)dB_s}{\|\theta-\theta_0\|^2} \leq \epsilon V(t)\right) \tag{37}$$

$$\leq P\left(\sup_{\|\theta-\theta_0\|\geq a} \frac{\int_0^t g(s,\theta)dB_s - \frac{1}{2}\int_0^t g^2(s,\theta)ds}{\|\theta-\theta_0\|^2} \leq \epsilon V(t) - \frac{1}{2}v(t)\right) \tag{38}$$

$$\leq P\left(\sup_{\|\theta-\theta_0\|\geq a} \frac{\int_0^t g(s,\theta)dB_s - \frac{1}{2}\int_0^t g^2(s,\theta)ds}{\|\theta-\theta_0\|^2} \leq v(t)\left(C\epsilon - \frac{1}{2}\right)\right) \tag{39}$$

□

We have the following proposition concerning the concentration of the conditional law:

Proposition 2.6 *Assume that (H1) is fulfilled,* $\lim_{t\to+\infty} V(t) = +\infty$, p_0 *is continuous, bounded. Then for all* $a > 0$

$$\frac{\int_{\Theta_a^c} f_t(\theta)\, d\theta}{\int_{\Theta_a} f_t(\theta)\, d\theta} \xrightarrow{P} 0 \quad as \quad t \to +\infty \tag{40}$$

Proof For all $\eta > 0$, by continuity of p_0 there exists a compact K such that $P(\theta_0 \in K) > 1 - \eta$ and p_0 is strictly positive on K. Then by uniform continuity there exists $a_0 > 0$ and $K_1 > 0$ such that on $\{\theta_0 \in K\} = \Omega_\eta$:

$$\inf_{\theta \in \Theta_{a_0}} p_0(\theta) \geq K_1. \tag{41}$$

p_0 is bounded, so let K_2 be such that

$$\sup_{\theta \in \mathbb{R}^d} p_0(\theta) \leq K_2 \tag{42}$$

Pick a_1 such that:

$$0 < a_1 < \frac{a}{\sqrt{C}} \wedge a_0 < a \wedge a_0 \tag{43}$$

Then for all ϵ such that

$$0 < \epsilon < \frac{a^2 - Ca_1^2}{2C(a^2+1)} \wedge \frac{1}{2C} \tag{44}$$

we note

$$A_\epsilon = \left\{ \inf_{\|\theta - \theta_0\| \leq a_1} \int_0^t g(s,\theta) dB_s - \frac{1}{2}\int_0^t g^2(s,\theta) ds \geq V(t)\left(-\frac{1}{2}a_1^2 - \epsilon\right)\right\} \tag{45}$$

$$B_\epsilon = \left\{ \sup_{\|\theta - \theta_0\| \geq a} \frac{\int_0^t g(s,\theta) dB_s - \frac{1}{2}\int_0^t g^2(s,\theta) ds}{\|\theta - \theta_0\|^2} \leq v(t)\left(C\epsilon - \frac{1}{2}\right)\right\} \tag{46}$$

The previous lemmas imply that for all $\eta > 0$ there exists $T > 0$ ($T > T_1 \vee T_2$ from the lemmas) such that for all $t > T$

$$P\left(A_\epsilon \cap B_\epsilon \cap \Omega_\eta\right) \geq 1 - 3\eta \tag{47}$$

Let us denote by μ the Lebesgue measure on $\mathbb{R}^d$. We get:

$$\Omega_\eta \cap A_\epsilon \subset \left\{ \int_{\Theta_a} f_s(\theta) d\theta \geq \mu\left(\Theta_{a_1}\right) K_1 \exp\left[V(t)\left(-\frac{1}{2}a_1^2 - \epsilon\right)\right]\right\} \tag{48}$$

and

$$B_\epsilon \subset \left\{ \int_{\Theta_a^c} f_s(\theta)d\theta \le K_2 \int_{\Theta_a^c} \exp\left[\|\theta-\theta_0\|^2 v(t) \left(C\epsilon - \frac{1}{2} \right) \right] d\theta \right\}. \tag{49}$$

Moreover

$$\begin{aligned} &\int_{\Theta_a^c} \exp\left[\|\theta-\theta_0\|^2 v(t) \left(C\epsilon - \frac{1}{2} \right) \right] d\theta \\ &= S_d \int_a^{+\infty} \exp\left[v(t) \left(C\epsilon - \frac{1}{2} \right) r^2 \right] r^{d-1} dr, \end{aligned} \tag{50}$$

using polar coordinates, where S_d is the area of the unit d-sphere. Then

$$\begin{aligned} &\int_{\Theta_a^c} \exp\left[\|\theta-\theta_0\|^2 v(t) \left(C\epsilon - \frac{1}{2} \right) \right] d\theta \\ &\le S_d \exp\left[a^2 v(t) \left(C\epsilon - \frac{1}{2} \right) \right] \int_a^{+\infty} \exp\left[v(t) \left(C\epsilon - \frac{1}{2} \right) \left(r^2 - a^2 \right) \right] r^{d-1} dr \end{aligned} \tag{51}$$

There exists $T_3 > 0$ such that for all $t > T_3$

$$v(t) \left(C\epsilon - \frac{1}{2} \right) \le -1 \tag{52}$$

So there exists a constant K_3 depending on ϵ but not on t such that:

$$\begin{aligned} &P(\Omega_\eta \cap A_\epsilon \cap B_\epsilon) \\ &\le P\left(\frac{\int_{\Theta_a^c} f_t(\theta)d\theta}{\int_{\Theta_a} f_t(\theta)d\theta} \le K_3 \exp\left[V(t) \left(\frac{1}{2} a_1^2 + \epsilon \right) + v(t) \left(C\epsilon - \frac{1}{2} \right) a^2 \right] \right) \end{aligned} \tag{53}$$

Let

$$\tilde{V}(t) = V(t) \left(\frac{1}{2} a_1^2 + \epsilon \right) + v(t) \left(C\epsilon - \frac{1}{2} \right) a^2 \tag{54}$$

Hypothesis on a_1, a and ϵ give:

$$\tilde{V}(t) \le v(t) \left(\frac{1}{2} \left(C a_1^2 - a^2 \right) + \epsilon \left(C a^2 + C \right) \right) \to -\infty \quad \text{as } t \to +\infty \tag{55}$$

So there exists $T_4 > 0$ such that for all $t > T_4$, $\tilde{V}(t) \le \log \frac{\epsilon}{K_3}$. We finally get that for all $\eta > 0$, for all ϵ small enough, say:

$$0 < \epsilon < \frac{a^2 - C a_1^2}{2C(a^2+1)} \wedge \frac{1}{2C} \tag{56}$$

there exists $T = T_1 \vee T_2 \vee T_3 \vee T_4$ such that for all $t > T$

$$1 - 3\eta \le P\left[\frac{\int_{\Theta_a^c} f_t(\theta)d\theta}{\int_{\Theta_a} f_t(\theta)d\theta} \le \epsilon \right] \tag{57}$$

Which gives the required convergence in probability. □

3 Dynamical noise free filtering: general case

We consider now a slightly different situation where we are given a dynamical noise free filtering problem:

$$\begin{cases} dX_t = b(X_t)dt \\ dY_t = h(X_t)dt + dB_t \end{cases} \tag{58}$$

where X_t takes values in $\mathbb{R}^d$ and Y_t in $\mathbb{R}^m$, X_0 is a random vector with the given law P_{X_0}, with density p_0, B_t is a m-dimensional Wiener process independent of X_0. In the sequel, $\Phi_t(x)$ will denote the (deterministic) flow of the state equation; it is the state reached by the system at time t, starting from x at $t = 0$. We will assume that it is well defined for every time t and every initial condition x_0. If we want to estimate the initial state, we have a particular case of the previous problem with:

$$\theta_0 = X_0 \quad \text{and} \quad S(t,.) = h(\Phi_t(.)) \tag{59}$$

Thus, in order to estimante X_0, we can apply the previous section and study the estimation problem. But here the term "filtering" means that we are interested in the current state X_t. So the object to be estimated is also moving, and we will say that the filtering algorithm gives good results if the conditional law of X_t concentrates on arbitrary balls centered at X_t (and also moving with it). The purpose of this section is to find sufficient conditions imposed on the system (58) to have such a behavior of the conditional law. So, to transpose the previous results on the current state X_t, we will assume in addition that the flow verifies:

(H2)

$$\|\Phi_t(x_1) - \Phi_t(x_2)\| \leq \alpha(t)\, \|x_1 - x_2\| \tag{60}$$

for all x_1, x_2 in $\mathbb{R}^d$, for all $t > 0$ and for some monotonous function α from $\mathbb{R}_+$ into $\mathbb{R}_+$.

We are interested in the concentration of the *a posteriori* law μ_t of X_t on sets of the form:

$$\Theta_a^t = \left\{ x \in \mathbb{R}^d / \|x - X_t\| \leq a \right\} \tag{61}$$

We can give an explicit expression for μ_t. Let $\{\mathcal{F}_t^Y,\, t \geq 0\}$ be the filtration of the process Y. For all test function ψ we have:

$$E\left[\psi(X_t) \,|\, \mathcal{F}_t^Y\right] = E\left[\psi \circ \Phi_t(X_0) \,|\, \mathcal{F}_t^Y\right] \tag{62}$$

$$= \frac{\int_{\mathbb{R}^d} \psi \circ \Phi_t(\xi) f_t(\xi)\, d\xi}{\int_{\mathbb{R}^d} f_t(\xi)\, d\xi} \tag{63}$$

$$= \frac{\int_{\mathbb{R}^d} \psi(x) f_t \circ \Phi_t^{-1}(x) \det \frac{\partial}{\partial x}\Phi_t^{-1}(x)\, dx}{\int_{\mathbb{R}^d} f_t \circ \Phi_t^{-1}(x) \det \frac{\partial}{\partial x}\Phi_t^{-1}(x)\, dx} \tag{64}$$

using the change of variables $x = \Phi_t(\xi)$, and where f_t is given by (6). So we have for μ_t the following expression:

$$\mu_t(x) = \frac{f_t \circ \Phi_t^{-1}(x) \det \frac{\partial}{\partial x}\Phi_t^{-1}(x)}{\int_{\mathbb{R}^d} f_t \circ \Phi_t^{-1}(x') \det \frac{\partial}{\partial x}\Phi_t^{-1}(x')\, dx'}. \tag{65}$$

In fact for technical reasons we will work with the following expression to study the concentration of μ_t:

$$\int_{\Theta_a^t} \mu_t(x)\, dx = \frac{\int_{\Phi_t^{-1}(\Theta_a^t)} f_t(x)\, dx}{\int_{\mathbb{R}^d} f_t(x)\, dx}. \tag{66}$$

This expression will converge to 1 in probability if and only if:

$$\frac{\int_{\Phi_t^{-1}(\Theta_a^t)^c} f_t(x)\, dx}{\int_{\Phi_t^{-1}(\Theta_a^t)} f_t(x)\, dx} \to 0 \quad \text{in probability as} \quad t \to +\infty \tag{67}$$

It is clear that in the case where the flow is contracting (i.e. $\lim_{t\to+\infty} \alpha(t) = 0$), the *a priori* law of X_t concentrates on the current true position, and also the conditional law does. We will come back to this case later; let us begin by the most interesting case where the flow is not concentrating but the observations are "good enough" so that the filter gives good results.

Theorem 3.1 *Assume that (H1) and (H2) are fulfilled, and p_0 is continuous, bounded. Assume moreover that*

$$\frac{V(t)}{\alpha(t)^2} \longrightarrow +\infty \tag{68}$$

and α is bounded away from zero. Then for all $a > 0$

$$\int_{\Theta_a^t} \mu_t(x)\, dx \xrightarrow{P} 1 \quad as \quad t \to +\infty \tag{69}$$

In order to prove this theorem we will need the

Lemma 3.2 *Let $K(t)$ and $a(t)$ two functions from $\mathbb{R}_+$ into $\mathbb{R}_+$ such that*

$$\lim_{t\to+\infty} K(t)a(t)^2 = +\infty \tag{70}$$

Then for all $d \in \mathbb{N}^*$, there exists $\Gamma_d > 0$ and $T_d > 0$ such that for all $t > T_d$

$$a(t)^{-d} \int_{a(t)}^{+\infty} \exp(-K(t)r^2) r^{d-1} dr \le \frac{\Gamma_d}{K(t)a(t)^2} \exp(-K(t)a(t)^2) \tag{71}$$

Proof We first show the property for $d = 1$, $d = 2$, and then generalize by recurrence.

Case $d = 1$. Simply by differentiating we get

$$\frac{1}{2K(t)a(t)} \exp(-K(t)a(t)^2) = \int_{a(t)}^{+\infty} \left(1 + \frac{1}{2K(t)r^2}\right) \exp(-K(t)r^2)dr \tag{72}$$

So

$$a(t)^{-1} \int_{a(t)}^{+\infty} \exp(-K(t)r^2)dr \leq \frac{1}{2K(t)a(t)^2} \exp(-K(t)a(t)^2) \tag{73}$$

In this case the inequality of the lemma is then verified by taking $\Gamma_1 = \frac{1}{2}$ and $T_1 = 0$.

Case $d = 2$. It is clear that

$$a(t)^{-2} \int_{a(t)}^{+\infty} \exp(-K(t)r^2)r dr = \frac{1}{2K(t)a(t)^2} \exp(-K(t)a(t)^2) \tag{74}$$

So here again $\Gamma_1 = \frac{1}{2}$ and $T_1 = 0$.

General case. Let $d > 2$. Assume that the relation is verified for all $d' < d$. An integration by parts gives:

$$\begin{aligned} &\int_{a(t)}^{+\infty} \exp(-K(t)r^2)r^{d-1}dr \\ &= \int_{a(t)}^{+\infty} \frac{d-2}{2K(t)} \exp(-K(t)r^2)r^{d-3}dr + \frac{a(t)^{d-2}}{2K(t)} \exp(K(t)a(t)^2) \end{aligned} \tag{75}$$

So for $t > T_{d-2}$

$$\begin{aligned} &a(t)^{-d} \int_{a(t)}^{+\infty} \exp(-K(t)r^2)r^{d-1}dr \\ &\leq \left[\frac{d-2}{2K(t)a(t)^2} \frac{\Gamma_{d-2}}{K(t)a(t)^2} + \frac{1}{2K(t)a(t)^2}\right] \exp(-K(t)a(t)^2) \end{aligned} \tag{76}$$

As $K(t)a(t)^2$ tends to $+\infty$ there exists $T' > 0$ such that for all $t > T'$ we get $K(t)a(t)^2 > (d-2)\Gamma_{d-2}$. By taking $\Gamma_d = 1$ and $T_d = T' \vee T_{d-2}$, we get for all $t > T_d$:

$$a(t)^{-d} \int_{a(t)}^{+\infty} \exp(-K(t)r^2)r^{d-1}dr \leq \frac{\Gamma_d}{K(t)a(t)^2} \exp(-K(t)a(t)^2), \tag{77}$$

which concludes the proof of the lemma. □

Proof of the theorem. To prove theorem 3.1 we will show the equivalent convergence:

$$\frac{\int_{\Phi_t^{-1}(\Theta_a)^c} f_t(x)dx}{\int_{\Phi_t^{-1}(\Theta_a)} f_t(x)dx} \xrightarrow{P} 0 \quad \text{as} \quad t \to +\infty. \tag{78}$$

We will not show directly the convergence in probability as we did for the proposition 2.6 but we will use the characterization of the convergence in probability in terms of a.s. convergence of subsequences. Just observe that using propositions 2.1 and 2.2, and (68) we have:

$$\frac{\alpha(t)^2}{V(t)} \sup_{\|x-X_0\| \leq \frac{a}{\alpha(t)}} \left| \int_0^t g(s,x)dB_s \right| \xrightarrow{P} 0 \quad \text{as} \quad t \to \infty \tag{79}$$

and

$$\frac{1}{V(t)} \sup_{\|x-X_0\| \geq \frac{a}{\alpha(t)}} \left| \frac{\int_0^t g(s,x)dB_s}{\|x-X_0\|^2} \right| \xrightarrow{P} 0 \quad \text{as} \quad t \to \infty. \tag{80}$$

Then proving the theorem is equivalent to show that the convergence (78) holds a.s. for every sequence t_n of positive real numbers such that:

$$\lim_{n \to +\infty} t_n = +\infty,$$

$$\frac{\alpha(t_n)^2}{V(t_n)} \sup_{\|x-X_0\| \leq \frac{a}{\alpha(t_n)}} \left| \int_0^{t_n} g(s,x)dB_s \right| \xrightarrow{p.s.} 0 \quad \text{as} \quad n \to \infty \tag{81}$$

and

$$\frac{1}{V(t_n)} \sup_{\|x-X_0\| \geq \frac{a}{\alpha(t_n)}} \left| \frac{\int_0^{t_n} g(s,x)dB_s}{\|x-X_0\|^2} \right| \xrightarrow{p.s.} 0 \quad \text{as} \quad n \to \infty. \tag{82}$$

We begin by seeing that:

$$\Phi_t^{-1}\left(\Theta_a^t\right)^c = \Phi_t^{-1}\left(\Theta_a^{t,c}\right) \subset \Theta^{0,c}_{\frac{a}{\alpha(t)}} \quad \text{and} \quad \Theta^0_{\frac{a}{\alpha(t)}} \subset \Phi_t^{-1}\left(\Theta_a^t\right) \tag{83}$$

Let

$$N_n = \int_{\Phi_{t_n}^{-1}\left(\Theta_a^{t_n,c}\right)} f_{t_n}(x)\,dx \tag{84}$$

and

$$D_n = \int_{\Phi_{t_n}^{-1}\left(\Theta_a^{t_n}\right)} f_{t_n}(x)\,dx \tag{85}$$

and pick a_1 and a_2 such that

$$a \geq a_1 > \frac{a_1}{\sqrt{6C}} > a_2 > 0 \tag{86}$$

Then we get

$$D_n \geq \int_{\Theta^0_{\frac{a_2}{\alpha(t_n)}}} f_{t_n}(x)\,dx \tag{87}$$

$$\geq \exp\left[-\frac{1}{2}\frac{V(t_n)a_2^2}{\alpha(t_n)^2} - \sup_{x'\in\Theta^0_{\frac{a_2}{\alpha(t_n)}}}\left|\int_0^{t_n} g(s,x')dB_s\right|\right]\int_{\Theta^0_{\frac{a_2}{\alpha(t_n)}}} p_0(x)\,dx \tag{88}$$

$$\geq \exp\left[-\frac{a_2^2V(t_n)}{\alpha(t_n)^2}\left[\frac{1}{2}+\frac{\alpha(t_n)^2}{a_2^2V(t_n)}\sup_{x'\in\Theta^0_{\frac{a_2}{\alpha(t_n)}}}\left|\int_0^{t_n} g(s,x')dB_s\right|\right]\right]\int_{\Theta^0_{\frac{a_2}{\alpha(t_n)}}} p_0(x)\,dx \tag{89}$$

$$\geq \exp\left[-\frac{a_2^2V(t_n)}{\alpha(t_n)^2}\right]\int_{\Theta^0_{\frac{a_2}{\alpha(t_n)}}} p_0(x)\,dx \tag{90}$$

For $n > n_1$ which does not depend on X_0, following (81).

Assume for the moment that $\lim_{t\to+\infty}\alpha(t) = +\infty$. As p_0 is continuous and a.s. non zero at X_0, there exists n_2 such that for all $n > n_2$ we have $p_0(x) > \frac{1}{2}p_0(X_0) > 0$ for all $x \in \Theta^0_{\frac{a_2}{\alpha(t_n)}}$. Then for $n > n_1 \vee n_2$

$$D_n \geq \frac{1}{2}\exp\left[-\frac{3}{2}\frac{V(t_n)a_2^2}{\alpha(t_n)^2}\right]p_0(X_0)\mu\left(\Theta^0_{\frac{a_2}{\alpha(t_n)}}\right) \tag{91}$$

where μ denotes the Lebesgue measure on $\mathbb{R}^d$. If we denote by K_d the volume of the unit ball in $\mathbb{R}^d$ we get

$$D_n \geq \frac{1}{2}K_d p_0(X_0)\left[\frac{a_2}{\alpha(t_n)}\right]^d \exp\left[-\frac{3}{2}\frac{V(t_n)a_2^2}{\alpha(t_n)^2}\right] \tag{92}$$

Consider now the numerator. p_0 is bounded, let K_0 be its supremum. Then we can write

$$N_n \leq K_0\int_{\Theta^{0,c}_{\frac{a_1}{\alpha(t_n)}}} \exp\left[-\frac{1}{2}v(t_n)\|x-X_0\|^2 + \left|\int_0^{t_n} g(s,x)dB_s\right|\right]dx \tag{93}$$

$$\leq K_0\int_{\Theta^{0,c}_{\frac{a_1}{\alpha(t_n)}}} \exp\left[\frac{V(t_n)}{C}\|x-X_0\|^2\left[-\frac{1}{2}+\frac{C}{V(t_n)}\sup_{x'\in\Theta^{0,c}_{\frac{a_1}{\alpha(t_n)}}}\frac{\left|\int_0^{t_n} g(s,x')dB_s\right|}{\|x'-X_0\|^2}\right]\right]dx \tag{94}$$

$$\leq K_0\int_{\Theta^{0,c}_{\frac{a_1}{\alpha(t_n)}}} \exp\left[-\frac{1}{4}\frac{V(t_n)}{C}\|x-X_0\|^2\right]dx \tag{95}$$

for $n > n_3$ which does not depend on X_0, following (82). Let S_d be the surface of the unit d-sphere, then polar coordinates permit us to write

$$N_n \leq K_0 S_d\int_{\frac{a_1}{\alpha(t_n)}}^{+\infty} \exp\left[-\frac{V(t_n)}{4Ca_1}r^2\right]r^{d-1}dr \tag{96}$$

Using (92) and (96) we get:

$$\frac{N_n}{D_n} \leq \frac{2K_0S_d}{K_d p_0(X_0)} \exp\left[\frac{3}{2}\frac{V(t_n)}{\alpha(t_n)^2}a_2^2\right].\left[\frac{a_1}{a_2}\right]^d.\left[\frac{\alpha(t_n)}{a_1}\right]^d \int_{\frac{a_1}{\alpha(t_n)}}^{+\infty} \exp\left[-\frac{V(t_n)}{4C}r^2\right] r^{d-1}dr \quad (97)$$

$$\leq \frac{2K_0S_d}{K_d p_0(X_0)}\left[\frac{a_1}{a_2}\right]^d.\exp\left[\frac{3}{2}\frac{V(t_n)}{\alpha(t_n)^2}a_2^2\right].\Gamma_d\left[\frac{\alpha(t_n)}{a_1}\right]^2\frac{4C}{V(t_n)}.\exp\left[-\frac{a_1^2}{4C}\frac{V(t_n)}{\alpha(t_n)^2}\right] \quad (98)$$

for $n > n_1 \vee n_2 \vee n_3$ by lemma 3.2. Let

$$K_1 = \frac{8CK_0S_d a_1^{d-2}\Gamma_d}{K_d a_2^d} \quad (99)$$

Then we obtain the following estimate:

$$\frac{N_n}{D_n} \leq \frac{K_1}{p_0(X_0)}\frac{\alpha(t_n)^2}{V(t_n)}\exp\left[\frac{V(t_n)}{\alpha(t_n)^2}\left(\frac{6Ca_2^2 - a_1^2}{4C}\right)\right] \quad (100)$$

which tend to 0 by virtue of (86).

Let us now come back to the case $\alpha(t)$ bounded. Let $A = \sup_{t\geq 0}\alpha(t) > 0$. We can then proceed is a very similar way. Until (3) the proof is the same. For n larger than some n_4, $\alpha(t_n) \geq \frac{A}{2}$. p_0 is continuous, so let a_2 be small enough such that $\forall x \in \Theta^0_{\frac{2a_2}{A}}$, $p_0(x) > \frac{1}{2}p_0(X_0)$ and $\frac{a_2}{\alpha t_n} < \frac{2a_2}{A} < \frac{a}{\alpha t_n}$. Then (91) is still valid. The end of the proof is unchanged, just replace n_2 by n_4. □

Some remarks on other cases. Now let us come back to the case $\alpha(t) \to 0$. With no observation the concentration speed around the true position follows the ratio:

$$\frac{\int_{\Phi_t^{-1}(\Theta_a^t)^c} p_0(x)\,dx}{\int_{\Phi_t^{-1}(\Theta_a^t)} p_0(x)\,dx} \simeq \int_{\Phi_t^{-1}(\Theta_a^t)^c} p_0(x)\,dx \quad (101)$$

because the denominator converges to 1 a.s. If we have observations, and (H1) is fulfilled, we can show a faster convergence in probability. Let t_n be a sequence verifying (81) and (82). Choose a_2 such that

$$a > \frac{a}{2\sqrt{C}} > a_2 > 0 \quad (102)$$

Using the same notations and arguments as in the proof of the previous theorem, we get, for n large enough:

$$N_n \leq \int_{\Phi_{t_n}^{-1}(\Theta_a^t)^c} \exp\left[\frac{v(t_n)\alpha(t_n)}{a}\|x - X_0\|^2\left(-\frac{1}{2}\frac{a}{\alpha(t_n)}\right.\right.$$

$$\left.\left. + \frac{a}{\alpha(t_n)}\frac{1}{v(t_n)}\sup_{x' \in \Theta^{0,c}_{\frac{a}{\alpha(t_n)}}}\frac{\left|\int_0^{t_n} g(s,x')dB_s\right|}{\|x' - X_0\|^2}\right)\right] p_0(x)\,dx \quad (103)$$

$$\leq \exp\left[-\frac{1}{4}v(t_n)\frac{a^2}{\alpha(t_n)}\right]\int_{\Phi_{t_n}^{-1}(\Theta_a^t)^c} p_0(x)\,dx \quad (104)$$

and

$$D_n \geq \int_{\Theta^0_{\frac{a_2}{\alpha(t_n)}}} \exp\left[-\frac{1}{2}\frac{V(t_n)a_2^2}{\alpha(t_n)^2} - \sup_{x'\in\Theta^0_{\frac{a_2}{\alpha(t_n)}}} \left|\int_0^{t_n} g(s,x')dB_s\right|\right] p_0(x)\,dx \tag{105}$$

$$\geq \int_{\Theta^0_{\frac{a_2}{\alpha(t_n)}}} \exp\left[\frac{V(t_n)a_2}{\alpha(t_n)}\left(-\frac{1}{2}\frac{a_2}{\alpha(t_n)} - \frac{\alpha(t_n)}{a_2 V(t_n)} \sup_{x'\in\Theta^0_{\frac{a_2}{\alpha(t_n)}}} \left|\int_0^{t_n} g(s,x')dB_s\right|\right)\right] p_0(x)\,dx \tag{106}$$

$$\geq \exp\left[-C\frac{v(t_n)a_2^2}{\alpha(t_n)^2}\right] \int_{\Theta^0_{\frac{a_2}{\alpha(t_n)}}} p_0(x)\,dx \tag{107}$$

$$\geq \frac{1}{2}\exp\left[-C\frac{v(t_n)a_2^2}{\alpha(t_n)^2}\right] \tag{108}$$

By taking the two last inequalities we have

$$\frac{N_n}{D_n} \leq 2\exp\left[\frac{v(t_n)}{\alpha(t_n)^2}\left(-\frac{a^2}{4} + Ca_2^2\right)\right] \int_{\Phi_{t_n}^{-1}(\Theta_a^t)^c} p_0(x)\,dx \tag{109}$$

So the convergence, this time in probability, is sped up by the exponential term.

Finally consider some cases when the filter does not give good results. More precisely, consider the opposite of (H2):

($\overline{\mathrm{H2}}$)

$$\|\Phi_t(x_1) - \Phi_t(x_2)\| \geq \alpha(t)\,\|x_1 - x_2\| \tag{110}$$

for all x_1, x_2 in $\mathbb{R}^d$, for all $t > 0$ and some non-decreasing function α from $\mathbb{R}_+$ into $\mathbb{R}_+$. We can show the following proposition (opposite of theorem 3.1):

Proposition 3.3 *Assume that (H1) and ($\overline{H2}$) are fulfilled, p_0 is continuous, bounded. Moreover assume that*

$$\frac{V(t)}{\alpha(t)^2} \longrightarrow 0 \tag{111}$$

and

$$\alpha(t) \longrightarrow +\infty \tag{112}$$

as $t \to +\infty$. Then for all $a > 0$

$$\frac{\int_{\Phi_t^{-1}(\Theta_a)^c} f_t(x)dx}{\int_{\Phi_t^{-1}(\Theta_a)} f_t(x)dx} \xrightarrow{P} +\infty \quad as \quad t \to +\infty \tag{113}$$

Proof Note that in this case it is reasonable to suppose (112), otherwise (111) would imply that $V(t) \to 0$, that is, by (H1), h is constant. Let

$$u(t) = \frac{\alpha(t)^{\frac{1}{2}}}{V(t)^{\frac{1}{4}}} \tag{114}$$

Obviously,

$$u(t) \to +\infty \tag{115}$$

and, as we can suppose that $V(t)$ is bounded away from 0, also does $u(t)\sqrt{V(t)}$ and using propositions 2.1 and 2.2 we get:

$$\sup_{\|x'-X_0\| \leq \frac{a}{\alpha(t)}} \left| \int_0^t g(s,x')dB_s \right| \xrightarrow{P} 0 \tag{116}$$

$$\sup_{\|x'-X_0\| \leq \frac{a}{u(t)\sqrt{V(t)}}} \left| \int_0^t g(s,x')dB_s \right| \xrightarrow{P} 0 \tag{117}$$

Let t_n a sequence of positive real numbers, increasing to $+\infty$, and such that the above convergences take place a.s. Pick any $\varepsilon > 0$. With the same notations we get for n large enough:

$$D_n \leq \int_{\Theta^0_{\frac{a}{\alpha(t_n)}}} \exp\left[-\frac{1}{2}v(t_n)\|x-X_0\|^2 + \int_0^{t_n} g(s,x)dB_s\right] p_0(x)\,dx \tag{118}$$

$$\leq \int_{\Theta^0_{\frac{a}{\alpha(t_n)}}} \exp\left[\sup_{\|x'-X_0\| \leq \frac{a}{\alpha(t_n)}} \left| \int_0^{t_n} g(s,x')dB_s \right|\right] p_0(x)\,dx \tag{119}$$

$$\leq \exp[\varepsilon] \int_{\Theta^0_{\frac{a}{\alpha(t_n)}}} p_0(x)\,dx \tag{120}$$

$$\leq \exp[\varepsilon] 2p_0(X_0)K_d \left[\frac{a}{\alpha(t_n)}\right]^d \tag{121}$$

Using the fact that p_0 is continuous, a.s. non zero at X_0, K_d being the volume of the unit d-ball. Let us denote, for all n:

$$\mathcal{A}_n = \left\{ \frac{a}{\alpha(t_n)} \leq \|x-X_0\| \leq \frac{a}{u(t)\sqrt{V(t_n)}} \right\} \tag{122}$$

Then for n large enough:

$$N_n \geq \int_{\Theta^{0,c}_{\frac{a}{\alpha(t_n)}}} \exp\left[-\frac{V(t_n)}{2}\|x-X_0\|^2 + \int_0^{t_n} g(s,x)dB_s\right] p_0(x)\,dx \tag{123}$$

$$\geq \int_{\mathcal{A}_n} \exp\left[-\frac{1}{2u(t_n)^2}\left[\frac{1}{2} + \sup_{\|x'-X_0\|\leq u(t_n)\sqrt{V(t_n)}} \frac{\left|\int_0^t g(s,x')dB_s\right|}{\alpha(t_n)^2 V(t_n)}\right]\right] p_0(x)\,dx \quad (124)$$

$$\geq \int_{\mathcal{A}_n} \exp[-\varepsilon] p_0(x)\,dx \quad (125)$$

$$\geq \exp[-2\varepsilon]\frac{p_0(X_0)}{2}\mu\left(\mathcal{A}_n\right) \quad (126)$$

$$\geq \exp[-2\varepsilon]\frac{p_0(X_0)}{2}K_d\left[\left[\frac{a}{u(t_n)\sqrt{V(t_n)}}\right]^d - \left[\frac{a}{\alpha(t_n)}\right]^d\right] \quad (127)$$

Then (115) clearly imply $\frac{N_n}{D_n} \to +\infty$. To conclude the proof, use again the characterization of the convergence in probability in terms of a.s. convergence of subsequences. □

4 Linear Dynamics

In this section we consider some systems for which we can observe the concentration of the conditional law, but without estimating the initial condition, and with non trivialy contracting dynamics.

4.1 Linear problem

The resolution of the Riccati equation below and some ideas of this small section are taken from [2]. We consider here the following linear system:

$$\begin{cases} dX_t = A X_t\,dt \\ dY_t = C\,X_t\,dt + dB_t, \qquad \text{with } X_0 \simeq N(m_0, P_0) \end{cases} \quad (128)$$

where A and C are matrices of respective dimensions $d \times d$ and $d \times n$, and X_0 and B are independent. We will investigate conditions insuring that the conditional covariance given by the Kalman-Bucy filter tends to 0 as $t \to +\infty$. It is well known that in this case the conditional law is Gaussian, so the above convergence is equivalent to the concentration of the law around the true position. See [7] or [11] for a general presentation of the Kalman-Bucy filter. Denote by P_t the conditional covariance. In our particular case we can write the Riccati equation for P_t:

$$\dot{P}_t = A\,P_t + P_t\,A^* - P_t\,C^*\,C\,P_t, \quad (129)$$

where $*$ denotes the transpose and the dot the differential w.r.t. t, P_0 being the initial condition. We can solve this equation explicitly. First we state the

Lemma 4.1 *Let M and N be two symmetric positive semidefinite $h \times h$ matrices (i.e. $\langle N\,x,\,x\rangle \geq 0$ and $\langle M\,x,\,x\rangle \geq 0$ for all x). Then if M or N is invertible, $I + MN$ is also invertible.*

Proof Suppose N is invertible (the case M invertible is similar), and $I + MN$ is not full rank. Take $x \in \ker(I + MN)$. Then $MNx = -x$, and

$$0 \leq \ \langle M\,N\,x,\,N\,x\rangle = -\langle x,\,N\,x\rangle \leq 0. \tag{130}$$

Thus, $\langle N\,x,\,x\rangle = 0$, and by the Cauchy-Schwartz inequality for the scalar product induced by N, and for all y we have:

$$0 \leq |\langle N\,x,\,y\rangle|^2 \leq \ \langle N\,x,\,x\rangle\,\langle N\,y,\,y\rangle = 0. \tag{131}$$

Then $Nx = 0$, that is $x = 0$ and we get that $\ker(I + MN) = \{0\}$. □

Now we can solve (129). This is the purpose of the

Lemma 4.2 *If P_0 is invertible, then also is P_t and*

$$P_t = e^{tA}\,\left(P_0^{-1} + Q(t)\right)^{-1}\,e^{tA^*} \tag{132}$$

with

$$Q(t) = \int_0^t e^{rA^*}\,C^*\,C\,e^{rA}\,dr \tag{133}$$

Proof The solution is unique because $P \mapsto F(P) = AP + PA^* - PC^*CP$ is locally Lipschitz. Then we construct a solution. Assume that P_t is invertible for all $t \geq 0$. Differentiating $P_t P_t^{-1} = I$ we get:

$$\frac{d}{dt}P_t^{-1} = -P_t^{-1}\,\frac{dP_t}{dt}\,P_t^{-1}. \tag{134}$$

By (129):

$$\frac{dP_t^{-1}}{dt} = -A^*\,P_t^{-1} - P_t^{-1}\,A + C^*\,C. \tag{135}$$

Using the variation of the constant method:

$$P_t^{-1} = e^{-tA^*}\,P_0^{-1}\,e^{-tA} + \int_0^t e^{-(t-s)A^*}\,C^*\,C\,e^{-(t-s)A}\,ds. \tag{136}$$

Then

$$P_t = e^{tA}\,\left(P_0^{-1} + Q(t)\right)^{-1}\,e^{tA^*} \tag{137}$$

with

$$Q(t) = \int_0^t e^{rA^*}\,C^*\,C\,e^{rA}\,dr. \tag{138}$$

Now assume that P_0 is invertible. By lemma 4.1, $I + Q(t)P_0$ is also invertible and it is easy to verify that P_t defined as

$$P_t = e^{tA}\, P_0\, \left(I + Q(t)\, P_0\right)^{-1}\, e^{tA^*} \tag{139}$$

is solution of (129) and is invertible. □

Now we can state the result:

Proposition 4.3 *Assume that P_0 is invertible. Then $P_t \to 0$ as $t \to +\infty$ if and only if the pair (A, C) is detectable, and for any eigenvalue λ of A, $\Re\mathrm{e}(\lambda) \leq 0$.*

Proof First recall that P_t is symmetric positive definite. As P_t is diagonalizable, with real positive eigenvalues, showing the convergence of P_t is equivalent to the following: $\forall x \in \mathbb{R}^d$, $x^* P_t^{-1} x \to +\infty$, or more simply for x in a basis of $\mathbb{R}^d$. Remark that it is also equivalent to consider complex vectors: take $x = x_1 + ix_2$ with x_1 and x_2 in $\mathbb{R}^d$, and write:

$$x^*\, P_t^{-1}\, x = (x_1^* - i x_2^*)\, P_t^{-1}\, (x_1 + i x_2) = x_1^*\, P_t^{-1}\, x_1 + x_2^*\, P_t^{-1}\, x_2 \tag{140}$$

So throughout the proof we will work in $\mathbf{C}^d$.

Consider the Jordan form of the matrix A, say J, and call T the corresponding basis transformation. Then $x^* P_t^{-1} x \to +\infty$, $\forall x \in \mathbf{C}^d$ iff $u_i^* P_t^{-1} u_i \to +\infty$, $\forall u_i$ in the Jordan basis.

We have two cases. First assume that u_i is in the stable subspace of A, i.e. associated with a submatrix J_k of the Jordan form and an eigenvalue λ_k with strictly negative real part. Then obviously:

$$u_i^*\, P_t^{-1}\, u_i \geq u_i^*\, e^{-tA^*}\, P_0^{-1}\, e^{-tA*}\, u_i = \left\| e^{-tA*}\, u_i \right\|^2_{P_0^{-1}} \to +\infty \tag{141}$$

Second case: u_i is in the unstable subspace of A, i.e. associated with a Jordan submatrix with pure imaginary eigenvalue. In this case we write:

$$u_i^*\, P_t^{-1}\, u_i \;\geq\; \int_0^t u_i^*\, e^{-(t-s)A^*}\, C^*\, C\, e^{-(t-s)A}\, u_i\, ds \tag{142}$$

$$= \int_0^t \left\| C\, e^{-(t-s)A}\, u_i \right\|^2\, ds \tag{143}$$

$$= \int_0^t \left\| C\, T\, e^{-(t-s)J}\, T^{-1}\, u_i \right\|^2\, ds \tag{144}$$

Let n_k be the dimension of the subblock corresponding to u_i, k_0 the rank of the first term of the subblock, and ℓ the rank of u_i in the corresponding basis of the eigenspace. J and

$e^{-(t-s)J}$ are block diagonal (see [7] or [3] for precisions on the Jordan form). Noting by J_k the corresponding subblock, we have:

$$e^{-(t-s)J_k} = e^{-\lambda_k(t-s)} \begin{bmatrix} 1 & \dots & \frac{-(t-s)^{n_k-1}}{(n_k-1)!} \\ & \ddots & \vdots \\ 0 & & 1 \end{bmatrix}. \tag{145}$$

Then observe that, m beeing the number of subblocks:

$$\left\| C\,T\,e^{-(t-s)J}\,(T^{-1}\,u_i) \right\|^2 = \left\| C\,T \begin{bmatrix} e^{-(t-s)J_1} & & & & \\ & \ddots & & 0 & \\ & & e^{-(t-s)J_k} & & \\ & 0 & & \ddots & \\ & & & & e^{-(t-s)J_m} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\|^2 \tag{146}$$

$$\geq \|C\,u_{k_0}\|^2 \left(\frac{(t-s)^{\ell-1}}{(\ell-1)!} \right)^2 - \sum_{q=1}^{l-1} \|C\,u_{k_0+q}\|^2 \left(\frac{(t-s)^{q-1}}{(q-1)!} \right)^2 \tag{147}$$

Remark that u_{k_0} is an eigenvector of A, so, by the dual version of the Hautus lemma [1] (see [12] p. 93 and 200), and the assumption that (A, C) is detectable, $u_{k_0} \notin \ker C$. Thus $\|Cu_{k_0}\| > 0$ and for t large enough, (144) is bigger that, say $K_i \int_0^t (t-s)^{2\ell-2}\,ds$ for some $K_i > 0$. Which gives the divergence to $+\infty$.

Now we prove the converse. We have again two cases. First assume that there is some eigenvalue of A, say λ_i, with $\Re\mathrm{e}\lambda_i > 0$. Let u_i an associated normalized eigenvectors. Then

$$u_i^*\,e^{-tA^*}\,P_0^{-1}\,e^{-tA}\,u_i \longrightarrow 0 \tag{148}$$

and

$$\int_0^t \left\| C\,e^{-(t-s)A}\,u_i \right\|^2\,ds \leq \|C\|^2 \int_0^t e^{-2(t-s)\Re\mathrm{e}\lambda_i}\,ds < +\infty \tag{149}$$

which proves that $P_t \not\to 0$.

Finally consider the second case: assume that all the eigenvalues of A have non-positive real parts but that the pair $(A,\,C)$ is not detectable. Then, using again the Hautus lemma, there exists an eigenvector u_i of A associated to some eigenvalue λ_i such that $u_i \in \ker C$ and $\Re\mathrm{e}\lambda_i = 0$. Then we have:

$$u_i^*\,e^{-tA^*}\,P_0^{-1}\,e^{-tA}\,u_i \leq \|u_i\|^2_{P_0^{-1}} \tag{150}$$

[1]Which tells us that (A, C) is observable if and only if $\begin{bmatrix} \lambda_i I - A \\ C \end{bmatrix}$ is full rank for all λ_i eigenvalue of A. Here we apply this to the unstable part of the system.

and

$$\int_0^t \left\| C\, e^{-(t-s)A} u_i \right\|^2 ds = 0, \tag{151}$$

which concludes the proof. □

4.2 Linear dynamics with non-linear observations

We will give here a class of non-trivial systems for which we observe the concentration of the conditional law on arbitrary small balls centered at the true position, but for which we cannot estimate the initial condition. More precisely, this section is mainly motivated by the following problem

$$\begin{cases} dX_t &= AX_t\, dt \\ dY_t &= h(X_t)\, dt + dB_t \end{cases} \tag{152}$$

where A is a $d \times d$ matrix (the other objects are the same as in (58)). We suppose that all the eigenvalues of A have non-positive real parts. In fact the question of the concentration of the conditional probability for exponentially growing dynamics is still open, even in dimension 1. More precisely, if the dynamics is given by $dX_t = a\, X_t\, dt$ with $a > 0$, and the observations by $dY_t = c\, X_t\, dt + dB_t$ with $c \neq 0$, and the initial condition is Gaussian, the asymptotic variance is given by $\frac{2a}{c^2}$, so we do not have concentration. Moreover, it can be shown that no non-linear observation $dY_t = h(X_t)\, dt + dB_t$ with h continuous, can fill assumptions (H1) and (H2). Even a numerical study of these cases is difficult because of the dynamical instability. So we restrict ourselves here to eigenvalues with non positive real parts, and let the problem with strictly positive ones to be solve by another method in the future.

Let $\bar{E}$ denote the stable subspace, that is the direct sum of the generalized eigenspaces whose eigenvalues have strictly negative real parts, and $\tilde{E}$ the direct sum for the eigenvalues with zero real parts. Then $\mathbb{R}^d = \bar{E} \oplus \tilde{E}$, so for all $x \in \mathbb{R}^d$ there is a unique decomposition $x = \bar{x} + \tilde{x}$, $\bar{x} \in \bar{E}$ and $\tilde{x} \in \tilde{E}$. Remark that these two subspaces are invariant by the flow Φ_t, and

$$\widetilde{\Phi_t(x)} = \Phi_t(\tilde{x}), \qquad \text{and} \qquad \overline{\Phi_t(x)} = \Phi_t(\bar{x}), \tag{153}$$

for all $x \in \mathbb{R}^d$ (as we will see below, only the first equality will be needed). This can be easily seen from the real canonical form of the matrix A: there exists a block diagonal matrix B, similar to A, each of its blocks corresponding to a real generalized eigenspace. This construction is done from the Jordan form by regrouping real and imaginary parts of complex eigenspaces with conjugate eigenvalues (as A is real, if $a + ib$ is an eigenvalue of A, that is a root of a polynomial with real coefficients, then $a - ib$ is also an eigenvalue of the same degree). See [3] pp126-136 for details. Thus we get a set of k independent differential equations, where k is the number of different real parts of the eigenvalues of A. Let $\tilde{d} = \dim \tilde{E}$ and $\bar{d} = \dim \bar{E}$ and assume that the system is written in a basis such that the $\tilde{d}$ first vectors form a basis of $\tilde{E}$, and the last $\bar{d}$ ones form a basis of $\bar{E}$.

Despite the title of this section, we can in fact consider the following non-linear system (which generalizes the linear case):

$$\begin{cases} dX_t &= b(X_t)\,dt \\ dY_t &= h(X_t)\,dt + dB_t \end{cases} \tag{154}$$

with the particular form for f :

$$b(x) = \begin{pmatrix} \tilde{b}(\tilde{x}) \\ \bar{b}(\tilde{x},\bar{x}) \end{pmatrix} \tag{155}$$

and $\bar{b}(\,.\,,0) = 0$. Then obviously $\tilde{E}$ is invariant by Φ and:

$$\widetilde{\Phi_t(x)} = \Phi_t(\tilde{x}) \tag{156}$$

As all the norms on $\mathbb{R}^d$ are equivalent, we will consider concentration of the conditional law on balls for the norm:

$$\|x\|_\oplus = \|\bar{x}\| \vee \|\tilde{x}\| \tag{157}$$

Assume:

($\tilde{\mathrm{H}}$1) There are two functions v and V from $\mathbb{R}_+$ into $\mathbb{R}_+$ and a positive constant C such that:

$$\forall(\tilde{x}_1,\tilde{x}_2) \in \tilde{E}^2 \quad v(t)\,\|\tilde{x}_1 - \tilde{x}_2\|^2 \le \int_0^t \|h(\Phi_t(\tilde{x}_1) - h(\Phi_t(\tilde{x}_2)\|^2\,ds \le V(t)\,\|\tilde{x}_1 - \tilde{x}_2\|^2$$

and

$$1 \le \frac{V(t)}{v(t)} \le C$$

To show that the above property is sufficient to extend the concentration of the conditional law on the whole space, we will need some more properties on the observation function h. Assume:

($\tilde{\mathrm{H}}$2) For $j = \tilde{d}+1,\dots,d$ and $i = 1,\dots,m$ the partial derivatives $\frac{\partial h_i}{\partial x_j}$ are bounded, and for $j = \tilde{d}+1,\dots,d$, $k = 1,\dots,\tilde{d}$ and $i = 1,\dots,m$ the real functions

$$\nu \longmapsto \frac{\partial^2 h_i}{\partial x_k \partial x_j}(x_1,\dots,x_{j-1},\nu,x_{j+1},\dots,x_d)$$

are in $L^1(\mathbb{R})$, and have their L^1-norms uniformly bounded w.r.t. the other variables of h.

Proposition 4.4 *Assume that ($\tilde{\mathrm{H}}$2) is fulfilled. Then there exists a constant $K > 0$ such that*

$$\forall(x_1,x_2) \in \mathbb{R}^d \times \mathbb{R}^d \quad \|h(x_1) - h(x_2)\| \le K\left(\|\tilde{x}_1 - \tilde{x}_2\| + \|\bar{x}_1 - \bar{x}_2\| + \|h(\tilde{x}_1) - h(\tilde{x}_2)\|\right) \tag{158}$$

Proof We note Dh the differential of the function h. For all $(x_1,x_2)\in\mathbb{R}^d\times\mathbb{R}^d$, for all $i=1,\ldots,d$ we have:

$$h_i(x_2)-h_i(x_1) = \int_0^1 Dh_i(x_1+\lambda(x_2-x_1))\,d\lambda \tag{159}$$

$$= \int_0^1 \sum_{j=1}^{d} \frac{\partial h_i}{\partial x_j}(x_1+\lambda(x_2-x_1))\,(x_2-x_1)_j\;d\lambda \tag{160}$$

$$= \sum_{j=1}^{\tilde d} \int_0^1 \frac{\partial h_i}{\partial x_j}(x_1+\lambda(x_2-x_1))\,(x_2-x_1)_j\;d\lambda + \sum_{j=\tilde d+1}^{d} \int_0^1 \frac{\partial h_i}{\partial x_j}(x_1+\lambda(x_2-x_1))\,(x_2-x_1)_j\;d\lambda \tag{161}$$

Consider the first sum in the last expression:

$$\begin{aligned}&\sum_{j=1}^{\tilde d} \int_0^1 \frac{\partial h_i}{\partial x_j}(x_1+\lambda(x_2-x_1))\,(x_2-x_1)_j\;d\lambda\\ &= \sum_{j=1}^{\tilde d}\int_0^1 \left(\frac{\partial h_i}{\partial x_j}(x_1+\lambda(x_2-x_1)) - \frac{\partial h_i}{\partial x_j}(\tilde x_1+\lambda(\tilde x_2-\tilde x_1))\right)\;(x_2-x_1)_j\;d\lambda\\ &+\sum_{j=1}^{\tilde d}\int_0^1 \frac{\partial h_i}{\partial x_j}(\tilde x_1+\lambda(\tilde x_2-\tilde x_1))\;(x_2-x_1)_j\;d\lambda\end{aligned} \tag{162}$$

Remark that for all $x\in\mathbb{R}^d$:

$$\frac{\partial h_i}{\partial x_j}(x)-\frac{\partial h_i}{\partial x_j}(\tilde x) = \sum_{k=1}^{\bar d}\frac{\partial h_i}{\partial x_j}(x_1,\ldots,x_{d-k+1},0,\ldots,0)-\frac{\partial h_i}{\partial x_j}(x_1,\ldots,x_{d-k},0,\ldots,0) \tag{163}$$

and for each term we have:

$$\left|\frac{\partial h_i}{\partial x_j}(x_1,\ldots,x_{d-k+1},0,\ldots,0)-\frac{\partial h_i}{\partial x_j}(x_1,\ldots,x_{d-k},0,\ldots,0)\right|$$

$$= \left|\int_0^{x_{d-k+1}} \frac{\partial^2}{\partial x_{d-k+1}\partial x_j}(x_1,\ldots,x_{d-k},\lambda,0,\ldots,0)\;d\lambda\right| \tag{164}$$

$$\le \left\|\frac{\partial^2}{\partial x_{d-k+1}\partial x_j}(x_1,\ldots,x_{d-k},\bullet,0,\ldots,0)\right\|_{L^1(\mathbb{R})} \tag{165}$$

$$\le K_1 \tag{166}$$

from the second part of ($\tilde{\mathrm{H}}2$), for some $K_1>0$. Then we get

$$\left\|\frac{\partial h_i}{\partial x_j}(x)-\frac{\partial h_i}{\partial x_j}(\tilde x)\right\|\le K_2 \tag{167}$$

for some $K_2 > 0$. Using this we obtain for the first term in (162):

$$\sum_{j=1}^{\tilde{d}} \int_0^1 \left(\frac{\partial h_i}{\partial x_j}(x_1 + \lambda(x_2 - x_1)) - \frac{\partial h_i}{\partial x_j}(\tilde{x}_1 + \lambda(\tilde{x}_2 - \tilde{x}_1)) \right) \, . \, (x_2 - x_1)_j \; d\lambda \tag{168}$$

$$\leq K_3 \, \|\tilde{x}_1 - \tilde{x}_2\| \tag{169}$$

for some $K_3 > 0$. On the other hand, the second term in (162) becomes:

$$\sum_{j=1}^{\tilde{d}} \int_0^1 \frac{\partial h_i}{\partial x_j}(\tilde{x}_1 + \lambda(\tilde{x}_2 - \tilde{x}_1)) \; (x_2 - x_1)_j \; d\lambda = h_i(\tilde{x}_2) - h_i(\tilde{x}_1) \tag{170}$$

Now we have for the second term in (162):

$$\left| \sum_{j=\tilde{d}+1}^{d} \int_0^1 \frac{\partial h_i}{\partial x_j}(x_1 + \lambda(x_2 - x_1)) \, (x_2 - x_1)_j \; d\lambda \right| \leq K_4 \|\bar{x}_1 - \bar{x}_2\| \tag{171}$$

for some $K_4 > 0$, from the first part of ($\tilde{\mathrm{H}}2$). And finally we get for some $K > 0$ the estimate:

$$\|h(x_1) - h(x_2)\| \leq K \left(\|\tilde{x}_1 - \tilde{x}_2\| + \|\bar{x}_1 - \bar{x}_2\| + \|h(\tilde{x}_1) - h(\tilde{x}_2)\| \right) \tag{172}$$

□

By a very similar argument we get the

Proposition 4.5 *Assume that ($\tilde{\mathrm{H}}2$) is fulfilled. Then there exists a constant $K > 0$ such that $\forall (x_1, x_2) \in \mathbb{R}^d \times \mathbb{R}^d$*

$$\|h(x_1) - h(x_2)\| \geq K \left(-\|\tilde{x}_1 - \tilde{x}_2\| - \|\bar{x}_1 - \bar{x}_2\| + \|h(\tilde{x}_1) - h(\tilde{x}_2)\| \right) \tag{173}$$

We now introduce the next assumption:

($\tilde{\mathrm{H}}3$) There exist two functions α and δ from $\mathbb{R}_+$ into $\mathbb{R}_+$ such that $\delta \in L^2(\mathbb{R}_+)$, α is non-decreasing and for all x_1, x_2 in $\mathbb{R}^d$ and for t large enough:

$$\|\Phi_t(\tilde{x}_1) - \Phi_t(\tilde{x}_2)\| \leq \alpha(t) \, \|\tilde{x}_1 - \tilde{x}_2\| \tag{174}$$

$$\|\overline{\Phi_t(x_1)} - \overline{\Phi_t(x_2)}\| \leq \delta(t) \, \|\bar{x}_1 - \bar{x}_2\|, \tag{175}$$

and as $t \to +\infty$:

$$\frac{V(t)}{\alpha(t)^2} \longrightarrow +\infty \tag{176}$$

$$\frac{\sqrt{V(t)}\,\delta(t)}{\alpha(t)} \longrightarrow 0 \tag{177}$$

$$\frac{V(t)}{A(t)} \longrightarrow +\infty \tag{178}$$

where $A(t) = \int_0^t \alpha(s)^2\, ds$.

We also consider the following notation:

$$\Delta(t) = \int_0^t \delta(s)^2\, ds \tag{179}$$

Remark 4.6 Consider the linear case. Let $\lambda_- = \max\{\mathrm{Re}(\lambda) < 0, \quad \lambda \text{ eigenvalue of } A\}$, then typically δ should be of the form $\delta(t) = K_\varepsilon \exp((\lambda_- + \varepsilon)t)$ with any $0 < \varepsilon < -\lambda_-$ and some $K_\varepsilon > 0$ depending on ε. Then δ^2 is clearly integrable. And the best choice for α should be a polynomial in t (eventually of degre zero) with positive coefficients (to see this use the Jordan form of A, see [7] theorem 1.10 for example, and recall that on $\tilde{E}$ all the eigenvalues of A have zero real parts). Then clearly (178) imply (176). Remark finally that (176) and the fact that the flow does not contract on $\tilde{E}$ imply that $V(t) \to +\infty$.

Remark 4.7 In all the cases, $\tilde{x}$ can be viewed as an asymptotic phase for x. This suggests to extend the following results using this notion in a more general way. In particular, when the asymptotic phases form a submanifold of $\mathbb{R}^d$, eventually with some "good" properties.

Applying the previous propositions to the observability condition ($\tilde{\mathrm{H}}1$) we obtain:

Proposition 4.8 *Assume that ($\tilde{\mathrm{H}}1$), ($\tilde{\mathrm{H}}2$) and ($\tilde{\mathrm{H}}3$) are fulfilled. Then there exists two constants $K_0 > 0$, $K_1 > 0$ and $K_2 > 0$ such that for all x_1 and x_2 in $\mathbb{R}^d$*

$$\begin{aligned} K_0 V(t)\, \|\tilde{x}_1 - \tilde{x}_2\|^2 - K_1 \|\bar{x}_1 - \bar{x}_2\|^2 &\le \int_0^t \|h(\Phi_s(x_1)) - h(\Phi_s(x_2))\|^2\, ds \\ &\le K_2 (V(t)\, \|\tilde{x}_1 - \tilde{x}_2\|^2 + \|\bar{x}_1 - \bar{x}_2\|^2) \end{aligned} \tag{180}$$

for t large enough.

Proof By proposition 4.4, (156) and (176)

$$\begin{aligned} &\int_0^t \|h(\Phi_s(x_1)) - h(\Phi_s(x_2))\|^2\, ds \\ &\le 4\, K^2 \left(\int_0^t \|h(\Phi_s(\tilde{x}_1)) - h(\Phi_s(\tilde{x}_2))\|^2\, ds + \int_0^t \|\Phi_s(\tilde{x}_1) - \Phi_s(\tilde{x}_2)\|^2\, ds \right. \end{aligned}$$

$$+ \int_0^t \|\overline{\Phi_s(x_1)} - \overline{\Phi_s(x_2)}\|^2 \, ds \Big) \tag{181}$$

$$\leq 4\, K^2 \, \Big((V(t) + A(t)) \, \|\tilde{x}_1 - \tilde{x}_2\|^2 + \Delta(t)^2 \, \|\bar{x}_1 - \bar{x}_2\|^2 \Big) \tag{182}$$

$$\leq 4\, K^2 \, \Big(2V(t) \, \|\tilde{x}_1 - \tilde{x}_2\|^2 + \|\delta\|_{L^2}^2 \, \|\bar{x}_1 - \bar{x}_2\|^2 \Big) \tag{183}$$

The other estimate follow by a similar use of proposition 4.5. □

In fact we also need a more precise estimate:

Proposition 4.9 *Assume that* ($\tilde{\text{H}}1$), ($\tilde{\text{H}}2$) *and* ($\tilde{\text{H}}3$) *are fulfilled. Then for all* $\varepsilon > 0$ *there exist* $K_\varepsilon > 0$ *and* $T_\varepsilon > 0$ *such that for all* $t > T_\varepsilon > 0$, *and all* x_1 *and* x_2 *in* $\mathbb{R}^d$

$$\int_0^t \|h(\Phi_s(x_1)) - h(\Phi_s(x_2))\|^2 \, ds \geq K_\varepsilon V(t) \, \|\tilde{x}_1 - \tilde{x}_2\|^2 - \varepsilon \, \|\bar{x}_1 - \bar{x}_2\|^2 \tag{184}$$

Proof By proposition 4.5 and (156) we have

$$K^2 \, \|h(\Phi_s(\tilde{x}_1)) - h(\Phi_s(\tilde{x}_2))\|^2 \, \leq \, 4(\|h(\Phi_s(x_1)) - h(\Phi_s(x_2))\|^2 + K^2(\|\Phi_s(\tilde{x}_1) - \Phi_s(\tilde{x}_1)\|^2$$

$$+ \|\overline{\Phi_s(x_1)} - \overline{\Phi_s(x_1)}\|^2)) \tag{185}$$

that is

$$\|h(\Phi_s(x_1)) - h(\Phi_s(x_2))\|^2 \, \geq \, \frac{K^2}{4} \|h(\Phi_s(\tilde{x}_1)) - h(\Phi_s(\tilde{x}_2))\|^2 - K^2(\|\Phi_s(\tilde{x}_1) - \Phi_s(\tilde{x}_1)\|^2$$

$$+ \|\overline{\Phi_s(x_1)} - \overline{\Phi_s(x_1)}\|^2) \tag{186}$$

Then for any $0 < t_0 < t$

$$\int_0^t \|h(\Phi_s(x_1)) - h(\Phi_s(x_2))\|^2 \, ds$$

$$\geq \int_{t_0}^t \|h(\Phi_s(x_1)) - h(\Phi_s(x_2))\|^2 \, ds \tag{187}$$

$$\geq \frac{K^2}{4} \int_{t_0}^t \|h(\Phi_s(\tilde{x}_1)) - h(\Phi_s(\tilde{x}_2))\|^2 \, ds$$

$$-K^2 \left(\int_{t_0}^t \|\Phi_s(\tilde{x}_1) - \Phi_s(\tilde{x}_1)\|^2 \, ds + \int_{t_0}^t \|\overline{\Phi_s(x_1)} - \overline{\Phi_s(x_2)}\|^2 \, ds \right) \tag{188}$$

$$\geq \left(\frac{K^2}{4} \, (v(t) - V(t_0)) - K^2 \, A(t) \right) \|\tilde{x}_1 - \tilde{x}_2\|^2 - K^2 \int_{t_0}^t \delta(s)^2 \, ds \, . \, \|\bar{x}_1 - \bar{x}_2\|^2 \tag{189}$$

$$\geq K_\varepsilon \, V(t) \, \|\tilde{x}_1 - \tilde{x}_2\|^2 - \varepsilon \, \|\bar{x}_1 - \bar{x}_2\|^2 \tag{190}$$

for t_0 and t large enough, using (178) and the integrability of δ^2. □

We now have the tools for estimating the stochastic integral which appears in the conditional density. Recall that μ_t being the conditional density:

$$\int_{\{\|x-X_t\|\leq a\}} \mu_t(x)\,dx = \frac{\int_{\Phi_t^{-1}(\{\|x-X_t\|\leq a\})} f_t(x)\,dx}{\int_{\mathbb{R}^d} f_t(x)\,dx}. \tag{191}$$

where

$$f_t(x) = \exp\left[-\frac{1}{2}\int_0^t \|h(\Phi_s(x)) - h(\Phi_s(X_0))\|^2\,ds + \int_0^t (h(\Phi_s(x)) - h(\Phi_s(X_0)))\,dB_s\right] p_0(x). \tag{192}$$

Proposition 4.10 *Assume that ($\tilde{\mathrm{H}}1$), ($\tilde{\mathrm{H}}2$) and ($\tilde{\mathrm{H}}3$) are fulfilled. Then there exists a constant $K>0$ such that for t large enough, for all $\alpha_0>0$, all $\delta_0>0$ and all $x_0\in\mathbb{R}^d$ the following estimate holds:*

$$E\left[\sup_{x\in D(\alpha_0,\delta_0,x_0)}\left|\int_0^t (h(\Phi_s(x)) - h(\Phi_s(x_0)))\,dB_s\right|\right] \leq K\,\left[\alpha_0\sqrt{V(t)}+\delta_0\right] \tag{193}$$

where

$$D(\alpha_0,\delta_0,x_0) = \left\{x\in\mathbb{R}^d \,/\, \|\tilde{x}-\tilde{x}_0\|\leq\alpha_0 \text{ and } \|\bar{x}-\bar{x}_0\|\leq\delta_0\right\} \tag{194}$$

Proof We use the same method and notations as in proposition 2.1. First we estimate the pseudo-metric given by the stochastic integral. In this case, by proposition 4.8

$$d_{U_t}(x_1,x_2) = \left(\int_0^t \|h(\Phi_s(x_1)) - h(\Phi_s(x_2))\|^2\,ds\right)^{\frac{1}{2}} \tag{195}$$

$$\leq K_1\left(\sqrt{V(t)}\,\|\tilde{x}_1-\tilde{x}_2\|_\infty \vee \|\bar{x}_1-\bar{x}_2\|_\infty\right) \tag{196}$$

for some $K_1>0$ and large t. Then it follows that:

$$N\left(D(\alpha_0,\delta_0),d_{U_t};\zeta\right) \leq N\left(D(\alpha_0,\delta_0),K_1\left(\sqrt{V(t)}\,\|\tilde{\cdot}\|_\infty \vee \|\bar{\cdot}\|_\infty\right)\right) \tag{197}$$

$$\leq \left[\left[\frac{2K_1\,\alpha_0\sqrt{V(t)}}{\zeta}\right]+1\right]^{\tilde{d}} \cdot \left[\left[\frac{2K_1\,\delta_0}{\zeta}\right]+1\right]^{\bar{d}} \tag{198}$$

Using this we get:

$$\int_0^{+\infty} \left(\log N\left(D(\alpha_0,\delta_0),d_{U_t};\zeta\right)\right)^{\frac{1}{2}}\,d\zeta$$

$$\leq \int_0^{+\infty} \left(\log N\left(D(\alpha_0,\delta_0), K_1\left(\sqrt{V(t)}\,\|\tilde{\ }\|_\infty \vee \|\bar{\ }\|_\infty\right); \zeta\right)\right)^{\frac{1}{2}} d\zeta \tag{199}$$

$$\leq \int_0^{2K_1\,\alpha_0\,\sqrt{V(t)}} \tilde{d}^{\frac{1}{2}} \log\left(\frac{2K_1\,\alpha_0\,\sqrt{V(t)}}{\zeta}+1\right)^{\frac{1}{2}} d\zeta$$

$$+\int_0^{2K_1\,\delta_0} \bar{d}^{\frac{1}{2}} \log\left(\frac{2K_1\,\delta_0}{\zeta}+1\right)^{\frac{1}{2}} d\zeta \tag{200}$$

$$\leq \tilde{d}^{\frac{1}{2}}\, 2K_1\,\alpha_0\,\sqrt{V(t)} \int_0^1 \log\left(1+\frac{1}{\zeta}\right)^{\frac{1}{2}} d\zeta + \bar{d}^{\frac{1}{2}}\, 2K_1\,\delta_0 \int_0^1 \log\left(1+\frac{1}{\zeta}\right)^{\frac{1}{2}} d\zeta \tag{201}$$

$$\leq \left[2\,d^{\frac{1}{2}}\,K_1 \int_0^1 \log\left(1+\frac{1}{\zeta}\right)^{\frac{1}{2}} d\zeta\right] . \left(\alpha_0\,\sqrt{V(t)}+\delta_0\right) \tag{202}$$

which proves the proposition. □

By independence of X_0 and B we have immediately the

Corollary 4.11 *Assume that ($\tilde{\mathrm{H}}1$), ($\tilde{\mathrm{H}}2$) and ($\tilde{\mathrm{H}}3$) are fulfilled. Then there exists a constant $K > 0$ such that for t large enough, for all $\alpha_0 > 0$ and all $\delta_0 > 0$ the following estimate holds:*

$$E\left[\sup_{x\in D(\alpha_0,\delta_0,X_0)} \left|\int_0^t (h(\Phi_s(x)) - h(\Phi_s(x_0)))\,dB_s\right|\right] \leq K\,\left[\alpha_0\sqrt{V(t)}+\delta_0\right] \tag{203}$$

Next we introduce the last assumption, which concerns the initial density:

($\tilde{\mathrm{H}}4$) p_0 is continuous and there are four strictly positive constants C_i, $i = 1\dots 4$ such that for all $x \in \mathbb{R}^d$

$$p_0(\tilde{x})\,C_1\,\exp(-C_2\|\bar{x}\|^2) \leq p_0(x) \leq p_0(\tilde{x})\,C_3\,\exp(-C_4\|\bar{x}\|^2) \tag{204}$$

We consider the following notations:

$$\begin{aligned} D_0(t,\alpha_0,\delta_0) &= D\left(\frac{\alpha_0}{\alpha(t)}, \frac{\delta_0}{\delta(t)}, X_0\right) \\ &= \left\{x \in \mathbb{R}^d \,/\, \|\tilde{x}-\tilde{X}_0\| \leq \frac{\alpha_0}{\alpha(t)} \text{ and } \|\bar{x}-\bar{X}_0\| \leq \frac{\delta_0}{\delta(t)}\right\} \end{aligned} \tag{205}$$

$$D_1(t,\alpha_0,\delta_0) = \left\{ x \in \mathbb{R}^d \,/\, \|\tilde{x}-\tilde{X}_0\| \le \frac{\alpha_0}{\alpha(t)} \text{ and } \|\bar{x}-\bar{X}_0\| \ge \frac{\delta_0}{\delta(t)} \right\} \tag{206}$$

$$D_2(t,\alpha_0) = \left\{ x \in \mathbb{R}^d \,/\, \|\tilde{x}-\tilde{X}_0\| \ge \frac{\alpha_0}{\alpha(t)} \text{ and } \|\bar{x}-\bar{X}_0\| \le 1 \right\} \tag{207}$$

$$D_3(t,\alpha_0) = \left\{ x \in \mathbb{R}^d \,/\, \|\tilde{x}-\tilde{X}_0\| \ge \frac{\alpha_0}{\alpha(t)} \text{ and } \|\bar{x}-\bar{X}_0\| \ge 1 \right\} \tag{208}$$

for any given $t \ge 0$, $\alpha_0 > 0$ and $\delta_0 > 0$. In the sequel we shall omit the dependence of these domains on the parameters when there is no possible confusion. Note also that $D_0 \cup D_1 \cup D_2 \cup D_3 = \mathbb{R}^d$ and $D_1 \cup D_2 \cup D_3 = D_0^c$. Now we are going to show some lemmas and propositions concerning the concentration of the conditional density on D_0. Until the end of this section, we will assume that ($\tilde{\mathrm{H}}1$), ($\tilde{\mathrm{H}}2$), ($\tilde{\mathrm{H}}3$), and ($\tilde{\mathrm{H}}4$) are fulfilled.

Lemma 4.12 *The following convergence holds:*

$$\left[\frac{\alpha_0^2\, V(t)}{\alpha(t)^2} + \frac{\delta_0^2}{\delta(t)^2}\right]^{-1} \sup_{x\in D_0} \left| \int_0^t (h(\Phi_s(x)) - h(\Phi_s(X_0)))\, dB_s \right| \xrightarrow{P} 0 \tag{209}$$

as $t \to +\infty$ for any $\alpha_0 > 0$ and $\delta_0 > 0$.

Proof By corollary 4.11

$$\left[\frac{\alpha_0^2\, V(t)}{\alpha(t)^2} + \frac{\delta_0^2}{\delta(t)^2}\right]^{-1} E\left[\sup_{x\in D_0} \left| \int_0^t (h(\Phi_s(x)) - h(\Phi_s(X_0)))\, dB_s \right| \right]$$

$$\le K \left[\frac{\alpha_0^2\, V(t)}{\alpha(t)^2}\right]^{-1} \frac{\alpha_0 \sqrt{V(t)}}{\alpha(t)} + K \left[\frac{\delta_0^2}{\delta(t)^2}\right]^{-1} \frac{\delta_0}{\delta(t)} \tag{210}$$

$$= K \left[\frac{\alpha(t)}{\alpha_0 \sqrt{V(t)}} + \frac{\delta(t)}{\delta_0} \right] \longrightarrow 0 \quad \text{as} \quad t \longrightarrow +\infty \tag{211}$$

using (176) and $\delta(t) \to 0$ by integrability of δ^2 in ($\tilde{\mathrm{H}}3$). □

Lemma 4.13

$$E\left[\sup_{x\in D_1} \frac{\left| \int_0^t (h(\Phi_s(x)) - h(\Phi_s(X_0)))\, dB_s \right|}{\|\bar{x}-\bar{X}_0\|^2} \right] \le \frac{K_1 \delta(t)}{\delta_0} + \frac{K_2\, \delta(t)^2\, \alpha_0 \sqrt{V(t)}}{\delta_0^2\, \alpha(t)} \tag{212}$$

for some constants $K_1 > 0$ and $K_2 > 0$, and t large enough.

Proof Let n_0 be the greatest relative integer such that $2^{n_0} \leq \frac{\delta_0}{\delta(t)}$. Then using proposition 4.10, we get:

$$E\left[\sup_{x\in D_1} \frac{|\int_0^t (h(\Phi_s(x)) - h(\Phi_s(X_0)))\, dB_s|}{\|\bar{x}-\bar{X}_0\|^2}\right]$$

$$\leq \sum_{n\geq n_0} \frac{1}{2^{2n}}\, E\left[\sup_{\substack{\|\tilde{x}-\tilde{X}_0\| \leq \frac{\alpha_0}{\alpha(t)} \\ 2^n \leq \|\bar{x}-\bar{X}_0\| \leq 2^{n+1}}} \left|\int_0^t (h(\Phi_s(x)) - h(\Phi_s(X_0)))\, dB_s\right|\right] \tag{213}$$

$$\leq K\left[\sum_{n\geq n_0} \frac{\alpha_0\sqrt{V(t)}}{\alpha(t)\, 2^{2n}} + \sum_{n\geq n_0} \frac{2^{n+1}}{2^{2n}}\right] \tag{214}$$

$$\leq K\left[2^{-2n_0}\sum_{p\geq 0} \frac{\alpha_0\sqrt{V(t)}}{\alpha(t) 2^{2p}} + 2^{-n_0}\sum_{p\geq 0}\frac{1}{2^{p-1}}\right] \tag{215}$$

$$\leq K\left[\frac{4\delta(t)^2}{\delta_0^2}\,\frac{\alpha_0\sqrt{V(t)}}{\alpha(t)}\sum_{p\geq 0}\frac{1}{2^{2p}} + \frac{2\delta(t)}{\delta_0}\sum_{p\geq 0}\frac{1}{2^{p-1}}\right] \tag{216}$$

for t large enough, which proves the lemma. □

Using (174) and the fact that $\delta(t) \to 0$ (because δ^2 is integrable) we immediately get the

Lemma 4.14

$$\sup_{x\in D_1} \frac{|\int_0^t (h(\Phi_s(x)) - h(\Phi_s(X_0)))\, dB_s|}{\|\bar{x}-\bar{X}_0\|^2} \xrightarrow{P} 0 \qquad as \qquad t\to+\infty \tag{217}$$

.

Now we can compare the conditional density over D_0 and over D_1.

Proposition 4.15 *The following convergence holds:*

$$\frac{\int_{D_1} f_t(x)\, dx}{\int_{D_0} f_t(x)\, dx} \xrightarrow{P} 0 \quad as \quad t\to+\infty \tag{218}$$

Proof Let $t_n \geq 0$ such that $\lim_{n\to+\infty} t_n = +\infty$ and for which the convergences (209) and (217) hold a.s. Then for n larger than some random N, using also proposition 4.8, for any $0 < \delta_1 \leq \delta_0$:

$$\int_{D_0} f_{t_n}(x)\,dx$$

$$\geq \int_{D_0(t_n,\alpha_0,\delta_1)} \exp\left[-\frac{1}{2}\,K_2(V(t_n)\,\|\tilde{x}-\tilde{X}_0\|^2 + \|\bar{x}-\bar{X}_0\|^2)\right.$$

$$\left. + \int_0^{t_n} (h(\Phi_s(x)) - h(\Phi_s(X_0)))\,dB_s\right] p_0(x)\,dx \tag{219}$$

$$\geq \int_{D_0(t_n,\alpha_0,\delta_1)} \exp\left[-\frac{1}{2}\,K_2\left(V(t_n)\,\frac{\alpha_0^2}{\alpha(t_n)^2} + \frac{\delta_1^2}{\delta(t_n)^2}\right)\right.$$

$$\left. + \int_0^{t_n} (h(\Phi_s(x)) - h(\Phi_s(X_0)))\,dB_s\right] p_0(x)\,dx \tag{220}$$

$$\geq \int_{D_0(t_n,\alpha_0,\delta_1)} \exp\left[-K_2\left(V(t_n)\,\frac{\alpha_0^2}{\alpha(t_n)^2} + \frac{\delta_1^2}{\delta(t_n)^2}\right)\right] p_0(x)\,dx \tag{221}$$

By ($\tilde{\mathrm{H}}4$) we get:

$$\int_{D_0} f_{t_n}(x)\,dx$$

$$\geq \exp\left[-K_2\left(V(t_n)\,\frac{\alpha_0^2}{\alpha(t_n)^2} + \frac{\delta_1^2}{\delta(t_n)^2}\right)\right].$$

$$\int_{\left\{\|\tilde{x}-\tilde{X}_0\|\leq\frac{\alpha_0}{\alpha(t_n)}\right\}} p_0(\tilde{x})\,d\tilde{x}\,.\int_{\left\{\|\bar{x}-\bar{X}_0\|\leq\frac{\delta_1}{\delta(t_n)}\right\}} C_1\exp\left[-C_2\|\bar{x}\|^2\right]\,d\bar{x} \tag{222}$$

As $\delta(t_n)\to 0$ the second integral converges to the same integral on the whole space $\bar{E}$, so it is bounded away from 0, say by some constant K, and we have:

$$\int_{D_0} f_{t_n}(x)\,dx \geq K\exp\left[-K_2\left(V(t_n)\,\frac{\alpha_0^2}{\alpha(t_n)^2} + \frac{\delta_1^2}{\delta(t_n)^2}\right)\right]\cdot\int_{\left\{\|\tilde{x}-\tilde{X}_0\|\leq\frac{\alpha_0}{\alpha(t_n)}\right\}} p_0(\tilde{x})\,d\tilde{x} \tag{223}$$

Furthermore, using (177) we have, for sufficiently large n:

$$\int_{D_0} f_{t_n}(x)\,dx \geq K\exp\left[-2K_2\frac{\delta_1^2}{\delta(t_n)^2}\right]\cdot\int_{\left\{\|\tilde{x}-\tilde{X}_0\|\leq\frac{\alpha_0}{\alpha(t_n)}\right\}} p_0(\tilde{x})\,d\tilde{x} \tag{224}$$

On the other hand, for any $\varepsilon > 0$:

$$\int_{D_1} f_{t_n}(x)\,dx$$

$$\leq \int_{D_1} \exp\left[\int_0^{t_n} (h(\Phi_s(x)) - h(\Phi_s(X_0)))\,dB_s\right] p_0(x)\,dx \tag{225}$$

$$\leq \int_{D_1} \exp\left[\sup_{x'\in D_1} \frac{|\int_0^t (h(\Phi_s(x')) - h(\Phi_s(X_0)))\,dB_s|}{\|\bar{x'} - \bar{X}_0\|^2}\,\|\bar{x} - \bar{X}_0\|^2\right]\,p_0(x)\,dx \tag{226}$$

$$\leq \int_{D_1} \exp\left[\varepsilon\|\bar{x} - \bar{X}_0\|^2\right]\,p_0(x)\,dx \tag{227}$$

for n larger than some random integer, by (217). Using again ($\tilde{\mathrm{H}}4$) we get:

$$\int_{D_1} f_{t_n}(x)\,dx \;\leq\; \int_{\left\{\|\tilde{x}-\tilde{X}_0\|\leq\frac{\alpha_0}{\alpha(t_n)}\right\}} p_0(\tilde{x})\,d\tilde{x}\,.$$

$$\int_{\left\{\|\bar{x}-\bar{X}_0\|\geq\frac{\delta_0}{\delta(t_n)}\right\}} C_3 \exp\left[-C_4\|\bar{x}\|^2 + \varepsilon\|\bar{x} - \bar{X}_0\|^2\right]\,d\bar{x} \tag{228}$$

Next we deal with the last integral in the previous inequality. Assume that we have chosen $\varepsilon < \frac{1}{4}C_4$, then:

$$\int_{\left\{\|\bar{x}-\bar{X}_0\|\geq\frac{\delta_0}{\delta(t_n)}\right\}} C_3 \exp\left[-C_4\|\bar{x}\|^2 + \varepsilon\|\bar{x} - \bar{X}_0\|^2\right]\,d\bar{x}$$

$$\leq\; C_3\,\exp\left[C_4\|\bar{X}_0\|^2\right]\,\int_{\left\{\|\bar{x}-\bar{X}_0\|\geq\frac{\delta_0}{\delta(t_n)}\right\}} C_3 \exp\left[-\frac{1}{4}C_4\|\bar{x} - \bar{X}_0\|^2\right]\,d\bar{x} \tag{229}$$

We denote by $S_{\bar{d}}$ the surface of the unit $\bar{d}$-sphere. Then using polar coordinates and lemma 3.2 we get:

$$\int_{\left\{\|\bar{x}-\bar{X}_0\|\geq\frac{\delta_0}{\delta(t_n)}\right\}} C_3 \exp\left[-\frac{1}{4}C_4\|\bar{x} - \bar{X}_0\|^2\right]\,d\bar{x}$$

$$=\; C_3\,S_{\bar{d}} \int_{\frac{\delta_0}{\delta(t_n)}}^{+\infty} \exp\left[-\frac{1}{4}C_4\,r^2\right]\,r^{\bar{d}-1}\,dr \tag{230}$$

$$\leq\; \frac{4C_3\,S_{\bar{d}}\,\Gamma_{\bar{d}}\,\delta(t_n)^{2-\bar{d}}}{C_4\,\delta_0^{2-\bar{d}}}\;\exp\left[-\frac{1}{4}C_4\,\frac{\delta_0^2}{\delta(t_n)^2}\right] \tag{231}$$

Finally by (224) and (228) we obtain:

$$\frac{\int_{D_1} f_{t_n}(x)\,dx}{\int_{D_0} f_{t_n}(x)\,dx} \;\leq\; C_3\,\exp\left[C_4\|\bar{X}_0\|^2\right]\,\frac{4C_3\,S_{\bar{d}}\,\Gamma_d\,\delta(t_n)^{2-\bar{d}}}{K\,C_4\,\delta_0^{2-\bar{d}}}$$

$$\exp\left[-\left(\frac{1}{4}C_4\,\delta_0^2 - 2\,K_2\,\delta_1^2\right)\,\frac{1}{\delta(t_n)^2}\right] \tag{232}$$

which tends to 0 provided we have chosen δ_1 such that $0 < \delta_1 < \delta_0\,\left(1 \wedge \sqrt{\frac{C_4}{4\,K_2}}\right)$.

Again we conclude the proof by using the characterization of the convergence in probability. □

Now we consider the domain D_2.

Lemma 4.16 *For all $\alpha_0 > 0$ and $\delta_0 > 0$ the following convergence holds:*

$$\sup_{x\in D_2} \frac{|\int_0^t (h(\Phi_s(x)) - h(\Phi_s(X_0)))\, dB_s|}{V(t)\, \|\tilde{x} - \tilde{X}_0\|^2} \xrightarrow{P} 0 \qquad as \qquad t \to +\infty. \tag{233}$$

Proof Let n_0 be the greatest relative integer such that $2^{n_0} \leq \frac{\alpha_0}{\alpha(t)}$. Then using proposition 4.10, we get:

$$E\left[\sup_{x\in D_2} \frac{|\int_0^t (h(\Phi_s(x)) - h(\Phi_s(X_0)))\, dB_s|}{\|\tilde{x} - \tilde{X}_0\|^2}\right]$$

$$\leq \sum_{n\geq n_0} \frac{1}{2^{2n}}\, E\left[\sup_{\substack{2^n\leq\ \|\tilde{x}-\tilde{X}_0\|\ \leq 2^{n+1} \\ \|\bar{x}-\bar{X}_0\|\ \leq 1}} \left|\int_0^t (h(\Phi_s(x)) - h(\Phi_s(X_0)))\, dB_s\right|\right] \tag{234}$$

$$\leq K\left[\sum_{n\geq n_0} \frac{2^{n+1}\sqrt{V(t)}}{2^{2n}} + \sum_{n\geq n_0} \frac{1}{2^{2n}}\right] \tag{235}$$

$$\leq K\left[2^{-n_0} \sum_{p\geq 0} \frac{\sqrt{V(t)}}{2^{p-1}} + 2^{-2n_0} \sum_{p\geq 0} \frac{1}{2^{2p}}\right] \tag{236}$$

$$\leq K\left[\frac{2\,\alpha(t)\sqrt{V(t)}}{\alpha_0} \sum_{p\geq 0} \frac{1}{2^{p-1}} + \frac{4\,\alpha(t)^2}{\alpha_0^2} \sum_{p\geq 0} \frac{1}{2^{2p}}\right] \tag{237}$$

Then using (176) we get:

$$E\left[\sup_{x\in D_2} \frac{|\int_0^t (h(\Phi_s(x)) - h(\Phi_s(X_0)))\, dB_s|}{V(t)\, \|\tilde{x} - \tilde{X}_0\|^2}\right] \longrightarrow 0 \tag{238}$$

which imply the convergence in probability. □

Lemma 4.17 *The following convergence holds:*

$$\frac{\alpha(t)^2}{V(t)} \sup_{x\in D\left(\frac{\alpha_0}{\alpha(t)},1,X_0\right)} \left|\int_0^t (h(\Phi_s(x)) - h(\Phi_s(X_0)))\, dB_s\right| \xrightarrow{P} 0 \tag{239}$$

for any $\alpha_0 > 0$.

Proof Recalling (176), and using corollary 4.11 we get:

$$\frac{\alpha(t)^2}{V(t)} E \left[\sup_{x \in D\left(\frac{\alpha_0}{\alpha(t)},1,X_0\right)} \left| \int_0^t (h(\Phi_s(x)) - h(\Phi_s(X_0)))\, dB_s \right| \right] \leq K \left[\frac{\alpha_0\, \alpha(t)}{\sqrt{V(t)}} + \frac{\alpha(t)^2}{V(t)} \right] \quad (240)$$

which gives the L^1-convergence, implying the convergence in probability. □

Lemma 4.18 *For any $\alpha_0 > 0$, $K > 0$, for $\alpha_1 > 0$ small enough and time t large enough, there is some $\tilde{K}$ not depending on t such that:*

$$\frac{\int_{\left\{\|\tilde{x}-\tilde{X}_0\| \geq \frac{\alpha_0}{\alpha(t)}\right\}} \exp\left[-K\, V(t)\, \|\tilde{x} - \tilde{X}_0\|^2\right]\, d\tilde{x}}{\int_{\left\{\|\tilde{x}-\tilde{X}_0\| \leq \frac{\alpha_1}{\alpha(t)}\right\}} p_0(\tilde{x})\, d\tilde{x}} \leq \tilde{K}\, \frac{\alpha(t)^2}{V(t)} \exp\left[-K\, V(t)\, \frac{\alpha_0^2}{\alpha(t)^2}\right] \quad (241)$$

Proof We first deal with the denominator. As $\alpha(t)$ is non decreasing, we will consider two cases: $\alpha(t)$ bonded and $\alpha(t) \to +\infty$. In the bounded case let $A = \sup_{t\geq 0} \alpha(t) = \lim_{t\to+\infty} \alpha(t)$, choose t large enough such that $\alpha(t) \geq \frac{A}{2}$, and by continuity of p_0 choose α_1 sufficiently small to have: for all $\tilde{x}$ such that $\|\tilde{x} - \tilde{X}_0\| \leq \frac{\alpha_1}{\alpha(t)} \leq \frac{2\alpha_1}{A}$ we have $p_0(\tilde{x}) \geq \frac{p_0(X_0)}{2}$. In the unbounded case just take t sufficiently large to have the same property provided that $\|\tilde{x} - \tilde{X}_0\| \leq \frac{\alpha_1}{\alpha(t)}$. Then we get:

$$\int_{\left\{\|\tilde{x}-\tilde{X}_0\| \leq \frac{\alpha_1}{\alpha(t)}\right\}} p_0(\tilde{x})\, d\tilde{x} \geq \frac{p_0(X_0)}{2} \cdot \frac{K_{\tilde{d}}\, \alpha_1^{\tilde{d}}}{\alpha(t)^{\tilde{d}}} \quad (242)$$

where $K_{\tilde{d}}$ denote the volume of the unit $\tilde{d}$-ball. Consider now the numerator. Using polar coordinates:

$$\int_{\left\{\|\tilde{x}-\tilde{X}_0\| \geq \frac{\alpha_0}{\alpha(t)}\right\}} \exp\left[-K\, V(t)\, \|\tilde{x} - \tilde{X}_0\|^2\right]\, d\tilde{x} = S_{\tilde{d}} \int_{\frac{\alpha_0}{\alpha(t)}}^{+\infty} \exp\left[-K\, V(t)\, r^2\right]\, r^{\tilde{d}-1}\, dr \quad (243)$$

where $S_{\tilde{d}}$ denote the surface of the unit $\tilde{d}$-sphere. Then by lemma 3.2 we get, for large t:

$$\begin{aligned} &\frac{\int_{\left\{\|\tilde{x}-\tilde{X}_0\| \geq \frac{\alpha_0}{\alpha(t)}\right\}} \exp\left[-K\, V(t)\, \|\tilde{x} - \tilde{X}_0\|^2\right]\, d\tilde{x}}{\int_{\left\{\|\tilde{x}-\tilde{X}_0\| \leq \frac{\alpha_1}{\alpha(t)}\right\}} p_0(\tilde{x})\, d\tilde{x}} \\ &\leq \frac{2}{p_0(X_0)} \cdot \frac{S_{\tilde{d}}}{K_{\tilde{d}}} \cdot \left(\frac{\alpha_0}{\alpha_1}\right)^{\tilde{d}} \cdot \frac{\Gamma_{\tilde{d}}\, \alpha(t)^2}{K\, V(t)\, \alpha_0^2} \exp\left[-K\, V(t)\, \frac{\alpha_0^2}{\alpha(t)^2}\right] \end{aligned} \quad (244)$$

which proves the lemma. □

Proposition 4.19 *The following convergence holds:*

$$\frac{\int_{D_2} f_t(x)\, dx}{\int_{D_0} f_t(x)\, dx} \xrightarrow{P} 0 \quad as \quad t \to +\infty \quad (245)$$

for all $\alpha_0 > 0$ and $\delta_0 > 0$.

Proof Let $t_n \geq 0$ be such that $\lim_{n\to+\infty} t_n = +\infty$ and for which the convergences (239) and (233) hold a.s. Then for n larger than some random N, using proposition 4.8, for any $0 < \alpha_1 < \alpha_0$ if $\alpha(t)$ is unbonded, and α_1 small enough in the bounded case:

$$\int_{D_0} f_{t_n}(x)\,dx \geq \int_{D\left(\frac{\alpha_1}{\alpha(t_n)},1,X_0\right)} f_{t_n}(x)\,dx \tag{246}$$

$$\geq \int_{D\left(\frac{\alpha_1}{\alpha(t_n)},1,X_0\right)} \exp\left[-\frac{1}{2}K_2(V(t_n)\,\|\tilde{x}-\tilde{X}_0\|^2+1) + \int_0^{t_n}(h(\Phi_s(x)) - h(\Phi_s(X_0)))\,dB_s\right] p_0(x)\,dx \tag{247}$$

$$\geq \int_{D\left(\frac{\alpha_1}{\alpha(t_n)},1,X_0\right)} \exp\left[-K_2\,V(t_n)\,\frac{\alpha_1^2}{\alpha(t_n)^2} - \frac{K_2}{2}\right] p_0(x)\,dx \tag{248}$$

Using now ($\tilde{\mathrm{H}}$4):

$$\int_{D_0} f_{t_n}(x)\,dx \geq \exp\left[-K_2\,V(t_n)\,\frac{\alpha_1^2}{\alpha(t_n)^2} - \frac{K_2}{2}\right]. \int_{\left\{\|\tilde{x}-\tilde{X}_0\|\leq\frac{\alpha_1}{\alpha(t_n)}\right\}} p_0(\tilde{x})\,d\tilde{x}\,.\int_{\{\|\bar{x}-\bar{X}_0\|\leq 1\}} C_1\,\exp\left[-C_2\,\|\bar{x}\|^2\right]\,d\bar{x} \tag{249}$$

We denote by I_0 this last integral multiplied by $\exp(-\frac{1}{2}K_2)$. Clearly I_0 depends on X_0 but not on n. Then

$$\int_{D_0} f_{t_n}(x)\,dx \geq I_0\,\exp\left[-K_2\,V(t_n)\,\frac{\alpha_1^2}{\alpha(t_n)^2}\right]. \int_{\left\{\|\tilde{x}-\tilde{X}_0\|\leq\frac{\alpha_1}{\alpha(t_n)}\right\}} p_0(\tilde{x})\,d\tilde{x} \tag{250}$$

On the other hand, using again proposition 4.8:

$$\int_{D_2} f_{t_n}(x)\,dx \leq \int_{D_2} \exp\left[-\frac{1}{2}K_0\,V(t_n)\,\|\tilde{x}-\tilde{X}_0\|^2 + \frac{1}{2}K_1 + \int_0^{t_n}(h(\Phi_s(x)) - h(\Phi_s(X_0)))\,dB_s\right] p_0(x)\,dx \tag{251}$$

$$\leq \int_{D_2} \exp\left[-\frac{1}{4}K_0\,V(t_n)\,\|\tilde{x}-\tilde{X}_0\|^2 + \frac{1}{2}K_1\right] p_0(x)\,dx \tag{252}$$

for n larger than some random N. Using ($\tilde{\mathrm{H}}$4):

$$\int_{D_2} f_{t_n}(x)\,dx \leq \int_{\left\{\|\tilde{x}-\tilde{X}_0\|\geq\frac{\alpha_0}{\alpha(t_n)}\right\}} \exp\left[-\frac{1}{4}K_0\,V(t_n)\,\|\tilde{x}-\tilde{X}_0\|^2 + \frac{1}{2}K_1\right] p_0(\tilde{x})\,d\tilde{x} \cdot \int_{\{\|\bar{x}-\bar{X}_0\|\leq 1\}} C_3\,\exp\left[-C_4\,\|\bar{x}\|^2\right]\,d\bar{x} \tag{253}$$

We denote by I_2 this last integral multiplied by $\exp(\frac{1}{2}K_1)$. Clearly I_2 depends on X_0 but not on n. Then

$$\int_{D_2} f_{t_n}(x)\,dx \leq I_2 \int_{\left\{\|\tilde{x}-\tilde{X}_0\|\geq \frac{\alpha_0}{\alpha(t_n)}\right\}} \exp\left[-\frac{1}{4}K_0\,V(t_n)\,\|\tilde{x}-\tilde{X}_0\|^2\right] p_0(\tilde{x})\,d\tilde{x} \tag{254}$$

Then, using (250), (254), lemma 4.18 and the boundedness of p_0 we obtain that there exists some $L>0$ not depending on n such that:

$$\frac{\int_{D_2} f_t(x)\,dx}{\int_{D_0} f_t(x)\,dx} \leq L\,\exp\left[K_2\,V(t_n)\,\frac{\alpha_1^2}{\alpha(t_n)^2}\right]\cdot\frac{\alpha(t_n)^2}{V(t_n)}\,\exp\left[-\frac{1}{4}K_0\,V(t_n)\,\frac{\alpha_0^2}{\alpha(t_n)^2}\right] \tag{255}$$

$$= L\,\frac{\alpha(t_n)^2}{V(t_n)}\,\exp\left[-\frac{V(t_n)}{\alpha(t_n)^2}\left(-K_2\,\alpha_1^2+\frac{1}{4}\,K_0\,\alpha_0^2\right)\right] \tag{256}$$

which tends to 0 provided we have chosen $\alpha_1 < \frac{1}{2}\sqrt{\frac{K_0}{K_2}}\,\alpha_0$. Then conclude using the characterization of the convergence in probability. □

Now it is time to consider the last domain D_3.

Lemma 4.20 *For all $\alpha_0>0$ and δ_0, all $K_1>0$ and $K_2>0$ the following convergence holds:*

$$\sup_{x\in D_3}\frac{\left|\int_0^t (h(\Phi_s(x))-h(\Phi_s(X_0)))\,dB_s\right|}{K_1\,V(t)\,\|\tilde{x}-\tilde{X}_0\|^2+K_2\,\|\bar{x}-\bar{X}_0\|^2}\xrightarrow{P} 0 \qquad as \qquad t\to+\infty. \tag{257}$$

Proof Let n_0 be the greatest relative integer such that $2^{n_0}\leq\frac{\alpha_0}{\alpha(t)}$. Then using proposition 4.10, we get:

$$E\left[\sup_{x\in D_3}\frac{\left|\int_0^t (h(\Phi_s(x))-h(\Phi_s(X_0)))\,dB_s\right|}{K_1\,V(t)\,\|\tilde{x}-\tilde{X}_0\|^2+K_2\,\|\bar{x}-\bar{X}_0\|^2}\right]$$

$$\leq \sum_{p\geq 0}\sum_{n\geq n_0}\frac{E\left[\sup_{\substack{2^n\leq\ \|\tilde{x}-\tilde{X}_0\|\ \leq 2^{n+1}\\ 2^p\ \|\bar{x}-\bar{X}_0\|\ \leq 2^{p+1}}}\left|\int_0^t (h(\Phi_s(x))-h(\Phi_s(X_0)))\,dB_s\right|\right]}{K_1\,V(t)\,2^{2n}+K_2\,2^{2p}} \tag{258}$$

$$\leq \sum_{p\geq 0}\sum_{n\geq n_0} K\,\frac{2^{n+1}\sqrt{V(t)}+2^{p+1}}{K_1\,V(t)\,2^{2n}+K_2\,2^{2p}} \tag{259}$$

$$\leq 4\,K\int_0^{+\infty}\int_{n_0}^{+\infty}\frac{2^{u+1}\sqrt{V(t)}+2^{v+1}}{K_1\,V(t)\,2^{2u}+K_2\,2^{2v}}\,du\,dv \tag{260}$$

$$= 4\,K\int_0^{+\infty}\int_{-\log_2\alpha(t)+\log_2\alpha_0-1}^{+\infty}\frac{2^{u+\frac{1}{2}\log_2 V(t)+1}+2^{v+1}}{K_1\,2^{2u+\log_2 V(t)}+K_2\,2^{2v}}\,du\,dv \tag{261}$$

$$= 4\,K \int_0^{+\infty} \int_{\frac{1}{2}\log_2 V(t) - \log_2 \alpha(t) + \log_2 \alpha_0 - 1}^{+\infty} \frac{2^{u'+1} + 2^{v+1}}{K_1\, 2^{2u'} + K_2\, 2^{2v}}\, du'\, dv \tag{262}$$

The last expression converges to 0 because the integrand is integrable on $\mathbb{R}_+ \times \mathbb{R}_+$ and by (176) $\lim_{t\to+\infty} \frac{1}{2}\log_2 V(t) - \log_2 \alpha(t) = +\infty$. Which proves the lemma, because the L^1-convergence implies the convergence in probability. □

And comparing $f_t(x)$ on D_0 and D_3 we get the

Proposition 4.21 *The following convergence holds:*

$$\frac{\int_{D_3} f_t(x)\, dx}{\int_{D_0} f_t(x)\, dx} \xrightarrow{P} 0 \quad as \quad t \to +\infty \tag{263}$$

for all $\alpha_0 > 0$ and $\delta_0 > 0$.

Proof Let $t_n \geq 0$ be such that $\lim_{n\to+\infty} t_n = +\infty$ and for which the convergences (239) and (257) hold a.s. Using proposition 4.9, for any $\varepsilon > 0$:

$$\begin{aligned}
&\int_{D_3} f_{t_n}(x)\, dx \\
&\leq \int_{D_3} \exp\Big[-\frac{1}{2}K_\varepsilon\, V(t_n)\, \|\tilde{x} - \tilde{X}_0\|^2 + \frac{1}{2}\varepsilon\, \|\bar{x} - \bar{X}_0\|^2 \\
&\qquad + \int_0^{t_n} (h(\Phi_s(x)) - h(\Phi_s(X_0)))\, dB_s\Big]\, p_0(x)\, dx \qquad (264)\\
&\leq \int_{D_3} \exp\Big[\Big(\frac{1}{2}K_\varepsilon\, V(t_n)\, \|\tilde{x} - \tilde{X}_0\|^2 + \frac{1}{2}\varepsilon\, \|\bar{x} - \bar{X}_0\|^2\Big) \cdot \\
&\qquad \Big(-1 + \frac{\int_0^{t_n} (h(\Phi_s(x)) - h(\Phi_s(X_0)))\, dB_s}{\frac{1}{2}K_\varepsilon\, V(t_n)\, \|\tilde{x} - \tilde{X}_0\|^2 + \frac{1}{2}\varepsilon\, \|\bar{x} - \bar{X}_0\|^2}\Big) + \varepsilon\, \|\bar{x} - \bar{X}_0\|^2\Big]\, p_0(x)\, dx \qquad (265)\\
&\leq \int_{D_3} \exp\Big[-\frac{1}{4}K_\varepsilon\, V(t_n)\, \|\tilde{x} - \tilde{X}_0\|^2 + \frac{3}{4}\,\varepsilon\, \|\bar{x} - \bar{X}_0\|^2\Big]\, p_0(x)\, dx \qquad (266)
\end{aligned}$$

for n large enough. Using ($\tilde{\mathrm{H}}4$):

$$\begin{aligned}
\int_{D_3} f_{t_n}(x)\, dx \;\leq\; &\int_{\left\{\|\tilde{x} - \tilde{X}_0\| \geq \frac{\alpha_0}{\alpha(t_n)}\right\}} \exp\Big[-\frac{1}{4}K_\varepsilon\, V(t_n)\, \|\tilde{x} - \tilde{X}_0\|^2\Big]\, p_0(\tilde{x})\, d\tilde{x} \\
&\cdot \int_{\{\|\bar{x} - \bar{X}_0\| \geq 1\}} C_3\, \exp\Big[-C_4\, \|\bar{x}\|^2 + \frac{3}{4}\,\varepsilon\, K_1\, \|\bar{x} - \bar{X}_0\|^2\Big]\, d\bar{x} \qquad (267)
\end{aligned}$$

The last integral is finite, provided we have chosen $\varepsilon < \frac{4}{3} C_4$. We denote it by I_3. Then, recalling that p_0 is bounded, say by K_{p_0}:

$$\int_{D_3} f_{t_n}(x)\, dx \leq K_{p_0}\, I_3 \int_{\left\{\|\tilde{x}-\tilde{X}_0\| \geq \frac{\alpha_0}{\alpha(t_n)}\right\}} \exp\left[-\frac{1}{4} K_\varepsilon\, V(t_n)\, \|\tilde{x}-\tilde{X}_0\|^2\right] d\tilde{x} \tag{268}$$

Using this, (250) and lemma 4.18 we get that there exists some $L > 0$ not depending on n such that:

$$\frac{\int_{D_3} f_t(x)\, dx}{\int_{D_0} f_t(x)\, dx} \leq L \exp\left[K_2\, V(t_n)\, \frac{\alpha_1^2}{\alpha(t_n)^2}\right] \cdot \frac{\alpha(t_n)^2}{V(t_n)} \exp\left[-\frac{1}{4} K_\varepsilon\, V(t_n)\, \frac{\alpha_0^2}{\alpha(t_n)^2}\right] \tag{269}$$

$$= L\, \frac{\alpha(t_n)^2}{V(t_n)} \exp\left[-\frac{V(t_n)}{\alpha(t_n)^2}\left(-K_2\, \alpha_1^2 + \frac{1}{4} K_\varepsilon\, \alpha_0^2\right)\right] \tag{270}$$

which tends to 0 provided we have chosen $\alpha_1 < \frac{1}{2}\sqrt{\frac{K_\varepsilon}{K_2}}\, \alpha_0$. Then conclude using the characterization of the convergence in probability. $\square$

Theorem 4.22 *We denote by Θ_a^t the ball centered at X_t, of radius a. Then for all $a > 0$ the following convergence holds:*

$$\int_{\Theta_a^t} \mu_t(x)\, dx \xrightarrow{P} 1 \quad as \quad t \to +\infty \tag{271}$$

Proof As all the norms on $\mathbb{R}^d$ are equivalent, we will consider Θ_a^t defined for the norm $\|\ .\ \|_\oplus$ (see (157)). Then (66) is still valid and the convergence of the theorem is equivalent to (67). Then remark that:

$$\Phi_t^{-1}\left(\Theta_a^t\right)^c = \Phi_t^{-1}\left(\Theta_a^{t,c}\right) \subset D_1(t,a,a) \cup D_2(t,a,a) \cup D_3(t,a,a) \tag{272}$$

and

$$D_0(t,a,a) \subset \Phi_t^{-1}\left(\Theta_a^t\right) \tag{273}$$

to get:

$$\frac{\int_{\Phi_t^{-1}(\Theta_a^t)^c} f_t(x)\, dx}{\int_{\Phi_t^{-1}(\Theta_a^t)} f_t(x)\, dx} \leq \frac{\int_{D_1} f_t(x)\, dx}{\int_{D_0} f_t(x)\, dx} + \frac{\int_{D_2} f_t(x)\, dx}{\int_{D_0} f_t(x)\, dx} + \frac{\int_{D_3} f_t(x)\, dx}{\int_{D_0} f_t(x)\, dx} \xrightarrow{P} 0 \tag{274}$$

as $t \to +\infty$ by propositions 4.15, 4.19, and 4.21. $\square$

5 Conclusion

The last section has shown that the observation function need to distinguish points only on the unstable part of the system, when the dynamics is linear or almost linear. This should be extended to more general classes of systems, using the tools of the theory of dynamical systems. In particular, the notions of asymptotic phase and attractor seem to be relevant here.

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