

# Some remarks about the Riemann problem of first order Hamilton–Jacobi equations on general geometries

Remi Abgrall

► **To cite this version:**

Remi Abgrall. Some remarks about the Riemann problem of first order Hamilton–Jacobi equations on general geometries. RR-2442, INRIA. 1994. <inria-00074233>

**HAL Id: inria-00074233**

**<https://hal.inria.fr/inria-00074233>**

Submitted on 24 May 2006

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

*Some remarks about the Riemann problem of  
first order Hamilton–Jacobi equations on  
general geometries*

Rémi Abgrall

**N° 2442**

Décembre 1994

PROGRAMME 6



*Rapport  
de recherche*



# Some remarks about the Riemann problem of first order Hamilton–Jacobi equations on general geometries

Rémi Abgrall \*

Programme 6 — Calcul scientifique, modélisation et logiciel numérique  
Projet Sinus

Rapport de recherche n° 2442 — Décembre 1994 — 9 pages

**Abstract:** We are interested in the most general Riemann problem for first order Hamilton–Jacobi equations. We propose two inequalities that the viscosity solution satisfies. This generalize a result of Bardi and Osher.

**Key-words:** First order hamilton–Jacobi equations, Riemann problem

*(Résumé : tsvp)*

\*abgrall@sophia.inria.fr

# Quelques remarques concernant le problème de Riemann des équations de Hamilton–Jacobi du premier ordre sur des géométries quelconques

**Résumé :** Nous nous intéressons au problème de Riemann le plus général des équations de Hamilton–Jacobi du premier ordre. Nous proposons deux inégalités que vérifie la solution de viscosité. Ces inégalités généralisent un résultat de Bardi et Osher.

**Mots-clé :** Equations de Hamilton–Jacobi du premier ordre, problème de Riemann

We are interested in the solutions of

$$\phi_t + H(\nabla\phi) = 0 \text{ in } \mathbb{R}^N \times (0, +\infty) \quad (1)$$

with an initial condition

$$\phi^0(x) = \phi_1^0(x) + \phi_2^0(x), \quad x \in \mathbb{R}^N \quad (2)$$

when

1.  $H \in C(\mathbb{R}^N)$ ,
2.  $\phi_1^0$  et  $\phi_2^0$  are respectively convex, concave and uniformly Lipschitz.

At  $t = 0$ ,  $\phi$  is also uniformly Lipschitz and continuous.

This Cauchy problem admits a unique viscosity solution in  $UC_x(\mathbb{R}^N \times [0, T])$ , the space of function which are uniformly continuous in  $x \in \mathbb{R}^N$  uniformly in  $t \in [0, T]$ . We want to generalize Bardi and Osher's inequalities [BO91] on the exact solution of problem (1–2). Their inequalities deal with the solution of problem (1–2) when  $\phi_1^0$  and  $\phi_2^0$  are :

$$\phi_1^0(x) = \phi_1(x_1, \dots, x_p) \quad , \quad \phi_2^0(x) = \phi_1(x_{p+1}, \dots, x_N)$$

for some  $p$ ,  $0 \leq p \leq N$ . The starting point is Hopf's formula that gives the solution when  $\psi^0$  is either convex or concave :

- for a convex initial condition,

$$\phi(x, t) = \sup_{v \in \mathbb{R}^N} [x \cdot v - \phi^{0*}(v) - tH(v)] ,$$

- and for a concave initial condition,

$$\phi(x, t) = \inf_{v \in \mathbb{R}^N} [x \cdot v - \phi^{0*}(-v) - tH(v)] .$$

# 1 General solution

We denote by  $\psi^*$  the Legendre transform of  $\psi$ . If  $\psi$  is convex,

$$\psi^*(v) = \sup_{x \in \mathbb{R}^N} [x \cdot v - \psi(x)],$$

if  $\psi$  is concave,

$$\psi^*(v) = \inf_{x \in \mathbb{R}^N} [-x \cdot v - \psi(x)].$$

In both cases,  $\psi$  is the Legendre transform of its Legendre transform. We denote by  $D_\phi$  the set defined, for a convex function by

$$D_\phi = \{v \in \mathbb{R}^N, \phi^*(v) < +\infty\}$$

and for a concave function by

$$D_\phi = \{v \in \mathbb{R}^N, \phi^*(v) > -\infty\}.$$

We start by solving (1–2) with

$$\Psi_v^0(x) = \phi_1^0(x) - x \cdot v - \phi_2^{0*}(v) \geq \phi^0(x). \quad (3)$$

This is a convex function because  $\phi_1^0$  is convex. We compute  $\Psi_v^{0*}$ . By definition,

$$\begin{aligned} \Psi_v^{0*}(w) &= \sup_{x \in \mathbb{R}^N} [x \cdot w - \Psi_v^0(x)] \\ &= \sup_{x \in \mathbb{R}^N} [x \cdot (v + w) - \phi_1^0(x) + \phi_2^{0*}(v)] \\ &= \begin{cases} \phi_2^{0*}(v) + \phi_1^{0*}(v + w) & \text{if } v \in D_{\phi_2^0} \\ +\infty & \text{else} \end{cases} \end{aligned}$$

$\Psi_v^{0*}$  is finite if and only if  $v \in D_{\phi_2^0}$  and  $v + w \in D_{\phi_1^0}$ . In the following, we say that  $\Psi_v^{0*}(w) = \phi_2^{0*}(v) + \phi_1^{0*}(v + w) = \infty$  if  $v \notin D_{\phi_2^0}$ . This changes nothing to the results.

Hopf's formula [BE84] says that the solution is

$$\Psi_v(x, t) = \sup_{w \in \mathbb{R}^N} [x \cdot w - \Psi_v^{*0}(w) - tH(w)].$$

Standard inequalities on viscosity solutions and (3) show that the viscosity solution  $\phi$  of (1–2) satisfies :

$$\inf_{v \in \mathbb{R}^N} \sup_{w \in \mathbb{R}^N} [x \cdot w - \phi_2^{0*}(v) - \phi_1^{*0}(v + w) - tH(w)] \geq \phi(x, t).$$

The same arguments show that

$$\sup_{v \in D_{\phi_1^0}} \inf_{w \in \mathbb{R}^N} [x \cdot v + w - \phi_1^{0*}(v) - tH(w)] \leq \phi(x, t).$$

This generalizes the result of [BO91].

**Proposition 1.1** *The viscosity solution  $\phi$  of the Cauchy problem (1–2) satisfies, when  $\phi^0 = \phi_1^0 + \phi_2^0$  for a convex  $\phi_1^0$  and a concave  $\phi_2^0$  :*

$$\begin{aligned} \inf_{v \in \mathbb{R}^N} \sup_{w \in \mathbb{R}^N} [x \cdot (w - v) - \phi_2^{0*}(w) - \phi_1^{*0}(v) - tH(w - v)] &\geq \phi(x, t) \\ &\geq \sup_{v \in \mathbb{R}^N} \inf_{w \in \mathbb{R}^N} [x \cdot (w - v) - \phi_2^{0*}(v) - \phi_1^{*0}(w) - tH(w - v)] \end{aligned}$$

**Remark 1.2** *When  $\phi_2^0$  is identically vanishing, the previous inequalities provide the viscosity solution because  $\phi_2^{0*} = -\infty$  except for  $v = 0$ . Similarly, with Bardi and Osher’s assumptions, we us set  $x = (x_A, x_B)$  when  $x_A \in \mathbb{R}^p$  and  $x_B \in \mathbb{R}^{N-p}$ . If  $\phi_1^0(x) \equiv \phi_1^0(x_A)$  and  $\phi_2^0(x) \equiv \phi_2^0(x_B)$ , clearly we have*

$$\phi_1^{0*}(v) = \begin{cases} +\infty & \text{if } v_B \neq 0 \\ \phi_1^{0*}(v_A) & \text{else.} \end{cases}$$

*A similar result can be obtained for  $\phi_2^{0*}$  : Bardi and Osher’s results can easily be deduced.*

## 2 Application to the solution of Riemann’s problem on general meshes

We consider (1–2) for a piecewise linear initial data. More precisely,  $\mathbb{R}^N$  is partitioned in angular sectors  $\Omega_1, \dots, \Omega_n$ . We define  $\phi$  by :

$$x \in \Omega_i, \phi^0(x) = A + U_i \cdot x. \tag{4}$$



A piecewise linear function is convex if and only if

$$\forall i \in \{0, \dots, n\}, \forall x \in \Omega_i, \forall j \in \{0, \dots, n\}, (U_j - U_i) \cdot x \leq 0. \quad (5)$$

For a piecewise linear concave function, the  $\leq$  of (5) is modified into a  $\geq$ .

We notice that  $\phi^0$  is the sum of two piecewise linear functions  $\phi_1$  and  $\phi_2$ , respectively convex and concave. This decomposition is not unique, but  $\phi'_1$  and  $\phi'_2$  is another one, there exists a linear function  $\psi$  such that  $\phi'_1 = \phi_1 + \psi$  and  $\phi'_2 = \phi_2 - \psi$ . This result is recalled in Appendix A.

What remains is to compute the Legendre transform of a piecewise linear convex function. We have

$$\forall x \in \Omega_i, x \cdot v - \phi(x) = A + x \cdot (v - U_i).$$

Since each  $\Omega_i$  is invariant by similarity, if  $v$  satisfies : there exists  $i \in \{0, \dots, n\}$  et  $x \in \Omega_i$  so that

$$(v - U_i) \cdot x > 0,$$

then  $\phi^*(v) = +\infty$ . Hence,

$$\phi^*(v) < +\infty \iff \forall i \in \{0, \dots, n\}, \forall x \in \Omega_i, (v - U_i) \cdot x \leq 0, \quad (6)$$

and we have  $\phi^*(v) = A$  because  $0 \in \Omega_i$ . We characterize this set. It is convex, it contains each  $U_j$ ,  $j \in \{0, \dots, n\}$  since  $\phi$  is convex, thus their convex envelope  $\mathcal{K}_\phi$ . Assume now there exists  $v \notin \mathcal{K}$  so  $\phi^*(v) < +\infty$ . There exists  $w$  and  $\alpha$  such that  $w \cdot v > \alpha$  and  $U_i \cdot w < \alpha$  for all  $i \in \{0, \dots, n\}$  thus  $w \cdot (v - U_i) > 0$  and then  $\phi^*(v) = +\infty$ . We have proved

$$\phi^*(v) = \begin{cases} A & \text{if } v \in \mathcal{K}_\phi \\ +\infty & \text{else.} \end{cases} \quad (7)$$

The formula (7) is still true for a concave function if we change the  $+\infty$  into  $-\infty$  and replace  $\mathcal{K}_\phi$  by  $\mathcal{K}_{-\phi}$ . Clearly,  $\mathcal{K}_{-\phi}$  is the symmetric of  $\mathcal{K}_\phi$  with respect to the origin. Thus, We obtain the following inequalities on the solution of the Riemann problem :

$$\begin{aligned} A + \inf_{v \in \mathcal{K}_{\phi_1}} \sup_{w \in \mathcal{K}_{\phi_2}} (x \cdot (w + v) - tH(w + v)) &\geq \phi(x, t) \\ A + \sup_{v \in \mathcal{K}_{\phi_1}} \inf_{w \in \mathcal{K}_{\phi_2}} (x \cdot (w + v) - tH(w + v)) &\leq \phi(x, t). \end{aligned} \quad (8)$$

Note that (8) is independent of the decomposition of  $\phi$ . This formula helps to understand how to discretize this type of PDE on an unstructured mesh. These equations are models for a wide variety of physical phenomena (propagation of flames for example), they are helpful for computer vision, mesh generation, etc. These problems have to be studied on complex geometries.

## References

- [BE84] M. Bardi and L. C. Evans. On Hopf’s formulas for solutions of Hamilton-Jacobi equations. *Nonlinear Analysis, Theory, Methods and Applications*, 8(11), 1984.
- [BO91] M. Bardi and S. Osher. The nonconvex multi-dimensional Riemann problem for Hamilton-Jacobi equations. *Siam J. Math. Anal.*, 22(2), March 1991.

## A Decomposition of a piecewise linear function

Let us assume that  $\mathbb{R}^2 = \Omega_1 \cup \dots \cup \Omega_n$ , each angular sectors meet at  $O$ . We recall how is built the decomposition of a piecewise linear function  $\phi$  as the sum of piecewise linear convex and concave functions, where both are continuous.

We consider the convex envelope of the graph of  $\phi$ . It is the graph of a continuous piecewise linear function  $\psi$ . We want to show that  $\psi' = \phi - \psi$  is the sum of two piecewise linear function, where one is convex whereas the other one is concave.

Since the graph of  $\psi$  is the convex envelope of  $\phi$ ,  $\psi'$  is the sum of functions that are zero everywhere except in a collection of angular sectors that meet at  $O$ . Moreover it is a negative function. Each of these angular sectors is the collection of several  $\Omega_i$ . In order to simplify the text, we assume that there are only two of them,  $\Omega_1$  and  $\Omega_2$ , as on Figure 1. We set  $\Omega = \mathbb{R}^2 - \Omega_1 \cup \Omega_2$ , and

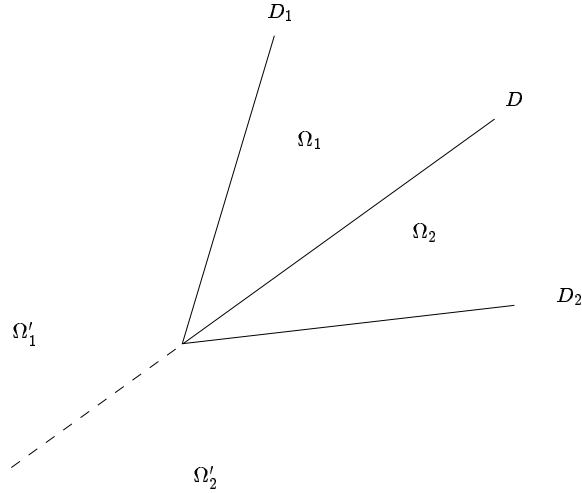


Figure 1:

we consider the case of a function  $\phi \geq 0$  defined by :

$$\phi(x) = \begin{cases} 0 & \text{in } \Omega \\ U_1 \cdot x & \text{in } \Omega_1 \\ U_2 \cdot x & \text{in } \Omega_2. \end{cases}$$

We define the lines  $D$ ,  $D_1$  et  $D_2$ , and the sets  $\Omega'_1$ ,  $\Omega'_2$  by :

- $U_1 \cdot x = U_2 \cdot x$  on  $D$ ,
- $U_1 \cdot x = 0$  on  $D_1$ ,
- $U_2 \cdot x = 0$  on  $D_2$ ,
- $U_1 \cdot x \leq U_2 \cdot x \leq 0$  on  $\Omega'_1$  and  $U_2 \cdot x \leq U_1 \cdot x \leq 0$  on  $\Omega'_2$ .

To say that  $\psi'$  is neither convex or concave means, because of the concavity-convexity criterion of §2 :

- $U_1 \cdot x \leq U_2 \cdot x$  on  $\Omega_1 \cup \Omega'_1$ , with  $0 \leq U_1 \cdot x \leq U_2 \cdot x$  on  $\Omega_1$ ,

- $U_2 \cdot x \leq U_1 \cdot x$  on  $\Omega_2 \cup \Omega'_2$ , with  $0 \leq U_2 \cdot x \leq U_1 \cdot x$  on  $\Omega_2$ .

Let us notice that  $\phi$  is concave on  $\Omega_1 \cup \Omega_2$ . Let  $\phi_1$  be :

$$\phi_1(x) = \begin{cases} 0 & \text{in } \Omega_1 \cup \Omega_2 \\ v_1 \cdot x & \text{in } \Omega'_1 \\ v_2 \cdot x & \text{in } \Omega'_2. \end{cases}$$

We are looking for a continuous  $\phi_1$  :  $\phi_1$  is vanishing on  $D_1$  and  $D_2$ , hence  $v_1 = \alpha_1 U_1$  and  $v_2 = \alpha_2 U_2$  on one hand and on the other hand we have  $v_1 \cdot x = v_2 \cdot x$  on  $D$  so that  $\alpha_1 = \alpha_2 \equiv \alpha$ . With the criterion of §2, we check that  $\phi_1$  is convex if  $\alpha < 0$  because  $U_1 \cdot x \leq U_2 \cdot x$  on  $\Omega'_1$  and we have the inverse inequality on  $\Omega'_2$ . Let us compute  $\phi_2 = \phi - \phi_1$  :

$$\phi_2(x) = \begin{cases} -\alpha U_1 \cdot x & \text{in } \Omega'_1 \\ -\alpha U_2 \cdot x & \text{in } \Omega'_2 \\ U_1 \cdot x & \text{in } \Omega_1 \\ U_2 \cdot x & \text{in } \Omega_2 \end{cases}$$

and check that  $\phi_2$  is concave for  $\alpha < -1$  :

- $x \in \Omega'_1$ , we have :
  1.  $-\alpha(U_2 - U_1) \cdot x \geq 0$  because  $\alpha < 0$  and by definition of  $\Omega'_1$ ,
  2.  $(1 + \alpha)U_1 \cdot x \geq 0$  because  $1 + \alpha < 0$  and by definition of  $\Omega'_1$ ,
  3.  $(U_2 + \alpha U_1) \cdot x \geq (U_1 + \alpha U_1) \cdot x \geq 0$  for the same reasons.
- $x \in \Omega_1$ , we have :
  1.  $-\alpha(U_2 - U_1) \cdot x \geq 0$  because  $\alpha < 0$  and by definition of  $\Omega_1$ ,
  2.  $(-U_1 - \alpha U_2) \cdot x(U_2 + \alpha U_2) \cdot x \geq 0$  because  $1 + \alpha < 0$  and by definition of  $\Omega_1$ ,
  3.  $(U_2 - U_1) \cdot x \geq 0$  for the same reasons.
- For  $x \in \Omega'_2$  ou  $\Omega_2$ , we have the same results for the same reasons.

Any continuous piecewise linear function is the sum of two continuous piecewise linear functions where one is convex and the other one is concave. If two decomposition are possible,  $\phi_1 + \phi_2 = \psi_1 + \psi_2$ , where  $\psi_1, \psi_2$  are convex and  $\phi_2, \psi_2$  concave, then the convex piecewise linear function  $\phi_1 - \psi_2$  is equal to the concave piecewise linear function  $\phi_2 - \psi_1$ , so that they are linear.



---

Unité de recherche INRIA Lorraine, Technopôle de Nancy-Brabois, Campus scientifique,  
615 rue du Jardin Botanique, BP 101, 54600 VILLERS LÈS NANCY  
Unité de recherche INRIA Rennes, Irisa, Campus universitaire de Beaulieu, 35042 RENNES Cedex  
Unité de recherche INRIA Rhône-Alpes, 46 avenue Félix Viallet, 38031 GRENOBLE Cedex 1  
Unité de recherche INRIA Rocquencourt, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex  
Unité de recherche INRIA Sophia-Antipolis, 2004 route des Lucioles, BP 93, 06902 SOPHIA-ANTIPOLIS Cedex

---

Éditeur

INRIA, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)

ISSN 0249-6399