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Remi Abgrall. Some remarks about the Riemann problem of first order Hamilton–Jacobi equations on general geometries. RR-2442, INRIA. 1994. <inria-00074233>

HAL Id: inria-00074233
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Submitted on 24 May 2006

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INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

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N° 2442

Décembre 1994

PROGRAMME 6

Rapport de recherche

ISSN 0249-6399

INRIA SOPHIA ANTIPOLIS

Some remarks about the Riemann problem of first order Hamilton–Jacobi equations on general geometries

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Programme 6 — Calcul scientifique, modélisation et logiciel numérique
Projet Sinus

Rapport de recherche n° 2442 — Décembre 1994 — 9 pages

Abstract: We are interested in the most general Riemann problem for first order Hamilton–Jacobi equations. We propose two inequalities that the viscosity solution satisfies. This generalize a result of Bardi and Osher.

Key-words: First order hamilton–Jacobi equations, Riemann problem

(Résumé : tsvp)

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Quelques remarques concernant le problème de Riemann des équations de Hamilton–Jacobi du premier ordre sur des géometries quelconques

Résumé : Nous nous intéressons au problème de Riemann le plus général des équations de Hamilton–Jacobi du premier ordre. Nous proposons deux inégalités que vérifie la solution de viscosité. Ces inégalités généralisent un résultat de Bardi et Osher.

Mots-clé : Equations de Hamilton–Jacobi du premier ordre, problème de Riemann

We are interested in the solutions of

$$\phi_t + H(\nabla\phi) = 0 \;\text{ in }\; \mathbb{R}^N \times (0, +\infty) \tag{1}$$

with an initial condition

$$\phi^0(x) = \phi_1^0(x) + \phi_2^0(x), \quad x \in \mathbb{R}^N \tag{2}$$

when

1. $H \in C(\mathbb{R}^N)$,
2. ϕ_1^0 et ϕ_2^0 are respectively convex, concave and uniformly Lipschitz.

At $t = 0$, ϕ is also uniformly Lipschitz and continuous.

This Cauchy problem admits a unique viscosity solution in $UC_x(\mathbb{R}^N \times [0, T])$, the space of function which are uniformly continuous in $x \in \mathbb{R}^N$ uniformly in $t \in [0, T]$. We want to generalize Bardi and Osher's inequalities [BO91] on the exact solution of problem (1–2). Their inequalities deal with the solution of problem (1–2) when ϕ_1^0 and ϕ_2^0 are :

$$\phi_1^0(x) = \phi_1(x_1, \cdots, x_p) \qquad , \qquad \phi_2^0(x) = \phi_1(x_{p+1}, \cdots, x_N)$$

for some p, $0 \leq p \leq N$. The starting point is Hopf's formula that gives the solution when ψ^0 is either convex or concave :

- for a convex initial condition,

$$\phi(x,t) = \sup_{v \in \mathbb{R}^N} \left[x \,.\, v - \phi^{0*}(v) - tH(v) \right],$$

- and for a concave initial condition,

$$\phi(x,t) = \inf_{v \in \mathbb{R}^N} \left[x \,.\, v - \phi^{0*}(-v) - tH(v) \right].$$

1 General solution

We denote by ψ^* the Legendre transform of ψ. If ψ is convex,

$$\psi^*(v) = \sup_{x\in\mathbb{R}^N} [x.v - \psi(x)],$$

if ψ is concave,

$$\psi^*(v) = \inf_{x\in\mathbb{R}^N} [-x.v - \psi(x)].$$

In both cases, ψ is the Legendre transform of its Legendre transform. We denote by D_ϕ the set defined, for a convex function by

$$D_\phi = \{v \in \mathbb{R}^N, \phi^*(v) < +\infty\}$$

and for a concave function by

$$D_\phi = \{v \in \mathbb{R}^N, \phi^*(v) > -\infty\}.$$

We start by solving (1–2) with

$$\Psi_v^0(x) = \phi_1^0(x) - x\,.\,v - \phi_2^{0*}(v) \geq \phi^0(x). \tag{3}$$

This is a convex function because ϕ_1^0 is convex. We compute Ψ_v^{0*}. By definition,

$$\begin{aligned}
\Psi_v^{0*}(w) &= \sup\nolimits_{x\in\mathbb{R}^N} [x\,.\,w - \Psi_v^0(x)] \\
&= \sup\nolimits_{x\in\mathbb{R}^N} [x.(v+w) - \phi_1^0(x) + \phi_2^{0*}(v)] \\
&= \begin{cases} \phi_2^{0*}(v) + \phi_1^{0*}(v+w) & \text{if} \quad v \in D_{\phi_2^0} \\ +\infty & \text{else} \quad . \end{cases}
\end{aligned}$$

Ψ_v^{0*} is finite if and only if $v \in D_{\phi_2^0}$ and $v + w \in D_{\phi_1^0}$. In the following, we say that $\Psi_v^{0*}(w) = \phi_2^{0*}(v) + \phi_1^{0*}(v+w) = \infty$ if $v \notin D_{\phi_0^2}$. This changes nothing to the results.

Hopf's formula [BE84] says that the solution is

$$\Psi_v(x,t) = \sup_{w\in\mathbb{R}^N} \left[x\,.\,w - \Psi_v^{*0}(w) - tH(w)\right].$$

Standard inequalities on viscosity solutions and (3) show that the viscosity solution ϕ of (1–2) satisfies :

$$\inf_{v\in\mathbb{R}^N}\sup_{w\in\mathbb{R}^N}\Big[x\,.\,w-\phi_2^{0*}(v)-\phi_1^{*0}(v+w)-tH(w)\Big]\geq\phi(x,t).$$

The same arguments show that

$$\sup_{v\in D_{\phi_1^0}}\inf_{w\in\mathbb{R}^N}\Big[x\,.\,v+w-\phi_1^{0*}(v)-tH(w)\Big]\leq\phi(x,t).$$

This generalizes the result of [BO91].

Proposition 1.1 *The viscosity solution ϕ of the Cauchy problem (1–2) satisfies, when $\phi^0=\phi_1^0+\phi_2^0$ for a convex ϕ_1^0 and a concave ϕ_2^0 :*

$$\inf_{v\in\mathbb{R}^N}\sup_{w\in\mathbb{R}^N}\left[x\,.\,(w-v)-\phi_2^{0*}(w)-\phi_1^{*0}(v)-tH(w-v)\right]\geq\phi(x,t)$$

$$\geq\sup_{v\in\mathbb{R}^N}\inf_{w\in\mathbb{R}^N}\left[x\,.\,(w-v)-\phi_2^{0*}(v)-\phi_1^{0*}(w)-tH(w-v)\right]$$

Remark 1.2 *When ϕ_2^0 is identically vanishing, the previous inequalities provide the viscosity solution because $\phi_2^{0*}=-\infty$ except for $v=0$. Similarly, with Bardi and Osher's assumptions, we us set $x=(x_A,x_B)$ when $x_A\in\mathbb{R}^p$ and $x_B\in\mathbb{R}^{N-p}$. If $\phi_1^0(x)\equiv\phi_1^0(x_A)$ and $\phi_2^0(x)\equiv\phi_2^0(x_B)$, clearly we have*

$$\phi_1^{0*}(v)=\begin{cases}+\infty & \text{if}\quad v_B\neq 0\\ \phi_1^{0*}(v_A) & \text{else.}\end{cases}$$

A similar result can be obtained for ϕ_2^{0*} : Bardi and Osher's results can easily be deduced.

2 Application to the solution of Riemann's problem on general meshes

We consider (1–2) for a piecewise linear initial data. More precisely, $\mathbb{R}^N$ is partitioned in angular sectors $\Omega_1,\cdots,\Omega_n$. We define ϕ by :

$$x\in\Omega_i,\phi^0(x)=A+U_i\,.\,x. \tag{4}$$

A piecewise linear function is convex if and only if

$$\forall i \in \{0, \cdots, n\}, \forall x \in \Omega_i, \forall j \in \{0, \cdots, n\}, (U_j - U_i) \,.\, x \leq 0. \tag{5}$$

For a piecewise linear concave function, the $\leq$ of (5) is modified into a $\geq$.

We notice that ϕ^0 is the sum of two piecewise linear functions ϕ_1 and ϕ_2, respectively convex and concave. This decomposition is not unique, but ϕ_1' and ϕ_2' is another one, there exists a linear function ψ such that $\phi_1' = \phi_1 + \psi$ and $\phi_2' = \phi_2 - \psi$. This result is recalled in Appendix A.

What remains is to compute the Legendre transform of a piecewise linear convex function. We have

$$\forall x \in \Omega_i, x \,.\, v - \phi(x) = A + x \,.\, (v - U_i).$$

Since each Ω_i is invariant by similarity, if v satisfies : there exists $i \in \{0, \cdots, n\}$ et $x \in \Omega_i$ so that

$$(v - U_i) \,.\, x > 0,$$

then $\phi^*(v) = +\infty$. Hence,

$$\phi^*(v) < +\infty \iff \forall i \in \{0, \cdots, n\}, \ \forall x \in \Omega_i, (v - U_i) \,.\, x \leq 0, \tag{6}$$

and we have $\phi^*(v) = A$ because $0 \in \Omega_i$. We characterize this set. It is convex, it contains each U_j, $j \in \{0, \cdots, n\}$ since ϕ is convex, thus their convex envelope $\mathcal{K}_\phi$. Assume now there exists $v \notin \mathcal{K}$ so $\phi^*(v) < +\infty$. Their exists w and α such that $w \,.\, v > \alpha$ and $U_i \,.\, w < \alpha$ for all $i \in \{0, \cdots, n\}$ thus $w \,.\, (v - U_i) > 0$ and then $\phi^*(v) = +\infty$. We have proved

$$\phi^*(v) = \begin{cases} A & \text{if } \ v \in \mathcal{K}_\phi \\ \\ +\infty & \text{else.} \end{cases} \tag{7}$$

The formula (7) is still true for a concave function if we change the $+\infty$ into $-\infty$ and replace $\mathcal{K}_\phi$ by $\mathcal{K}_{-\phi}$. Clearly, $\mathcal{K}_{-\phi}$ is the symetric of $\mathcal{K}_\phi$ with respect to the origin. Thus, We obtain the following inequalities on the solution of the Riemann problem :

$$\begin{array}{rcl} A + \inf_{v \in \mathcal{K}_{\phi_1}} \sup_{w \in \mathcal{K}_{\phi_2}} (x \,.\, (w + v) - tH(w + v)) & \geq & \phi(x, t) \\ A + \sup_{v \in \mathcal{K}_{\phi_1}} \inf_{w \in \mathcal{K}_{\phi_2}} (x \,.\, (w + v) - tH(w + v)) & \leq & \phi(x, t). \end{array} \tag{8}$$

Note that (8) is independent of the decomposition of ϕ. This formula helps to understand how to discretize this type of PDE on an unstructured mesh. These equations are models for a wide variety of physical phenomenons (propagation of flames for example), they are helpful for computer vision, mesh generation, etc. These problems have to be studied on complex geometries.

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A Decomposition of a piecewise linear function

Let us assume that $\mathbb{R}^2 = \Omega_1 \cup \cdots \cup \Omega_n$, each angular sectors meet at O. We recall how is built the decomposition of a piecewise linear function ϕ as the sum of piecewise linear convex and concave functions, where both are continuous.

We consider the convex envelope of the graph of ϕ. It is the graph of a continuous piecewise linear function ψ. We want to show that $\psi' = \phi - \psi$ is the sum of two piecewise linear function, where one is convex wheras the other one is concave.

Since the graph of ψ is the convex envelope of ϕ, ψ' is the sum of functions that are zero everywhere except in a collection of angular sectors that meet at O. Moreover it is a negative function. Each of these angular sectors is the collection of several Ω_i. In order to simplify the text, we assume that there are only two of them, Ω_1 and Ω_2, as on Figure 1. We set $\Omega = \mathbb{R}^2 - \Omega_1 \cup \Omega_2$, and

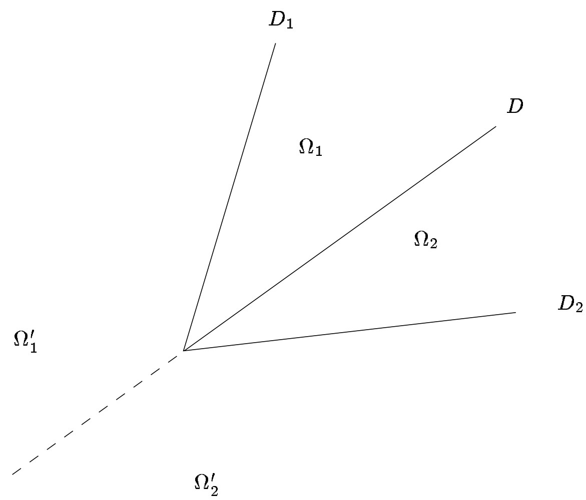

Figure 1:

we consider the case of a function $\phi \geq 0$ defined by :

$$\phi(x) = \begin{cases} 0 & \text{in} \quad \Omega \\ U_1 \,.\, x & \text{in} \quad \Omega_1 \\ U_2 \,.\, x & \text{in} \quad \Omega_2. \end{cases}$$

We define the lines D, D_1 et D_2, and the sets Ω_1', Ω_2' by :

- $U_1 \,.\, x = U_2 \,.\, x$ on D,
- $U_1 \,.\, x = 0$ on D_1,
- $U_2 \,.\, x = 0$ on D_2,
- $U_1 \,.\, x \leq U_2 \,.\, x \leq 0$ on Ω_1' and $U_2 \,.\, x \leq U_1 \,.\, x \leq 0$ on Ω_2'.

To say that ψ' is neither convex or concave means, because of the concavity–convexity criterion of §2 :

- $U_1 \,.\, x \leq U_2 \,.\, x$ on $\Omega_1 \cup \Omega_1'$, with $0 \leq U_1 \,.\, x \leq U_2 \,.\, x$ on Ω_1,

- $U_2 \,.\, x \leq U_1 \,.\, x$ on $\Omega_2 \cup \Omega_2'$, with $0 \leq U_2 \,.\, x \leq U_1 \,.\, x$ on Ω_2.

Let us notice that ϕ is concave on $\Omega_1 \cup \Omega_2$. Let ϕ_1 be :

$$\phi_1(x) = \begin{cases} 0 & \text{in} \quad \Omega_1 \cup \Omega_2 \\ v_1 \,.\, x & \text{in} \quad \Omega_1' \\ v_2 \,.\, x & \text{in} \quad \Omega_2'. \end{cases}$$

We are looking for a continuous ϕ_1 : ϕ_1 is vanishing on D_1 and D_2, hence $v_1 = \alpha_1 U_1$ and $v_2 = \alpha_2 U_2$ on one hand and on the other hand we have $v_1 \,.\, x = v_2 \,.\, x$ on D so that $\alpha_1 = \alpha_2 \equiv \alpha$. With the criterion of §2, we check that ϕ_1 is convex if $\alpha < 0$ because $U_1 \,.\, x \leq U_2 \,.\, x$ on Ω_1' and we have the inverse inequality on Ω_2'. Let us compute $\phi_2 = \phi - \psi$:

$$\phi_2(x) = \begin{cases} -\alpha U_1 \,.\, x & \text{in} \quad \Omega_1' \\ -\alpha U_2 \,.\, x & \text{in} \quad \Omega_2' \\ U_1 \,.\, x & \text{in} \quad \Omega_1 \\ U_2 \,.\, x & \text{in} \quad \Omega_2 \end{cases}$$

and check that ϕ_2 is concave for $\alpha < -1$:

- $x \in \Omega_1'$, we have :
  1. $-\alpha(U_2 - U_1) \,.\, x \geq 0$ because $\alpha < 0$ and by definition of Ω_1',
  2. $(1 + \alpha) U_1 \,.\, x \geq 0$ because $1 + \alpha < 0$ and by definition of Ω_1',
  3. $(U_2 + \alpha U_1) \,.\, x \geq (U_1 + \alpha U_1) \,.\, x \geq 0$ for the same reasons.
- $x \in \Omega_1$, we have :
  1. $-\alpha(U_2 - U_1) \,.\, x \geq 0$ because $\alpha < 0$ and by definition of Ω_1,
  2. $(-U_1 - \alpha U_2) \,.\, x (U_2 + \alpha U_2) \,.\, x \geq 0$ because $1+\alpha < 0$ and by definition of Ω_1,
  3. $(U_2 - U_1) \,.\, x \geq 0$ for the same reasons.
- For $x \in \Omega_2'$ ou Ω_2, we have the same results for the same reasons.

Any continuous piecewise linear function is the sum of two continuous piecewise linear functions where one is convex and the other one is concave. If two decomposition are possible, $\phi_1 + \phi_2 = \psi_1 + psi_2$, where phi_1, ψ_1 are convex and ϕ_2, ψ_2 concave, then the convex piecewise linear function $\phi_1 - \psi_2$ is equal to the concave piecewise linear function $\phi_2 - \psi_1$, so that they are linear.

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Éditeur

INRIA, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)

ISSN 0249-6399