Convergence Analysis of Domain Decomposition algorithms with full overlapping for the advection-diffusion problems

Patrick Le Tallec, Moulay D. Tidriri

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET AUTOMATIQUE

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Abstract: The aim of this paper is to study the convergence properties of a Time Marching Algorithm solving Advection-Diffusion problems on two domains using incompatible discretizations. The basic algorithm is first presented, and theoretical or numerical results illustrate its convergence properties. This study is based on spectral theory, a priori estimates and a Di-Giorgi-Nash maximum principle .

Key-words: non symmetric elliptic operators, domain decomposition, incompatible grids, advection-diffusion, time marching algorithms, overlapping, maximun principle.

(Résumé : tsvp)

*Université Paris-Dauphine and INRIA, Domaine de Voluceau- Rocquencourt- B.P. 105- Le Chesnay Cedex (France) *email:* Patrick.LeTallec@inria.fr

**Department of Mechanical Engineering, Yale University, New Haven, CT 06520 *email:* tidriri@icase.edu. This work has been done in 1991 at INRIA and has been supported by the Hermes Research program under grant number RDAN 86.1/3. The author is presently supported by the National Science Foundation under contract number ECS-8957475 and by the United Technologies Research Center.

Unité de recherche INRIA Rocquencourt
Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)
Téléphone : (33 1) 39 63 55 11 – Télécopie : (33 1) 39 63 53 30

Analyse de Convergence des Méthodes de Décomposition de Domaines avec Recouvrement total

Résumé : Le but de l'article proposé est d'étudier les propriétés de convergence d'un algorithme de marche en temps appliqué à la solution d'un problème d'advection diffusion posè sur deux sous-domaines recouvrants. Les discrétisations utilisées sur chaque sous-domaine peuvent être totalement indépendantes. L'article rappelle les principes de l'algorithme utilisé, en présente une analyse théorique, et une illustration numérique. L'étude théorique s'appuie sur des estimations apriori, sur le principe du maximun de Di Giorgi-Nash, et sur une analyse spectrale unidimensionnelle.

Mots-clé : opérateurs elliptiques nonsymétriques, décomposition de domaines, grilles incompatibles, advection-diffusion, algorithmes de marche en temps, recouvrement, principe du maximum.

1 Introduction

Domain decomposition methods have recently appear as an efficient strategy for solving large scale problems on parallel computers ([1], [2], [3], [4], [5], [6]). Nevertheless, they can also be used in order to couple different models [11], [17]. In this paper we will examine a domain decomposition strategy which can be applied to such situations.

This approach was introduced in order to solve several difficulties that occur in fluid mechanics. In particular, our aim is to introduce several sub-domains in order to do one of the following :

- Solve different problems on each subdomain.
- Use different kinds of approximation methods on each subdomain [7].
- Use "local refinement techniques" or "mesh adaptive techniques", locally, per subdomain ([10]).

The subdomains fully overlap and the coupling is achieved through "friction" forces acting on the internal boundary of the domain, these friction forces being updated by an explicit time marching algorithm.

The theoretical study of our method will be done on an Advection-Diffusion problem, which will serve as our model problem from now on. The analysis will be made at the continuous level, independently of any discretization strategy, which means that the derived results will be mesh independent.

In the next section we will describe this model problem. In the third section we will present our algorithm for some special cases. The fourth section will treat the one-dimensional stationary problem.

We will show from this study that we can improve the convergence of this method by introducing a relaxation parameter (see [5]) and by replacing one of the Dirichlet condition by a Neumann type condition. The fifth section will focus on the resulting multidimensional case. We first study the proposed strategy for the time implicit problem. We then establish a priori estimates and show from these estimates exponential convergence of our algorithm. A detailed study of the semi-discrete algorithm will be done in section 6. In the last section we study the numerical stability of the explicit algorithm. Practical applications of the proposed algorithm to real life CFD problems can be found in [14], [18], [19], and [20].

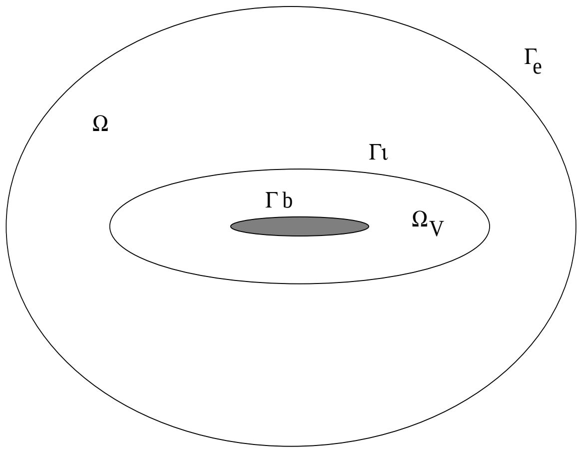

Figure 1: description of computational domain.

2 The model problem

Consider a bounded domain, Ω of $\mathbb{R}^n$ such that its boundary $\partial\Omega$ is lipschitzian and $\Omega_{loc} = \Omega_v$ a connected domain of $\mathbb{R}^n$ with $\Omega_{loc} \subset \Omega$ (fig. 1). Let :

$$\Gamma_b = \partial\Omega \cap \partial\Omega_{loc}, (\text{ internal boundary})$$

$$\Gamma_i = \partial\Omega_{loc} \cap \Omega, (\text{ interface})$$

$$\Gamma_\infty = \partial\Omega \backslash \Gamma_b .(\text{farfield boundary})$$

We denote by n the external unit normal vector to $\partial\Omega$ or $\partial\Omega_{loc}$.

Let v a velocity field given such that :

$$\left\{ \begin{array}{rcll} \mathrm{div} v & = & 0 & \text{in} \quad \Omega, \\ v.n & = & 0 & \text{on} \quad \Gamma_b. \end{array} \right. \tag{1}$$

The model problem we want to solve is the following:

Find φ, a real valued function, defined on Ω and satisfying :

$$\left\{\begin{array}{rcl} \mathrm{div}(v\varphi) - \nu\Delta\varphi &=& 0 \text{ in } \Omega, \\ \varphi &=& \varphi^{\infty} \text{ on } \Gamma_{\infty}, \\ \varphi &=& 0 \text{ on } \Gamma_b. \end{array}\right. \tag{2}$$

Above v is the flow velocity and ν is the diffusion coefficient.

Most CFD algorithms will in fact consider the solution of this problem as the stationary solution of the evolution problem (3) described below :

$$\text{Find } \phi : \Omega \times (0,T) \rightarrow \mathbb{R} \text{ such that,}$$

$$\left\{\begin{array}{rcl} \dfrac{\partial\phi}{\partial t} + \mathrm{div}(v\phi) - \nu\Delta\phi &=& 0 \;\text{ in }\; \Omega \times (0,T), \\ \phi &=& \phi^{\infty} \;\text{ on }\; \Gamma_{\infty} \times (0,T), \\ \phi &=& 0 \;\text{ on }\; \Gamma_b \times (0,T), \\ \phi(0) &=& \phi_0 \;\text{ in }\; \Omega. \end{array}\right. \tag{3}$$

The general CFD algorithm consists then in integrating (3) in time until reaching a stationary solution.

3 General Algorithm

3.1 Time-Continuous case

Let us introduce the local sub-domain Ω_{loc} (see fig. 1) which has as external boundary Γ_i, and let us consider the trace ϕ_{loc} of ϕ on the sub-domain Ω_{loc}, as an independent variable, to which we associate an arbitrary independent initial value $\phi_{ol} \neq \phi_o|_{\Omega_{loc}}$. We now replace the evolution problem (3) by the following evolution system :

Find ϕ (resp. ϕ_{loc}) : $\Omega \rightarrow \mathbb{R}$ (resp. $\Omega_{loc} \rightarrow \mathbb{R}$) satisfying :

$$\left\{\begin{array}{rcl} \dfrac{\partial\phi}{\partial t} + \mathrm{div}(v\phi) - \nu\Delta\phi &=& 0 \;\text{ in }\; \Omega \times (0,T), \\ \phi &=& \phi^{\infty} \;\text{ on }\; \Gamma_{\infty} \times (0,T), \\ \nu\dfrac{\partial\phi}{\partial n} &=& \nu\dfrac{\partial\phi_{loc}}{\partial n} \;\text{ on }\; \Gamma_b \times (0,T), \end{array}\right. \tag{4}$$

$$\left\{ \begin{aligned} \frac{\partial \phi_{loc}}{\partial t} + \mathrm{div}(v\phi_{loc}) - \nu \Delta \phi_{loc} &= 0 \text{ in } \Omega_{loc} \times (0,T), \\ \phi_{loc} &= 0 \text{ on } \Gamma_b \times (0,T), \\ \phi_{loc} &= \phi \text{ on } \Gamma_i \times (0,T), \end{aligned} \right. \tag{5}$$

$$\phi(0) = \phi_o \text{ in } \Omega, \ \ \phi_{loc}(0) = \phi_{ol} \text{ in } \Omega_{loc}. \tag{6}$$

Remark 3.1 *The global problem (4)-(6) has no no-slip boundary condition. This suppresses the boundary layer which appears at low viscosity and facilitates the numerical solution of this problem. The boundary layers are modeled by the local problems (5) (6) which are only to be solved on a small domain Ω_{loc}, with a very fine discretisation if needed. The two problems are only coupled by their boundary conditions and not by volumic interpolation.*

Remark 3.2 *The "Neumann" version of this method consists in replacing in (5) the boundary condition on Γ_i by the corresponding Neumann condition :*

$$\frac{\partial \phi_{loc}}{\partial n} = \frac{\partial \phi}{\partial n} \ \ on \ \Gamma_i.$$

3.2 Time Discrete case

The general algorithm that we propose for the solution of our model problem (2) is as usual to integrate in time the evolution problem (4)-(5)-(6) until we reach a stationary solution. This integration in time is then achieved by the following uncoupled semi-explicit algorithm, where the operators are treated implicitely inside each subdomain and where the coupling boundary conditions are treated explicitely :

- set $\phi^0_{loc} = \phi_{ol}$ and $\phi^0 = \phi_0$,
- then, for $n \geq 0, \phi^n_{loc}$ and ϕ^n being known, solve successively

$$\left\{ \begin{aligned} \frac{\phi^{n+1}_{loc} - \phi^n_{loc}}{\Delta t} + \mathrm{div}(v\phi^{n+1}_{loc}) - \nu \Delta \phi^{n+1}_{loc} &= 0 \text{ in } \Omega_{loc}, \\ \phi^{n+1}_{loc} &= \phi^n \text{ on } \Gamma_i, \\ \phi^{n+1}_{loc} &= 0 \text{ on } \Gamma_b, \end{aligned} \right. \tag{7}$$

$$\left\{ \begin{array}{rcl} \dfrac{\phi^{n+1}-\phi^n}{\Delta t} + \operatorname{div}(v\phi^{n+1}) - \nu\Delta\phi^{n+1} & = & 0 \text{ in } \Omega, \\ \phi^{n+1} & = & \phi^\infty \text{ on } \Gamma_\infty, \\ \nu\dfrac{\partial\phi^{n+1}}{\partial n} & = & \nu\dfrac{\partial\phi^{n+1}_{loc}}{\partial n} \text{ on } \Gamma_b. \end{array} \right. \tag{8}$$

Remark 3.3 *We have a full decoupling between (7) and (8). They can be discretized and solved by two independent solution techniques.*

Remark 3.4 *The "Neumann" version of this method consists in replacing the Dirichlet boundary condition on Γ_i by the corresponding Neumann condition :*

$$\frac{\partial\phi^{n+1}_{loc}}{\partial n} = \frac{\partial\phi^n}{\partial n} \quad on \ \Gamma_i.$$

Remark 3.5 *The fully implicit version of this method consists in replacing the condition :*

$$\phi^{n+1}_{loc} = \phi^n \quad on \ \Gamma_i$$

(resp. $\dfrac{\partial\phi^{n+1}_{loc}}{\partial n} = \dfrac{\partial\phi^n}{\partial n}$ on Γ_i) by the condition :

$$\phi^{n+1}_{loc} = \phi^{n+1} \quad on \ \Gamma_i,$$

(resp. $\dfrac{\partial\phi^{n+1}_{loc}}{\partial n} = \dfrac{\partial\phi^{n+1}}{\partial n}$ on Γ_i).

The two subproblems are then coupled at each time step.

Remark 3.6 *If we replace in (8) Ω by Ω_E defined as follows :*

$$\Omega_E = \Omega\backslash\Omega_{loc},$$

and Γ_b by Γ_i, and if we set $\Delta t = \infty$, we obtain a nonoverlapping version of our strategy, which is a standard Dirichlet-Neumann algorithm [15], [16].

Remark 3.7 *The initial condition ϕ_{ol} is not assumed to be equal to ϕ_0 on the local subdomain Ω_{loc} because in most cases this condition is impossible to impose at the discrete level since the grid used on Ω_{loc} will be in general different from the grid used on Ω. In addition, even if we assume $\phi_{ol} = \phi_0$, we will not have $\phi^n_{loc} = \phi^n$ on Ω_{loc} unless we use the fully implicit algorithm on compatible grids.*

4 Stationary one-dimensional case

For $\Delta t = +\infty$, the above algorithm can be written :

- set $\phi_{loc}^0 = \phi_0$ and $\phi^0 = \phi_0$,
- then, for $n \geq 0, \phi_{loc}^n$ and ϕ^n being known, solve

$$\left\{ \begin{array}{rcl} \mathrm{div}(v\phi_{loc}^{n+1}) - \nu\Delta\phi_{loc}^{n+1} & = & 0 \text{ in } \Omega_{loc}, \\ \phi_{loc}^{n+1} & = & \phi^n \text{ on } \Gamma_i, \\ \phi_{loc}^{n+1} & = & 0 \text{ on } \Gamma_b, \end{array} \right. \tag{9}$$

$$\left\{ \begin{array}{rcl} \mathrm{div}(v\phi^{n+1}) - \nu\Delta\phi^{n+1} & = & 0 \text{ in } \Omega, \\ \phi^{n+1} & = & \phi^\infty \text{ on } \Gamma_\infty, \\ \nu\dfrac{\partial\phi^{n+1}}{\partial n} & = & \nu\dfrac{\partial\phi_{loc}^{n+1}}{\partial n} \text{ on } \Gamma_b. \end{array} \right. \tag{10}$$

In one space dimension, if we take as domain Ω the interval $]0,1[$ of $\mathbb{R}$ decomposed into two sub-domains $\Omega =]0,1[$ and $\Omega_{loc} =]h_2, 1[$

$$0 < h_2 < 1, \tag{11}$$

then we obtain

$$\left\{ \begin{array}{rcl} v\varphi_2^{(n)'} - \nu\varphi_2^{(n)"} & = & 0 \quad \text{on} \quad]h_2, 1[, \\ \varphi_2^{(n)}(h_2) & = & \varphi_1^{(n-1)}(h_2), \\ \varphi_2^{(n)}(1) & = & b, \end{array} \right. \tag{12}$$

$$\left\{ \begin{array}{rcl} v\varphi_1^{(n)'} - \nu\varphi_1^{(n)"} & = & 0 \quad \text{on} \quad]0, 1[, \\ \varphi_1^{(n)}(0) & = & a, \\ \varphi_1^{(n)'}(1) & = & \varphi_2^{(n)'}(1). \end{array} \right. \tag{13}$$

By introducing relaxation parameters, we can also introduce the following variants of the above algorithm :

$$\left\{\begin{array}{rcl} v\varphi_2^{(n)'} - \nu\varphi_2^{(n)"} & = & 0 \quad \text{on} \quad]h_2,1[, \\ \varphi_2^{(n)}(1) & = & b, \\ \varphi_2^{(n)}(h_2) & = & \theta_2\varphi_1^{(n-1)}(h_2) + (1-\theta_2)\varphi_2^{(n-1)}(h_2), \end{array}\right. \tag{14}$$

$$\left\{\begin{array}{rcl} v\varphi_1^{(n)'} - \nu\varphi_1^{(n)"} & = & 0 \quad \text{on} \quad]0,1[, \\ \varphi_1^{(n)}(0) & = & a, \\ \varphi_1^{(n)'}(1) & = & \theta_1\varphi_2^{(n)'}(1) + (1-\theta_1)\varphi_1^{(n-1)'}(1). \end{array}\right. \tag{15}$$

We will now try to produce conditions under which algorithm (15)-(14) converges, and those for which this convergence is optimal. For this purpose, we write the interface solution under the form

$$\varphi_1^{(n)'}(1) = \psi'(1) + \gamma^n, \tag{16}$$

$$\varphi_2^{(n)}(h_2) = \psi(h_2) + \delta^n, \tag{17}$$

where ψ is the converged solution we are looking for. Using the analytical solutions of problems (14) (15), we obtain the following induction formula

$$\begin{pmatrix} \delta^n \\ \gamma^n \end{pmatrix} = M_{IN} \begin{pmatrix} \delta^{(n-1)} \\ \gamma^{(n-1)} \end{pmatrix} . \tag{18}$$

with

$$M_{IN} = \begin{pmatrix} 1-\theta_2 & \theta_2 \dfrac{\nu}{v} e^{-(\frac{v}{\nu})}(e^{\frac{v}{\nu}h_2} - 1) \\ \dfrac{\theta_1(1-\theta_2)(\frac{v}{\nu})}{e^{(\frac{v}{\nu})(h_2-1)} - 1} & \dfrac{e^{(\frac{-v}{\nu})}\theta_1\theta_2(e^{(\frac{v}{\nu})h_2} - 1)}{(e^{(\frac{v}{\nu})(h_2-1)} - 1)} + (1-\theta_1) \end{pmatrix} \tag{19}$$

This iterative process converges if the spectral radius of matrix M_{IN} is less than 1. A direct but tedious calculation then yields :

Lemma 4.1 *The spectral radius of the transfer matrix of algorithm (15)-(14) is :*

$$\rho(M_{IN}) = max[\frac{1}{2}|D \pm \sqrt{D^2 - 4R}|] \tag{20}$$

with

$$D = 2 - (\theta_1 + \theta_2) + \theta_1\theta_2 e^{(-v/\nu)}(e^{(v/\nu)h_2} - 1)\frac{1}{e^{-v/\nu}e^{v(h_2/\nu)} - 1} \tag{21}$$

$$R = (1-\theta_1)(1-\theta_2). \tag{22}$$

From this calculation, we can get the following results :

i) When h_2 goes to 1 (nonoverlapping), D goes to $+\infty$, and then, $\rho(M_{IN})$ goes to $+\infty$. There is no-convergence at this limit.

ii) The optimal convergence is obtained in the case where all the eigenvalues of the matrix M_{IN} are zero, i.e., when : $D = 0$ and $R = 0$. The latter conditions imply in particular

$$\theta_1 = 1 \text{ or } \theta_2 = 1.$$

If we choose, in addition, $\theta_1 = \theta_2$, the condition $D = 0$ implies $h_2 = 0$. In this case the sub-domain Ω_{loc} is equal to the whole domain, and the associate algorithm has no interest.

iii) The convergence of the method depends symmetrically on both relaxation parameters.

From ii), it is reasonable to take one of the θ_i equal to 1 and call the other θ. By setting :

$$A = 1 - \frac{e^{(-v/\nu)}(e^{(v/\nu)h_2} - 1)}{e^{(v/\nu)(h_2-1)} - 1} \tag{23}$$

we have then

$$\rho(M_{IN}) = |1 - \theta A|. \tag{24}$$

In this case, we get the following convergence results :

Theorem 4.1 *1) There is optimal convergence (in 1 iteration) if*

$$\theta = \theta_{opt} = \{1 - \frac{e^{(-v/\nu)}(e^{(v/\nu)h_2} - 1)}{e^{(v/\nu)(h_2-1)} - 1}\}^{-1} < 1. \tag{25}$$

2) The algorithm converges for all θ in $]0, \frac{2}{A}[$.

Corollary 4.1 *1) The case without relaxation ($\theta = 1$) converges only if :*

$$\frac{2}{A} \geq 1,$$

i.e., by setting $d = 1 - h_2$ (overlapping length), only if :

$$d \geq \frac{\nu}{v} Log \frac{2}{(1 + e^{-v/\nu})} \quad \textit{(stability condition)}.$$

2) When v goes to zero, we must have $d \geq \frac{1}{2}$.

Remark 4.1 *This theorem states that the application of algorithm (15)-(14) to the steady case converges only if the overlapping d is sufficiently large. In the same situation, we will see that the unsteady approach will converge to the same steady solution but with less restrictions on d. This motivates the introduction of the time marching algorithm of section 3. Moreover, this time marching technique is well adapted to nonlinear problems such as those encountered in fluid mechanics.*

5 Implicit time discretization

5.1 The general algorithm

This section deals with the convergence analysis of the proposed algorithm in multiple dimension when one uses the fully implicit version of our strategy (4)-(6) :

- Set $\phi^0_{loc} = \phi_{ol}$ and $\phi^0 = \phi_0$;
- then, for $n \geq 0, \phi^n_{loc}$ and ϕ^n being known, solve

$$\left\{ \begin{array}{rcll} \dfrac{\phi^{n+1} - \phi^n}{\Delta t} + \operatorname{div}(v\phi^{n+1}) - \nu\Delta\phi^{n+1} & = & 0 & \text{in} \quad \Omega, \\ \phi^{n+1} & = & \phi_\infty & \text{on} \quad \Gamma_\infty, \\ \nu \dfrac{\partial \phi^{n+1}}{\partial n} & = & \nu \dfrac{\partial \phi^{n+1}_{loc}}{\partial n} & \text{on} \quad \Gamma_b, \end{array} \right. \tag{26}$$

$$\left\{ \begin{array}{rcll} \dfrac{\phi^{n+1}_{loc} - \phi^n_{loc}}{\Delta t} + \operatorname{div}(v\phi^{n+1}_{loc}) - \nu\Delta\phi^{n+1}_{loc} & = & 0 & \text{in} \quad \Omega_{loc}, \\ \phi^{n+1}_{loc} & = & \phi^{n+1} & \text{on} \quad \Gamma_i, \\ \phi^{n+1}_{loc} & = & 0 & \text{on} \quad \Gamma_b. \end{array} \right. \tag{27}$$

5.2 Convergence analysis

We have the following convergence result :

Theorem 5.1 *The solution of algorithm (26)-(27) converges linearly in $H^1(\Omega)$ towards the solution of the stationary problem (2), for all values of Δt and all choices of Ω_{loc}.*

Proof

Step 1: Local L^2 estimates of $\phi - \phi_{\mathbf{loc}}$

Substracting (27) from (26), multiplying the result by $\phi^{n+1} - \phi_{loc}^{n+1}$ and integrating by parts, we obtain the following relation :

$$\begin{aligned}\int_{\Omega_{loc}} \frac{1}{\Delta t}(\phi^{n+1} - \phi_{loc}^{n+1})^2 - \int_{\Omega_{loc}} \frac{1}{\Delta t}(\phi^n - \phi_{loc}^n)(\phi^{n+1} - \phi_{loc}^{n+1}) \\ + \int_{\Omega_{loc}} \nu|\nabla(\phi^{n+1} - \phi_{loc}^{n+1})|^2 = 0.\end{aligned} \tag{28}$$

By using Cauchy-Schwarz, we deduce

$$\begin{aligned}\frac{1}{2\Delta t}\|\phi^{n+1} - \phi_{loc}^{n+1}\|_{0,2,\Omega_{loc}}^2 + \nu|\phi^{n+1} - \phi_{loc}^{n+1}|_{1,2,\Omega_{loc}}^2 \\ \leq \frac{1}{2\Delta t}\|\phi^n - \phi_{loc}^n\|_{0,2,\Omega_{loc}}^2.\end{aligned} \tag{29}$$

From the Poincaré inequality, it follows that

$$\|\phi^{n+1} - \phi_{loc}^{n+1}\|_{0,2,\Omega_{loc}}^2 \leq \frac{1}{1 + 2\nu\Delta tc}\|\phi^n - \phi_{loc}^n\|_{0,2,\Omega_{loc}}^2 \tag{30}$$

where c is the Poincaré constant. By setting $c_o = \|\phi^o - \phi_{loc}^o\|_{0,2,\Omega_{loc}}^2$, we get by induction our basic L^2 estimate

$$\|\phi^{n+1} - \phi_{loc}^{n+1}\|_{0,2,\Omega_{loc}}^2 \leq (\frac{1}{1 + 2\nu\Delta tc})^{n+1}c_o. \tag{31}$$

By summing (29), we also obtain

$$\begin{aligned}\|\phi^{n+1} - \phi_{loc}^{n+1}\|_{0,2,\Omega_{loc}}^2 + 2\nu\Delta t\sum_{i=p}^{n}|\phi^{i+1} - \phi_{loc}^{i+1}|_{1,2,\Omega_{loc}}^2 \\ \leq \|\phi^p - \phi_{loc}^p\|_{0,2,\Omega_{loc}}^2.\end{aligned} \tag{32}$$

In particular, we obtain for $p = 0$

$$\|\phi^{n+1} - \phi_{loc}^{n+1}\|_{0,2,\Omega_{loc}}^2 + 2\nu\Delta t \sum_{i=0}^{n} |\phi^{i+1} - \phi_{loc}^{i+1}|_{1,2,\Omega_{loc}}^2 \leq \|\phi^o - \phi_{loc}^o\|_{0,2,\Omega_{loc}}^2. \tag{33}$$

Step 2: Local H^1 estimates of $\phi^{\mathbf{n}} - \phi_{\mathbf{loc}}^{\mathbf{n}}$

Substracting the two first equations in (26) and (27), multiplying the result by

$$\delta x^n = \frac{(\phi^{n+1} - \phi_{loc}^{n+1}) - (\phi^n - \phi_{loc}^n)}{\Delta t}, \tag{34}$$

and integrating on Ω_{loc}, we obtain

$$\begin{aligned} 0 \quad = \quad & \int_{\Omega_{loc}} |\delta x^n|^2 + \int_{\Omega_{loc}} \operatorname{div}(v(\phi^{n+1} - \phi_{loc}^{n+1}))\delta x^n \\ & +\nu \int_{\Omega_{loc}} \nabla(\phi^{n+1} - \phi_{loc}^{n+1})\nabla\delta x^n \\ & -\nu \int_{\partial\Omega_{loc}} \frac{\partial}{\partial n}(\phi^{n+1} - \phi_{loc}^{n+1})\delta x^n. \end{aligned} \tag{35}$$

The use of the third relation in (26) and the second relation in (27) then yields

$$\int_{\Omega_{loc}} |\delta x^n|^2 + \int_{\Omega_{loc}} \operatorname{div}(v(\phi^{n+1} - \phi_{loc}^{n+1}))\delta x^n + \nu \int_{\Omega_{loc}} \nabla(\phi^{n+1} - \phi_{loc}^{n+1})\nabla\delta x^n = 0.$$

By using now Cauchy-Schwarz, we obtain

$$\begin{aligned} \int_{\Omega_{loc}} |\delta x^n|^2 + \nu \int_{\Omega_{loc}} \nabla(\phi^{n+1} - \phi_{loc}^{n+1})\nabla\delta x^n \quad \leq \quad & \frac{\|v\|_\infty^2}{2}|\phi^{n+1} - \phi_{loc}^{n+1}|_{1,2,\Omega_{loc}}^2 \\ & +\frac{1}{2}\|\delta x^n\|_{0,2,\Omega_{loc}}^2. \end{aligned}$$

By construction of δx^n and Cauchy-Schwartz, the last inequality becomes

$$\begin{aligned}\frac{1}{2}\|\delta x^n\|^2_{0,2,\Omega_{loc}} &\leq \frac{\|v\|^2_\infty}{2}|\phi^{n+1}-\phi^{n+1}_{loc}|^2_{1,2,\Omega_{loc}}+\frac{\nu}{2\Delta t}|\phi^n-\phi^n_{loc}|^2_{1,2,\Omega_{loc}}\\ &\quad -\frac{\nu}{2\Delta t}|\phi^{n+1}-\phi^{n+1}_{loc}|^2_{1,2,\Omega_{loc}}.\end{aligned} \tag{36}$$

By using again relation (29), it follows that

$$\begin{aligned}\|\delta x^n\|^2_{0,2,\Omega_{loc}} &\leq \frac{\|v\|^2_\infty}{2\nu\Delta t}(\|\phi^n-\phi^n_{loc}\|^2_{0,2,\Omega_{loc}}-\|\phi^{n+1}-\phi^{n+1}_{loc}\|^2_{0,2,\Omega_{loc}})\\ &\quad +\frac{\nu}{\Delta t}(|\phi^n-\phi^n_{loc}|^2_{1,2,\Omega_{loc}}-|\phi^{n+1}-\phi^{n+1}_{loc}|^2_{1,2,\Omega_{loc}})\\ &\leq \frac{\nu}{\Delta t}(G(n)-G(n+1))\end{aligned} \tag{37}$$

with G defined by

$$G(n)=\frac{\|v\|^2_\infty}{2\nu^2}\|\phi^n-\phi^n_{loc}\|^2_{0,2,\Omega_{loc}}+|\phi^n-\phi^n_{loc}|^2_{1,2,\Omega_{loc}}. \tag{38}$$

From (37), G is a decreasing function. This last property then yields

$$\begin{aligned}(n+2-p)G(n+1) &\leq \sum_{i=p}^{n+1}G(i)\\ &\leq \sum_{i=p}^{n+1}|\phi^i-\phi^i_{loc}|^2_{1,2,\Omega_{loc}}+\frac{\|v\|^2_\infty}{2\nu^2}\sum_{i=p}^{n+1}\|\phi^i-\phi^i_{loc}\|^2_{0,2,\Omega_{loc}}\end{aligned} \tag{39}$$

By using (29) again, we obtain

$$(n+2-p)G(n+1)\leq\frac{1}{2\nu\Delta t}\sum_{i=p}^{n+1}\|\phi^{i-1}-\phi^{i-1}_{loc}\|^2_{0,2,\Omega_{loc}}+\frac{\|v\|^2_\infty}{2\nu^2}\sum_{i=p}^{n+1}\|\phi^i-\phi^i_{loc}\|^2_{0,2,\Omega_{loc}}$$

But since $\|\phi^i-\phi^i_{loc}\|_{0,2,\Omega_{loc}}$ is a decreasing function, we finally have

$$\begin{aligned}&(n+2-p)G(n+1)\leq\\ &\quad (n+2-p)\left(\frac{1}{2\nu\Delta t}+\frac{\|v\|^2_\infty}{2\nu^2}\right)\|\phi^{p-1}-\phi^{p-1}_{loc}\|^2_{0,2,\Omega_{loc}},\end{aligned} \tag{40}$$

that is

$$G(n+1) \leq \left(\frac{1}{2\nu\Delta t} + \frac{\|v\|_\infty^2}{2\nu^2}\right)\|\phi^{p-1} - \phi_{loc}^{p-1}\|_{0,2,\Omega_{loc}}^2, \forall p \leq n. \tag{41}$$

Step 3: Global L^2 estimate of $\phi^{\mathbf{n}} - \phi_{\mathbf{sta}}$

We now define $\bar{\phi}^{n+1}$ by

$$\bar{\phi}^{n+1} = \begin{cases} \phi^{n+1} - \phi_{sta} & \text{in} \quad \Omega\backslash\Omega_{loc}, \\ \phi_{loc}^{n+1} - \phi_{sta} & \text{in} \quad \Omega_{loc}, \end{cases} \tag{42}$$

with ϕ_{sta} the solution of the stationary problem (2). By construction, $\bar{\phi}^{n+1}$ satisfies the following equations :

$$\begin{cases} \dfrac{\bar{\phi}^{n+1} - \bar{\phi}^n}{\Delta t} + \operatorname{div}(v\bar{\phi}^{n+1}) - \nu\Delta\bar{\phi}^{n+1} &= 0 \quad \text{in} \quad \Omega_{loc} \cup (\Omega\backslash\Omega_{loc}), \\ \bar{\phi}^{n+1} &= 0 \quad \text{on} \quad \partial\Omega, \\ \bar{\phi}^{n+1} \quad \text{continuous across} \quad \Gamma_i. \end{cases} \tag{43}$$

By multiplying this equation by $\bar{\phi}^{n+1}$ and integrating by parts over Ω_{loc} and $\Omega\backslash\Omega_{loc}$, we get

$$\begin{aligned} &\int_\Omega \frac{\bar{\phi}^{n+1} - \bar{\phi}^n}{\Delta t}\bar{\phi}^{n+1} + \nu\int_\Omega |\nabla\bar{\phi}^{n+1}|^2 \\ &+ \int_{\partial\Omega_{loc}} \left(\frac{1}{2}(\bar{\phi}^{n+1})^2 v.n - \nu\bar{\phi}^{n+1}\frac{\partial}{\partial n}\bar{\phi}^{n+1}\right) \\ &+ \int_{\partial(\Omega\backslash\Omega_{loc})} \left(\frac{1}{2}(\bar{\phi}^{n+1})^2(v.n) - \nu\bar{\phi}^{n+1}\frac{\partial}{\partial n}\bar{\phi}^{n+1}\right) = 0. \end{aligned} \tag{44}$$

By taking into account the boundary conditions in (43) we obtain the following relation :

$$\begin{aligned} &\int_\Omega \frac{\bar{\phi}^{n+1} - \bar{\phi}^n}{\Delta t}\bar{\phi}^{n+1} + \nu\int_\Omega |\nabla\bar{\phi}^{n+1}|^2 \\ &-\nu\int_{\Gamma_i} \frac{\partial}{\partial n}(\phi_{loc}^{n+1} - \phi^{n+1})\bar{\phi}^{n+1} = 0. \end{aligned} \tag{45}$$

But on $\Omega_{loc}, \phi_{loc}^{n+1} - \phi^{n+1}$ satisfies the equation

$$\frac{(\phi_{loc}^{n+1} - \phi^{n+1}) - (\phi_{loc}^{n} - \phi^{n})}{\Delta t} + \mathrm{div}[v(\phi_{loc}^{n+1} - \phi^{n+1})] - \nu\Delta(\phi_{loc}^{n+1} - \phi^{n+1}) = 0. \quad (46)$$

Therefore, after multiplication by $\bar{\phi}^{n+1}$, integration by parts and from Cauchy-Schwarz, we obtain

$$\begin{aligned} \nu\left|\int_{\Gamma_i} \frac{\partial}{\partial n}(\phi_{loc}^{n+1} - \phi^{n+1})\bar{\phi}^{n+1}\right| &\leq \frac{1}{2c_1^2}\|\delta x^n\|_{0,2,\Omega_{loc}}^2 + \frac{1}{2}c_1^2\|\bar{\phi}^{n+1}\|_{0,2,\Omega_{loc}}^2 \\ &+ \frac{\|v\|_\infty^2}{2c_1^2}|\phi_{loc}^{n+1} - \phi^{n+1}|_{1,2,\Omega_{loc}}^2 + \frac{1}{2}c_1^2\|\bar{\phi}^{n+1}\|_{0,2,\Omega_{loc}}^2 \\ &+ \frac{\nu}{2}|\phi_{loc}^{n+1} - \phi^{n+1}|_{1,2,\Omega_{loc}}^2 + \frac{\nu}{2}|\bar{\phi}^{n+1}|_{1,2,\Omega_{loc}}^2, \end{aligned} \quad (47)$$

with $c_1 > 0$ arbitrary.

This latter inequality combined with (45) yields

$$\begin{aligned} &\int_\Omega \frac{\bar{\phi}^{n+1} - \bar{\phi}^n}{\Delta t}\bar{\phi}^{n+1} + \frac{\nu}{2}|\bar{\phi}^{n+1}|_{1,2,\Omega}^2 - c_1^2\|\bar{\phi}^{n+1}\|_{0,2,\Omega_{loc}}^2 \\ &\qquad \leq \frac{1}{2c_1^2}\|\delta x^n\|_{0,2,\Omega_{loc}}^2 + (\frac{\|v\|_\infty^2}{2c_1^2} + \frac{\nu}{2})|\phi_{loc}^{n+1} - \phi^{n+1}|_{1,2,\Omega_{loc}}^2. \end{aligned}$$

By using Cauchy-Schwarz again we deduce :

$$\begin{aligned} (\frac{1}{2\Delta t} - c_1^2)\|\bar{\phi}^{n+1}\|_{0,2,\Omega}^2 + \frac{\nu}{2}|\bar{\phi}^{n+1}|_{1,2,\Omega}^2 \quad &\leq \quad \frac{1}{2\Delta t}\|\bar{\phi}^n\|_{0,2,\Omega}^2 + \frac{1}{2c_1^2}\|\delta x^n\|_{0,2,\Omega_{loc}}^2 + \\ &(\frac{\|v\|_\infty^2}{2c_1^2} + \frac{\nu}{2})|\phi_{loc}^{n+1} - \phi^{n+1}|_{1,2,\Omega_{loc}}^2 \end{aligned} \quad (48)$$

By using now Poincaré inequality, we have

$$\begin{aligned} (1 + (\nu c - 2c_1^2)\Delta t)\|\bar{\phi}^{n+1}\|_{0,2,\Omega}^2 \quad &\leq \quad \|\bar{\phi}^n\|_{0,2,\Omega}^2 + \frac{\Delta t}{c_1^2}\|\delta x^n\|_{0,2,\Omega_{loc}}^2 \\ &+ \Delta t(\frac{\|v\|_\infty^2}{c_1^2} + \nu)|\phi_{loc}^{n+1} - \phi^{n+1}|_{1,2,\Omega_{loc}}^2. \end{aligned} \quad (49)$$

This writes

$$\begin{aligned} \|\bar{\phi}^{n+1}\|^2_{0,2,\Omega} \leq A\|\bar{\phi}^{n}\|^2_{0,2,\Omega} + B_1\|\delta x^n\|^2_{0,2,\Omega_{loc}} \\ +B_2|\phi^{n+1}_{loc} - \phi^{n+1}|^2_{1,2,\Omega_{loc}} \end{aligned} \tag{50}$$

with

$$\begin{aligned} A &= \frac{1}{1+(\nu c - 2c_1^2)\Delta t}, \\ B_1 &= \frac{\Delta t}{c_1^2} A, \\ B_2 &= \Delta t(\frac{\|v\|^2_\infty}{c_1^2} + \nu)A. \end{aligned}$$

Step 4: Final result

Let us choose c_1 such that $\nu c - 2c_1^2 > 0$. By induction, it follows

$$\begin{aligned} \|\bar{\phi}^{n+1}\|^2_{0,2,\Omega} \quad \leq \quad & A^p\|\bar{\phi}^{n+1-p}\|^2_{0,2,\Omega} + \textstyle\sum_{i=0}^{p-1} A^i(B_1\|\delta x^{n-i}\|^2_{0,2,\Omega_{loc}} + \\ & B_2|\phi^{n+1-i}_{loc} - \phi^{n+1-i}|^2_{1,2,\Omega_{loc}}). \end{aligned} \tag{51}$$

Since $A < 1$ by assumption on c_1, this implies

$$\begin{aligned} \|\bar{\phi}^{n+1}\|^2_{0,2,\Omega} \quad \leq \quad & A^p\|\bar{\phi}^{n+1-p}\|^2_{0,2,\Omega} + A(B_1 \sum_{i=n+1-p}^{n} \|\delta x^i\|^2_{0,2,\Omega_{loc}} + \\ & B_2 \sum_{i=n+1-p}^{n} |\phi^{i+1}_{loc} - \phi^{i+1}|^2_{1,2,\Omega_{loc}}). \end{aligned} \tag{52}$$

Now, by using (37) and (32), we obtain

$$\begin{aligned} \|\bar{\phi}^{n+1}\|^2_{0,2,\Omega} \quad \leq \quad & A^p\|\bar{\phi}^{n+1-p}\|^2_{0,2,\Omega} + A(B_1\frac{\nu}{\Delta t}(G(n+1-p) - G(n+1)) + \\ & B_2\frac{1}{2\nu\Delta t}\|\phi^{n+1-p}_{loc} - \phi^{n+1-p}\|^2_{0,2,\Omega_{loc}}). \end{aligned} \tag{53}$$

The same relation written between 0 and $n+1-p$ yields

$$\begin{aligned}\|\bar{\phi}^{n+1-p}\|^2_{0,2,\Omega} &\leq A^{n+1-p}\|\bar{\phi}^o\|^2_{0,2,\Omega_{loc}} + A(B_1\frac{\nu}{\Delta t}(G(0)-G(n-p+1)) \\ &\quad + B_2\frac{1}{2\nu\Delta t}\|\phi^0-\phi^0_{loc}\|^2_{0,2,\Omega_{loc}}).\end{aligned}$$

By combining this relation with (53), we finally obtain

$$\begin{aligned}\|\bar{\phi}^{n+1}\|^2_{0,2,\Omega} &\leq A^{n+1}\|\bar{\phi}^0\|^2_{0,2,\Omega} \\ &+ A^{p+1}(B_1\frac{\nu}{\Delta t}G(0) + B_2\frac{1}{2\nu\Delta t}\|\phi^0-\phi^0_{loc}\|^2_{0,2,\Omega_{loc}}) \\ &+ A(B_1\frac{\nu}{\Delta t}G(n+1-p) + \frac{B_2}{2\nu\Delta t}\|\phi^{n+1-p}_{loc}-\phi^{n+1-p}\|^2_{0,2,\Omega_{loc}}).\end{aligned}$$

By taking p such that $n = 2p + q$ and by using (41) we conclude that

$$\|\bar{\phi}^{n+1}\|^2_{0,2,\Omega} \leq A^{n+1}C_2 + A^{p+1}C_3 + C_4\|\phi^p_{loc}-\phi^p\|^2_{0,2,\Omega_{loc}} \tag{54}$$

which implies from (31) the linear convergence of $\|\bar{\phi}^{n+1}\|^2_{0,2,\Omega}$ towards 0.

On the other hand by using (37) in (48) we obtain

$$\begin{aligned}&(\frac{1}{2\Delta t}-c_1^2)\|\bar{\phi}^{n+1}\|^2_{0,2,\Omega} + \frac{\nu}{2}|\bar{\phi}^{n+1}|^2_{1,2,\Omega} \leq \frac{1}{2\Delta t}\|\bar{\phi}^n\|^2_{0,2,\Omega} \\ &+\frac{\nu}{2c_1^2\Delta t}(G(n)-G(n+1)) + (\frac{\|v\|^2_\infty}{2c_1^2}+\frac{\nu}{2})|\phi^{n+1}_{loc}-\phi^{n+1}|^2_{1,2,\Omega_{loc}}.\end{aligned} \tag{55}$$

Therefore by using (29) we obtain

$$\begin{aligned}&(\frac{1}{2\Delta t}-c_1^2)\|\bar{\phi}^{n+1}\|^2_{0,2,\Omega} + \frac{\nu}{2}|\bar{\phi}^{n+1}|^2_{1,2,\Omega} \leq \frac{1}{2\Delta t}\|\bar{\phi}^n\|^2_{0,2,\Omega} \\ &+\frac{\nu}{2c_1^2\Delta t}(G(n)-G(n+1)) + (\frac{\|v\|^2_\infty}{2c_1^2}+\frac{\nu}{2})(\frac{\|\phi^n_{loc}-\phi^n\|^2_{0,2,\Omega_{loc}}}{2\nu\Delta t}).\end{aligned} \tag{56}$$

Our result follows then from (41), (31) and the linear convergence of $\|\bar{\phi}^n\|_{0,2,\Omega}$.

6 Theoretical analysis of the explicit method

6.1 Local estimates

In this section, we will try to establish local estimates and a maximum principle for an arbitrary elliptic operator of second order. These tools are then needed in the convergence analysis of the explicit version of our coupling method.

Let L be an operator written under the form

$$Lu = a^{ij}(x)D_{ij}u + b^i(x)D_iu + c(x)u,$$

for any u in $W^{2,n}(\Omega)$, with Ω a bounded domain of $\mathbb{R}^n$. The coefficients a^{ij}, b^i and $c, i, j = 1, ..., n$ are defined on Ω. As usual, the repeated indices indicate a summation from 1 to n.

We suppose that the operator L is strictly elliptic in Ω in the sense that the matrix $\mathcal{A}$ of coefficients $[a^{ij}]$ is strictly positive everywhere in Ω. Let λ and Λ denote respectively the smallest and the biggest eigenvalue of $\mathcal{A}$. Let $\mathcal{D}$ denote the determinant of the matrix $\mathcal{A}$ and $\mathcal{D}^* = \mathcal{D}^{1/n}$. We have

$$0 < \lambda \leq \mathcal{D}^* \leq \Lambda.$$

We suppose in addition that the coefficients a^{ij}, b^i and c are bounded in Ω, and that there exists two positive real numbers γ and δ such that :

$$\Lambda/\lambda \leq \gamma, \qquad (L \text{ is uniformly elliptic}) \tag{57}$$

$$(|b|/\lambda)^2 \leq \delta. \tag{58}$$

Now, we are in a position to state the principal result of this section, proved in annex.

Theorem 6.1 *Let $u \in W^{2,n}(\Omega)$ and suppose that $Lu \geq f$ with $f \in L^n(\Omega)$ and $c \leq 0$. Then for all spheres $B = B_{2R}(y)$ of center y and radius $2R$ included in Ω and for all $p > 0$, we have :*

$$sup_{B_R(y)} u \leq cte\{(\frac{1}{|B|}\int_B (u^+)^p)^{1/p} + \frac{R}{\lambda}\|f\|_{n,B}\}, \tag{59}$$

where the constant cte depends on $(n, \gamma, \delta R^2, p)$, but is independent of c. Above $u^+ = max(u, 0)$.

Remark 6.1 *The statement of the same theorem can be found in [12], under the assumption*

$$|c|/\lambda \leq \delta. \tag{60}$$

So, there the constant cte depends indirectly on c through δ. That is exactly what we want to avoid, since we need this constant to be independent of c, that is of Δt.

6.2 Technical results

In this section, we will try to apply the "maximum principle" of the previous section to study the model subproblems used in our algorithm (7)-(8).

6.2.1 Passing from the local to the global solution

Let W be the sub-space of $H^1(\Omega)$ defined by

$$W = \{w \in H^1(\Omega), w = 0 \text{ on } \Gamma_\infty\}. \tag{61}$$

We then define the bilinear forms on W

$$a(v, w) = \int_\Omega \nu \nabla v \nabla w + \int_\Omega \mathrm{div}(Vv)w, \tag{62}$$

$$(v, w) = \int_\Omega vw. \tag{63}$$

In the above equation, V is the velocity field defined by the relation (1), i.e., V satisfies :

$$\begin{cases} \mathrm{div} V = 0 \text{ in } \Omega, \\ V.n = 0 \text{ on } \Gamma_b. \end{cases} \tag{64}$$

The first basic problem associated to (8), can be written as follows :
Find $v \in W$ satisfying :

$$a(v, w) + (1/\tau)(v, w) = \int_{\Gamma_b} gwd\Gamma, \ \ \forall w \in W. \tag{65}$$

Here the function g is given in $H^{-1/2}(\Gamma_b)$ and the coefficient τ is strictly positive. Moreover, from now on, we suppose that the coefficients ν and τ satisfy the following relation :

$$\nu\tau \leq 1. \tag{66}$$

This hypothesis is not necessary but simplifies the proofs to come. Moreover, it is not restrictive, since we want the convergence for small τ corresponding to Δt small.

Let d denotes the overlapping distance as in figure 2. Let then β be a real number such that

$$0 < \beta < 3\sqrt{\nu}/d,$$

and set

$$k = \beta/(\nu\sqrt{\tau}).$$

We then have the following result :

Theorem 6.2 *Let v the solution of problem (65). If τ is sufficiently small, we have*

$$\begin{aligned} \|v\|_{1/2,\Gamma_i} &\leq C_1\sqrt{C_2/d}\left(1+\frac{1}{\nu}\|V\|_\infty\sqrt{d/C_2}\right)^{1/2} \\ & (1/\nu)exp(-kd^2/36)\|g\|_{-1/2,\Gamma_b}, \end{aligned} \tag{67}$$

where C_1 and C_2 are constants, with C_1 depending only on d, ν, γ and δ, but not on τ.

The proof of this theorem will be done in five steps and is based on the local estimates described in Theorem 6.1 and on the maximum principle.

Step 1. Global estimates

By using relation (64), we have the equality :

$$\begin{aligned} \int_\Omega v\operatorname{div}(Vv) &= 1/2\int_\Omega \operatorname{div}(Vv^2) \\ &= 1/2\int_\Gamma V.nv^2 \\ &= 0,\ \forall\, v\in W. \end{aligned}$$

By taking $w = v$ in (65), we then obtain

$$\int_\Omega\{\nu|\nabla v|^2 + (1/\tau)v^2\} = \int_{\Gamma_b} gv. \tag{68}$$

From this equality we deduce the following bound :

$$\nu\|v\|^2_{1,2,\Omega} \leq \|g\|_{-1/2,\Gamma_b}\|v\|_{1/2,\Gamma_b}.$$

Now by using the trace theorem, we obtain :

$$\|v\|_{1,2,\Omega} \leq (c_o/\nu)\|g\|_{-1/2,\Gamma_b}, \tag{69}$$

which implies in particular

$$\|v\|_{0,2,\Omega} \leq (c_o/\nu)\|g\|_{-1/2,\Gamma_b}. \tag{70}$$

Step 2. Nearby local estimates

Let L be the second order operator associate to problem (65) :

$$Lv = -\nu\Delta v + V.\nabla v + (1/\tau)v. \tag{71}$$

The operator

$$L' = -L$$

satisfies the assumptions of Theorem 6.1, with $c = -1/\tau, f = 0$. Applying then Theorem 6.1 for $p = 2,\ y \in \Omega_i$ (the subdomain of width $\frac{d}{3}$ externally delimited by Γ_i) (fig. 2) and $Ky = B_{d/3}(y)$, the sphere of center y and radius $d/3$ we obtain

$$\|v\|_{\infty,Ky} \leq c_1\|v\|_{0,2,B_{2d/3}(y)}.$$

Therefore

$$\|v\|_{\infty,Ky} \leq c_1\|v\|_{0,2,\Omega}, \tag{72}$$

where c_1 is a constant depending only on $\nu, \gamma, \delta d^2$ and $(3/2d)^{n/2}$.

Moreover, there exist $y_1, \ldots, y_l$ belonging to Ω_i such that

$$\Omega_{2i} = \cup_{y\in\Omega_i} B_{\frac{d}{6}}(y) \subset \cup_{j=1}^{l} K_{y_j} = K$$

with

$$K_{y_j} = B_{d/3}(y_j).$$

By applying the relation (72) to each K_{y_j} we obtain :

$$\|v\|_{\infty,K} \leq \sup_{j=1,\ldots,l} c_{1j}\|v\|_{0,2,\Omega}.$$

By setting $c_1 = \sup\limits_{j=1,\ldots,l} c_{1j}$, we finally get

$$\|v\|_{\infty,K} \leq c_1\|v\|_{0,2,\Omega}. \tag{73}$$

Step 3. Use of maximum principle

For any M_i in Ω_i, we introduce (see fig. (2)) :

- B_i = the ball centered on M_i of radius $d/6$,
- $v_i = exp[k(r^2 - d^2/36)]\|v\|_{\infty,\partial B_i}$.

The operator L applied to v_i, can be written in polar coordinates (with $r = M_iM$) :

$$Lv_i = 4(-k^2\nu r^2 - k\nu + \frac{k}{2}V.e_r r + \frac{1}{4\tau})v_i.$$

Therefore

$$Lv_i \geq 4(-k^2\nu r^2 - \frac{k}{2}|V.e_r|r + (\frac{1}{4\tau} - k\nu))v_i. \tag{74}$$

We set then :

$$\varphi(r,k) = a(k)r^2 + b(k)r + c(k), \tag{75}$$

with

$$a(k) = -k^2\nu \tag{76}$$

$$b(k) = -\frac{k}{2}|V.e_r| \tag{77}$$

$$c(k) = \frac{1}{4\tau} - k\nu. \tag{78}$$

We seek to satisfy the following relation :

$$0 \leq \inf \varphi(r,k) \text{ for } 0 \leq r \leq \frac{d}{6}.$$

As $\varphi(r,k)$ decreases on R^+, this will be satisfied if and only if

$$\varphi(d/6) \geq 0,$$

i.e. if and only if

$$-\frac{k^2\nu d^2}{36} - \frac{kd\|V\|}{12} + \frac{1}{4\tau} - k\nu \geq 0.$$

We replace k by its value. We therefore have to satisfy

$$-\frac{\beta^2 d^2}{(36\nu\tau)} - \frac{\beta d\|V\|}{12\nu\sqrt{\tau}} + \frac{1}{4\tau} - \frac{\beta\nu}{\nu\sqrt{\tau}} \geq 0.$$

Multiplying by $\sqrt{\tau}$, it remains

$$\frac{1}{4\sqrt{\tau}}(1-\frac{\beta^2 d^2}{9\nu}) \geq \beta(1+\frac{d\|V\|}{12\nu}).$$

From the constraint $\beta < 3\sqrt{\nu}/d$, we finally obtain after division

$$\varphi(r,k) \geq 0 \text{ iff } \frac{1}{4\sqrt{\tau}} \geq \beta[1+\frac{d\|V\|}{12\nu}][1-\frac{\beta^2 d^2}{9\nu}]^{-1}. \tag{79}$$

From the relation (74) and the previous calculation, we deduce that for $\beta < 3\sqrt{\nu}/d$ and τ satisfying the above inequality, we have

$$Lv_i \geq 0 = Lv.$$

In addition, by construction

$$v_i \geq v \text{ on } \partial B_i.$$

Consequently, by using the maximum principle we end up at the following relation :

$$v \leq v_i \text{ on } B_i.$$

In particular

$$v(M_i) \leq exp(-kd^2/36)\|v\|_{\infty,\partial B_i}.$$

We do the same for $-v$, and finally we have

$$|v(M_i)| \leq \exp(-kd^2/36)\|v\|_{\infty,\partial B_i},\ \forall M_i \in \Omega_i. \tag{80}$$

Step 4 : H^1 estimate

Let $\xi \in H^1(\Omega)$ be such that :

$$\begin{cases} \xi = 1 \text{ in } \Omega_\infty = \Omega - \Omega_{loc}, \\ \text{supp}\xi \subset \Omega_i \cup \Omega_\infty. \end{cases}$$

We have from (65),

$$\int_\Omega (-\nu\Delta v + \text{div}(Vv) + v/\tau)\xi^2 v = 0. \tag{81}$$

By using the Green's formula we deduce :

$$\int_\Omega -\nu\Delta v\xi^2 v = \int_\Omega \nu|\nabla(\xi v)|^2 - \int_\Omega \nu|\nabla\xi|^2 v^2. \tag{82}$$

On the other hand, we have :

$$\int_\Omega \mathrm{div}(Vv)\xi^2 v = \int_\Omega \mathrm{div}(V\xi^2 v^2/2) - \int_\Omega V.\nabla\xi\xi v^2. \tag{83}$$

With the relations (82)-(83), (81) becomes

$$\begin{aligned}
0 &= \int_\Omega (\nu|\nabla(\xi v)|^2 + \mathrm{div}(V\xi^2 v^2/2) + \xi^2 v^2/\tau) - \int_\Omega (\nu v^2|\nabla\xi|^2 + v^2\xi V\cdot\nabla\xi) \\
&= \int_\Omega \nu(|\nabla(\xi v)|^2 + |\xi v|^2) + \int_\Omega (1/\tau - \nu)\xi^2 v^2 - \int_\Omega (\nu v^2|\nabla\xi|^2 + v^2\xi V.\nabla\xi) \\
&= \int_{\Omega_\infty} \nu(|\nabla v|^2 + |v|^2) + \int_{\Omega_i} \nu(|\nabla(\xi v)|^2 + |\xi v|^2) + \int_\Omega (1/\tau - \nu)\xi^2 v^2 \\
&\quad - \int_{\Omega_i} (\nu v^2|\nabla\xi|^2 + v^2\xi V.\nabla\xi).
\end{aligned}$$

Hence, we obtain :

$$\nu\|v\|^2_{1,2,\Omega_\infty} + \int_{\Omega_i} \nu(|\nabla(\xi v)|^2 + |\xi v|^2) + \int_\Omega (1/\tau - \nu)\xi^2 v^2 = \int_{\Omega_i} (\nu v^2|\nabla\xi|^2 + v^2\xi V\cdot\nabla\xi).$$

With the relation (66), it follows

$$\begin{aligned}
\nu\|\xi v\|^2_{1,2,\Omega_\infty\cup\Omega_i} &\leq \int_{\Omega_i} (\nu v^2|\nabla\xi|^2 + v^2\xi V.\nabla\xi) && (84) \\
&\leq \|v\|^2_{\infty,\Omega_i} \int_{\Omega_i} (\nu|\nabla\xi|^2 + \xi V.\nabla\xi) && (85) \\
&\leq \|v\|^2_{\infty,\Omega_i} (\nu|\xi|^2_{1,2,\Omega_i} + \|\xi\|_{0,2,\Omega_i}\|V\||\xi|_{1,2,\Omega_i}) \\
&\leq \|v\|^2_{\infty,\Omega_i} |\xi|^2_{1,2,\Omega_i} (\nu + \|\xi\|_{0,2,\Omega_i}\|V\|/|\xi|_{1,2,\Omega_i}). && (86)
\end{aligned}$$

If we take ξ such that

$$|\xi|_{0,2,\Omega_i} \leq 1,$$

$$|\xi|^2_{1,2,\Omega_i} = c_2/d,$$

where c_2 is a constant, (86) then becomes

$$\|v\|_{1,2,\Omega_\infty} \leq \|v\|_{\infty,\Omega_i}\sqrt{c_2/d}\left(1+\frac{\|V\|}{\nu}\sqrt{d/c_2}\right)^{1/2}. \tag{87}$$

Step 5: Getting the final estimate.

We have

$$\|v\|_{\infty,\partial B_i} \leq \|v\|_{\infty,K}, \tag{88}$$

since $\partial B_i \subset K$. From (73), this implies

$$\|v\|_{\infty,\partial B_i} \leq c_1\|v\|_{0,2,\Omega}. \tag{89}$$

By using the relations (80) and (89) it follows :

$$|v(M_i)| \leq exp(-kd^2/36)c_1\|v\|_{0,2,\Omega},\ \forall M_i \in \Omega_i.$$

Consequently we have

$$\|v\|_{\infty,\Omega_i} \leq exp(-kd^2/36)c_1\|v\|_{0,2,\Omega}. \tag{90}$$

By using now the relation (70), we obtain :

$$\|v\|_{\infty,\Omega_i} \leq \frac{c_1c_o}{\nu}exp(-kd^2/36)\|g\|_{-1/2,\Gamma_b}. \tag{91}$$

Now, by using the relation (87) we obtain :

$$\|v\|_{1,2,\Omega_\infty} \leq c_oc_1\sqrt{c_2/d}\left(1+\frac{\|V\|}{\nu}\sqrt{d/c_2}\right)^{1/2} \tag{92}$$

$$(1/\nu)exp(-kd^2/36)\|g\|_{-1/2,\Gamma_b}.$$

To conclude we use the trace theorem which yields

$$\|v\|_{1/2,\Gamma_i} \le c_3 \|v\|_{1,2,\Omega_\infty}.$$

Consequently, we arrive at the following relation :

$$\|v\|_{1/2,\Gamma_i} \le c_o c_1 c_3 \sqrt{c_2/d}(1 + \frac{\|V\|}{\nu}\sqrt{d/c_2})^{1/2} \quad (93)$$

$$(1/\nu) exp(-kd^2/36)\|g\|_{-1/2,\Gamma_b}.$$

This is our theorem with $C_1 = c_o c_1 c_3$ and $C_2 = c_2$.

■

6.2.2 Passing from the global to the local solution

Let W denote now the sub-space of $H^1(\Omega_{loc})$ defined by

$$W = \{w \in H^1(\Omega_{loc}) / w = 0 \text{ on } \Gamma_b\}.$$

We then set for any v and w in W

$$a(v, w) = \nu \int_{\Omega_{loc}} \nabla v . \nabla w + \int_{\Omega_{loc}} \text{div}(Vv) w, \quad (94)$$

$$(v, w) = \int_{\Omega_{loc}} vw, \quad (95)$$

where V is the velocity field defined by the relation (64).

The second basic problem associated to (7) can be written as :

Find $v \in W$ such that

$$a(v, w) + (1/\tau)(v, w) = \int_{\Gamma_i} \nu \frac{\partial v}{\partial n} w, \ \ \forall w \in W, \quad (96)$$

$$v|_{\Gamma_i} = h, \quad (97)$$

where h is given in $H^{1/2}(\Gamma_i)$. We first have the following lemma ;

Lemma 6.1 *For τ sufficiently small, we have*

$$a(w, w) + (1/\tau)(w, w) \ge (\nu/2)\|w\|^2_{1,2,\Omega_{loc}}, \ \ \forall w \in W.$$

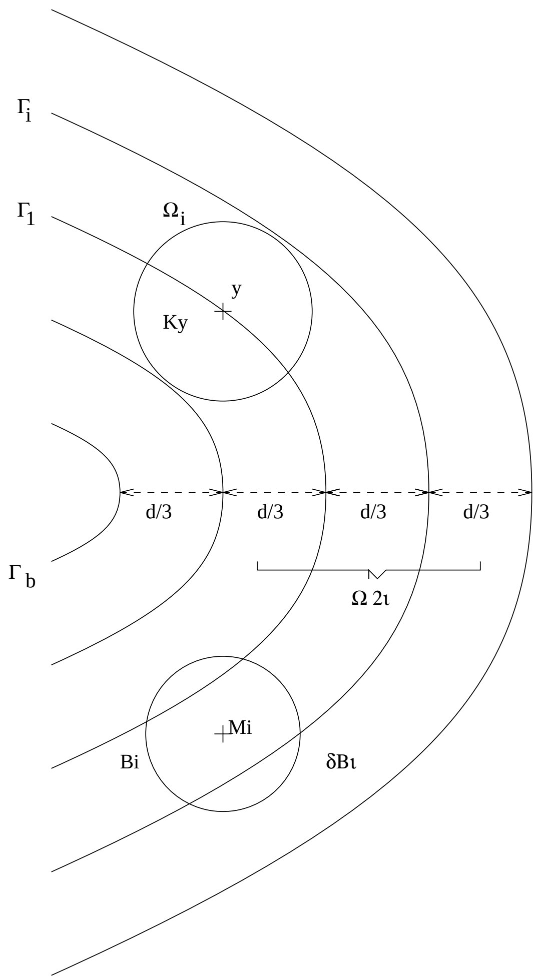

Figure 2: Description of the Domain Ω_{loc} and of the splitting used in the majoration of the local solution.

Proof :

Under the hypothesis $1/\tau \geq \nu/2 + (1/2\nu)\|V\|_\infty^2$, and using the Cauchy-Schwarz inequality, we obtain :

$$
\begin{aligned}
a(v,v) + (1/\tau)(v,v) &= \int_{\Omega_{loc}} \nu \nabla v . \nabla v + \int_{\Omega_{loc}} V . \nabla v v + (1/\tau) \int_{\Omega_{loc}} v^2 \\
&\geq \nu \|\nabla v\|_{0,2}^2 + (1/\tau)\|v\|_{0,2}^2 - \|V\|_\infty \|\nabla v\|_{0,2} \|v\|_{0,2} \\
&\geq \nu \|\nabla v\|_{0,2}^2 + (1/\tau)\|v\|_{0,2}^2 - (\nu/2)\|\nabla v\|_{0,2}^2 \\
&\quad -(1/2\nu)\|V\|_\infty^2 \|v\|_{0,2}^2 \\
&\geq (\nu/2)(\|\nabla v\|_{0,2}^2 + \|v\|_{0,2}^2).
\end{aligned}
$$

■

We will also make the simplifying assumption (66). We have then the following result :

Theorem 6.3 *Under the notation of Theorem 6.2 and for τ sufficiently small, the solution of problem (96) is bounded by*

$$
\begin{aligned}
\|\partial v/\partial n\|_{-1/2,\Gamma_b} &\leq C_1 \sqrt{C_2/d} \left(1 + \frac{1 + \|V\|_\infty^2}{\nu^2}\right) \\
&\quad \left(1 + \frac{\|V\|_\infty}{\nu}\sqrt{d/C_2}\right)^{1/2} \\
&\quad (1 + 1/\tau^2) exp(-kd^2/36)\|h\|_{1/2,\Gamma_i}.
\end{aligned} \tag{98}
$$

Here C_1 and C_2 are constants with C_1 depending on d, v, ν and δ.

Proof

To prove this theorem, we will proceed as in the proof of Theorem 6.2.

Step 1 : Global estimates.

By taking $w = v$ in (96), we obtain :

$$\nu \int_{\Omega_{loc}} |\nabla v|^2 + \int_{\Omega_{loc}} (\mathrm{div}(Vv)v + (1/\tau)v^2) = \int_{\Gamma_i} \nu \frac{\partial v}{\partial n} h. \tag{99}$$

Using the above Lemma and setting $\alpha = \nu/2$, this implies

$$\alpha \|v\|^2_{1,2,\Omega_{loc}} \leq \nu \|\partial v/\partial n\|_{-1/2,\Gamma_i} \|h\|_{1/2,\Gamma_i}. \tag{100}$$

We will try now to get estimates on $\|\partial v/\partial n\|_{-1/2,\Gamma_i}$. From (96) and (64) we obtain :

$$\int_{\Gamma_i} \frac{\partial v}{\partial n} w = \int_{\Omega_{loc}} (\nabla v \nabla w + (1/\nu) V.\nabla v w + \frac{1}{\nu\tau} v w).$$

Therefore, for any w in W, we have

$$\begin{aligned}
|\int_{\Gamma_i} \frac{\partial v}{\partial n} w| \quad &\leq \quad \|\nabla v\|_{0,2,\Omega_{loc}} \|\nabla w\|_{0,2,\Omega_{loc}} + (1/\nu)\|V\|_\infty \|\nabla v\|_{0,2,\Omega_{loc}} \|w\|_{0,2,\Omega_{loc}} \\
&\quad + \frac{1}{\nu\tau} \|v\|_{0,2,\Omega_{loc}} \|w\|_{0,2,\Omega_{loc}} \\
&\leq \quad (\|\nabla v\|^2_{0,2,\Omega_{loc}} + (1/\nu^2)\|V\|^2_\infty \|\nabla v\|^2_{0,2,\Omega_{loc}} + (1/\nu^2)\|v\|^2_{0,2,\Omega_{loc}})^{1/2} \\
&\quad (\|\nabla w\|^2_{0,2,\Omega_{loc}} + \|w\|^2_{0,2,\Omega_{loc}} + (1/\tau^2)\|w\|^2_{0,2,\Omega_{loc}})^{1/2} \\
&\leq \quad [1 + \frac{(1 + \|V\|^2_\infty)}{\nu^2}]^{1/2} \|v\|_{1,2,\Omega_{loc}} (1 + \tau^{-2})^{1/2} \|w\|_{1,2,\Omega_{loc}}.
\end{aligned}$$

From the trace theorem, this yields

$$\|\partial v/\partial n\|_{-1/2,\Gamma_i} \leq (1 + \tau^{-2})^{1/2} \left(1 + \frac{1 + \|V\|^2_\infty}{\nu^2}\right)^{1/2} \|v\|_{1,2,\Omega_{loc}}. \tag{101}$$

From the relations (100)-(101), we end up with

$$\alpha \|v\|_{1,2,\Omega_{loc}} \leq \nu (1 + \tau^{-2})^{1/2} \left(1 + \frac{1 + \|V\|^2_\infty}{\nu^2}\right)^{1/2} \|h\|_{1/2,\Gamma_i} \tag{102}$$

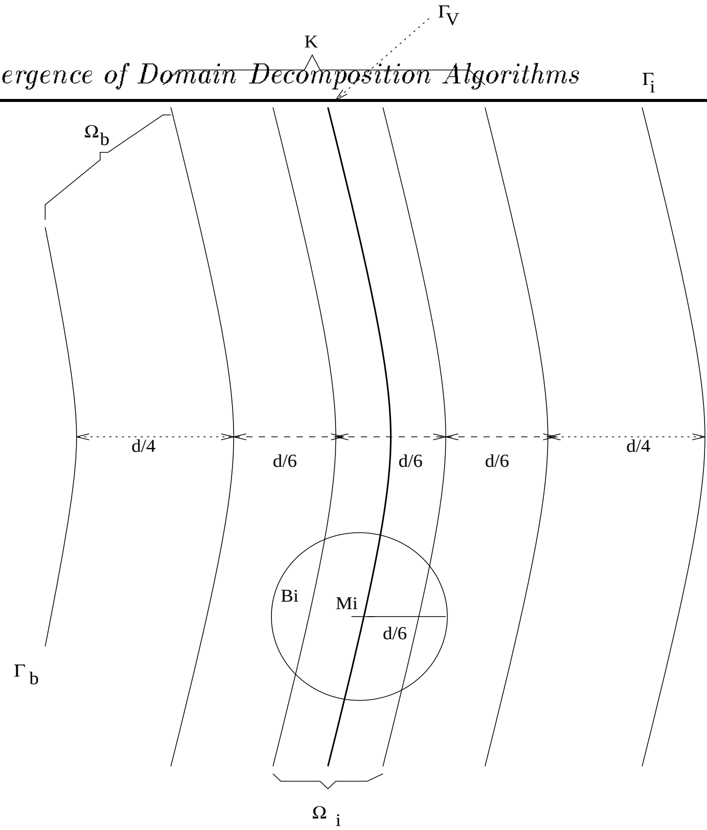

Figure 3: Description of the local domain Ω_{loc} and of the splitting used in the majoration of the global solution.

and hence in particular

$$\alpha\|v\|_{0,2,\Omega_{loc}} \leq \nu(1+\tau^{-2})^{1/2}\left(1+\frac{1+\|V\|_{\infty}^{2}}{\nu^{2}}\right)^{1/2}\|h\|_{1/2,\Gamma_i}. \tag{103}$$

Step 2 : Nearby local estimates

Let $K_y = B_{d/4}(y)$ be the sphere centered on y and of radius $d/4$, with y belonging to Γ_V (see fig. 3). By construction, Γ_V is the center surface of Ω_{loc} and Ω_i is subdomain of width $\frac{d}{6}$ centered on Γ_V .

Then by following the same argument as in step 2 of the proof of Theorem 6.2, we obtain :

$$\|v\|_{\infty,K_y} \leq c_1\|v\|_{0,2,\Omega_{loc}}, \tag{104}$$

where c_1 is a constant depending only on d, ν, γ and δ. Let us consider now $y_1, \ldots, y_l$ in Ω_i such that

$$\Omega_{2i} = \bigcup_{y\in\Omega_i} B_{\frac{d}{6}}(y) \subset \bigcup_{j=1}^{l} K_{y_j} = K.$$

By applying the relation (104) to each K_{y_j}, we obtain

$$\|v\|_{\infty,K} \le \sup_{j=1,\dots,l} c_{1j}\|v\|_{0,2,\Omega_{loc}} = c_1\|v\|_{0,2,\Omega_{loc}}. \tag{105}$$

Step 3: Using the maximum principle

For any $M_i \in \Omega_i$, we introduce (see fig. 3) :

- a ball B_i centered on M_i and of radius $d/6$,
- the function $v_i = exp[k(r^2 - d^2/36)]\|v\|_{\infty,\partial B_i}$.

By construction of k (see previous §), $\varphi(r,k)$ is positive for all $r \in [0, d/6]$. Then by following the same argument as in step 3 of the proof of Theorem 6.2, we obtain the following inequality :

$$|v(M_i)| \le exp[-kd^2/36]\|v\|_{\infty,\partial B_i}. \tag{106}$$

Step 4 : H^1 estimates

Let us consider $\xi \in H^1(\Omega_{loc})$, such that

$$\xi = 1 \quad \text{in} \quad \Omega_b, \quad \text{(Figure 3)}$$
$$supp\xi \subset \Omega_b \cup \Omega_i.$$

By choosing $w = \xi^2 v$ in (96) we obtain :

$$\int_{\Omega_{loc}} (-\nu\Delta v + \mathrm{div}(Vv) + (v/\tau))\xi^2 v = 0. \tag{107}$$

By using the Green's formula, as in step 4 of the previous theorem, we obtain :

$$\nu\|\xi v\|_{1,2,\Omega_b\cup\Omega_i} \le \int_{\Omega_i} (\nu v^2|\nabla\xi|^2 + v^2\xi V.\nabla\xi).$$

Then, if we take ξ such that

$$\|\xi\|_{0,2,\Omega_i} \le 1$$

and

$$|\xi|^2_{1,2,\Omega_i} = c_2/d,$$

we obtain finally as in the proof of Theorem 6.2 :

$$\|v\|_{1,2,\Omega_b\cup\Omega_i} \le \|v\|_{\infty,\Omega_i}\sqrt{c_2/d}\left(1+\frac{\|V\|_\infty}{\nu}\sqrt{d/c_2}\right)^{1/2}. \tag{108}$$

Step 5 : Getting the result

Since $\partial B_i \subset K$ by construction, (105) and (106) imply

$$\|v\|_{\infty,\Omega_i} \le exp(-kd^2/36)c_1\|v\|_{0,2,\Omega_{loc}}. \tag{109}$$

Furthermore by using the relation (103), it follows :

$$\begin{aligned}\|v\|_{\infty,\Omega_i} \le \frac{\nu}{\alpha}&\left(1+\frac{1+\|V\|^2_\infty}{\nu^2}\right)^{1/2}\\ &c_1(1+1/\tau^2)^{1/2}exp(-kd^2/36)\|h\|_{1/2,\Gamma_i}.\end{aligned} \tag{110}$$

By using the relation (108), we then obtain :

$$\begin{aligned}\|v\|_{1,2,\Omega_b\cup\Omega_i} \le \frac{\nu}{\alpha}&\left(1+\frac{1+\|V\|^2_\infty}{\nu^2}\right)^{1/2}\\ &c_1\sqrt{c_2/d}\left(1+\frac{\|V\|_\infty}{\nu}\sqrt{d/c_2}\right)^{1/2}\\ &(1+1/\tau^2)^{1/2}exp(-kd^2/36)\|h\|_{1/2,\Gamma_i}.\end{aligned} \tag{111}$$

Before concluding we will establish an estimate of the quantity

$$\|\partial v/\partial n\|_{-1/2,\Gamma_b}.$$

From (96), and by choosing

$$w \in H^1(\Omega_{loc}), \text{ with } w=0 \text{ on } \partial\Omega_b\cap\partial\Omega_i,$$

we obtain :

$$\int_{\Omega_b}(-\nu\Delta v+\operatorname{div}(Vv)+v/\tau)w = 0.$$

After application of the Green's formula and from (64), we obtain :

$$\int_{\Gamma_b} \frac{\partial v}{\partial n} w = \int_{\Omega_b} (\nabla v \nabla w + (1/\nu) V.\nabla v w + \frac{1}{\nu\tau} v w).$$

By proceeding as in step 1 of the current proof, we obtain the following estimate :

$$\|\partial v/\partial n\|_{-1/2,\Gamma_b} \leq (1 + 1/\tau^2)^{1/2} \left(1 + \frac{1 + \|V\|_\infty^2}{\nu^2}\right)^{1/2} \|v\|_{1,2,\Omega_b}. \tag{112}$$

The relations (111)-(112) finally yield the desired result.

■

6.3 Convergence of the explicit algorithm with $\Delta t = \infty$ and α small.

Consider the following elliptic problem :

$$\left\{ \begin{array}{rcl} \dfrac{\phi}{\alpha} + V.\nabla\phi - \nu\Delta\phi & = & 0 \text{ in } \Omega, \\ \phi & = & \phi_\infty \text{ on } \Gamma_\infty, \\ \phi & = & 0 \text{ on } \Gamma_b, \end{array} \right. \tag{113}$$

that we want to solve by the fundamental algorithm (7)-(8) with "$\Delta t = +\infty$". The proposed algorithm can be written in this case as

- set $\phi^o_{loc} = \phi_{ol}$ and $\phi^o = \phi_o$.
- then, for $n \geq 0, \phi^n_{loc}$ and ϕ^n being known,

  solve

$$\left\{ \begin{array}{rcl} \dfrac{\phi^{n+1}_{loc}}{\alpha} + V.\nabla\phi^{n+1}_{loc} - \nu\Delta\phi^{n+1}_{loc} & = & 0 \text{ in } \Omega_{loc}, \\ \phi^{n+1}_{loc} & = & \phi^n \text{ on } \Gamma_i, \\ \phi^{n+1}_{loc} & = & 0 \text{ on } \Gamma_b, \end{array} \right. \tag{114}$$

$$\left\{ \begin{array}{rcl} \dfrac{\phi^{n+1}}{\alpha} + V.\nabla\phi^{n+1} - \nu\Delta\phi^{n+1} & = & 0 \text{ in } \Omega, \\ \phi^{n+1} & = & \phi_\infty \text{ on } \Gamma_\infty, \\ \nu\dfrac{\partial\phi^{n+1}}{\partial n} & = & \nu\dfrac{\partial\phi^{n+1}_{loc}}{\partial n} \text{ on } \Gamma_b. \end{array} \right. \tag{115}$$

We will show in this section that the algorithm described above converges. More precisely we have the following theorem.

Theorem 6.4 *For α sufficiently small, ϕ being the solution of the stationary problem (113), we have :*

i) $\frac{\partial \phi_{loc}^{n+1}}{\partial n}$ *converges towards* $\frac{\partial \phi}{\partial n}$ *in* $H^{-1/2}(\Gamma_b)$,

ii) ϕ^{n+1} *converges towards* ϕ *in* $H^{1/2}(\Gamma_i)$,

iii) ϕ^{n+1} *converges towards* ϕ *in* $H^1(\Omega)$,

iv) ϕ_{loc}^{n+1} *converges towards* ϕ *in* $H^1(\Omega_{loc})$.

Proof:

In the rest of this section, we will take $\alpha = \tau$. By the transformation $\phi^{n+1} \rightarrow \phi^{n+1} - \phi$ with ϕ the solution of the stationary problem, we will reduce the problem to the case $\phi_\infty = 0$.

After multiplication and integration by parts in (114), we obtain that $\phi_{loc}^{n+1} = \phi^n$ on Γ_i and satisfies :

$$\int_{\Omega_{loc}} \frac{\phi_{loc}^{n+1}}{\tau} w + \int_{\Omega_{loc}} V.\nabla \phi_{loc}^{n+1} w + \nu \int_{\Omega_{loc}} \nabla \phi_{loc}^{n+1} \nabla w$$

$$= \nu \int_{\Gamma_i} \frac{\partial \phi_{loc}^{n+1}}{\partial n} w, \quad \forall w \in W. \tag{116}$$

We can apply then Theorem 6.3 and obtain

$$\begin{aligned} \|\frac{\partial \phi_{loc}^{n+1}}{\partial n}\|_{-1/2,\Gamma_b} \quad \leq \quad & c_1' \sqrt{c_2'/d} \left(1 + \frac{1}{\nu^2}(1 + \|V\|_\infty^2)\right) \\ & \left(1 + \frac{1}{\nu}\|V\|_\infty \sqrt{d/c_2'}\right)^{1/2} \\ & (1 + 1/\tau^2) exp(-kd^2/36) \|\phi^n\|_{1/2,\Gamma_i}. \end{aligned} \tag{117}$$

On the other hand, after multiplication by w and integration by parts in (115), we obtain the equality

$$\int_\Omega \frac{\phi^{n+1}}{\tau} w + \int_\Omega V.\nabla\phi^{n+1} w + \nu \int_\Omega \nabla\phi^{n+1}\nabla w = \nu \int_{\Gamma_b} \frac{\partial \phi^{n+1}_{loc}}{\partial n} w \tag{118}$$

with $w \in H^1(\Omega)$ and $w = 0$ on Γ_∞. By applying Theorem 6.2, we obtain :

$$\begin{aligned} \|\phi^{n+1}\|_{1/2,\Gamma_i} \quad \le \quad & c_1\sqrt{c_2/d}\left(1+\frac{1}{\nu}\|V\|_\infty\sqrt{d/c_2}\right)^{1/2} \\ & exp(-kd^2/36)\|\partial\phi^{n+1}_{loc}/\partial n\|_{-1/2,\Gamma_b}. \end{aligned} \tag{119}$$

By using (117) and (119), we then get :

$$\begin{aligned} \|\partial\phi^{n+1}_{loc}/\partial n\|_{-1/2,\Gamma_b} \quad \le \quad & c_1'\sqrt{c_2'/d}(1+\frac{1}{\nu^2}\left(1+\|V\|^2_\infty)\right) \\ & \left(1+\frac{1}{\nu}\|V\|_\infty\sqrt{d/c_2'}\right)^{1/2} \\ & c_1\sqrt{c_2/d}\left(1+\frac{1}{\nu}\|V\|_\infty\sqrt{d/c_2}\right)^{1/2} \\ & (1+1/\tau^2)exp\left(-k\frac{d^2}{18}\right)\|\partial\phi^{n}_{loc}/\partial n\|_{-1/2,\Gamma_b}, \end{aligned}$$

with $k = \dfrac{\beta}{\nu\sqrt{\tau}}$. Therefore for τ sufficiently small, the coefficient of reduction will be dominated by the exponential term and will be then strictly less than 1, implying the linear convergence towards zero of

$$\|\partial\phi^{n+1}_{loc}/\partial n\|_{-1/2,\Gamma_b}.$$

This is exactly assertion (i).

By using (119), it also follows that ϕ^{n+1} tends to 0 in $H^{1/2}(\Gamma_i)$. From (69) applied with $g = \partial\phi^{n+1}_{loc}/\partial n$, we have in addition

$$\|\phi^{n+1}\|_{1,2,\Omega} \le \frac{c_o}{\nu}\|\partial\phi^{n+1}_{loc}/\partial n\|_{-1/2,\Gamma_b},$$

and therefore $\|\phi^{n+1}\|_{1,2,\Omega}$ converges to zero at the speed of $\|\partial\phi_{loc}^{n+1}/\partial n\|_{-1/2,\Gamma_b}$. From (102) we also have

$$\frac{\nu}{2}\|\phi_{loc}^{n+1}\|_{1,2,\Omega_{loc}} \leq \nu(1+1/\tau^2)^{1/2}\left(1+\frac{1}{\nu^2}(1+\|V\|_\infty^2)\right)^{1/2}\|\phi^n\|_{1/2,\Gamma_i}$$

and then $\|\phi^{n+1}\|_{1,2,\Omega}$ also converges to zero at the speed of $\|\phi^n\|_{1/2,\Gamma_i}$.

■

6.4 Convergence of a fixed point method for the implicit scheme.

The implicit scheme of section (5) couples the global and the local problem. To uncouple them, it is advisable to use the fixed point algorithm below :

- set $\phi_{loc,0}^o = \psi_{ol}$ and $\phi^o = \psi_o$,
- then for $k \geq 0, \phi_{k|\Gamma_i}^{n+1}$ being known,

  solve

$$\left\{\begin{array}{rcl} \dfrac{\phi_{loc,k+1}^{n+1}-\phi_{loc}^n}{\Delta t} + \mathrm{div}(v\phi_{loc,k+1}^{n+1}) - \nu\Delta\phi_{loc,k+1}^{n+1} & = & 0 \quad \text{in} \quad \Omega_{loc}, \\ \phi_{loc,k+1}^{n+1} & = & \phi_k^{n+1} \quad \text{on} \quad \Gamma_i, \\ \phi_{loc,k+1}^{n+1} & = & 0 \quad \text{on} \quad \Gamma_b, \end{array}\right. \tag{120}$$

$$\left\{\begin{array}{rcl} \dfrac{\phi_{k+1}^{n+1}-\phi^n}{\Delta t} + \mathrm{div}(v\phi_{k+1}^{n+1}) - \nu\Delta\phi_{k+1}^{n+1} & = & 0 \quad \text{in} \quad \Omega, \\ \phi_{k+1}^{n+1} & = & \phi_\infty \quad \text{on} \quad \Gamma_\infty, \\ \nu\partial\phi_{k+1}^{n+1}/\partial n & = & \nu\partial\phi_{loc,k+1}^{n+1}/\partial n \quad \text{on} \quad \Gamma_b. \end{array}\right. \tag{121}$$

We will study now the algorithm (120)-(121). By setting

$$\psi_{loc,k,q} = \phi_{loc,k+1}^{n+1} - \phi_{loc,q+1}^{n+1}, \tag{122}$$

$$\psi_{k,q} = (\phi_k^{n+1} - \phi_q^{n+1}), \tag{123}$$

we have that $\psi_{loc,k,q}$ and $\psi_{k,q}$ verify the following equations :

$$\left\{\begin{array}{rcl} \psi_{loc,k,q}/\Delta t + div(v\psi_{loc,k,q}) - \nu\Delta\psi_{loc,k,q} & = & 0 \quad \text{in} \quad \Omega_{loc}, \\ \psi_{loc,k,q} & = & \psi_{k-1,q-1} \quad \text{on} \quad \Gamma_i, \\ \psi_{loc,k,q} & = & 0 \quad \text{on} \quad \Gamma_b, \end{array}\right. \tag{124}$$

$$\left\{\begin{array}{rcl} \psi_{k,q}/\Delta t + div(v\psi_{k,q}) - \nu\Delta\psi_{k,q} & = & 0 \quad \text{in} \quad \Omega, \\ \psi_{k,q} & = & 0 \quad \text{on} \quad \Gamma_\infty, \\ \nu\dfrac{\partial\psi_{k,q}}{\partial n} & = & \nu\dfrac{\partial\psi_{loc,k,q}}{\partial n} \quad \text{on} \quad \Gamma_b. \end{array}\right. \tag{125}$$

If Δt is sufficiently small, we have from the previous section that $\psi_{k,q}$ and $\psi_{loc,k,q}$ converge linearly to zero. Hence the sequences ϕ_k^{n+1} and $\phi_{loc,k}^{n+1}$ are Cauchy sequences which converge linearly towards the unique solutions ϕ^{n+1} and ϕ_{loc}^{n+1} of the implicit scheme. This guarantees the convergence of the above fixed point algorithm.

7 Numerical analysis of the stability of the algorithm (7)-(8)

In this section we focus our attention on the numerical solution of the steady problem (2), using now the explicit time marching algorithm (7)-(8) studied in the previous paragraphs. We first assume here that the boundary condition on Γ_b in (8) is explicit

$$(\nu \frac{\partial \phi^{n+1}}{\partial n} = \nu \frac{\partial \phi^{n}_{loc}}{\partial n})$$

so that the resulting algorithm is parallel (Jacobi type).

Here, Ω denotes the domain surrounding the obstacle (an ellipse in our numerical study) as described in Figure 1. The global and local domains are discretized by fully overlapping compatible finite element grids. The global mesh contains 1378 nodes and 2662 elements (see figure 4). Further the time marching algorithm is being initialized by setting ϕ_0 to zero.

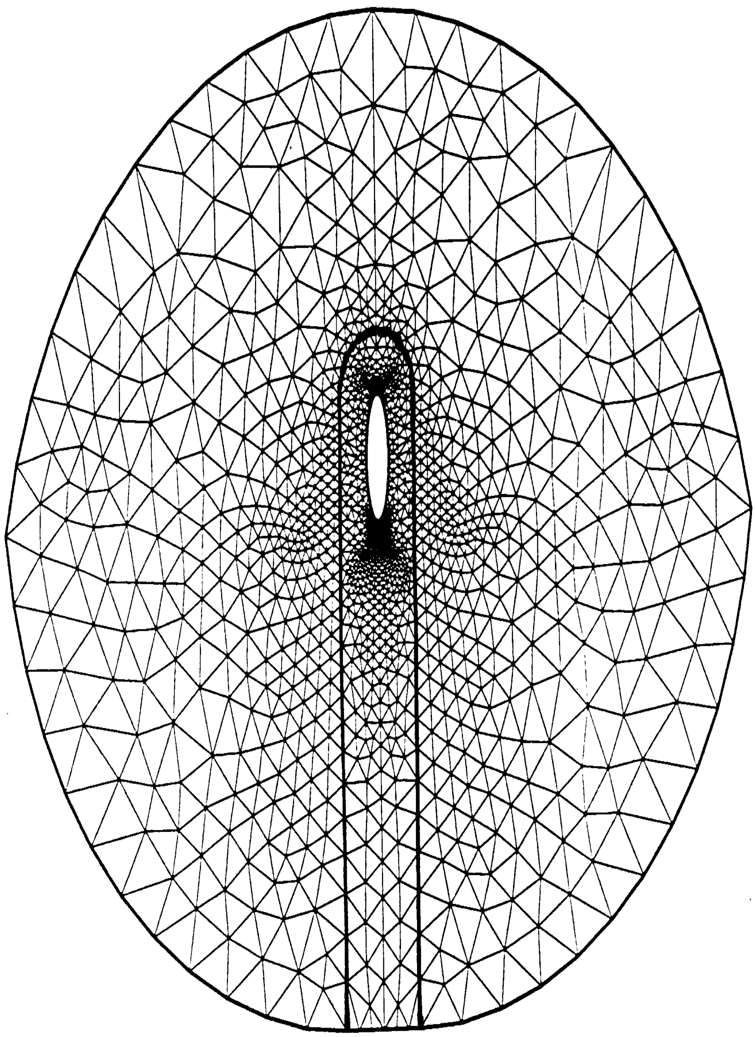

Figure 4: Description of the finite element mesh and of the local subdomain.

In a first step, the velocity field is obtained by solving the following problem :

$$\begin{aligned}
&\text{div } v = \text{ curl } v = 0 \text{ (inviscid incompressible flow)},\\
&v_\infty = (1,0),\\
&v.n = 0 \text{ on the body} \Gamma_b,
\end{aligned}$$

with a first order finite element method using the same global mesh.

If we set $v = 0$, the algorithm **may or may not converge** depending on the values of $\nu\Delta t$. More precisely, we observe that the algorithm converges linearly when $\nu\Delta t < \alpha_0$ and is divergent otherwise. This is graphically shown in figures (4-7) where the values of $\frac{||\phi^{n+1}-\phi^n||}{\Delta t||\phi_0||_\infty}$ are plotted versus the iteration count n for $\nu\Delta t$ taken on the values 10^{-6}, 10^{-1}, 1, and 10. Further, when the velocity is taken sufficiently large, the algorithm becomes unconditionnally stable. In particular, initializing our algorithm by $\phi_0 = 0$ with $||v_\infty|| = 1$, $\nu = 0.1$ and $\Delta t = 100$ results in a converging scheme (fig. 8).

By intuition such a behavior seems natural. An overestimation of the solution ϕ^n at the interface Γ_i implies an overestimation of the friction forces on Γ_b. For sufficiently small time steps, this overestimation will not affect the value of ϕ^{n+1} on Γ_i and can therefore be ignored at the next time step. If the Reynolds is sufficiently large, this error will only affect the wake region but will not have any influence at the interface Γ_i. To the contrary, for large Δt and small ν, the introduced error does affect the value of ϕ^{n+1} on Γ_i. The influence of the error on ϕ^{n+1} may be amplified throughout the iteration process.

Another variant of the algorithm consists of replacing the explicit Dirichlet condition

$$\phi_{loc}^{n+1} = \phi^n \text{ on } \Gamma_i \text{ in the algorithm (7)-(8)}$$

by the following semi-implicit condition

$$\phi_{loc}^{n+1} = \phi^{n+1} \text{ on } \Gamma_i.$$

In fact, this implies replacing the previously parallel algorithm (Jacobi like) by the sequential algorithm (Gauss-Seidel like).

When we solve the pure diffusion problem (i.e. with flow velocity $v = 0$) with $\nu = 1$ and $\Delta t = 1$ (respectively $\Delta t = 2$) we obtain a better convergence history :

- the speed of the new algorithm is linear and clearly faster than the parallel algorithm.
- the domain of convergence is moderately larger (see table 1).

To study experimentally in more details the convergence behavior of both algorithms we assume that we have a linear behavior of our algorithm, and hence that the error at the iteration n will satisfy the following inequality

$$\|\phi^{n+1} - \phi^n\|_\infty \approx K^n \|\phi^1 - \phi^0\|_\infty.$$

The algorithm converges if $K < 1$. An estimate for K can be found by considering as in table 1 the ratio

$$-\frac{1}{n}\log\frac{\|\phi^{n+1} - \phi^n\|_\infty}{\|\phi^1 - \phi^0\|_\infty} = -\log K$$

which is displayed as a function of $(\nu\Delta t)$ for $n = 14$ and different values of $V = \frac{v}{\nu}$. A negative value of this ratio means divergence of the algorithm. As expected, this ratio is positive for sufficiently small values of Δt and converges towards zero as Δt goes to zero.

In this table, we observe that for $V = 0, \nu\Delta t < \alpha_0 \approx 2$, the algorithm converges. However the convergence is slow since the minimal contraction constant K_{min} (for the optimal value of $\nu\Delta t$) is close to one (see table 2). For $V = 10$, then the algorithm appears to converge for a much larger range of values of $\nu\Delta t$ and the optimal contraction constant is much smaller. This is summarized on table 2 where we have displayed the best possible contraction constants for each of the coupling algorithms and different values of the Reynolds $V = \frac{v}{\nu}$.

$\nu\Delta t$	1/1000	1/10	1/2	2	5	10	50	1000
Gauss-Seidel $V=0$	0.06	0.1	0.22	0.5	-0.27	-0.5	-0.75	-0.8
Jacobi $V=0$	-	0.1	0.22	0	-0.09	-0.25	-0.4	-0.41
Gauss-Seidel $V=10$	0.03	0.25	1.46	2.12	2.8	2.6	2.4	2.4
Jacobi $V=10$	0.03	0.28	1.15	1.15	1.15	1.15	1.14	1.14
Jacobi $V=1000$	0.23	2.79	2.8	2.7	2.75	2.8	-	-

Table 1: Contraction constant (in fact minus its logarithm) in function of $\nu\Delta t$ for the explicit (Jacobi) and semi-explicit (Gauss-Seidel) version of our coupling algorithm. We observe divergence for $V=0$ and $\nu\Delta t>2$ and convergence otherwise.

Jacobi (parallel)		Gauss-Seidel (sequential)	
V	K_{min}	V	K_{min}
0	0.85	0	0.68
10	0.50	10	0.11
10^3	0.14		

Table 2: Minimal contraction constant versus the Reynolds V for both sequential and parallel versions of the algorithm.

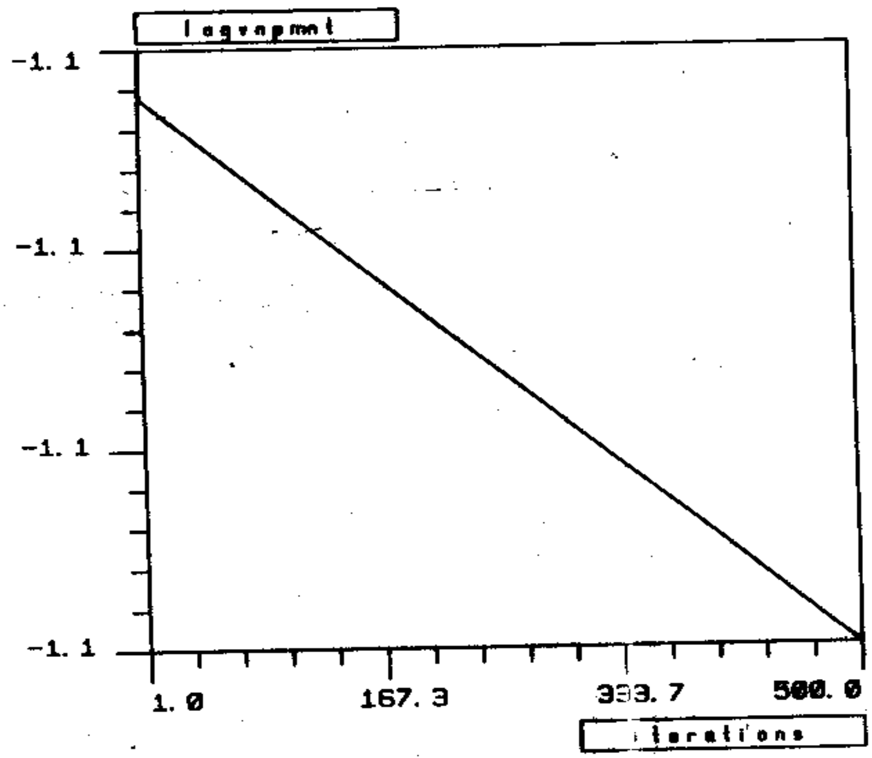

Figure 5: Convergence of the Time Marching Algorithm: $\frac{||\phi^{n+1}-\phi^{n}||}{\Delta t||\phi_0||_\infty}$ are plotted versus the iteration count n for $\nu\Delta t = 10^{-6}$, $v = 0$ (Jacobi). Observe the very slow convergence.

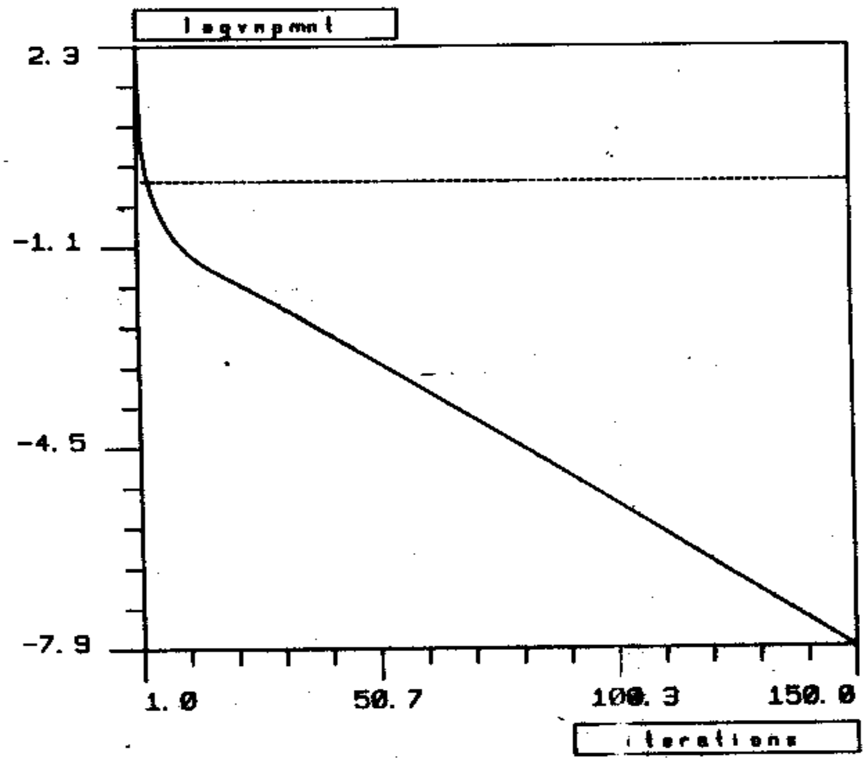

Figure 6: Convergence of the Time Marching Algorithm: $\frac{||\phi^{n+1}-\phi^{n}||}{\Delta t||\phi_0||_\infty}$ are plotted versus the iteration count n for $\nu\Delta t = 10^{-1}$, $v = 0$ (Jacobi).

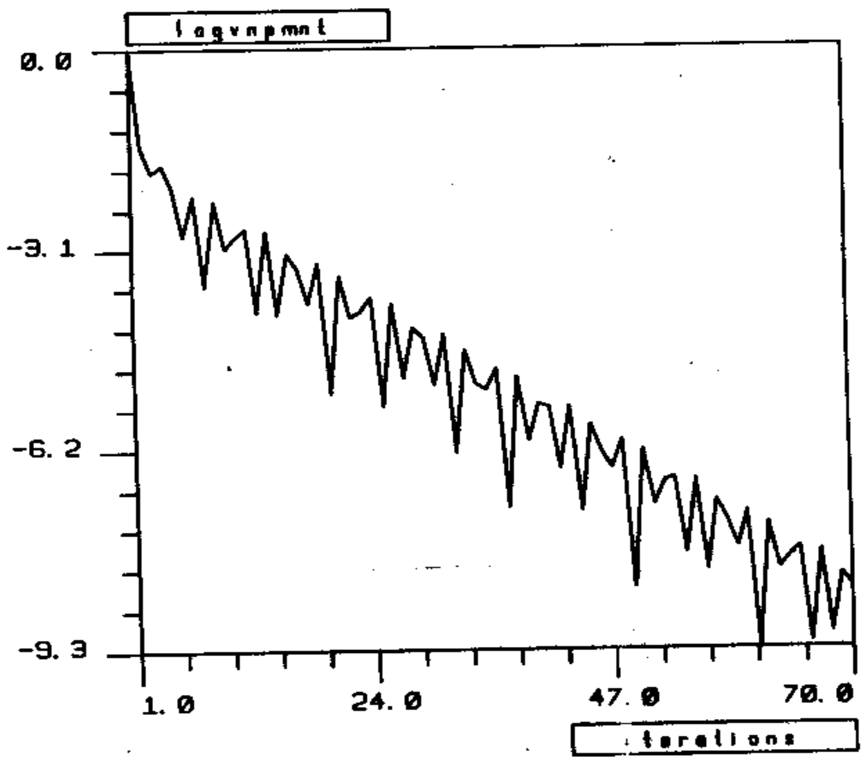

Figure 7: Convergence of the Time Marching Algorithm: $\frac{||\phi^{n+1}-\phi^n||}{\Delta t||\phi_0||_\infty}$ are plotted versus the iteration count n for $\nu\Delta t = 1$, $v = 0$ (Jacobi).

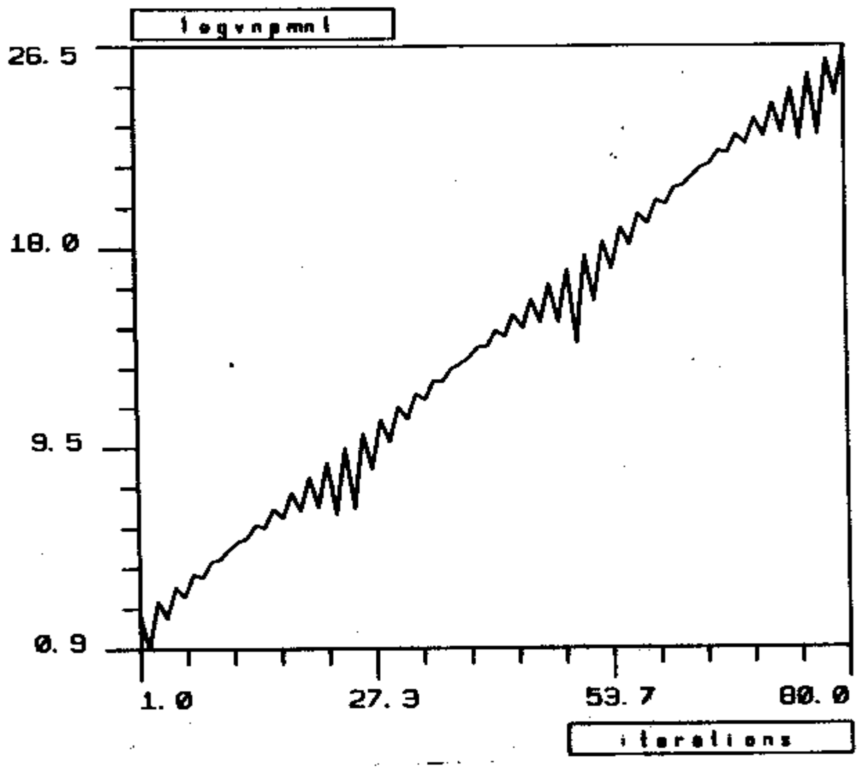

Figure 8: Divergence of the Time Marching Algorithm: $\frac{||\phi^{n+1}-\phi^{n}||}{\Delta t||\phi_0||_\infty}$ are plotted versus the iteration count n for $\nu\Delta t = 10$, $v = 0$ (Jacobi).

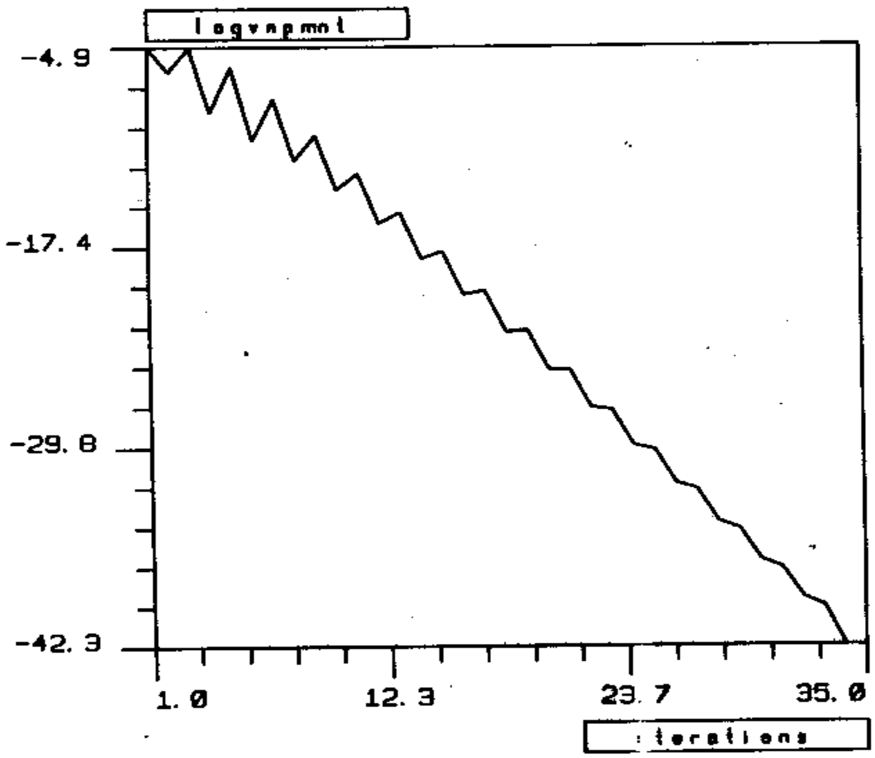

Figure 9: Convergence of the Time Marching Algorithm: $\frac{||\phi^{n+1}-\phi^{n}||}{\Delta t||\phi_0||_\infty}$ are plotted versus the iteration count n for $\nu\Delta t = 10$ and the flow velocity is equal to 1 (Jacobi).

8 Conclusion

We have analysed the convergence properties of a standard time marching algorithm for solving a domain decomposed advection-diffusion problem. We were able to prove theoretically the unconditional stability and linear convergence of the fully implicit algorithm (§5). Using the maximum principle, we were also able to prove in §6 the linear convergence of the semi explicit algorithm used with $\Delta t = +\infty$ on the operator $\mathcal{L}u = -\Delta u + \mathrm{div}(vu) + \alpha u$ if α was sufficiently large, and the linear convergence of the fixed point algorithm applied at each time step to the solution of the implicit problem, independently of any discretization strategy.

When using the uncoupled semi-explicit algorithm in the general case, numerical evidence indicate that this algorithm is unstable for large values of Δt and small overlapping, and that it becomes linearly convergent when Δt is below a Reynolds dependent threshold (§7). This conditional stability is not a real issue for practical CFD problems because most solvers already require to use small time steps inside each domain. Nevertheless, it would be nicer to derive an uncoupled unconditionnally stable version of the present time marching algorithm.

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Appendix

The main result of this section relies on the notion of a contact set. If u is a continuous arbitrary function on Ω, the upper contact set, denoted Γ^+ or Γ_u^+, is the sub-set of Ω, defined by

$$\Gamma^+ = \{y \in \Omega, \exists p(y) \in \mathbb{R}^n \text{such that } u(x) \leq u(y) + p \cdot (x-y) \; \forall x \in \Omega \;\}. \tag{126}$$

We see that u is a concave function on Ω iff $\Gamma^+ = \Omega$. When $u \in C^1(\Omega)$ we must have $p = Du(y)$ in the relation (126). In addition, when $u \in C^2(\Omega)$, the Hessian matrix $D^2u = [D_{ij}u]$ is negative on Γ^+. In general, Γ^+ is closed in Ω.

If u is a continuous arbitrary function on Ω, we define the "normal mapping" $\chi(y) = \chi_u(y)$ at point $y \in \Omega$ by

$$\chi(y) = \{p \in \mathbb{R}^n, u(x) \leq u(y) + p.(x-y) \; \forall x \in \Omega\}. \tag{127}$$

We can see that $\chi(y)$ is non empty iff $y \in \Gamma^+$. In addition when $u \in C^1(\Omega)$, we have $\chi(y) = Du(y)$ on Γ^+; in other words χ is the gradient field of u on Γ^+.

As a particular case of the Bakelman-Alexandrov ([8] and [9]) maximum principle, we have under the above notation.

Lemma .1 *For $u \in C^2(\Omega) \cap C^o(\bar{\Omega})$, we have :*

$$\sup_{\Omega} u \leq sup_{\partial\Omega} u + \frac{d}{n w_n^{1/n}} \|a^{ij} D_{ij} u / \mathcal{D}^*\|_{n,\Gamma^+}$$

with d the diameter of Ω and w_n the volume of a unit sphere in R^n.

For further details see [12].

We now proceed to the proof of Theorem 6.1, by following the steps of [12]. We take $\hat{B} = B_1(0)$ and the general case will be deduced by considering the coordinate transform, $x \rightarrow \hat{x} = (x-y)/2R$.

We will begin, in first step, by showing this result for $u \in C^2(\Omega) \cap W^{2,n}(\Omega)$ and then in a second step we will deduce the result for $u \in W^{2,n}(\Omega)$.

Step 1:

We suppose that $u \in C^2(\Omega) \cap W^{2,n}(\Omega)$. For $\beta \geq 1$, we consider the cut off function η defined by

$$\eta(\hat{x}) = (1 - |\hat{x}|^2)^\beta.$$

By differentiation, we obtain

$$\hat{D}_i \eta = -2\beta \hat{x}_i (1 - |\hat{x}|^2)^{\beta - 1},$$

$$\hat{D}_{ij}\eta = -2\beta\delta_{ij}(1-|\hat{x}|^2)^{\beta-1} + 4\beta(\beta-1)\hat{x}_i\hat{x}_j(1-|\hat{x}|^2)^{\beta-2}.$$

By setting

$$v = \eta u,$$

we then obtain

$$\begin{aligned} \hat{a}^{ij}\hat{D}_{ij}v &= \eta\hat{a}^{ij}\hat{D}_{ij}u + 2\hat{a}^{ij}\hat{D}_i\eta\hat{D}_j u + u\hat{a}^{ij}\hat{D}_{ij}\eta \\ &\geq \eta(\hat{f} - \hat{b}^i\hat{D}_i u - \hat{c}u) + 2\hat{a}^{ij}\hat{D}_i\eta\hat{D}_j u + u\hat{a}^{ij}\hat{D}_{ij}\eta. \end{aligned}$$

Let $\Gamma^+ = \Gamma_v^+$ be the upper contact set v, in the sphere $\hat{B}$; we have :

$$u > 0 \text{ on } \Gamma^+.$$

If $x \in \partial\hat{B}$ such that $p.(x-y) < 0$ we indeed have $v(x) = 0$. Consequently

$$v(y) + p.(x-y) \geq v(x) = 0.$$

Moreover, using the concavity of v on Γ^+, we can estimate the following quantity :

$$|\hat{D}u| = (1/\eta)|\hat{D}v - u\hat{D}\eta|.$$

Indeed,

$$\begin{aligned} \left|\hat{D}u\right| &\leq (1/\eta)(\left|\hat{D}v\right| + u\left|\hat{D}\eta\right|) \\ &\leq (1/\eta)(\frac{v}{1-|\hat{x}|} + u|\hat{D}\eta|) \\ &\leq 2(1+\beta)\eta^{-1/\beta}u. \end{aligned}$$

In that way, we have on Γ^+ the following inequality :

$$\begin{aligned} -\hat{a}^{ij}\hat{D}_{ij}v \leq \quad & \{(16\beta^2 + 2\eta\beta)\hat{\Lambda}\eta^{-2/\beta} + \\ & 2\beta|\hat{b}|\eta^{-1/\beta} + \hat{c}\}v + \eta|\hat{f}|. \end{aligned}$$

Since $\hat{c} \leq 0$, we deduce the inequality

$$\begin{aligned} -\hat{a}^{ij}\hat{D}_{ij}v &\leq \{(16\beta^2 + 2\eta\beta)\hat{\Lambda}\eta^{-2/\beta} + \\ & \quad 2\beta|\hat{b}|\eta^{-1/\beta}\}v + \eta|\hat{f}| \\ &\leq c_1\hat{\lambda}\eta^{-2/\beta}v + |\hat{f}|, \end{aligned} \tag{128}$$

with $c_1 = c(n, \beta, \gamma, \hat{\delta})$ independent of $\hat{c}$.

Consequently, by applying Lemma 6.1 on $\hat{B}$, we obtain, for $\beta \geq 2$:

$$\sup_{\hat{B}} v \leq (\frac{\hat{d}}{nw_n^{1/n}})(\frac{1}{\hat{\mathcal{D}}^*})||c_1\hat{\lambda}\eta^{-2/\beta}v + |\hat{f}|||_{n,\hat{B}}.$$

By using the relation (6.1), it comes

$$\begin{aligned}
\sup_{\hat{B}} v &\leq (\frac{\hat{d}}{nw_n^{1/n}})c_1||\eta^{-2/\beta}v||_{n,\hat{B}} + (\frac{\hat{d}}{nw_n^{1/n}})(\frac{1}{\hat{\lambda}})||\hat{f}||_{n,\hat{B}} \\
&\leq c_1\hat{d}(||\eta^{-2/\beta}v||_{n,\hat{B}} + (1/\hat{\lambda})||\hat{f}||_{n,\hat{B}}) \\
&\leq c_1\hat{d}(||\eta^{-2/\beta}v^+||_{n,\hat{B}} + (1/\hat{\lambda})||\hat{f}||_{n,\hat{B}}) \\
&\leq c_1\hat{d}((supv^+)^{1-2/\beta}||(u^+)^{2/\beta}||_{n,\hat{B}} + (1/\hat{\lambda})||\hat{f}||_{n,\hat{B}}),
\end{aligned}$$

where c_1 is a constant depending only on n, β, γ and $\hat{\delta}$. Here, $\hat{d}$ is the diameter of $\hat{B}(\hat{d} = 2)$.

By using the Young inequality under the form

$$ab \leq \varepsilon a^q + \varepsilon^{-r/q} b^r$$

for $q = (1 - 2/\beta)^{-1}$ and $r = \beta/2$, we have

$$(supv^+)^{1-2/\beta}||(u^+)^{2/\beta}||_{n,\hat{B}} \leq \varepsilon supv^+ + \varepsilon^{1-\beta/2}||(u^+)^{2/\beta}||_{n,\hat{B}}^{\beta/2}, \quad \forall\varepsilon > 0.$$

By taking $\varepsilon = \dfrac{1}{2c_1\hat{d}}$ and plugging in our inequality on v, we obtain :

$$\begin{aligned}
\sup_{\hat{B}} v &\leq (1/2)\sup v^+ + (1/2)^{1-\beta/2}(c_1\hat{d})^{\beta/2}||(u^+)^{2/\beta}||_{n,\hat{B}}^{\beta/2} \\
&\quad +(c_1\hat{d}/\hat{\lambda})||\hat{f}||_{n,\hat{B}}.
\end{aligned} \tag{129}$$

We want to prove the theorem for all $p > 0$. We will treat separately the cases $p \leq n$ and $p > n$.

If $p \leq n$, we set $\beta = 2n/p$. In this case we have

$$||(u^+)^{2/\beta}||_{n,\hat{B}}^{\beta/2} = ||(u^+)||_{p,\hat{B}}.$$

Plugging this in our inequality on v, we obtain :

$$(1/2)\sup_{\hat{B}} v \leq (1/2)^{1-\beta/2}(c_1\hat{d})^{\beta/2}||(u^+)||_{p,\hat{B}} + (c_1\hat{d}/\hat{\lambda})||\hat{f}||_{n,\hat{B}}.$$

Consequently, we obtain the following inequality ;

$$\sup_{\hat{B}} v \leq c_2\{(\int_{\hat{B}}(u^+)^p)^{1/p} + (\hat{d}/2\hat{\lambda})\|\hat{f}\|_{n,\hat{B}}\}.$$

On the sphere $B_{1/2}(0)$, the cut off function satisfies

$$1/\eta \leq (1/2)^\beta.$$

It follows, then

$$\begin{aligned} \sup_{B_{1/2}(0)} u &\leq \sup_{B_{1/2}(0)} (v/\eta) \\ &\leq 2^\beta \sup_{\hat{B}} v. \end{aligned}$$

Finally we end up at the desired estimate

$$\sup_{B_{1/2}(0)} u \leq c_3\{(\int_{\hat{B}}(u^+)^p)^{1/p} + (\hat{d}/2\hat{\lambda})\|\hat{f}\|_{n,\hat{B}}\}.$$

for u in $W^{2,n}(\Omega) \cap C^2(\bar{\Omega})$. The constant c_3 above depend only on n, β, γ and $\hat{\delta}$, but is independent of $\hat{c}$.

On the other hand if $p > n$, we have :

$$2n/\beta < p, \ \forall \beta \geq 2.$$

Then, it follows (by assuming $\beta \geq 2$)

$$|\hat{B}|^{-1/(2n/\beta)}\|(u^+)\|_{2n/\beta,\hat{B}} \leq |\hat{B}|^{-1/p}\|u^+\|_{p,\hat{B}}.$$

But

$$\|u^+\|_{2n/\beta} = \|(u^+)^{2/\beta}\|_{n,\hat{B}}^{\beta/2},$$

and therefore, by processing as before, we obtain the desired estimate

$$\sup_{B_{1/2}(0)} u \leq c_4\{(\int_{\hat{B}}(u^+)^p)^{1/p} + (\hat{d}/2\hat{\lambda})\|\hat{f}\|_{n,\hat{B}}\}$$

for u in $W^{2,n}(\Omega) \cap C^2(\bar{\Omega})$. The constant c_4 above depends only on n, β, γ and $\hat{\delta}$, but is independent of $\hat{c}$.

Transformation $\hat{x} \to x$.

By construction, $\hat{D}_{ij} = R^{-2}D_{ij}$, thus $\hat{\lambda} = R^{-2}\lambda$ and $\hat{\delta} = \delta R^2$. In addition, we have $|B| = w_n(2R)^n$ and $|g|_{p,\hat{B}} = R^{-n/p}|g|_{p,B}$.

Written in term of x, the last inequality becomes

$$\sup_{B_{R(y)}} u \leq c_4\{(\frac{2^n w_n}{|B|}\int_B (u^+)^p\, dx)^{1/p} + (\frac{2w_n^{1/n}R}{\lambda})\|f\|_{n,B}\},$$

with c_4 a function of $n, \gamma, \hat{\delta} = \delta R^2$ and p. This is the desired estimate for $u \in W^{2,n}(\Omega) \cap C^o(\bar{\Omega})$.

Step 2:

Now, let $u \in W^{2,n}(\Omega)$. By density, let (u_m) be a sequence of functions of $C^2(\bar{B})$, converging towards u in $W^{2,n}(B)$. The injection of $W^{2,n}(B)$ in $C^o(B)$ is continuous, consequently (u_m) converges uniformly towards u in B. We have

$$\begin{aligned} Lu_m &= L(u_m - u) + Lu \\ &\geq f + L(u_m - u). \end{aligned}$$

By setting, $f_m = L(u_m - u)$, we observe by construction that f_m converges towards 0 in $L^n(\Omega)$. As $u_m \in W^{2,n}(\Omega) \cap C^2(\Omega)$ and $\tilde{f}_m = f + f_m$ is in $L^n(\Omega)$, the estimate (59) is valid also for u_m, so that we have

$$\sup_{B_R(y)} u_m \leq cte\{(\frac{1}{|B|}\int_B (u_m^+)^p)^{1/p} + \frac{R}{\lambda}\|\tilde{f}\|_{n,B}\}. \tag{130}$$

Using previous results and taking the limit, we have :

$$\sup_{B_R(y)} u \leq cte\{(\frac{1}{|B|}\int_B (u^+)^p)^{1/p} + \frac{R}{\lambda}\|f\|_{n,B}\}.$$

■

Observe also that by replacing u by $-u$, the theorem can be extended easily to the case of supersolutions and solutions of the equation :

$$Lu = f.$$

Unité de recherche Inria Lorraine, Technopôle de Nancy-Brabois, Campus scientifique,
615 rue du Jardin Botanique, BP 101, 54600 Villers Lès Nancy
Unité de recherche Inria Rennes, Irisa, Campus universitaire de Beaulieu, 35042 Rennes Cedex
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