A self-dual modality for ”before” in the category of coherence spaces and in the category of hypercoherences

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Abstract: In his paper "A new constructive logic: classical logic" Jean-Yves Girard brought up the question of a self-dual modality. This note provides a semantical solution with respect to the self-dual connective `before` in the category of coherence spaces, and in the category of hypercoherences.

Key-words: Denotational semantics. Logic, proof theory, linear logic.

(Résumé : tsvp)

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A more "categorical" version is to appear in french: [Ret94b]

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Une modalité autoduale pour “précède” dans la catégorie des espaces cohérents et dans celle des hypercohérences

Résumé : Dans son article “A new constructive logic: classical logic” Jean-Yves Girard pose la question d’une modalité autoduale. Cette note fournit une solution sémantique relative au connecteur autodual `précède` dans la catégorie des espaces cohérents et dans celle des hypercohérences.

Mots-clé : Sémantique dénotationnelle. Logique, théorie de la démonstration, logique linéaire.

1 Presentation

This note strongly relies on [Gir91], where the question was raised. This section is just a concise reminder.

The structural rules of classical logic are responsible for the non-determinism of classical logic, and linear logic which carefully handles these rules is especially adequate for a constructive treatment of classical logic, as the afore mentioned paper shows. Linear logic handles structural rules by the modalities (a.k.a. exponentials) "?" and "!". The modality "?" allows contraction and weakening in positive position, and the modality "!" in negative position. The formula $!A$ linearly implies A while the formula $?A$ is linearly implied by A.

In semantical words this means we have the linear morphisms:

$$
\begin{tikzcd}
!A \arrow[r, "\multimap"] \arrow[d, "\multimap"] \arrow[dr, "\multimap"] & !A \otimes !A \\
\mathbf{1} & A
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
?A & ?A \wp ?A \arrow[l, "\multimap"'] \\
\mathbf{1} \arrow[u, "\multimap"] & A \arrow[ul, "\multimap"']
\end{tikzcd}
$$

The major difficulty when dealing with classical logic are the cross-cuts, appearing in the cut elimination theorem of Gentzen as a rule called MIX [Gen34]. This rule is a generalised cut between several occurences of A and several occurences of $\neg A$, which is in fact a cut between two formulae coming both from a contraction. This is a major cause of non-determinism: an example can be found in [Gir91], Appendix B, Example 2, p 294.

In linear logic, such a cut may not happen, since contraction applies on $?A$ formulae, while their negation for applying a cut is $!A^\perp$ *which can not come from a contraction*.

Let us now quote the precise paragraph of [Gir91] p 257 which motivates this note:

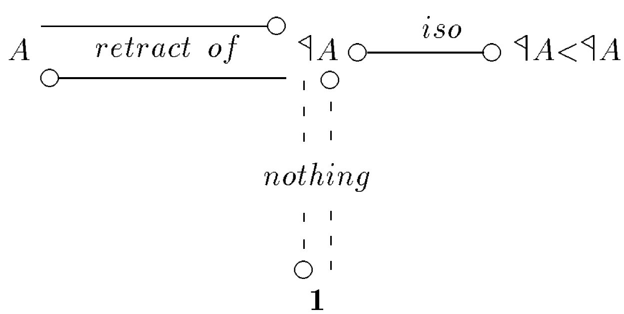
> *"The obvious candidate for a classical semantics was of course coherence spaces which had already given birth to linear logic; the main reason for choosing them was the presence of the involutive* linear *negation. However the difficulty with classical logic is to accommodate structural rules (*weakening *and* contraction*); in linear logic, this is possible by considering*

coherent spaces $?X$. But since classical logic allows contraction and weakening both on a formula and its negation, the solution seemed to require the linear negation of $?X$ to be of the form $?Y$, which is a nonsense (the negation of $?X$ is $!X^\perp$ which is by no means of isomorphic to a space $?Y$). Attempts to find a self-dual variant $\S Y$ of $?Y$ (enjoying $(\S Y)^\perp = \S Y^\perp$) systematically failed. The semantical study of classical logic stumbled on this problem of self-duality for years."

Here too we focus on coherence spaces because of their tight relation to linear logic [Gir87, Tro92, Ret94a]. Once the modality is found in the category of coherence spaces, we briefly show that this modality also exists in the category of hypercoherences that Thomas Ehrhard introduced in[Ehr93]. This is encouraging since hypercoherences, which may be viewed as a refinement of coherence spaces, are a different semantics, also very close to linear logic.

In our previous work on pomset logic [Ret93, Ret95], we studied a self-dual connective `before` , together with partially ordered multisets of formulae. This lead us to the modality $\triangleleft$ to be described. It is a functor, it is self-dual, and it enjoys both left and right contraction with respect to `before` , and A is a retract of $\triangleleft A$. Fortunately it does not interprets weakening, otherwise we would have (semantically) some strange phenomena like $\mathbf{1} \multimap A$ and $A \multimap \mathbf{1}$ for all A!

Here is a picture of the canonical linear morphisms that the functorial modality $\triangleleft$ enjoys:



There is not yet any syntax extending pomset logic to this modality. We firstly need to study the basic steps of cut-elimination, in particular the contraction/contraction case and the commutative diagrams it requires, as Myriam Quatrini did in [Qua95] for the logical calculus LC of [Gir91].

2 Preliminary remarks

2.1 Before

We refer the reader to [Gir87, Tro92] for the definition of coherence spaces. Let us simply recall the multiplicative connective `before` studied in [Ret93, Ret95], written $A{<}B$:

Definition 1 *Given two coherence spaces A and B, the coherence space $A{<}B$ is defined by:*

web $|A{<}B| = |A| \times |B|$

coherence $(a,b) \frown (a',b')[A{<}B]$ *whenever* $(a \frown a'[A]$ and $b = b')$ or $b \frown b'[B]$

From [Ret93] we know that the following easy proposition holds:

Proposition 1 *This connective is:*

non-commutative, $A{<}B \not\equiv B{<}A$,

self-dual, $(A{<}B)^\perp \equiv A^\perp{<}B^\perp$,

associative $A{<}(B{<}C) \equiv (A{<}B){<}C$,

admits $\mathbf{1}$ *as a unit,* $A{<}\mathbf{1} \equiv A \equiv \mathbf{1}{<}A$

in between $\wp$ *and* $\otimes$*: for all formulae A and B, we have*

$$A \otimes B \multimap A{<}B \text{ and } A{<}B \multimap A \wp B.$$

2.2 Several trivial remarks on trees

Definition 2 *We write 2 for $\{0,1\}$, 2^* for the set of finite words on 2, including the empty word, 2^ω for the set of infinite words on 2, with the usual lexicographical order and product topology. Letters like w, v, u range over 2^ω, while m range over 2^*. We use the standard notation $w = m(m')^*$ for $w = mm'm'm'm'm'....$*

Proposition 2 ($\mathtt{gt}_M$ generic trees on M) *The set $\mathtt{gt}_M$ of continuous functions from 2^ω to a set M (discrete topology) is in a one-to-one correspondence with the set of finite binary trees on M such that any two sister leaves have distinct labels.*

Thus, an element f of $\mathtt{gt}_M$ may be described:

(1) as a finite set $\{(m_1, a_1), ..., (m_k, a_k)\} \subset \mathcal{P}_{fin}(2^* \times M)$ *satisfying:*

(a) $\forall w \in 2^\omega\ \exists! i\ \exists w' \in 2^\omega\ w = m_i w'$

(b) $\forall i, j\ \ [\exists m \in 2^*\ m_i = m0\ \text{ and }\ m_j = m1] \Rightarrow a_i \neq a_j$

In this formalism $f(w)$ *is computed as follows: applying (a), there exists a unique* i *such that there exists* w' *with* $w = m_i w'$*, let* $f(w) = f(m_i(0)^*)$*.*

(2) as the normal form *of a term of the grammar:*

$$\mathcal{T}_M \ :: \ \underline{M} \Big| \langle \mathcal{T}_M\ \mathcal{T}_M \rangle$$

where the reduction is $\forall x \in M\ t\left(\langle \underline{x}\ \underline{x} \rangle\right) \longrightarrow t\left(\underline{x}\right)$ *where* $t\left(u\right)$, $u \in \mathcal{T}_M$ *means a term of* $\mathcal{T}_M$ *having an occurrence of the subterm* $u \in \mathcal{T}_M$*.*

In this formalism $f(w)$ *is computed as follows:*

$\underline{a}(w) = a \qquad \langle t_0 t_1 \rangle (0w) = t_0(w) \qquad \langle t_0 t_1 \rangle (1w) = t_1(w)$

Proof: Straightforward, see appendix if not familiar with 2^ω. $\diamond$

As it is very simple, an example will avoid us to be too formal:

Example 1 *Let* $M = \{a, b, c\}$*.*
Here are the three description of the same element of $\mathtt{gt}_M$*:*

(0) as a function

$$\begin{aligned} f(000w) = f(001w) = f(010w) = f(011w) &= a \\ f(100w) &= a \\ f(101w) &= b \\ f(110w) &= a \\ f(111w) &= b \end{aligned}$$

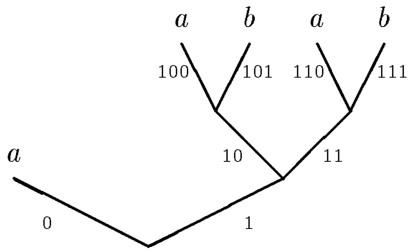
(1) as a finite set of pairs $\{(m_i, a_i)/m_i \in 2^*\ \text{ and }\ a_i \in M\}$:

$$f = \{(0, a), (100, a), (101, b), (110, a), (111, b)\}$$

(2) as a normal term of $\mathcal{T}_M$: $f = \left\langle \underline{a}\ \left\langle \langle \underline{a}\ \underline{b} \rangle\ \langle \underline{a}\ \underline{b} \rangle \right\rangle \right\rangle$

such a term is the normal form of, e.g. $\left\langle \langle \underline{a}\ \underline{a} \rangle\ \left\langle \langle \underline{a}\ \underline{b} \rangle\ \left\langle \langle \underline{a}\ \underline{a} \rangle\ \ \underline{b} \right\rangle \right\rangle \right\rangle$

(3) the best way to think of it is the most unpleasant to type:



One more easy remark:

Proposition 3 *Let $f, g \in \mathtt{gt}_M$. If $f \neq g$, then there exists $w \in 2^\omega$ such that*

$$f(w) \neq g(w) \text{ and } \forall w' > w\ f(w') = g(w')$$

<u>*Proof:*</u> As for proposition 2., see appendix if necessary. ⋄

3 The modality and its properties

3.1 The modality

The first attempt to find such a modality, inspired by the product of Q copy of A lead me to consider sequences of finitely many tokens in A, indexed by the rational numbers of $[0, 1[$. It works but has too many ($2^{\aleph_0}$!) contraction isomorphisms. Achim Jung told me binary trees should work and avoid this drawback, and thanks to his suggestion, I arrived to the following:

Definition 3 *Let A be a coherence space. We define $\triangleleft A$ as follows:*

web *The set $\mathtt{gt}_{|A|}$ of continuous functions from 2^ω to $|A|$ (discrete topology).*

coherence *Two functions f and g of $\mathtt{gt}_{|A|} = |\triangleleft A|$ are said to be strictly coherent whenever*

$$\exists w \in 2^\omega\ f(w) \frown g(w)[A] \text{ and } \forall w' > w\ f(w') = g(w')$$

3.2 Properties

The following is clear from proposition 2:

Proposition 4 (denumerable web) *If $|A|$ is denumerable, so is $|\triangleleft A|$.*

And next come the key property:

Proposition 5 (self-duality) *The modality $\triangleleft$ is self-dual, i.e. $(\triangleleft A)^{\perp} \equiv \triangleleft(A^{\perp})$.*

Proof: These two coherence spaces obviously have the same web.

Hence it is equivalent to show that, given two distinct tokens f, g in $|\triangleleft A|$ the following exclusive disjunction holds:

$$f \frown g[\triangleleft A] \mathbin{⩔} f \frown g[\triangleleft(A^{\perp})]$$

If $f \neq g$ because of the previous proposition 3, there is an infinite word of 2^{ω} such that $f(w) \neq g(w)$ and $\forall w' > w \;\; f(w') = g(w')$. The following exclusive disjunction holds:

$$f(w) \frown g(w)[A] \mathbin{⩔} f(w) \frown g(w)[A^{\perp}]$$

and is part-wise equivalent to the previous exclusive disjunction. $\diamond$

Proposition 6 (contraction isomorphism) *There is a canonical linear isomorphism between $\triangleleft A$ and $\triangleleft A < \triangleleft A$*

Proof: Consider the following subset of $|\triangleleft A| \times |\triangleleft A < \triangleleft A|$:

$$\mathfrak{C} = \{(h, (h_0, h_1)) / \forall w \in 2^{\omega} \; h(0w) = h_0(w) \;\text{ and }\; h(1w) = h_1(w)\}$$

We show that it is the trace of a linear isomorphism between $\triangleleft A$ and $\triangleleft A < \triangleleft A$.

Firstly, $\mathfrak{C}$ clearly defines a bijection, between the webs $|\triangleleft A|$ and $|\triangleleft A < \triangleleft A| = |\triangleleft A| \times |\triangleleft A|$.

Secondly, let us see that, given $(h, (h_0, h_1))$ and $(g, (g_0, g_1))$ in $\mathfrak{C}$ we have

$$h \frown h'[\triangleleft A] \Longleftrightarrow (h_0, h_1) \frown (h'_0, h'_1)[\triangleleft A < \triangleleft A]$$

$\boxed{\Rightarrow}$ We assume that $h\frown h'[\lhd A]$, i.e. $\exists w \in 2^\omega\ h(w)\frown g(w)$ and $\forall v{>}w\ \ h(v) = g(v)$. Two cases may occur: either $w = 0w'$ or $w = 1w'$, and we show that in both cases we have $(h_0, h_1)\frown(g_0, g_1)[\lhd A{<}\lhd A]$

(0) If $w = 0w'$ we have $h_0\frown g_0[\lhd A]$ and $h_1 = g_1$, and thus $(h_0, h_1)\frown(g_0, g_1)[\lhd A{<}\lhd A]$

$\underline{h_0\frown g_0[\lhd A]}$

$\underline{h_0(w')\frown g_0(w')[A]}$ Indeed we have $h_0(w') = h(0w') = h(w)$ and $g_0(w') = g(0w') = g(w)$ and we know that $h(w)\frown g(w)[A]$.

$\underline{\forall v' > w'\ h_0(v') = g_0(v')}$ Indeed for all $v' > w'$ we have $0v' > 0w' = w$ and therefore $h_0(v') = h(0v') = g(0v') = g_0(v')$.

$\underline{h_1 = g_1}$ Indeed for all u, we have $1u > 0w' = w$ and therefore $h_1(u) = h(1u) = g(1u) = g_1(u)$.

(1) If $w = 1w'$, then we have $h_1\frown g_1[\lhd A]$ and thus $(h_0, h_1)\frown(g_0, g_1)[\lhd A{<}\lhd A]$.

$\underline{h_1(w')\frown g_1(w')[A]}$ Indeed we have $h_1(w') = h(1w') = h(w)$ and $g_1(w') = g(1w') = g(w)$ and we know that $h(w)\frown g(w)[A]$.

$\underline{\forall v' > w'\ h_1(v') = g_1(v')}$ Indeed for all $v' > w'$ we have $1v' > 1w' = w$ and therefore $h_1(v') = h(1v') = g(1v') = g_1(v')$.

$\boxed{\Leftarrow}$ We assume that $(h_0, h_1)\frown(g_0, g_1)[\lhd A{<}\lhd A]$ and therefore we either have $(h_0\frown g_0[\lhd A]$ and $h_1 = g_1)$ or $h_1\frown g_1[\lhd A]$. We show that in both cases we have $h\frown g[\lhd A]$.

(0) If $h_0\frown g_0[\lhd A]$ and $h_1 = g_1$ then there exists w' such that $h_0(w')\frown g_0(w')[A]$ and $h_0(v') = g_0(v')$ for all $v' > w'$. Let $w = 0w'$.

$\underline{h(w)\frown g(w)[A]}$ Indeed we have $h(w) = h(0w') = h_0(w')$ and $g(w) = g(0w') = g_0(w')$ and we know that $h_0(w')\frown g_0(w')[A]$

$\underline{\forall v > w\ h(v) = g(w)}$ Let $v > w$.

$\underline{\text{If } v = 0v'}$ then $v' > w'$ and therefore $h(v) = h(0v') = h_0(v') = g_0(v') = g(0v') = g(v)$.

$\underline{\text{If } v = 1v'}$ then $h(v) = h(1v') = h_1(v') = g_1(v') = g(1v') = g(v)$.

(1) If $h_1\frown g_1[\lhd A]$ then there exists w' such that $h_1(w')\frown g_1(w')[A]$ and $h_1(v') = g_1(v')$ for all $v' > w'$. Let $w = 1w'$.

$\underline{h(w)\frown g(w)[A]}$ Indeed we have $h(w) = h(1w') = h_1(w')$ and $g(w) = g(1w') = g_1(w')$ and we know that $h_1(w')\frown g_1(w')[A]$.

$\underline{\forall v > w\ h(v) = g(v)}$ If $v > w$ then $v = 1v'$ with $v' > w'$. Therefore $h(v) = h(1v') = h_1(v') = g_1(v') = g(1v') = g(v)$.

$\diamond$

Proposition 7 (A retract of $\triangleleft A$) *Any coherence space A is a linear retract of $\triangleleft A$:*

$$\triangleleft A \underset{\mathfrak{f}_A}{\overset{\mathfrak{t}_A}{\rightleftarrows}} A$$

Proof: Consider the following subset of $|A| \times |\triangleleft A|$, where $\underline{a} \in |\triangleleft A|$ stands for the constant function mapping any element of 2^ω to a.

$$\{(a, \underline{a}) / a \in |A|\}$$

It is a linear trace both from A to $\triangleleft A$ and from $\triangleleft A$ to A. The compound $\mathfrak{f}_A \circ \mathfrak{t}_A$ is $\{(\underline{a}, \underline{a}) / a \in |A|\}$ which is a strict subset of $Id_{\triangleleft A}$, while the compound $\mathfrak{t}_A \circ \mathfrak{f}_A$ is exactly Id_A. $\diamond$

Proposition 8 ($\triangleleft$ is a functor) *If $\ell : A \multimap B$ defines $\triangleleft \ell : \triangleleft A \multimap \triangleleft B$ by the following trace:*

$$\triangleleft \ell = \{(f, g) / \forall w \in 2^\omega \ \ (f(w), g(w)) \in \ell\}$$

This makes $\triangleleft$ an endo-functor.

Proof:

(1) Firstly,let us show that $\triangleleft \ell$ defines a linear map from $\triangleleft A$ to $\triangleleft B$. Let (f, g) and (f', g') be in $\triangleleft \ell$.

- Assume that $f \frown f'[\triangleleft A]$. Thus there exists w such that $f(w) \frown f'(w)[A]$ and $f(v) = f'(v)$ for all $v > w$. Since we know that $(f(w), g(w))$ and $(f'(w), g'(w))$ are in ℓ which is linear, we have $g(w) \frown g'(w)[B]$.
Now let $v > w$. We have $f(v) = a = f'(v)$ and since both $(a, g(v))$ and $(a, g'(v))$ are in ℓ which is linear we have $g(v) \mathrel{\substack{\frown \\ \smile}} g'(v)[B]$.
Applying proposition 3, there exists an u such that $g(u) \neq g'(u)$ and $g(t) = g'(t)$ for all $t > u$. We necessarily have $u \geqslant w$ and therefore $g(u) \frown g'(u)[B]$. Hence $g \frown g'[\triangleleft B]$.
- Assume $f = f'$. For all w, both $(f(w), g(w))$ and $(f(w), g'(w))$ are in ℓ which is linear. Therefore, for all w, $g(w) \mathrel{\substack{\frown \\ \smile}} g'(w)[B]$. Applying proposition 3, either $g = g'$ or there exists an u such that $g(u) \neq g'(u)$ and $g(v) = g'(v)$ for all $v > u$. In the second case we have $g(u) \frown g'(u)[B]$ since $g(w) \mathrel{\substack{\frown \\ \smile}} g'(w)[B]$ for all w. In both case we have $g \mathrel{\substack{\frown \\ \smile}} g'[\triangleleft B]$

(2) It is easily seen that $\triangleleft Id_A = Id_{\triangleleft A}$.

(3) Let us now show that $\triangleleft$ commutes with linear composition. Let $\ell : A \multimap B$ and $\ell' : B \multimap C$, be two linear functions.

- Assuming that (f,h) is in $(\triangleleft\ell')\circ(\triangleleft\ell)$ it is easily seen that (f,h) is in $\triangleleft(\ell'\circ\ell)$. Indeed there exists a g in $|\triangleleft B|$ such that (f,g) is in $\triangleleft\ell$ and (g,h) is in $\triangleleft\ell'$. Thus, for all w the pair $(f(w),g(w))$ is in ℓ and the pair $(g(w),h(w))$ is in ℓ', so that $(f(w),h(w))$ is in $\ell'\circ\ell$ for all w, i.e. (f,h) is in $\triangleleft(\ell'\circ\ell)$.
- We now assume that (f,h) is in $\triangleleft(\ell'\circ\ell)$ and show that it is in $(\triangleleft\ell')\circ(\triangleleft\ell)$ too. If (f,h) is in $\triangleleft(\ell'\circ\ell)$ then $(f(w),h(w)$ is in $\ell'\circ\ell$ for all w, i.e. for all w there *exists* some $g(w)$ such that $(f(w),g(w))$ is in ℓ and $((g(w),h(w))$ is in ℓ'. The point is to show that among the functions from 2^ω to $|B|$ that this existential quantifier defines, one is an element of $|\triangleleft B|$.
  Consider the function (f,h) from 2^ω to $|A|\times|C|$ with the discrete topology on $|A|\times|C|$. It is continuous, because the product topology of a finite product of discrete spaces is the discrete topology on the (finite) product of the involved sets. Therefore it may be described as a finite binary tree, with leaves in $|A|\times|C|$. We write it as a finite set $\{(m_i,(a_i,c_i))\}$ with the properties (a) and (b), like in (1) of proposition 2. For all i, there exists w such that $f(w)=a_i$ and $h(w)=c_i$ take e.g. $w=m_i(0)^*$. Therefore for all i there exists b_i in $|B|$ such that (a_i,b_i) is in ℓ and (b_i,c_i) is in ℓ', and for each i we choose one (there are finitely many i). Now, we can define $g(w)$ with the help of (a). For each w there exists a unique i such that $w=m_iw'$, and we define $g(w)$ to be b_i, and thus g is clearly a continuous function from 2^ω to $|B|$. Notice that the generic tree of g is not necessarily $\{(m_i,b_i)\}$ but its normal form according to (2) of proposition 2: the property (b) may fail since there possibly exist i, j and m such that $m_i=m0$, $m_j=m1$ while $b_i=b_j$.
  Now it is easily seen that (f,g) is in $\triangleleft\ell$ and (g,h) in $\triangleleft\ell'$. Indeed for all w there exists a unique i such that $w=m_iw'$ and we then have $f(w)=a_i$, $g(w)=b_i$, $h(w)=c_i$ and thus $(f(w),g(w))=(a_i,b_i)$ is in ℓ and $(g(w),h(w))=(b_i,c_i)$ is in ℓ'.

$\diamond$

4 The modality in the category of hypercoherences

Since the construction is very similar, we shall be brief, and closely refer to [Ehr93].

Definition 4 *Let X and Y be hypercoherences. The hypercoherence $X{<}Y$ is the hypercoherence whose web is $|X|\times|Y|$ and whose* strict *atomic coherence is defined by:*

$$w \in \Gamma^*(X) \text{ . iff . } \Big((\pi_1(w) \in \Gamma^*(X) \text{ and } \#\pi_2(x) = 1) \text{ or } \pi_2(w) \in \Gamma^*(X)\Big)$$

It is easily seen that this connective is associative, self-dual, non-commutative and in between the tensor product and the par — just like in the category of coherence spaces.

Now, the self-dual modality enjoying the wanted properties is defined in the category of hypercoherences by:

Definition 5 *Let X be an hypercoherence. The hypercoherence $\triangleleft X$ is the hypercoherence whose web is:*

$$|\triangleleft X| = \{f \in \mathsf{C}(2^\omega, |X|) \ / \ f(2^\omega) \in \mathcal{P}_{fin}(|X|)\}$$

and whose strict *atomic coherence is defined by:*

$$\{f_1, ..., f_k\} \in \Gamma^*(X) \text{ . iff . } \exists m \in 2^\omega \begin{cases} \{f_1(m), ..., f_k(m)\} \in \Gamma^*(X) \\ \qquad\qquad \text{and} \\ \forall m' > m \ \#\{f_1(m'), ..., f_k(m')\} = 1 \end{cases}$$

The proofs that it enjoys the same properties as $\triangleleft$ in the category of coherence spaces are so similar that we skip them.

Appendix

Here we write down the proofs of proposition 2. and 3., which are completely straightforward for anybody familiar with 2^ω.

General remarks

Since we consider the product topology on 2^ω, the subsets $U_m = \{w/\exists w' \;\; w = mw'\}$ where m range over 2^*, are an open basis of 2^ω. Given two open sets of the basis U_m and $U_{m'}$ we either have $U_m \cap U'_m = \emptyset$ or $U_m \subset U_{m'}$ or $U_{m'} \subset U_m$. These sets are clopen sets: the complement of $U_{a_1..a_l}$ is $\bigcup_{m'\in\overline{m}} U_{m'}$ where $\overline{m} = 2^l - \{m\}$. Notice that $U_{m0} \cup U_{m1} = U_m$.

Any clopen set may be written as the finite union of disjoint U_m: as it is open it is the union of U_{m_i}, as it is closed in a compact it is compact, hence only a finite number of the U_{m_i} are needed, and because the U_{m_i} are either disjoint or included one in the other, we can assume that they are disjoint.

A clopen set always possesses a maximum element: $S = \cup_{1\leqslant i\leqslant p} U_{m_i}$, thus $\max S = \max_{1\leqslant i\leqslant p} m_i(1)^*$.

Proof of proposition 2

We prove that any continuous function from 2^ω to a set M with the discrete topology may be described by a unique binary tree whose leaves are labelled by elements of M, such that no sister leaves have the same label:

$[f] = \{(m_q, x_q)\} \subset \mathcal{P}_{fin}(2^* \times M)$ with

(a) $\forall w \in 2^\omega \exists ! p \, \exists w' \;\; w = m_p w'$

(b) $\forall n, p \, \forall m (m_n = m0$ and $m_p = m1 \Rightarrow x_n \neq x_p)$.

It is obvious that such binary trees correspond to normal terms of the grammar given in (2) of proposition 2.

FOR ALL SUCH f THERE EXISTS SUCH A BINARY TREE $[f]$ DESCRIBING f

We have $2^\omega = \cup_{x\in M} f^{-1}(x)$. But each $f^{-1}(x)$ is open ($\{x\}$ is an open of the discrete topology on M) and 2^ω is compact, hence there exists a finite number of elements of M say $x_1, ..., x_k$ such that $2^\omega = \cup_{1\leqslant i\leqslant k} f^{-1}(x_i)$ — by the way f takes finitely many values.

As $f^{-1}(x_i)$ is a clopen set ($\{x\}$ is closed too) it is the union of finitely many disjoint clopen sets of the basis, say $U_{m_i^1}, ..., U_{m_i^{p_i}}$, and we can assume that for all n, p with $1 \leqslant n, p \leqslant p_i$, for all m we do not have both $m0 = m_i^n$ and $m1 = m_i^p$ — because $U_{m0} \cup U_{m1} = U_m$.

Hence $2^\omega = f^{-1}(x_1) \cup ... \cup f^{-1}(x_i)$ becomes $2^\omega = \bigcup_{1\leqslant i\leqslant k} \bigcup_{1\leqslant s\leqslant p_i} U_{m_i^s}$

Let $[f] = \{(m_i^s, x_i)\}_{1\leqslant i\leqslant k}^{1\leqslant s\leqslant p_i}$.

These $U_{m_i^s}$ are easily seen to be pairwise disjoint: indeed we have seen that $U_{m_i^n}$ is disjoint from $U_{m_i^p}$, and $f(U_{m_i^n}) = x_i$ while $f(U_{m_j^p}) = x_j$. Hence for all w there exists a unique $U_{m_i^s}$ such that $w \in U_{m_i^s}$, i.e. for all w there exists a unique m_i^s such that there exists w' with $w = m_i^s w'$, i.e. (a) is fulfilled. Moreover, as we have seen, for all n, p with $1 \leqslant n, p \leqslant p_i$, for all m we do not have both $m0 = m_i^n$ and $m1 = m_i^p$, i.e. (b) is fulfilled too.

IF TWO SUCH BINARY TREES DIFFER
EITHER DO THE CONTINUOUS FUNCTIONS THEY DEFINE

Assume now we have two such binary trees, i.e. two finite sets $[f] = \{(m_q, x_q)\}$ and $[g] = \{(m'_r, x'_r)\}$ both satisfying (a) and (b). Write T_f for the projection of $[f]$ to 2^*.

This defines two continuous functions from 2^ω to M, by setting $f(w) = x_i$ where i is the unique index for which there exists m_i and w' such that $w = m_i w'$. Indeed (a) makes sure that for a given m_q in T_f there exists exactly one x such that (m_q, x) is in $[f]$, and the inverse image of a subset N of M is the finite union $\cup_{x \in N \wedge (m_q, x) \in [f]} U_{m_q}$, which is a clopen set of 2^ω, hence an open set.

Assume that $[f] \neq [f']$, i.e. that there exists p such that (m'_p, x'_p) is in $[f']$ but not in $[f]$.

It is easily noticed that the U_{m_q} (resp. U_r) are the biggest basis open on which f (resp. g) is constant. Indeed, assuming the contrary, say $U_m \supset U_{m_r}$ and $f(U_m) = \{x_r\}$, we obtain that $m_r = mm'$ and therefore there are two sister leaves of $[f]$ above m, say $mm''0$ and $mm''1$, but as their labels are the images by f of any infinite word extending them, their labels are the same, namely x_r, and this conflicts with (b).

Because of the previous paragraph, if $T_f \neq T_g$ then $f \neq g$. Otherwise, if $T_f = T_g$, i.e. if the sets $\{m_q\}$ and $\{m'_r\}$ are equal, up to some renumbering, let us say that $m_q = m'_q$. Since $[f] \neq [g]$, there exists p such that $x_p \neq x'_p$. Thus $f(m_p(0)^*) = x_p \neq x'_p = f'(m_p(0)^*)$, and $f \neq g$.

Proof of proposition 3

Now, let us see that whenever two continuous functions from 2^ω to a set M (discrete topology) differ, there exists w such that $f(w) \neq g(w)$ and for all $v > w$ $f(v) = g(v)$.

The product of the discrete topological space M by itself, is the discrete topological space $M \times M$ too. Hence the function Δ from $M \times M$ to 2 defined by $\Delta(x, y) = 1$ iff $x = y$ is continuous. The function (f, g) from 2^ω to $M \times M$ defined by $(f, g)(w) = (f(w), g(w))$ is continuous — product topology on $M \times M$. Therefore $(\Delta \circ (f, g))^{-1}(0)$ is a clopen set, which has a greatest element w (of the shape $m(1)^*$). Thus $f(w) \neq g(w)$ and $f(v) = g(v)$ whenever $v > w$.

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