On a Generalization of Kingman's Bounds

Zhen Liu, Philippe Nain, Don Towsley

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

On a Generalization of Kingman's Bounds

Zhen LIU Philippe NAIN Don TOWSLEY

N° 2423

Décembre 1994

PROGRAMME 1

Architectures parallèles,

bases de données,

réseaux et systèmes distribués

apport
de recherche

1994

ISSN 0249-6399

On a Generalization of Kingman's Bounds*

Zhen LIU Philippe NAIN Don TOWSLEY

Programme 1 — Architectures parallèles, bases de données, réseaux et systèmes distribués

Projet MISTRAL

Rapport de recherche n° 2423 — Décembre 1994 — 12 pages

Abstract: In this paper we develop a framework for computing upper and lower bounds of an exponential form for a class of single server queueing systems with non-renewal inputs. These bounds generalize Kingman's bounds for queues with renewal inputs.

Key-words: Queues; Exponential bounds; General state space Markov chain.

(Résumé : tsvp)

***Correspondence**: Zhen LIU and Philippe NAIN, INRIA, Centre Sophia Antipolis, 2004 route des Lucioles, B.P. 93, 06902 Sophia-Antipolis, France. e-mail: liu@sophia.inria.fr and nain@sophia.inria.fr

Sur une généralisation des bornes de Kingman

Résumé : Dans cet article nous développons un cadre pour le calcul de bornes exponentielles supérieures et inférieures pour une classe de systèmes de files d'attente à serveur unique, où les processus d'entrée sont modulés par une chaîne de Markov à espace d'état général. Ces bornes généralisent celles obtenues par Kingman dans le cas où les processus d'entrée sont des processus de renouvellement.

Mots-clé : Files d'attente; Bornes exponentielles; Chaîne de Markov à espace d'état général.

1 Introduction

Let $(\Omega, \mathcal{F})$ be a measurable space large enough to carry a $\mathbb{R}_+$-valued random variable (r.v.) X_0, a sequence of $\mathbb{E}$-valued r.v.'s $\{Y_n,\, n = 0, 1, \ldots\}$ and a sequence of $\mathbb{R}$-valued r.v.'s $\{\xi_n,\, n = 0, 1, \ldots\}$. The set $\mathbb{R}$ (resp. $\mathbb{R}_+$) denotes the set of all real numbers (resp. the set of all nonnegative real numbers) endowed with the σ-algebra $\mathcal{R}$ (resp. $\mathcal{R}_+$). We will assume that $\mathbb{E}$ is a general space endowed with the σ-algebra $\mathcal{E}$.

Let $\{F(x, \cdot),\, x \in \mathbb{E}\}$ and $\{Q(x, \cdot),\, x \in \mathbb{E}\}$ be two fixed families of probability measures on $\mathbb{R}$ and $\mathbb{E}$, respectively, and let μ and ν be probability measures on $\mathbb{E}$ and $\mathbb{R}_+$, respectively.

We postulate the existence of a probability measure P_ν^μ on $(\Omega, \mathcal{F})$ such that for all $A \in \mathcal{E}$, $B \in \mathcal{R}$, $C \in \mathcal{R}_+$,

$$P_\nu^\mu\,(X_0 \in C, Y_0 \in A) = \nu(C)\,\mu(A) \tag{1}$$

and

$$P_\nu^\mu\,(Y_{n+1} \in A, \xi_n \in B | X_0, Y_0, \ldots, Y_n, \xi_0, \ldots, \xi_{n-1}) = Q(Y_n, A) F(Y_n, B), \quad n \geq 0. \tag{2}$$

The definition (1) implies that under P_ν^μ, X_0 and Y_0 are independent r.v.'s with probability distribution ν and μ, respectively. The definition (2) implies, in particular, that $\mathbf{Y} = \{Y_n\}_n$ is a time-homogeneous Markov chain on $\mathbb{E}$ with transition kernel Q and initial probability distribution μ. Also observe from (2) that $P_\nu^\mu(\xi_n \in B \,|\, Y_n = x) = F(x, B)$ and that Y_{n+1} and ξ_n, conditioned on Y_n, are independent r.v.'s under P_ν^μ.

We will assume that the Markov chain $\mathbf{Y}$ is aperiodic, positive Harris recurrent (see [15, Theorem 13.0.1]) and will denote by π its invariant probability measure. Let $\{\pi_n^\mu\}_n$ be a family of probability measures on $\mathbb{E}$ recursively defined by $\pi_n^\mu(B) = \int_{\mathbb{E}} Q(x, B)\, \pi_{n-1}^\mu(dx)$ for $n = 1, 2, \ldots$, with $\pi_0^\mu \equiv \mu$, so that π_n^μ is the probability distribution of Y_n given the initial probability distribution μ for Y_0.

On $(\Omega, \mathcal{F})$ we define the new sequence $\{X_n,\, n = 1, 2, \ldots\}$ of $\mathbb{R}_+$-valued r.v.'s by

$$X_{n+1} = \max\,(0,\, X_n + \xi_n)\,, \quad n \geq 0. \tag{3}$$

The aim of this note is to compute exponential upper and lower bounds for the tail distribution of X_n, both for every $n = 1, 2, \ldots$ and for the stationary regime X of X_n (when it exists), namely, to find real numbers $a, c \geq 0$, $b, d, \theta > 0$ such that

$$a\, e^{-\theta s} \leq P_\nu^\mu(X_n \geq s) \leq b\, e^{-\theta s} \tag{4}$$

for all $s > 0$, $n \geq 1$, and

$$c\, e^{-\theta s} \leq P_\nu^\mu(X \geq s) \leq d\, e^{-\theta s} \tag{5}$$

for all $s > 0$.

Consider the case that $\mathbb{E}$ is a singleton. Observe that in this case $\{\xi_n,\, n = 0,1,\ldots\}$ is a renewal sequence of $\mathbb{R}$-valued r.v.'s. If the stability condition $E[\xi_0] < 0$ is satisfied, then Kingman [12] showed that the upper bound in (4) (resp. in (5)) holds with $b = 1$ (resp. $d = 1$) for every $0 < \theta \leq \theta^* = \sup\{\alpha > 0 \,:\, E[\exp(\alpha\xi_0)] \leq 1\}$, and that the lower bounds in (4)-(5) hold with $a = c = \inf_{s>0} \int_s^\infty F(du) / \int_s^\infty \exp(\theta^*(u - s)F(du)$ and $\theta = \theta^*$, where F is the common distribution function of the ξ_n's.

Consider now the case that $\mathbb{E}$ is a finite set. If the stability condition $E^\pi[\xi_0] < 0$ holds, then Liu et al. [14] showed that the upper bounds in (4)-(5) hold for every $0 < \theta \leq \theta^* = \sup\{\alpha > 0,\, \rho(\alpha) \leq 1\}$ with

$$b = d = \sup_{\substack{s>0\\ n\geq 0\\ j\in\mathbf{E}}} \frac{\sum_{k\in\mathbf{E}} Q(k,j)\, \pi_n^\mu(k)\, (1 - F(k,(-\infty,s)))}{\sum_{k\in\mathbf{E}} Q(k,j)\, r(k;\theta)\, \int_s^\infty \exp(\theta(u-s))\, F(k,du)} \tag{6}$$

where $(r(k;\theta),\, k \in \mathbb{E})^T$ is the right eigenvector of the matrix

$$H(\theta) := Q^T \mathrm{diag}\,(E[\exp(\theta\xi_n)\,|\,Y_n = k],\, k \in \mathbb{E})$$

associated with its largest eigenvalue $\rho(\theta)$ such that $\sum_{k\in\mathbf{E}} r(k;\theta) = 1$, with $Q = [Q(k,j)]$. The lower bound is obtained in [14] for $\theta = \theta^*$. In this case, the coefficient a in (4) (resp. c in (5)) is obtained by substituting "sup" for "inf" and θ for θ^* in the right-hand side of (6). The bounds reported above for the tail distribution of X_n will hold for $n \geq 1$ only if ν satisfies some conditions (see [12, 14] for details).

The results in this note generalize Kingman's bounds to the case when $(\mathbb{E},\mathcal{E})$ is a general measurable space.

Let us introduce further notation. Let $\Phi(x;\theta) = \int_{\mathbf{R}} \exp(\theta u)\, F(x,du)$, $G(x, A \times B) = P(Y_{n+1} \in A, \xi_n \in B\,|\,Y_n = x)$, and define the transformed kernel $\hat{Q}(\theta)$ by

$$\begin{aligned} \hat{Q}(x,A;\theta) &:= \int_{\mathbf{R}} e^{\theta u}\, G(x, A \times du) \\ &= Q(x,A)\, \Phi(x;\theta) \end{aligned} \tag{7}$$

for all $x \in \mathbb{E}$, $A \in \mathcal{E}$, where the latter equality comes from (2). In the case that $\mathbb{E}$ is a finite set then $\hat{Q}(k,j;\theta)$ is the (k,j)-entry of the matrix $H(\theta)$.

Let $G^{(1)}(x, A \times B) = G(x, A \times B)$ and $G^{(i)}(x, A \times B) = \int_{\mathbf{E}} \int_{\mathbf{R}} G^{(i-1)}(x, du \times ds)\, G(u, A \times B - s)$, $i \geq 2$, and define $\hat{Q}^{(i)}(x,A;\theta) = \int \exp(\theta s) G^{(i)}(x, A \times ds)$ for all $i \geq 1$.

From now on we will assume that there exist a probability measure m on $\mathbb{E} \times \mathbb{R}$, an integer i, and real numbers $0 < a_1 \leq a_2 < \infty$ such that

$$a_1\, m(A \times B) \leq G^{(i)}(x, A \times B) \leq a_2\, m(A \times B), \quad \forall x \in \mathbb{E},\, A \in \mathcal{E},\, B \in \mathcal{R}. \tag{8}$$

Define $\hat{m}(A;\theta) = \int_{\mathbb{R}} \exp(\theta s)\, m(A \times ds)$ and let $\mathcal{D} = \{\theta \in \mathbb{R} : \hat{m}(\mathbb{E};\theta) < \infty\}$.

By applying Lemma 3.1 in [8] we deduce that for each $\theta \in \mathcal{D}$, $\hat{Q}(\theta)$ has a maximal simple eigenvalue $\rho(\theta) > 0$ with uniformly positive and bounded associated right eigenfunction and left eigenmeasure $r(\theta) = \{r(x;\theta),\ x \in \mathbb{E}\}$ and $l(\theta) = \{l(A;\theta),\ A \in \mathcal{A}\}$, respectively. Recall that the right eigenfunction $r(\theta)$ and the left eigenmeasure $l(\theta)$ satisfy the relationships $\rho(\theta)\, r(x;\theta) = \int_{\mathbb{E}} r(y;\theta)\, \hat{Q}(x, dy;\theta)$ and $\rho(\theta)\, l(A;\theta) = \int_{\mathbb{E}} \hat{Q}(x, A;\theta)\, l(dx;\theta)$, respectively, for all $x \in \mathbb{E}$, $A \in \mathcal{E}$. We will assume without loss of generality that the left eigenmeasure is chosen so that

$$l(\mathbb{E};\theta) = 1, \quad \forall \theta \in \mathcal{D}. \tag{9}$$

Recently, Duffield [5] found an exponential *upper* bound for the tail distribution of the *stationary regime* X in the case when, like in our setting, the joint distribution of (Y_{n+1}, ξ_n), conditioned on $(X_0, Y_0, \ldots, Y_n, \xi_0, \ldots, \xi_{n-1})$ depends only on Y_n for all $n \geq 0$, but without imposing the product-form condition in the right-hand side of (2) (the latter case is referred to as the "uncoupled" case in [8, Section 7i]). Specializing Duffield's result to the uncoupled case and by assuming, as in [5], that $\mathbf{Y}$ is ergodic and stationary, $E^\pi[\xi_0] < 0$ (stability condition), and $F(x, \mathbb{R}_+) > 0$ for all $x \in \mathbb{E}$, we get that, for every $0 < \theta \leq \theta^* = \sup\{\alpha > 0 : \rho(\alpha) \leq 1\}$, the upper bound in (5) holds with

$$d = \frac{E^\pi[r(Y_0;\theta)]}{\inf_{x \in \mathbb{E}} r(x;\theta)}. \tag{10}$$

Besides the theoretical interest of obtaining bounds like those in (4)-(5), bounds on the tail distribution of quantities such as buffer occupancy and response times can be used in the design of high speed networks. In addition, bounds can also be used to develop policies for controlling the admission of new applications or sessions to the network. The interested reader is referred to [2, 3, 4, 6, 7, 9, 10, 13, 14, 16] where these issues have been addressed.

2 Exponential Bounds

In what follows, for any probability measure q on $\mathbb{R}$ we will substitute the notation $q_c(s)$ for $q((s,\infty))$ for all $s \in \mathbb{R}$.

Let $\{\gamma(s,\cdot),\ s \geq 0\}$ be a collection of measures on $\mathbb{E}$ such that

$$\int_{\mathbb{E}} Q(x, A) \left[\int_{-\infty}^{s} \gamma(s-u, dx)\, F(x, du) + F_c(x, s)\, \pi_n^\mu(dx)\right] \leq \gamma(s, A) \tag{11}$$

for all $s \in \mathbb{R}$, $A \in \mathcal{E}$, $n \geq 0$.

The following technical lemma holds:

Lemma 2.1 *Let $\mathcal{P}_n$ denote the property that*

$$P_\nu^\mu(X_n \geq s, Y_n \in A) \leq \gamma(s, A) \tag{12}$$

for all $s > 0$, $A \in \mathcal{E}$, $n \geq 0$.

If $\mathcal{P}_0$ is true, then $\mathcal{P}_n$ is true for all $n \geq 0$.

Proof. We will use an induction argument on n. Assume that $\mathcal{P}_i$ is true for $i = 0, 1, \ldots, n$ and let us show that $\mathcal{P}_{n+1}$ is true.

We have for all $s > 0$, $A \in \mathcal{E}$,

$$\begin{aligned}
&P_\nu^\mu(X_{n+1} \geq s,\, Y_{n+1} \in A) \\
&\quad = \int_{\mathbf{E}} P_\nu^\mu(X_n + \xi_n \geq s,\, Y_{n+1} \in A \,|\, Y_n = x)\, \pi_n^\mu(dx) \\
&\quad = \int_{\mathbf{E}} Q(x, A)\, P_\nu^\mu(X_n + \xi_n \geq s \,|\, Y_n = x)\, \pi_n^\mu(dx) && (13) \\
&\quad = \int_{\mathbf{E}} Q(x, A) \left[\int_{-\infty}^{\infty} P_\nu^\mu(X_n \geq s - u \,|\, Y_n = x, \xi_n = u)\, F(x, du)\right] \pi_n^\mu(dx) && (14) \\
&\quad = \int_{\mathbf{E}} Q(x, A) \left[\int_{-\infty}^{s} P_\nu^\mu(X_n \geq s - u, Y_n \in dx)\, F(x, du) + F_c(x, s)\, \pi_n^\mu(dx)\right] \\
&\quad \leq \int_{\mathbf{E}} Q(x, A) \left[\int_{-\infty}^{s} \gamma(s - u, dx)\, F(x, du) + F_c(x, s)\, \pi_n^\mu(dx)\right] && (15) \\
&\quad \leq \gamma(s, A). && (16)
\end{aligned}$$

The relations (13)-(14) are easy consequences of the definition (2). The inequalities (15) and (16) follow from the induction hypothesis and from the definition (11), respectively. ♣

We are now in position to prove the main result of this note:

Proposition 2.1 *Let $\theta \in \mathcal{G} := \{\alpha > 0 \,:\, \rho(\alpha) \leq 1\} \cap \mathcal{D}$. If*

$$\nu_c(s) \leq b(\theta)\, e^{-\theta s}, \quad \forall s > 0 \tag{17}$$

then, for any initial probability measure μ for Y_0,

$$P_\nu^\mu(X_n \geq s) \leq b(\theta)\, e^{-\theta s}, \quad \forall s > 0,\, n \geq 0 \tag{18}$$

where

$$b(\theta) = \sup_{\substack{A \in \mathcal{E} \\ s > 0 \\ n \geq 0}} \frac{\int_{x \in \mathbf{E}} Q(x, A)\, F_c(x, s)\, \pi_n^\mu(dx)}{\int_{x \in \mathbf{E}} Q(x, A)\, (\int_s^\infty e^{\theta(u - s)}\, F(x, du))\, l(dx; \theta)}. \tag{19}$$

Proof. Observe first that the set $\mathcal{G}$ is nonempty since $\rho(0) = 1$ and $\hat{m}(\mathbb{E}; 0) = 1$ which implies that $0 \in \mathcal{G}$.

Define $\gamma(s, A) = b(\theta)\, \exp(-\theta s)\, l(A; \theta)$ for all $s \geq 0$, $A \in \mathcal{E}$.

If $\{\gamma(s, A),\, s \geq 0,\, A \in \mathcal{E}\}$ satisfies (11) then, by Lemma (2.1),

$$\begin{aligned} P_\nu^\mu(X_n \geq s) &= P_\nu^\mu(X_n \geq s, Y_n \in \mathbb{E}) \\ &\leq \gamma(s, \mathbb{E}) \\ &= b(\theta)\, e^{-\theta s} \end{aligned}$$

for all $s > 0$, $n \geq 1$, by using the normalizing condition (9).

Let us check that $\{\gamma(s, A),\, s \geq 0,\, A \in \mathcal{E}\}$ satisfies (11). For all $s \geq 0$, $A \in \mathcal{E}$, we have

$$\begin{aligned} &\int_{\mathbb{E}} Q(x, A) \left[\int_{-\infty}^{s} \gamma(s-u, dx)\, F(x, du) + F_c(x, s)\, \pi_n^\mu(dx)\right] \\ &= \int_{\mathbb{E}} Q(x, A) \left[\left(\int_{-\infty}^{s} b(\theta)\, e^{\theta(u-s)}\, F(x, du)\right) l(dx; \theta) + F_c(x, s)\, \pi_n^\mu(dx)\right] \\ &= \int_{\mathbb{E}} Q(x, A) \Bigg[b(\theta)\, e^{-\theta s}\, \Phi(x; \theta)\, l(dx; \theta) \\ &\quad - \int_{s}^{\infty} \left(b(\theta)\, e^{\theta(u-s)}\, l(dx; \theta) - \pi_n^\mu(dx)\right) F(x, du)\Bigg] \\ &\leq b(\theta)\, e^{-\theta s} \int_{\mathbb{E}} \hat{Q}(x, A; \theta)\, l(dx; \theta) && (20) \\ &= b(\theta)\, e^{-\theta s}\, \rho(\theta)\, l(A; \theta) && (21) \\ &\leq \gamma(s, A). \end{aligned}$$

The inequality (20) follows from (7) and from the definition of $b(\theta)$ while (21) follows from the definition of the left eigenmeasure $l(\cdot\,; \theta)$. ♣

From now on we will assume that there exists $0 < B < \infty$ such that $m(\mathbb{E}, (B, \infty))$ is strictly positive. If this assumption does not hold then it can be shown from (8) and (3) that $X_n \rightarrow_n 0$ almost surely and the system becomes trivial.

Proposition 2.1 says that an exponential upper bound with a strictly positive exponential decay rate exists for the tail distribution of X_n if $\mathcal{G}$ is non-empty. If so, then $\theta^* := \sup\{\theta \in \mathcal{G}\}$ is the largest exponential decay rate over all $\theta \in \mathcal{G}$. However, this leaves open the questions (a) under what condition(s) $\theta^* > 0$ and (b) whether θ^* the largest possible exponential over all $\theta \geq 0$?

The answer to question (a) is provided in Lemma 2.2.

Lemma 2.2 *Assume that $\mathcal{D}$ is an open set. Then,*

$$\theta^* > 0 \qquad \textit{if and only if} \qquad E^\pi[\xi_0] < 0. \tag{22}$$

If $E^\pi[\xi_0] < 0$ then θ^* is the unique strictly positive solution of the equation $\rho(\theta) = 1$.

Proof. It is shown in [8, Lemma 3.4i] that $\log(\rho(\theta))$ is differentiable and strictly convex on $\mathcal{D}$. This implies that $\rho(\theta)$ is also differentiable and strictly convex on $\mathcal{D}$. On the other hand, specializing the differentiation formula (4.3) in [8] to our model yields $\rho'(0) = E^\pi[\xi_0]$, where $\rho'(0)$ denotes the derivative of $\rho(\theta)$ at the point $\theta = 0$. Combining the above results together with the obvious identity $\rho(0) = 1$ gives (22).

Let us now prove the second statement of the lemma. Assume that $E^\pi[\xi_0] < 0$. Since $\rho(0) = 1$, $\rho'(0) = E^\pi[\xi_0] < 0$, and since $\rho(\theta)$ is convex on $\mathcal{D}$, it is enough to show that

$$\lim_{\theta \to \theta_0 \in \partial D \cap (0,\infty)} \rho(\theta) = \infty. \tag{23}$$

¿From $\rho(\theta)^i\, l(\mathbb{E};\theta) = \int_{\mathbb{E}} Q^{(i)}(x,\mathbb{E};\theta)\, l(dx;\theta)$ and (8) we obtain that

$$\rho(\theta)^i \geq a_1\, \hat{m}(\mathbb{E};\theta). \tag{24}$$

If $\theta_0 < \infty$ then $\lim_{\theta\to\theta_0} \hat{m}(\mathbb{E};\theta) = \infty$ from Fatou's lemma and the definition of the set $\mathcal{D}$. Consequently, (23) is proved for $0 < \theta_0 < \infty$ by letting $\theta \to \theta_0$ in (24).

Recall the assumption that $m(\mathbb{E},(B,\infty)) > 0$ for some $B > 0$. Pick such a B and note that $\hat{m}(\mathbb{E};\theta) \geq \exp(\theta B)\, m(\mathbb{E},(-\infty,B)$ for $\theta > 0$, so that $\rho(\theta)^i \geq a_1 \exp(\theta B)\, m(\mathbb{E},(-\infty,B))$ for $\theta > 0$. Letting $\theta \to \infty$ in the latter inequality yields (23) when $\theta_0 = \infty$. This concludes the proof. ♣

We now establish the lower bound.

Proposition 2.2 *Assume that $E^\pi[\xi_0] < 0$ and that $\mathcal{D}$ is an open set. If*

$$\nu_c(s) \geq a\, e^{-\theta^* s}, \quad \forall s > 0 \tag{25}$$

then, for any initial probability measure μ of Y_0,

$$P^\mu_\nu(X_n \geq s) \geq a\, e^{-\theta^* s}, \quad \forall s > 0,\, n = 1,2,\ldots \tag{26}$$

where

$$a = \inf_{\substack{A\in\mathcal{E}\\ s>0\\ n\geq 0}} \frac{\int_{x\in\mathbb{E}} Q(x,A)\, F_c(x,s)\, \pi^\mu_n(dx)}{\int_{x\in\mathbb{E}} Q(x,A)\, (\int_s^\infty e^{\theta^*(u-s)}\, F(x,du))\, l(dx;\theta^*)}. \tag{27}$$

Proof. Let $\{\delta(s,\cdot),\ s\geq 0\}$ be a collection of measures on $\mathbb{E}$ such that

$$\int_{\mathbf{E}} Q(x,A)\left[\int_{-\infty}^{s} \delta(s-u,dx)\,F(x,du)+F_c(x,s)\,\pi_n^{\mu}(dx)\right]\geq \delta(s,A) \qquad (28)$$

for all $s\in\mathbb{R}$, $A\in\mathcal{E}$, $n\geq 1$. As in Lemma 2.1 we can easily show that if the property

$$P(X_n\geq s, Y_n\in A)\geq \delta(s,A), \qquad \forall s>0, \quad A\in\mathcal{E} \qquad (29)$$

holds for $n=0$ then it holds for all $n\geq 0$.

Define $\delta(s,A)=a\,\exp(-\theta^* s)\,l(A;\theta^*)$ for all $s\geq 0$, $A\in\mathcal{E}$.

If $\{\delta(s,A),\ s\geq 0,\ A\in\mathcal{E}\}$ satisfies (28) then, according to (29),

$$\begin{aligned} P_\nu^\mu(X_n\geq s) &= P_\nu^\mu(X_n\geq s, Y_n\in\mathbb{E}) \\ &\geq \delta(s,\mathbb{E}) \\ &= a\,e^{-\theta^* s} \end{aligned}$$

for all $s>0$, $n\geq 1$. It remains to check that $\{\delta(s,A),\ s\geq 0,\ A\in\mathcal{E}\}$ satisfies (28). This can be done by mimicking the proof of Proposition 2.1 and by using the property that $\rho(\theta^*)=1$ (see Lemma 2.2). The proof is omitted. ♣

We now address the stability of the system, namely the existence of a proper probability distribution function G on $\mathbb{R}_+$ such that $P_\nu^\mu(X_n<s)\to_n G(s)$ for all $s\geq 0$, independently of the initial distributions μ and ν. Using a coupling result due to Borovkov and Foss [1, Theorem 5], it can be shown that the system is stable if $E^\pi[\xi_0]<0$ and that $X_n\to_n\infty$ a.s. if $E^\pi[\xi_0]>0$ (see [14, Lemma 2.2] for details). Let X be the stationary regime of the process $\{X_n\}_n$ when it exists.

We are now in position to answer question (b) when $E^\pi[\xi_0]<0$ which is the only interesting case. In this case, $\lim_{s\to\infty}(1/s)\log(P(X\geq s))=-\theta^*$ [5, Theorem 3], which readily implies that θ^* is the best exponential decay over all $\theta\geq 0$.

The next results are direct consequences of Propositions 2.1 and 2.2.

Corollary 2.1 *Assume that* $E^\pi[\xi_0]<0$. *Then,* $\forall\theta\in\mathcal{G}$,

$$P(X\geq s)\leq d(\theta)\,e^{-\theta s}, \quad \forall s>0 \qquad (30)$$

where

$$d(\theta)=\sup_{\substack{A\in\mathcal{E}\\ s>0}} \frac{\int_{x\in\mathbf{E}} Q(x,A)\,F_c(x,s)\,\pi(dx)}{\int_{x\in\mathbf{E}} Q(x,A)\left(\int_s^\infty e^{\theta(u-s)}\,F(x,du)\right)l(dx;\theta)}. \qquad (31)$$

If we further assume that $\mathcal{D}$ is an open set, then

$$P(X \geq s) \geq c\, e^{-\theta^* s}, \quad \forall s > 0 \tag{32}$$

where

$$c = \inf_{\substack{A \in \mathcal{E} \\ s>0}} \frac{\int_{x \in \mathbf{E}} Q(x, A)\, F_c(x, s)\, \pi(dx)}{\int_{x \in \mathbf{E}} Q(x, A) \left(\int_s^\infty e^{\theta^*(u-s)}\, F(x, du)\right) l(dx; \theta^*)}. \tag{33}$$

3 Concluding Remarks

The results in the present paper extend Kingman's bounds to queues with Markovian environmenent. Our bounds reduce to Kingman's when the underlying Markov chain has a single state and they agree with the bounds found in [14] when the state space $\mathbb{E}$ is countable.

It is also worth noticing that the upper bound in (30) (with $\theta = \theta^*$) and the lower bound in (32) are identical when $(X_n)_n$ represents the waiting time process in a $GI/M/1$ queue (cf. [14]).

For computational reasons it may be interesting to derive an upper bound for the coefficient $b(\theta)$ in (19) with a simpler form than $b(\theta)$. We have

$$\begin{aligned} b(\theta) &\leq \sup_{\substack{A \in \mathcal{E} \\ s>0 \\ n \geq 0}} \frac{\int_{\mathbf{E}} Q(x, A) \left(\int_s^\infty e^{\theta(u-s)}\, F(x, du)\right) \pi_n^\mu(dx)}{\int_{\mathbf{E}} Q(x, A) \left(\int_s^\infty e^{\theta(u-s)}\, F(x, du)\right) l(dx; \theta)} && (34) \\ &\leq \sup_{\substack{A \in \mathcal{E} \\ n \geq 0}} \frac{\pi_n^\mu(A)}{l(A; \theta)} && (35) \end{aligned}$$

where (34) follows from the inequality $F_c(x, s) \leq \int_s^\infty \exp(\theta(u - s))\, F(x, du)$. To derive (35), fix $A \in \mathcal{E}$, $s > 0$, and define $h(x) := Q(x, A) \int_s^\infty e^{\theta(u-s)}\, F(x, du)$. Assume first that h is a simple function, namely, $h = \sum_{i \in I} \lambda_i \mathbf{1}_{B_i}$, with $(B_i,\, i \in I)$ a partition of $\mathbb{E}$. Then, $\int_{\mathbf{E}} h(x)\, \pi_n^\mu(dx) / \int_{\mathbf{E}} h(x)\, l(dx; \theta) = \sum_{i \in I} \lambda_i\, \pi_n^\mu(B_i) / \sum_{i \in I} \lambda_i\, l(B_i; \theta) \leq \sup_{i \in I} \pi_n^\mu(B_i)/l(B_i; \theta)$. By monotone convergence we conclude that

$$\frac{\int_{\mathbf{E}} h(x)\, \pi_n^\mu(dx)}{\int_{\mathbf{E}} h(x)\, l(dx; \theta)} \leq \sup_{B \in \mathcal{E}} \frac{\pi_n^\mu(B)}{l(B; \theta)}$$

for any measurable and positive function h, from which (35) follows. In particular, cf. (31),

$$d(\theta) \leq d_1(\theta) := \sup_{A \in \mathcal{E}} \frac{\pi(A)}{l(A; \theta)} \tag{36}$$

when the Markov chain $\mathbf{Y}$ is stationary.

As mentioned in the introduction another previous work closely related to ours is that of Duffield [5] who uses a martingale approach (similar to that in [11] for the GI/GI/1 queue) to obtain upper bounds. This approach that does not require the "decoupling" condition (2), does not appear easily to yield lower bounds or transient results.

It is interesting to compare our bound in (30) to that of Duffield, at least in the case that $F(x, \mathbb{R}_+) > 0$ for all $x \in \mathbb{E}$. Unfortunately, we have not been able to find a general relationship between $d(\theta)$ and Duffield's corresponding coefficient $\tilde{d}(\theta) := E^\pi[r(Y_0;\theta)]/\inf_{x\in\mathbb{E}} r(x;\theta)$ (see (10)) nor between $d_1(\theta)$ defined in (36) and $\tilde{d}(\theta)$.

We only have a few partial results. First, note that $\tilde{d}(\theta) \geq 1$. Therefore, $d(\theta) \leq \tilde{d}(\theta)$ when the input sequence is the superposition of independent on-off sources since we have shown in [14, Corollary 3.1] that $d(\theta) \leq 1$ in this case.

It is also easily shown that $d_1(\theta) = \tilde{d}(\theta)$ when $\mathbb{E}$ is a finite set and when the probability transition matrix Q is equal to its transpose (hint: use the fact that $\pi(k) = 1/K$ for all $k \in \mathbb{E}$ if $\mathbb{E}$ has K elements), which in turn implies from (36) that $d(\theta) \leq \tilde{d}(\theta)$.

References

[1] A. A. Borovkov, S. G. Foss, "Stochastically Recursive Sequences and Their Generalizations", *Siberian Advances in Mathematics*, **2**, pp. 16–81, 1992.

[2] C.-S. Chang, "Stability, Queue Length and Delay of Deterministic and Stochastic Queueing Networks", *IEEE Trans. Aut. Contr.*, **39**, No. 5, pp. 913-931, May 1994.

[3] R. L. Cruz, "A Calculus for Network Delay, Part I: Network Elements in Isolation", *IEEE Trans. Inf. Theory*, **37**, No. 1, pp. 114-131, Jan. 1991.

[4] R. L. Cruz, "A Calculus for Network Delay, Part II: Network Analysis", *IEEE Trans. Inf. Theory*, **37**, No. 1, pp. 132-141, Jan. 1991.

[5] N. G. Duffield, "Exponential Bounds for Queues with Markovian Arrivals", Technical Report DIAS-APG-93-01, 1993.

[6] A. I. Elwalid and D. Mitra, "Effective Bandwidth of General Markovian Traffic Sources and Admission Control of High Speed Networks", *IEEE/ACM Trans. on Networking*, **1**, No. 3, pp. 329-343, Jun. 1993.

[7] R. J. Gibbens and P. J. Hunt, "Effective Bandwidths for the Multi-Type UAS Channel", *Queueing Systems*, **9**, pp. 17-28, 1991.

[8] I. Iscoe, P. Ney and E. Nummelin, "Large Deviations of Uniformly Recurrent Markov Additive Processes", *Adv. in Appl. Math.*, **6**, pp. 373-412, 1985.

[9] F. P. Kelly, "Effective Bandwidths at Multi-Class Queues", *Queueing Systems*, **9**, pp. 5-16, 1991.

[10] G. Kesidis, J. Walrand and C.-S. Chang, "Effective Bandwidths for Multiclass Markov Fluids and Other ATM Sources", *IEEE/ACM Trans. Networking*, **1**, No. 4, pp. 424-428, Aug. 1993.

[11] J. F. C. Kingman, "A Martingale Inequality in the Theory of Queues", *Camb. Phil. Soc.*, **59**, pp. 359–361, 1964.

[12] J. F. C. Kingman, "Inequalities in the Theory of Queues", *J. Roy. Stat. Soc.*, Series B, **32**, pp. 102-110, 1970.

[13] J. F. Kurose, "On Computing per-Session Performance Bounds in High-Speed Multi-Hop Computer Networks", *Proc. ACM SIGMETRICS and PERFORMANCE'92*, Newport, RI, pp. 128-139, Jun. 1992.

[14] Z. Liu, P. Nain and D. Towsley, "Exponential Bounds with an Application to Call Admission", Technical Report, University of Massachusetts, CMPSCI 94-63, Oct. 1994. Submitted to *J.A.C.M.*

[15] S. P. Meyn and R. L. Tweedie, *Markov Chains and Stochastic Stability*. Springer Verlag, London, 1993.

[16] O. Yaron and M. Sidi, "Performance and Stability of Communication Networks Via Robust Exponential Bounds", *IEEE/ACM Trans. Networking*, **1**, No. 3, pp. 372-385, Jun. 1993.

Unité de recherche INRIA Lorraine, Technopôle de Nancy-Brabois, Campus scientifique,
615 rue du Jardin Botanique, BP 101, 54600 VILLERS LÈS NANCY
Unité de recherche INRIA Rennes, Irisa, Campus universitaire de Beaulieu, 35042 RENNES Cedex
Unité de recherche INRIA Rhône-Alpes, 46 avenue Félix Viallet, 38031 GRENOBLE Cedex 1
Unité de recherche INRIA Rocquencourt, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex
Unité de recherche INRIA Sophia-Antipolis, 2004 route des Lucioles, BP 93, 06902 SOPHIA-ANTIPOLIS Cedex

Éditeur
INRIA, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)
ISSN 0249-6399