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N ° RR-2416

Novembre 1994

PROGRAMME 2

Calcul symbolique,
programmation
et génie logiciel

Rapport de recherche

1994

ISSN 0249-6399

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Projet PARA

Rapport de recherche n° RR-2416 — Novembre 1994 — 18 pages

Abstract: The purpose of this article is to establish a new intrinsic characterization of the principal type schemes (or pts's for short) of the approximate normal λ-terms (or approximate normal forms). This is done by defining a correspondance between cut-free proof nets, β-normal λ-terms and their principal type schemes, and better, by defining a correspondance between approximate cut-free proof nets, approximate normal terms and their principal type schemes.

Key-words: lambda-calculus, proof nets, principal type schemes

(Résumé : tsvp)

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Une nouvelle caractérisation intrinsèque du type principal

Résumé : L'objet de ce rapport est d'établir une nouvelle caractérisation intrinsèque du typage principal (ou brièvement en anglais pts) des approximants. La méthode consiste en la définition d'une bijection entre réseaux de preuve sans coupures, λ-termes β-normaux et leurs schémas de typage principal, et mieux encore, en la définition d'une bijection entre approximants de réseaux de preuve sans coupures, formes normales approximantes et leurs schémas de typage principal.

Mots-clé : lambda-calcul, réseaux de preuve, type principal

1 Introduction

In relation with the extended type system of [1] as extension of Curry's type system, was defined in the same article the notion of *principal type scheme* of *approximate normal terms* (or *approximate normal form* in [2]). The principal type schemes characterize the *approximate normal terms* in the following senses:
let t be a β-normal λ-term (notation: $t \in \Lambda^+$),
i) $pts(t)$ is unique,
ii) if a λ-term u is typable by $pts(t)$ (that is $x_1 : A_1, \ldots, x_n : A_n \vdash u : Z$ and $(A_1, \ldots, A_n, Z) = pts(t)$), then there exists $v \in \Lambda^+$ such that u β-reduces to v and v is a η-reduct of t ([11], [9]);
iii) the set $\{x_1 : A_1, \ldots, x_n : A_n \vdash t : Z\}$ is definable by suitable operations on $pts(t)$ [10].

In [10], Ronchi found the following intrinsic characterization of the set PTS of the principal types schemes of the approximate normal terms a (notation: $a \in \Lambda_{Ap}$)
($PTS = \{(A_1, \ldots, A_n, Z)/(A_1, \ldots, A_n, Z) = pts(a)$ with $a \in \Lambda_{Ap}\}$):

Theorem 1 • *$(A_1, \ldots, A_n, Z) \in PTS$ iff $(A_1, \ldots, A_n, Z)$ is of one of the four following forms:*

- *$(\emptyset, \ldots, \emptyset, \emptyset)$;*
- *$(\emptyset, \ldots, \emptyset, \{\alpha\}, \emptyset, \cdots, \emptyset, \alpha)$;*
- *$(A_1, \ldots, A_n, A_{n+1} \rightarrow X)$ if $(A_1, \ldots, A_n, A_{n+1}, X) \in PTS$;*
- *$(\{Z_1, \ldots, Z_p \rightarrow \alpha\} \wedge \bigwedge_{i=1}^{p} A_1^i, \bigwedge_{i=1}^{p} A_2^i, \ldots, \bigwedge_{i=1}^{p} A_n^i, \alpha)$ if $(A_1^i, \ldots, A_n^i, Z_i) \in PTS$ for all i, $1 \leq i \leq p$, such that $at((A_1^i, \ldots, A_n^i, Z_i)) \cap at((A_1^{i'}, \ldots, A_n^{i'}, Z_{i'})) = \emptyset$ for all i, i', $1 \leq i$, $i' \leq p$, $i \neq i'$ and $\alpha \notin at((A_1^i, \ldots, A_n^i, Z_i))$ for all i, $1 \leq i \leq p$.*

- *The sets PTS and Λ_{Ap} are isomorphic.*

The object of the present article is to give a different intrinsic characterization of the set PTS by defining properties directly on the set $at((A_1, \ldots, A_n, Z))$ of the atoms of $(A_1, \ldots, A_n, Z)$. The first two conditions (Cond 1,2 of *Property* 1) were found by Ronchi in [10]. The two others (Cond 3,4 of *Property* 2), which are not so intuitive on sequences of types, become clear when they are translated in linear logic terms: in fact, these conditions were firstly found for proof nets in terms of *criterion of correction* on graphs [7], the two previous conditions on sequences of types corresponding to the definition of the graphs involved. We prove then, in Section 4, the correspondance between PTS and the set PS of the sequences of types which satisfy the four conditions, and, in Section 5, the isomorphism between PS and the set PN_{Ap} of the approximate cut-free proof nets to get some intuition about the conditions.
So the main results of the article are contained in the following theorem:

Theorem 2 • *The sets Λ_{Ap}, PTS, PN_{Ap} and PS are isomorphic.*

- *The sets* Λ^+, PN *and* PS^+ *are isomorphic.*

whose proof is done by steps in the different propositions below; the proof of the isomorphism between Λ^+ and PN is done in [7].

2 β-normal λ-terms and approximate normal terms

Definition 1 • V *is the set of term variables* ($V = \{x_i / i \in N\}$).

- *The set* $\Lambda^+[x_1, \ldots, x_n]$ *of the* β-normal λ-terms defined in the context $\{x_1, \ldots, x_n\}$ *is defined by induction as follows from the finite subset* $\{x_1, \ldots, x_n\}$ *of* V *in which the free variables of the* λ*-terms belong:*
  - x_i *belongs to* $\Lambda^+[x_1, \ldots, x_n]$ ($1 \leq i \leq n$);
  - *if* t *belongs to* $\Lambda^+[x_1, \ldots, x_n, x_{n+1}]$, *then* $\lambda x_{n+1}.t$ *belongs to* $\Lambda^+[x_1, \ldots, x_n]$;
  - *if* $t_1, \ldots, t_p$ *belong to* $\Lambda^+[x_1, \ldots, x_n]$, *then* $x_1 t_1 \ldots t_p$ *belongs to* $\Lambda^+[x_1, \ldots, x_n]$.
- *The set* Λ^+ *of the* β-normal λ-terms *is the following disjoint union:*

$$\Lambda^+ = \coprod_{\{x_1,\ldots,x_n\} \in \mathcal{P}_{fin}(V)} \Lambda^+[x_1, \ldots, x_n].$$

Each β-normal λ-term is an *approximate normal term*, so Λ^+ is a subset of the following set.

Definition 2 • *The set* $\Lambda_{Ap}[x_1, \ldots, x_n]$ *of the* approximate normal forms defined in the context $\{x_1, \ldots, x_n\}$ *is defined by induction as follows from the finite subset* $\{x_1, \ldots, x_n\}$ *of* V *plus a new constant symbol* ω:

  - $\omega \in \Lambda_{Ap}[x_1, \ldots, x_n]$;
  - $x_i \in \Lambda_{Ap}[x_1, \ldots, x_n]$ ($1 \leq i \leq n$);
  - *if* a *belongs to* $\Lambda_{Ap}[x_1, \ldots, x_n, x_{n+1}]$ *such that* $a \neq \omega$, *then* $\lambda x_{n+1}.a$ *belongs to* $\Lambda_{Ap}[x_1, \ldots, x_n]$;
  - *if* $a_1, \ldots, a_p$ *belong to* $\Lambda_{Ap}[x_1, \ldots, x_n]$, *then* $x_1 a_1 \ldots a_p$ *belongs to* $\Lambda_{Ap}[x_1, \ldots, x_n]$.
- *The set* Λ_{Ap} *of the* approximate normal forms *is the following disjoint union:*

$$\Lambda_{Ap} = \coprod_{\{x_1,\ldots,x_n\} \in \mathcal{P}_{fin}(V)} \Lambda_{Ap}[x_1, \ldots, x_n].$$

Remark that Λ_{Ap} is isomorphic to the set of the Böhm trees.

3 Principal type schemes of approximate normal terms and principal type schemes of β-normal λ-terms

3.1 The sets of types

Definition 3 *Let At be an infinite and countable set of propositional variables.*
The sets of types $D\Omega P$ and $D\Omega P^*$ are defined as follows:

- $At \subset D\Omega P$;
- $D\Omega P^* = \mathcal{P}_{fin}(D\Omega P)$, *the set of the finite subsets of* $D\Omega P$;
- *If* $A \in D\Omega P^*$ *and* $X \in D\Omega P$, *then* $A \to X \in D\Omega P$.

Convention 1 **i)** *The notion of equality of types (or formulas) is the notion of existence of isomorphism between them, so types are defined up to renamming of atoms.*
ii) *If* $X_1, \ldots, X_n \in D\Omega P$, *the subset* $\{X_1, \ldots, X_n\}$ *will be denoted by* $\bigwedge_{i=1}^n X_i$. *We agree upon the notation* $X_i \in \bigwedge_{i=1}^n X_i$ *for all* $i, 1 \leq i \leq n$.
In case $n = 1$, *the notation* $\bigwedge_{i=1}^1 X_i$ *will be simplified in* X_1; *in other words, we will identify each singleton with the unique element of it.*
In case $n = 0$, *the notation will be* $\emptyset$ *obviously, and we have the property:* $\emptyset \wedge A = A$.
iii) $A_1, \ldots, A_n \to \alpha$ *will denote* $A_1 \to (\ldots \to (A_n \to \alpha) \ldots)$.
iv) *The capital letters of the beginning of the latin alphabet (A, B, etc) will denote the elements of* $D\Omega P^*$; *the capital letters of the end of the latin alphabet* (X, Y, *etc except* Z), *the elements of* $D\Omega P$. *The capital letters of the beginning of the greek alphabet except* ω (α, β, *etc) will denote the elements of* At.
We keep the latin letters Z, Z', *etc to denote any element of* $D\Omega P$ *or of* $D\Omega P^*$ *as well.*
v) *We say that* Z is a subtype of Z' *(notation:* $Z \in St(Z')$*) iff*
$\cdot$ $Z' = Z$,
$\cdot$ Z *is a subtype of* A *or of* X, *in case* $Z' = A \to X$,
$\cdot$ Z *is a subtype of* X_i *for one* $i, 1 \leq i \leq n$, *in case* $Z' = \bigwedge_{i=1}^n X_i$.
vi) *The notation* $X(\alpha)$ *of the type* X *specifies that the atom* α *ends the type* X:
$\cdot$ *if* $X = \alpha$ *then* $X = X(\alpha) = \alpha$;
$\cdot$ *if* $X = A \to X'(\alpha)$, *then* $X = X(\alpha)$.
To say that α *is an atom which ends one of the elements of* $A \in D\Omega P^*$ *(i.e.* $X(\alpha) \in A$*), the notation* $A(\alpha)$ *will be used.*
If Z *is a type, the set of the atoms which occur in* Z *is denoted* At(Z).
vii) $D\Omega P^+$, $D\Omega P^-$, $D\Omega P^{+*}$ *and* $D\Omega P^{-*}$ *will denote the sets of the types defined above with a positive or negative signature. Precisely,* $D\Omega P^+ = D\Omega P \times \{+\}$, $D\Omega P^{+*} = D\Omega P^{*+}$, *the sets of the* positive types, $D\Omega P^- = D\Omega P \times \{-\}$, $D\Omega P^{-*} = D\Omega P^{*-}$, *the sets of the* negative types.
We recall that the type $A \to X$ *is a positive type (that is* $A \to X \in D\Omega P^+$*) iff*
$A \in D\Omega P^{-*} \wedge X \in D\Omega P^+$, *the type* $\bigwedge_{i=1}^n X_i$ *is a positive type* ($\bigwedge_{i=1}^n X_i \in D\Omega P^{+*}$) *iff* $X_i \in D\Omega P^+$ *for all* $i, 1 \leq i \leq n$.

The type $A \to X$ is a negative type (that is $A \to X \in D\Omega P^-$) iff $A \in D\Omega P^{+*} \wedge X \in D\Omega P^-$, the type $\bigwedge_{i=1}^n X_i$ is a negative type ($\bigwedge_{i=1}^n X_i \in D\Omega P^{-*}$) iff $X_i \in D\Omega P^-$ for all $i, 1 \leq i \leq n$. We extend on the more natural way the previous conventions to these sets of types. In particular, α^σ will denote an atom, α in the present case, with a signature σ; At(Z) *will denote the set of the atoms which occur in Z (if α^σ occurs in Z, then $\alpha \in At(Z)$).*

3.2 Principal type schemes

In the following definition, the *principal type schemes* are defined up to renamming the atoms of type (see *Convention* 1 i)):

Definition 4 *Let a be element of $\Lambda_{Ap}[x_1, \ldots, x_n] \subset \Lambda_{Ap}$. $x_1 : A_1, \ldots, x_n : A_n \vdash a : Z$ (or $(A_1, \ldots, A_n, Z)$) is the* principal type scheme of *a ($pts(a)$) inductively defined as follows: if a is*

- *ω, then $x_1 : \emptyset, \ldots, x_n : \emptyset \vdash \omega : \emptyset$;*
- *x_i, then $x_1 : \emptyset, \ldots, x_{i-1} : \emptyset, x_i : \alpha, x_{i+1} : \emptyset, \ldots, x_n : \emptyset \vdash x_i : \alpha$ (or $(\emptyset, \ldots, \emptyset, \alpha, \emptyset, \ldots, \emptyset, \alpha)$) is $pts(x_i)$;*
- *$\lambda x_{n+1}.a'$ ($a' \neq \omega$), and if $x_1 : A_1, \ldots, x_n : A_n, x_{n+1} : A_{n+1} \vdash a' : Z$ (or $(A_1, \ldots, A_n, A_{n+1}, Z)$) is $pts(a')$, then $x_1 : A_1, \ldots, x_n : A_n \vdash \lambda x_{n+1}.a' : A_{n+1} \to Z$ (or $(A_1, \ldots, A_n, A_{n+1} \to Z)$) is $pts(\lambda x_{n+1}.a')$;*
- *$x_1 a_1 \ldots a_p$, and if $x_1 : A_1^i, \ldots, x_n : A_n^i \vdash a_i : Z_i$ (or $(A_1^i, \ldots, A_n^i, Z_i)$) is $pts(a_i)$ ($1 \leq i \leq p$), then*
  $x_1 : (Z_1, \ldots, Z_p \to \alpha) \wedge \bigwedge_{i=1}^p A_1^i, x_2 : \bigwedge_{i=1}^p A_2^i, \ldots, x_n : \bigwedge_{i=1}^p A_n^i \vdash x_1 a_1 \ldots a_p : \alpha$ (or $(Z_1, \ldots, Z_p \to \alpha) \wedge \bigwedge_{i=1}^p A_1^i, \bigwedge_{i=1}^p A_2^i, \ldots, \bigwedge_{i=1}^p A_n^i, \alpha)$) is $pts(x_1 a_1 \ldots a_p)$ using Convention *1 ii) if $Z_i \neq \emptyset$*
  (we suppose that $at(a_i) \cap at(a_{i'}) = \emptyset$ if $i \neq i'$ and that α is a fresh atom*).*

Remark 1 i) *In case of β-normal λ-terms, the definition of* pts *is restricted to the three last rules with the fact that, in the last one, $Z_i \neq \emptyset$ for all $i, 1 \leq i \leq p$.*
ii) *In the second and the last points, we have used the identification of a singleton and its unique element.*
iii) *The definition of* pts *corresponds to Coppo, Dezani, Venneri's one ([3]).*
It differs from Krivine's one ([9]) by the second rule which is
$x_1 : \alpha_1, \ldots, x_{i-1} : \alpha_{i-1}, x_i : \alpha_i, x_{i+1} : \alpha_{i+1}, \ldots, x_n : \alpha_n \vdash x_i : \alpha_i$
(or $(\alpha_1, \ldots, \alpha_{i-1}, \alpha_i, \alpha_{i+1}, \ldots, \alpha_n, \alpha_i)$) and by the fact that pts *are defined only for β-normal terms.*
iv) *Remark that if $(A_1, \ldots, A_n, Z)) = pts(a)$ then Z belongs to $D\Omega P^*$ iff $a = \omega$; if $a \neq \omega$, then Z belongs to $D\Omega P$.*
v) *In case $t \in \Lambda^+$, then $pts(t) \in ||t||$ in the modelization of Λ by a coherent set (see [4]). This is why we consider Coppo, etc's definition of* pts*: in case t is a variable x_i, then $(\emptyset, \ldots, \emptyset, \alpha, \emptyset, \ldots, \emptyset, \alpha) \in ||x_i||$ (precisely,*

$(\emptyset, \ldots, \emptyset, \alpha, \emptyset, \ldots, \emptyset, \alpha) \in Tr_n\|(u_1, \ldots, u_n) \mapsto x_i(u_1, \ldots, u_n)\|$ *which is the* trace *of the n-ary stable function projection*$_i$*:* $(x_1, \ldots, x_n) \mapsto x_i$*).*
vi) *As usual in sequent calculus terms, in the sequent* $x_1 : A_1, \ldots, x_n : A_n \vdash a : Z$*, the type* Z *is a positive type (or formally* Z *is element of* $D\Omega P^+$*); the types* $A_1, \ldots, A_n$ *are negative types (or formally* $A_1, \ldots, A_n$ *are elements of* $D\Omega P^{-*}$*).*

The two following conditions were presented in [10]; Krivine formalizes an analogous condition to Cond 2+ ([9]). Cond 1,2 are satisfied by the *pts* of approximate normal terms, and Cond 1,2+ by the *pts* of β-normal λ-terms:

Property 1 *The* pts*'s satisfy the following conditions:*

- **Cond 1** *each atom, if occurs, appears exactly twice, once with positive signature, once with negative signature;*
- **Cond 2** *the cardinality of each positive (sub)type is less or equal to 1 (that is, if* A *is a positive (sub)set of pts, then* $\#A \leq 1$*).*

Moreover the pts*'s of* β*-normal terms satisfy also the stronger following condition:*

- **Cond 2+** *each positive (sub)type is a singleton.*

We define a binary relation on the set of the atoms of the *pts* of an approximate normal term a:
let α, β be elements of $at(pts(a))$,
$\alpha <_1 \beta$ iff there exists a negative subtype of $pts(t)$ of the form $X(\alpha^+) \rightarrow Y(\beta^-)$.
$<$ (resp. $\leq$) denotes the *transitive closure* (resp. the *transitive and reflexive closure*) of $<_1$.

The two following conditions which give, with the previous conditions, the announced characterization of the *pts*, will appear clearer in *Section* 5.2:

Property 2 *The* pts*'s satisfy also the following conditions:*

- **Cond 3** *the transitive closure of the relation* $<_1$ *is a strict ordering with greatest element;*
- **Cond 4** *for each positive subtype* $A(\alpha^-) \rightarrow Z(\beta^+)$ *of the pts, one has* $\alpha \leq \beta$*.*

The proof is obvious by induction on a in $pts(a)$ or on the number of atoms in $pts(a)$ ($\#at(pts(a))$).

4 Principal sequences

We extend the definition of the binary relation $<_1$ to the the set of the atoms of the elements of $D\Omega P^{-*n} \times (D\Omega P^+ \cup D\Omega P^{+*})$.

We denote by
- $PPS^o = \{(A_1, \ldots, A_n, X) \in D\Omega P^{-*n} \times D\Omega P^{+}/(A_1, \ldots, A_n, X)$ satisfies Cond 1,2 $\}$,
- $PPS^* = \{(A_1, \ldots, A_n, B) \in D\Omega P^{-*n} \times D\Omega P^{+*}/(A_1, \ldots, A_n, B)$ satisfies Cond 1,2 $\}$,
- PPS^{o+} (resp. PPS^{*+}) is the subset of PPS^o (resp. PPS^*) of the sequences that satisfy also Cond 2+,
- $PS^o = \{(A_1, \ldots, A_n, X) \in D\Omega P^{-*n} \times D\Omega P^{+}/(A_1, \ldots, A_n, X)$ satisfies Cond 1-4 $\}$,
- $PS^* = \{(A_1, \ldots, A_n, B) \in D\Omega P^{-*n} \times D\Omega P^{+*}/(A_1, \ldots, A_n, B)$ satisfies Cond 1-4 $\}$,
- PS^{o+} (resp. PS^{*+}) the subset of PS^o (resp. PS^*) of the sequences that satisfy also Cond 2+.

Remark that in PPS^* and PS^*, there exists the particular sequence $(\emptyset, \ldots, \emptyset, \emptyset) \in D\Omega P^{-*n} \times D\Omega P^{+*}$; this particular element will be denoted by $\Omega^n\Omega$.

With the identification between a singleton and its unique element (*Convention* 1 ii)), we have the following obvious lemma:

Lemma 1 *Let* $(A_1, \ldots, A_n, Z) \in D\Omega P^{-*n} \times (D\Omega P^{+} \cup D\Omega P^{+*})$.
If $Z \neq \Omega^n\Omega$, *then*

- $(A_1, \ldots, A_n, Z) \in PPS^o$ *iff* $(A_1, \ldots, A_n, Z) \in PPS^*$;
- $(A_1, \ldots, A_n, Z) \in PPS^{o+}$ *iff* $(A_1, \ldots, A_n, Z) \in PPS^{*+}$;
- $(A_1, \ldots, A_n, Z) \in PS^o$ *iff* $(A_1, \ldots, A_n, Z) \in PS^*$.

The equivalences established in the previous lemma allow us to define the *principal sequences* as elements of the following subsets of $D\Omega P^{-*n} \times (D\Omega P^{+} \cup D\Omega P^{+*})$ (the symbol $+$ will denote the disjoint sum of sets):

Definition 5
- $PPS = PPS^o$;
- $PPS^{+} = PPS^{o+}$;
- $PS = PS^o$;
- $PS^{+} = PS^{o+}$ *(which is isomorphic to* $PS^{o*} \setminus \{\Omega^n\Omega\}$*)*.

Lemma 2 *Let* $S = (A_1, \ldots, A_n, Z) \in PS$. *Let* $\alpha, \beta, \beta' \in At(S)$.

- *If* $\alpha <_1 \beta$ *and* $\alpha <_1 \beta'$, *then* $\beta = \beta'$.
- *If* $Z = Z(\alpha^{+})$, *then* $Z'(\alpha^{-})$ *is a subtype of* Z *or there exists* i $(1 \leq i \leq n)$, *such that* $Z'(\alpha^{-}) \in A_i$.
- *If* β^{-} *or* β^{+} *is a subtype of* $Z'(\gamma)$ *and if* $Z'(\gamma)$ *is a subtype of* S, *then* $\beta \leq \gamma$.

The proof is obvious.

Proposition 1 • *The sets PS and PTS are isomorphic.*

- *PS^+ and the set PTS^+ of the principal type schemes of β-normal terms are isomorphic.*

Proof
We prove the first point, the proof of the second point being analogous and simplier.
By *Properties* 1, 2, each element of PTS satisfies Cond 1-4.
Conversely, let $S = (A_1, \ldots, A_n, Z) \in PS$. We prove, by lexicographical induction on $(\#at(S), complexity(Z))$, that there exists an approximate normal term $a \in \Lambda_{Ap}[x_1, \cdots, x_n]$ such that $pts(a) = (A_1, \ldots, A_n, Z)$:
if $\#at(S)$ is
• equal to 0, then $S = \Omega^n\Omega$, which is $pts(\omega)$;
• strictly greatest than 0, then we consider the complexity of the type Z:
∘ if $Z = \alpha$, then there exists i, $1 \leq i \leq n$, such that $Z'(\alpha^-) \in A_i$; $Z'(\alpha^-)$ is of the form $Z'(\alpha^-) = B_1, \ldots, B_p \to \alpha^-$.
- Or $p = 0$, then $S = (\emptyset, \ldots, \emptyset, \alpha, \emptyset, \ldots, \emptyset, \alpha)$ which is $pts(x_i)$ if $A_i = \alpha$;
- or $p > 0$, then, for all j, $1 \leq j \leq p$, $\#B_j \leq 1$. We define the following partition of $\{1, \cdots, p\}$: $Ind_0 = \{j \in \{1, \cdots, p\} / \#B_j = 0\}, Ind_1 = \{j \in \{1, \cdots, p\} / \#B_j = 1\}$.
∘ For all $j \in Ind_0$, we set $S_j = \Omega^n\Omega$, so $S_j = pts(\omega)$.
∘ For all $j \in Ind_1$, we pose $X_j(\alpha_j^+) = B_j$ with *Convention* 1 ii), so we define
$A_i^j = \bigwedge\{X(\beta) \in A_i / \beta \leq \alpha_j\}$ for all i, $1 \leq i \leq n$.
We prove that *for all* $j \in Ind_1$, $S_j = (A_1^j, \ldots, A_n^j, X_j(\alpha_j^+)) \in PS$:
Cond 1: let $\beta \in at(S) \setminus \{\alpha\}$ such that there exists $j, j' \in Ind_1, j \neq j', \beta \in at(S_j) \cap at(S_{j'})$.
Suppose that β^+ occurs in S_j and β^- occurs in $S_{j'}$.
β^+ occurs in S_j implies that there exists a subtype $Y_j(\gamma^\sigma)$ of S_j such that β^+ occurs in $Y_j(\gamma^\sigma)$ and such that $Y_j(\gamma^\sigma) = X(\alpha_j^+)$ or $Y_j(\gamma^\sigma) \in A_k^j$ for some $k, 1 \leq k \leq n$.
β^- occurs in $S_{j'}$ implies that there exists a subtype $Y_{j'}(\gamma'^{\sigma'})$ of $S_{j'}$ such that β^- occurs in $Y_{j'}(\gamma'^{\sigma'})$ and such that $Y_{j'}(\gamma'^{\sigma'}) = X(\alpha_{j'}^+)$ or $Y_{j'}(\gamma'^{\sigma'}) \in A_{k'}^{j'}$ for some $k', 1 \leq k' \leq n$.
So, in S, there exist $Y_j(\gamma^\sigma)$, $Y_{j'}(\gamma'^{\sigma'})$ subtypes of S such that
i) β^+ occurs in $Y_j(\gamma^\sigma)$ and β^- occurs in $Y_{j'}(\gamma'^{\sigma'})$ (that is $\beta \leq \gamma$ and $\beta \leq \gamma'$),
ii) $Y_j(\gamma^\sigma)$ satisfies $Y_j(\gamma^\sigma) = X(\alpha_j^+)$ (that is $\gamma = \alpha_j$) or $Y_j(\gamma^\sigma) \in A_k$ for some $k, 1 \leq k \leq n$, since $Y_j(\gamma^\sigma) \in A_k^j$ (that is $\gamma \leq \alpha_j$), and
iii) $Y_{j'}(\gamma'^{\sigma'})$ satisfies $Y_{j'}(\gamma'^{\sigma'}) = X(\alpha_{j'}^+)$ (that is $\gamma' = \alpha_{j'}$) or $Y_{j'}(\gamma'^{\sigma'}) \in A_{k'}$ for some $k', 1 \leq k' \leq n$, since $Y_{j'}(\gamma'^{\sigma'}) \in A_{k'}^{j'}$ (that is $\gamma \leq \alpha_{j'}$).
Then, in $at(S)$, $\beta \leq \gamma \leq \alpha_j$ and $\beta \leq \gamma' \leq \alpha_{j'}$. So $\beta \leq \alpha_j$ and $\beta \leq \alpha_{j'}$ in $at(S)$, which is impossible for an element of PS by the previous lemma and by the definition of the relation $<_1$.
Cond 2-4 are obviously satisfied by S_j for all $j \in Ind_1$.
So $S_j \in PS$ ($j \in Ind_1$). By hypothesis of induction, there exists $a_j \in \Lambda_{Ap}[x_1, \ldots, x_n]$ such that $pts(a_j) = S_j$.

Remark that for all $i, 1 \leq i \leq n, \bigwedge\{A_i^j / j \in Ind_1\} = A_i$.
So $S = pts(x_i a_1 \dots a_p)$ with $a_j = \omega$ if $j \in Ind_0$ and $a_j \neq \omega$ if $j \in Ind_1$.
$\circ$ if $Z = A \rightarrow Z'$, then $S' = (A_1, \dots, A_n, A, Z')$ satisfies Cond 1-4. By hypothesis of induction, there exists $a' \in \Lambda_{Ap}[x_1, \dots, x_n, x_{n+1}]$ such that $S' = pts(a')$; moreover $a' \neq \omega$ since $Z \neq \emptyset$. So $S = pts(\lambda x_{n+1}.a')$.
$\square$

5 Cut-free proof nets

5.1 Definition

This section will repeat some contents of [7].

Definition 6 *An* approximate proof graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ *is a finite graph such that*

- *the set* $\mathcal{V}$ *of* vertices *is a set (of occurrences) of the following linear connectors:*
  - AX *or* axiom-connector*,*
  - CUT *or* cut-connector*,*
  - $\wp$ *or* parallelization-connector*,*
  - $\otimes$ *or* tensor-connector*, the (semantic) dual of the* $\wp$*-connector,*
  - $!$ *or* positive exponential-connector*,*
  - $?$ *or* negative exponential-connector*, the (semantic) dual of the positive exponential-connector,*
- *the set* $\mathcal{E}$ *of* edges *is a set (of occurrences) of the following formulas:*
  - *or* O *(for* output*),*
  - *or* I *(for* input*), the (semantic) orthogonal (or dual)* $O^\perp$ *of the* O*-formula,*
  - *or* $!O$*, the exponential of* O*,*
  - *or* $?I$*, the (semantic) orthogonal (or dual)* $(!O)^\perp$ *of the* $!O$*-formula.*

and such that the elements of $\mathcal{V}$ *and* $\mathcal{E}$ *define linear links:*

- $\overline{}_{I \qquad O}$ *or* AX*-link,*
- $\underline{O \qquad I}$ *or* CUT*-link,*
- $\dfrac{?I \qquad O}{O}$ *or* $\wp$*-link,*
- $\dfrac{I \qquad !O}{I}$ *or* $\otimes$*-link,*
- $\dfrac{O}{!O}$ *or* $!$*-link,* $\dfrac{}{!O}$ *or* $!_{Ap}$*-link,*

- $\frac{I \quad \cdots \quad I}{?I}$ *or ?-link.*

Moreover, a proof graph *is an approximate proof graph without any $!_{Ap}$-link.*

Remark 2 *As it was seen in [4], [5] and [7], a proof graph is a finite model of a first order theory. We recall it and define the first order theory of the approximate proof graph:* $(V, E, s, b, h, c, AX, CUT, \wp, \otimes, !, !_{Ap}, ?, O, I, !O, ?I, c^I_{AX}, c^O_{AX}, p^I_{CUT}, p^O_{CUT}, p^{?I}_{\wp}, p^O_{\wp}, c^O_{\wp}, p^I_{\otimes}, p^{!O}_{\otimes}, c^I_{\otimes}, c^{!O}_{!_{Ap}}, p^O_{!}, c^{!O}_{!}, c^{?I}_{?})$ *is the language of approximate proof graphs, the language of proof graphs is the same one without the symbols $!_{Ap}$ and $c^{!O}_{!_{Ap}}$, where*
- *V, E are two types of distinct objects;*
- *s, g are two symbols of unary function of type (E, V), s for* source, *g for* goal *(in graph terminology, one say* target *rather than goal);*
- *h, c are two symbols of constant of type V, h for* hypothesis, *c for* conclusion*;*
- *$O, I, !O, ?I$ are four symbols of unary predicate of type V (we could also say that $O, I, !O, ?I$ are subtypes of the type V);*
- *$c^I_{AX}, c^O_{AX}, p^I_{CUT}, p^O_{CUT}, p^{?I}_{\wp}, p^O_{\wp}, c^O_{\wp}, p^I_{\otimes}, p^{!O}_{\otimes}, c^I_{\otimes}, c^{!O}_{!_{Ap}}, p^O_{!}, c^{!O}_{!}, c^{?I}_{?}$ are forteen symbols of unary function of type (E, V).*

The axioms of the theory of the approximate proof graphs are G1-G14 *and those of the theory of the proof graphs are the same ones except* G5,G6 *which are replaced by the axioms* G5+,G6+ *respectively, where*

(G1) $\forall\varphi, g(\varphi) \neq h$;

(G2) $\forall\varphi, s(\varphi) \neq c$;

(G3) $\forall\varphi, O(\varphi) \vee I(\varphi) \vee !O(\varphi) \vee ?I(\varphi)$ *(which means that the only objects of type E are objects of its four subtypes (or predicates));*

(G4) $\forall\varphi, A(\varphi) \Rightarrow \neg B(\varphi)$ *where A designates one of the four symbols of predicate of type E and B one of the others (which means that the different predicates of type E designate distinct objects);*

(G5) $\forall\kappa, AX(\kappa) \vee CUT(\kappa) \vee \wp(\kappa) \vee \otimes(\kappa) \vee !_{Ap}(\kappa) \vee !(\kappa) \vee ?(\kappa) \vee (\kappa = h) \vee (\kappa = c)$ *(which means that the only objects of type V are objects of the seven subtypes (or predicates) or the* hypothesis *vertex h or the* conclusion *vertex c);*

(G5+) $\forall\kappa, AX(\kappa) \vee CUT(\kappa) \vee \wp(\kappa) \vee \otimes(\kappa) \vee !(\kappa) \vee ?(\kappa) \vee (\kappa = h) \vee (\kappa = c)$ *(which means that the only objects of type V are objects of the six subtypes (or predicates) or the* hypothesis *vertex h or the* conclusion *vertex c);*

(G6)
$\forall\kappa, (\kappa = h) \vee (\kappa = c) \Rightarrow \neg[AX(\kappa) \vee CUT(\kappa) \vee \wp(\kappa) \vee \otimes(\kappa) \vee !_{Ap}(\kappa) \vee !(\kappa) \vee ?(\kappa)]$ *(which means that the* hypothesis *h and the* conclusion *c, which are objects of type V, are not objects of the seven subtypes (or predicates) of the type V);*

(G6+) $\forall\kappa, (\kappa = h) \vee (\kappa = c) \Rightarrow \neg[AX(\kappa) \vee CUT(\kappa) \vee \wp(\kappa) \vee \otimes(\kappa) \vee !(\kappa) \vee ?(\kappa)]$ *(which means that the* hypothesis *h and the* conclusion *c, which are objects of type V, are not objects of the six subtypes (or predicates) of the type V);*

(G7) $\forall\kappa, N(\kappa) \Rightarrow \neg M(\kappa)$ *where N designates one of the symbols of predicate of type V and M one of the others (which means that the different predicates of type V designate distinct objects);*

in G8-G13*, N will designate one of the symbols of predicate of type V (in case of the axioms* G8, G10, G12*, N is not the predicates $?$ and $!_{Ap}$), A one of the four symbols of predicate of type E, and p_N^A, c_N^A two symbols of function of type (E, V):*
(G8) $\forall\kappa, N(\kappa) \Rightarrow A(p_N^A(\kappa))$ *(which means that the premise p_N^A of the vertex κ of subtype N is of subtype A),*
(G9) $\forall\kappa, N(\kappa) \Rightarrow A(c_N^A(\kappa))$ *(which means that the conclusion c_N^A of the vertex κ of subtype N is of subtype A),*
(G10) $\forall\varphi, \forall\kappa, [A(\varphi) \ \wedge \ N(\kappa) \ \wedge \ \varphi = p_N^A] \Rightarrow g(\varphi) = \kappa$,
(G11) $\forall\varphi, \forall\kappa, [A(\varphi) \ \wedge \ N(\kappa) \ \wedge \ \varphi = c_N^A] \Rightarrow s(\varphi) = \kappa$,
(G12) $\forall\varphi, \forall\kappa, [A(\varphi) \ \wedge \ N(\kappa) \ \wedge \ g(\varphi) = \kappa] \Rightarrow \varphi = p_N^A$,
(G13) $\forall\varphi, \forall\kappa, [A(\varphi) \ \wedge \ N(\kappa) \ \wedge \ s(\varphi) = \kappa] \Rightarrow \varphi = c_N^A$
(the axioms G8-G13 *define completely the linear links except the $?$-link, the $!_{Ap}$-links are completely defined by the axioms* G9, G11, G13*),*
(G14) $\forall\varphi, ?(g(\varphi)) \Rightarrow I(\varphi)$ *(this axiom defines the $?$-links, it precises the type of the premises; the cardinality of the premises of a $?$-link is not fixed* a priori*).*

We will only consider *cut-free* proof graph, that is proof graph with no CUT-link.

Terminology 1 i) *The* bottom *(resp.* top*) of an edge is the extremity of the edge which points up (resp. down).*
ii) *An* hypothesis *(resp.* conclusion*) in a proof graph is an edge whose top (resp. bottom) points at no link.*

Definition 7 *A* cut-free approximate proof structure *(or* approximate proof structure *for short) is an approximate proof graph $\mathcal{G}$ such that:*
i) there is no CUT-link,
ii) there is no hypothesis,
iii) no I-formula is conclusion of $\mathcal{G}$.
Moreover a cut-free proof structure *(or* proof structure *for short) is a (cut-free) approximate proof structure with no $!_{Ap}$-link.*

Definition 8 *Let $\mathcal{G}(\mathcal{V}, \mathcal{E})$ be an approximate proof structure.*

- $\mathcal{E}^* = \{\varphi \in \mathcal{E} / \varphi$ *is not a $?I$-formula*$\}$ *and* $\mathcal{V}^* = \{\kappa \in \mathcal{V} / \kappa$ *is not a $?$-connector*$\}$.
- *To every $\varphi \in \mathcal{E}^*$, we add the following orientation $o(\varphi) \in \{\uparrow, \downarrow\}$: $o(\varphi) = \uparrow$ if φ is a I-formula, and $o(\varphi) = \downarrow$ if φ is a O-formula or a $!O$-formula.*
  So we get $(I, \uparrow), (O \downarrow), (!O, \downarrow) \in \mathcal{E}^* \times \{\uparrow, \downarrow\}$, which will be written $I \uparrow, O \downarrow$ and $!O \downarrow$ respectively.
- *This orientation induces a binary relation $\mathcal{T}$ on $\mathcal{E}^*$: let $\varphi, \psi \in \mathcal{E}^*$,*
  $\varphi \mathcal{T} \psi$ iff there exists a vertex κ such that the orientation added to φ points at κ and the orientation added to ψ goes out κ;
  explicitely,

- $\dfrac{}{I\uparrow \qquad \downarrow O}$,
- $\dfrac{?I \qquad \swarrow O}{\downarrow O}$,
- $\dfrac{I\nwarrow \qquad \swarrow !O}{\uparrow I}$,
- $\dfrac{\downarrow O}{\downarrow !O}$, $\dfrac{}{\downarrow !O}$
- $\dfrac{\uparrow I \quad \dots \quad I\uparrow}{?I}$.

$\mathcal{T}'$ *is the* transitive closure *of* $\mathcal{T}$, $\mathcal{T}''$ *the* transitive and reflexive closure *of* $\mathcal{T}$.

Remark 3 i) *The relation $\mathcal{T}$ is as follows for each link κ: if κ is a:*
- *AX-link* ($\dfrac{}{I \qquad O}$): $I\ \mathcal{T}\ O$,
- *$\wp$-link* ($\dfrac{?I \qquad O}{O}$): $O_{premise}\ \mathcal{T}\ O_{conclusion}$,
- *$\otimes$-link* ($\dfrac{I \qquad !O}{I}$): $I_{conclusion}\ \mathcal{T}\ I_{premise}, !O\ \mathcal{T}\ I_{premise}$,
- *!-link* ($\dfrac{O}{!O}$): $O\ \mathcal{T}\ !O$.

ii) *An obvious formalization of the notions of the previous definition can be done in an analogous way as in* Remark *2 (See [4] for it).*

Condition N1
$\mathcal{T}'$ is a strict ordering relation (that is, $\neg(\varphi\mathcal{T}'\varphi)$, for all $\varphi \in \mathcal{E}^*$) with greatest element.

Condition N2
$\dfrac{\dfrac{I\cdots I_\varphi \cdots I}{?I} \qquad O}{O_\psi}$. For any such subgraph, if I_φ is a premise of the ?-link, then $I_\varphi\ \mathcal{T}''\ O_\psi$.

Remark 4 *If an approximate proof structure satisfies Condition N1, N2, then its conclusions are only $?I$-formulas except one which is a O or $!O$-formula, this conclusion being the greatest element of the relation $\mathcal{T}$.*
The smallest elements of $\mathcal{T}$ are the premises I of the ?-links and the conclusion $!O$ of the eventually $!_{Ap}$-links.

Definition 9
- *A* cut-free approximate proof net with a O or a $!O$-conclusion *(or* approximate proof net with a O or a $!O$-conclusion *for short) is a (cut-free) approximate proof structure which satisfies Conditions N1 and N2 with a O or a $!O$-conclusion respectively.*
- *A* cut-free proof net with a O or a $!O$-conclusion *(or* proof net with a O or a $!O$-conclusion *for short) is a (cut-free) proof structure which satisfies Conditions N1 and N2 with a O or a $!O$-conclusion respectively.*

We denote by
- $PN(O)_{Ap}$, the set of the approximate proof nets with a O-conclusion;
- $PN(!O)_{Ap}$, the set of the approximate proof nets with a $!O$-conclusion;
- $PN(O)$, the set of the proof nets with a O-conclusion;
- $PN(!O)$, the set of the proof nets with a $!O$-conclusion.

Remark that in $PN(!O)_{Ap}$, there exists the particular element $\overline{?I} \dots \overline{?I}\ \overline{!O}$; it will be denoted by N^{Ω}. If the $!O$-conclusion of an element $\mathcal{G}(\mathcal{V}, \mathcal{E})$ of $PN(!O)_{Ap}$ is conclusion of a $!_{Ap}$-link, then $\mathcal{G}(\mathcal{V}, \mathcal{E}) = N^{\Omega}$.

Lemma 3 • *$PN(O)_{Ap}$ and $PN(!O)_{Ap} \setminus \{N^{\Omega}\}$ are isomorphic.*

- *$PN(O)$ and $PN(!O)$ are isomorphic.*

Proof
Transform each approximate proof net with a O-conclusion $\mathcal{G}$ into an approximate proof net with a $!O$-conclusion by adding a new !-link κ whose premise is the O-conclusion of $\mathcal{G}$; the $!O$-conclusion of $\mathcal{G}'$ is the $!O$-formula conclusion of the !-link κ.
Conversely, each approximate proof net with a $!O$-conclusion $\mathcal{G}$ can be transformed in an approximate proof net with a O-conclusion by suppressing the link whose conclusion is the $!O$-conclusion of $\mathcal{G}$: necessarly, this link is a !-link, since $\mathcal{G} \neq N^{\Omega}$.
□

We define then the set of the *cut-free proof nets* (or *proof nets* for short) as follows:

Definition 10 • *$PN_{Ap} = PN(O)_{Ap} + \{N^{\Omega}\}$ (which is isomorphic to $PN(!O)_{Ap}$) is the set of the* (cut-free) approximate proof nets.

- *$PN = PN(O)$ is the set of the* (cut-free) proof nets.

5.2 Other definitions

Definition 11 *Let $\mathcal{G}(\mathcal{V}, \mathcal{E})$ be an approximate proof graph.*
$\mathcal{R}$ is a binary relation defined on $\mathcal{E}$:
$\varphi \mathcal{R} \psi$ iff there exists a vertex $\kappa \in \mathcal{V}$ such that φ is a premise of κ and ψ is a conclusion of κ.
$\mathcal{G}$ is said $\mathcal{R}$-well-founded *iff the transitive closure of $\mathcal{R}$ is a well-founded strictly ordering relation.*

Remark 5 i) *Using the axiomatization of* Remark *2, the binary relation $\mathcal{R}$ can be defined as follows:*
$\varphi \mathcal{R} \psi$ iff there exists a vertex $\kappa \in \mathcal{V}$ such that $g(\varphi) = \kappa$ and $s(\psi) = \kappa$.
ii) *The elements of PN_{Ap} are $\mathcal{R}$-well-founded (see* Remark *7 below).*
In [4], the $\mathcal{R}$-well foundation of the elements of $PN(O)$ is proved.

Proposition 2 *The elements of PPS are (isomorphic to) the $\mathcal{R}$-well-founded approximate proof structures with a O-conclusion or a $!O$-conclusion. PPS^{+}'s ones are (isomorphic to) the $\mathcal{R}$-well-founded proof structures with a O-conclusion or a $!O$-conclusion.*

Proof
Let $S \in PPS$. We define the approximate proof structure $\mathcal{G}(S) = (\mathcal{V}, \mathcal{E})$ which is (isomorphic to) S:
• $\mathcal{E} = St(S)$ with
- $||O|| = St(S) \cap D\Omega P^{+}$,
- $||I|| = St(S) \cap D\Omega P^{-}$,
- $||!O|| = St(S) \cap D\Omega P^{+*}$,
- $||?I|| = St(S) \cap D\Omega P^{-*}$
• $\mathcal{V}$ is the set of the occurrences of the constructors of the types. Formally $\mathcal{V}$ is the union of the following sets:
- AX is (isomorphic to) $At(S)$,
- $||\wp|| = \{\rightarrow / \rightarrow \text{ in } A \rightarrow X \in ||O||\}$,
- $||\otimes|| = \{\rightarrow / \rightarrow \text{ in } A \rightarrow X \in ||I||\}$,
- $||!_{Ap}|| = \{\bigwedge / \bigwedge \text{ in } \bigwedge_{i=1}^{0} Z_i \in ||!O||\}$ $(\bigwedge_{i=1}^{0} Z_i = \emptyset)$,
- $||!|| = \{\bigwedge / \bigwedge \text{ in } \bigwedge_{i=1}^{1} X_i \in ||!O||\}$ $(\bigwedge_{i=1}^{1} X_i = X_i)$,
- $||?|| = \{\bigwedge / \bigwedge \text{ in } \bigwedge_{i=1}^{k} X_i \in ||?I||\}$
and such that $(\mathcal{V}, \mathcal{E})$ satisfies
- $\{(\alpha^{-}, \alpha^{+}) / \alpha \in At(S)\}$ is the set of the AX-links,
- $\{(A, X, A \rightarrow X) \in D\Omega P^{-*} \times D\Omega P^{+} \times D\Omega P^{+} / A \rightarrow X \in St(S)\}$ is the set of the $\wp$-links,
- $\{(A, X, A \rightarrow X) \in D\Omega P^{+*} \times D\Omega P^{-} \times D\Omega P^{-} / A \rightarrow X \in St(S)\}$ is the set of the $\otimes$-links,
- $\{(\emptyset) \in D\Omega P^{+*} / \emptyset \in St(S)\}$ is the set of the $!_{Ap}$-links,
- $\{(X_i, \bigwedge_{i=1}^{1} X_i) / \bigwedge_{i=1}^{1} X_i \in St(S)\}$ is the set of the $!$-links,
- $\{(X_1, \cdots, X_k, \bigwedge_{i=1}^{k} X_i) \in D\Omega P^{-k} \times D\Omega P^{-*} / \bigwedge_{i=1}^{k} X_i \in St(t)\}$ is the set of the $?$-links.
• $\mathcal{G}(\mathcal{V}, \mathcal{E})$ is $\mathcal{R}$-well founded, since the types are well founded.
□

Remark 6 *This is analogous to the interpretation of the proof nets by coherent sets of types (see [8] and [6]):*

- $\dfrac{}{\alpha^{-}:I \qquad\qquad \alpha^{+}:O}$,
- $\dfrac{X^{-}:O \qquad\qquad X^{+}:I}{}$,
- $\dfrac{A:?I \qquad\qquad X:O}{A \rightarrow X:O}$,
- $\dfrac{X:I \qquad\qquad !O:A}{A \rightarrow X:I}$,
- $\dfrac{X:O}{\bigwedge X:!O}$,
- $\dfrac{X_1:I \quad \ldots \quad X_n:I}{\bigwedge_{I=1}^{n} X_i:?I}$.

By extension, we could define the following interpretation of the $!_{Ap}$-links:

$\dfrac{}{\emptyset:!O}$

Definition 12 *Let $\mathcal{G}(\mathcal{V},\mathcal{E})$ be an approximate proof structure.*
i) $\mathcal{T}_\mathcal{A}$ is the sub-relation of $\mathcal{T}$ defined on the set of the (occurrences of) I-formulas which are conclusions of the AX-links:
$\varphi\mathcal{T}_\mathcal{A}\psi$ iff φ,ψ are I-formulas conclusions of AX-links, $\varphi\mathcal{T}'\psi$ and
(for all I-formula χ conclusion of a AX-link, if $\varphi\mathcal{T}''\chi\mathcal{T}''\psi$, then $\chi=\varphi$ or $\chi=\psi$).
ii) $\mathcal{T}'_\mathcal{A}$ (resp. $\mathcal{T}''_\mathcal{A}$) is the transitive *(resp.* transitive and reflexive*) closure of the relation $\mathcal{T}_\mathcal{A}$.*

Lemma 4 *Let $\mathcal{G}(\mathcal{V},\mathcal{E})$ be a $\mathcal{R}$-well-founded approximate proof structure.*
$\mathcal{T}'$ is a strict ordering relation with greatest element iff $\mathcal{T}'_\mathcal{A}$ also.

Proof
• Since $\mathcal{T}_\mathcal{A}$ is a sub-relation of $\mathcal{T}$, $\mathcal{T}'_\mathcal{A}$ is a strict ordering relation if $\mathcal{T}'$ is also.
Conversely, suppose that $\mathcal{T}'$ contains a cycle, we prove that $\mathcal{T}'_\mathcal{A}$ does also:
since the approximate proof structure $\mathcal{G}$ is $\mathcal{R}$-well-founded, a cycle relatively to $\mathcal{T}'$ $(f_i)_{0\leq i<m}$, with $f_i\mathcal{T}f_{i+1}(0\leq i<m-1)$ and $f_{m-1}\mathcal{T}f_0$, cannot contain only O-formulas or only I-formulas. So $(f_i)_{0\leq i<m}$ must contain $O, !O$ and I-formulas and it must have the following form:
$(f_i)_{0\leq i<m}=$
$(g^1_{1,k},\cdots,g^{p_k}_{1,k},g_{2,k},g^1_{3,k},\cdots,g^{q_k}_{3,k})_{0\leq k<n-1}$ (with $n-1<m$) such that the sequence
$(g^1_{1,k}\cdots g^{p_k}_{1,k},g_{2,k},g^1_{3,k}\cdots g^{q_k}_{3,k})$ satisfies,
for all index $k\in Z/nZ$,
- $g^1_{1,k},\cdots,g^{p_k}_{1,k}$ are O-formulas, $g_{2,k}$ is a $!O$-formula, $g^1_{3,k},\cdots,g^{q_k}_{3,k}$ are I-formulas,
- $g^1_{1,k}$ is conclusion of an AX-link,
- $g^i_{1,k}$ is premise and $g^{i+1}_{1,k}$ is conclusion of the same $\wp$-link $(1\leq i<p_k-1)$,
- $g^{p_k}_{1,k}$ is the premise and $g_{2,k}$ is the conclusion of the same !-link,
- $g_{2,k}$ and $g^1_{3,k}$ are the premises of the same $\otimes$-link,
- $g^i_{3,k}$ is the conclusion and $g^{i+1}_{3,k}$ is premise of the same $\otimes$-link $(1\leq i<p_k-1)$,
- $g^{q_k}_{3,k}$ and $g^1_{1,k+1}$ are the conclusions of the same AX-link.
So $(g^1_{3,0},\cdots,g^1_{3,n-1})$ is a $\mathcal{T}'_\mathcal{A}$-cycle, with $g^1_{3,i}\mathcal{T}_\mathcal{A}g^1_{3,i+1}$ $(0\leq i<n-1)$ and $g^1_{3,n}\mathcal{T}_\mathcal{A}g^1_{3,0}$.
• Let φ be the O-conclusion or the $!O$-conclusion of $\mathcal{G}$, that is the greatest element of $\mathcal{T}$, then the conclusion ψ of the AX-link which is hereditary premise of φ is the greatest element of $\mathcal{T}_\mathcal{A}$.
□

Proposition 3 *Let $S\in PPS$.*

- *Let $\alpha,\beta\in At(S)$. $\alpha<_1\beta$ iff $\alpha^-\mathcal{T}_\mathcal{A}\beta^-$.*
- *S satisfies* Cond 3 *iff the approximate proof structure $\mathcal{G}(S)$ (isomorphic to S) satisfies* Condition N1.
- *Suppose that S satisfies* Cond 3.
  S satisfies Cond 4 *iff $\mathcal{G}(S)$ satisfies* Condition N2.

So $S \in PS$ *iff* $\mathcal{G}(S) \in PN_{Ap}$.
And, so $S \in PS^+$ *iff* $\mathcal{G}(S) \in PN$.

Proof
• Suppose that $\alpha <_1 \beta$, that is $X(\alpha^+) \to Y(\beta^-) \in St(S)$.
By *Remark* 3 we get:
$\alpha^+ \; \mathcal{T}' \; X(\alpha^+) \; \mathcal{T} \; \bigwedge X(\alpha^+) \; \mathcal{T} \; Y(\beta^-)$; since $\alpha^- \mathcal{T} \alpha^+$ and $Y(\beta^-) \mathcal{T}' \beta^-$, we can conclude $\alpha^- \mathcal{T}' \beta^-$.
The two following properties imply the result $\alpha^- \mathcal{T}_{\mathcal{A}} \beta^-$: let $\varphi \in At(S)$,
$\alpha^+ \; \mathcal{T}'' \; \varphi^+ \; \mathcal{T}'' \; X(\alpha^+)$ *implies* $\varphi^+ = \alpha^+$,
and $Y(\beta^-) \; \mathcal{T}'' \; \varphi^- \; \mathcal{T}'' \; \beta^-$ *implies* $\varphi^- = \beta^-$.
The proof of the reverse assertion is analogous.
• Since $\mathcal{G}(S)$ is $\mathcal{R}$-well-founded, the previous lemma and the property $\alpha <_1 \beta$ *iff* $\alpha^- \mathcal{T}_{\mathcal{A}} \beta^-$ imply that S *satisfies* Cond 3 *iff* $\mathcal{G}(S)$ *satisfies* Condition N1.
• The last point is a consequence of the first point of the proposition.
□

Remark 7 i) *The notion of equality of* sequences *(or of* pts*) coincide to the equality of (approximate) proof nets (isomorphism between (approximate) proof nets).*
ii) *The property of $\mathcal{R}$-well foundation of the approximate cut-free proof nets is a consequence of the well foundation of the elements of the principal sequences: the types of $D\Omega P$ are well founded.*

6 Conclusion

This present article gives identities between syntactical and semantical notions such that proof nets and principal sequences. The notions of syntax and semantics are no more so clear. The second author defined in [12] the elements of the interpretation of the λ-terms as $D\Omega$-graphs (proof nets with a new definition of !-links analogous to our definition of ?-link): the formulas are seen as particular graphs. These particular graphs are $\mathcal{R}$-well founded. Forgetting the $\mathcal{R}$-well foundation allows to create cyclic formulas and allows to think of defining new kinds of models of the lambda-calculus.

As in [10], next works should study the suitable operations on *pts* (or on proof nets) in the present framework, and should create analogous operations for coherent models of lambda-calculus.

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Éditeur
Inria, Domaine de Voluceau, Rocquencourt, BP 105, 78153 Le Chesnay Cedex (France)
ISSN 0249-6399