Convergence Rate for the Approximation of the Limit Law of Weakly Interacting Particles 2: Application to the Burgers Equation

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N° 2410

Novembre 1994

PROGRAMME 6

Calcul scientifique,

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Rapport de recherche

1994

ISSN 0249-6399

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Projet Omega

Rapport de recherche n°2410 — Novembre 1994 — 49 pages

Abstract: In this paper, we construct a stochastic particles method for the Burgers equation with a monotonic initial condition; we prove that the convergence rate is $O\left(\frac{1}{\sqrt{N}}+\sqrt{\Delta t}\right)$ for the $L^1(\mathbb{R}\times\Omega)$-norm of the error. To obtain that result, we link the PDE and the algorithm to a system of weakly interacting stochastic particles; the difficulty of the analysis comes from the discontinuity of the interaction kernel, equal to the Heaviside function.

In [4], we show how the algorithm and the result extend to the case of non monotonic initial conditions for the Burgers equation; we also treat the case of nonlinear PDE's related to particles systems with Lipschitz interaction kernels.

Our next objective is to adapt our methodology to the (more difficult) case of the 2-D inviscid Navier-Stokes equation.

(Résumé : tsvp)

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Vitesse de convergence pour l'approximation de la loi limite de particules en interaction faible
2: Application à l'Equation de Burgers

Résumé : Dans cet article, nous construisons une méthode particulaire stochastique pour l'équation de Burgers unidimensionnelle avec condition initiale monotone ; nous montrons que la vitesse de convergence est d'ordre $O\left(\frac{1}{\sqrt{N}} + \sqrt{\Delta t}\right)$ pour la norme $L^1(I\!R \times \Omega)$ de l'erreur. Pour obtenir ce résultat, nous interprétons l'E.D.P. et l'algorithme à l'aide d'un système de particules en interaction ; les difficultés de l'analyse proviennent de la discontinuité du noyau d'interaction, égal à la fonction de Heaviside.

Dans [4], nous montrons comment l'algorithme et le résultat s'étendent au cas des conditions initiales non monotones pour l'équation de Burgers ; nous traitons aussi le cas des équations aux dérivées partielles non linéaires reliées à des systèmes particulaires avec noyaux d'interaction lipschitziens.

Notre objectif est ensuite d'adapter notre méthodologie au cas (plus difficile) de l'équation de Navier-Stokes 2-D incompressible.

1 Introduction

In this paper and in [4], we study the convergence rate of a stochastic particles method for the numerical solving of McKean-Vlasov nonlinear Partial Differential Equations

$$\frac{d}{dt} < \mu_t, f > = < \mu_t, L(\mu_t) f > \quad ,$$

where μ_t is a probability measure, f is any real function of class $\mathcal{C}^\infty$ with a compact support, and the operator L_μ is defined by

$$L_{(\mu)} f(x) \; = \; \frac{1}{2} \left(\int_{I\!R} s(x,y) d\mu(y) \right)^2 f''(x) \; + \; \left(\int_{I\!R} b(x,y) d\mu(y) \right) f'(x) \quad .$$

The method is based upon the simulation of a weakly interacting particles system.

Here, our concern is the Burgers equation

$$\frac{\partial V}{\partial t} \; = \; \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial x^2} \; - \; V \, \frac{\partial V}{\partial x} \quad ,$$

with an initial condition equal to a distribution function. The corresponding interaction kernels are as follows: $s(\cdot,\cdot)$ is a constant function, $b(x,y) = H(x-y)$, H being the Heaviside function, whose discontinuity makes difficult the error analysis. Our objective is to give an estimate for the convergence rate in $L^1(\Omega \times I\!R)$ norm of the empirical distribution function of the particles to the solution of the Burgers equation.

Our motivation to focus on the Burgers equation is two-fold: first, in the numerical analysis literature, this equation is used as a test-case for deterministic algorithms solving some nonlinear PDE's of this type (in particular the Navier-Stokes equation), especially to test their performances when the viscosity term tends to 0; second, our objective is to extend our method and error analysis to the 2-D Navier-Stokes equation: in that case, the kernel $b(\cdot,\cdot)$ is still less regular than the Heaviside function (it is even singular), thus we found useful to have an intermediate step between smooth kernels and singular kernels.

In Bossy & Talay [4], we extend the algorithm and the estimate of the convergence rate of the present paper to the case where the initial condition of the Burgers equation is non monotonic; we also treat the case where the interaction kernels $b(\cdot,\cdot)$ and $s(\cdot,\cdot)$ are bounded and Lipschitz, and $s(\cdot,\cdot)$ is bounded below by a strictly positive constant; under additional hypotheses, an estimate is also given for an approximation of the density of μ_t.

The construction of our algorithm is not based upon a splitting of the nonlinear differential operator, as it is the case for the well-known random vortex methods developed by Chorin, Hald, etc (recent publications on these methods are those of Chorin [5], Chorin and Marsden [6], Goodman [11], Hald ([13] and [14]), Puckett ([22] and [21]), Roberts [23] and Long [17], e.g.; see also the bibliography in [5] and in the different contributions of [12], in particular those by Chorin and Hald; the convergence rate of the Puckett algorithm, and of its extension for convection-reaction-diffusion equations with a nonlinear reaction term, has been established by Bernard-Talay-Tubaro [2]). Nevertheless, the Chorin's method applied to the inviscid 2-D Navier-Stokes equation (not the Burgers one) is identical to ours. Thus our error estimates, which seem to be what one expects for this family of methods, give a new theoretical insight to the Chorin's method; the novelty of our approach consists in forgetting the splitting and using a link between the nonlinear PDE and an interacting particles system.

We now fix some notation.

Consider the Burgers equation:

$$\begin{cases} \dfrac{\partial V}{\partial t} = \dfrac{1}{2}\sigma^2 \dfrac{\partial^2 V}{\partial x^2} - V\,\dfrac{\partial V}{\partial x} \ , \text{ in } [0,T]\times \mathbb{R}\,, \\ \\ V(0,x) = V_0(x) \ . \end{cases} \tag{1}$$

In the whole paper, we suppose that the initial condition, V_0, is the distribution function of a probability measure U_0 on $\mathbb{R}$:

$$V_0(x) = \int_{-\infty}^{x} U_0(dy) \ .$$

For such an initial condition, we interpret the solution of the Burgers equation as the distribution function of the probability measure U_t solution to the following PDE of McKean-Vlasov type:

$$\begin{cases} \dfrac{\partial U}{\partial t} = \dfrac{1}{2}\sigma^2 \dfrac{\partial^2 U}{\partial x^2} - \dfrac{\partial}{\partial x}\left(\left(\int_{\mathbb{R}} H(x-y) U_t(dy) \right) U_t \right) \ , \\ \\ U_{t=0} = U_0 \ . \end{cases} \tag{2}$$

Note that the above PDE is understood in the distribution sense (U operates on smooth functions with a compact support in $]0,T[\times\mathbb{R}$); its nonlinear part makes appear the discontinuous interaction kernel $b(x,y) = H(x-y)$, where H is the Heaviside function ($H(z)=0$ if $z<0$, $H(z)=1$ if $z\geq 1$).

To this McKean-Vlasov equation, is associated the nonlinear stochastic differential equation

$$\begin{cases} dX_t = \sigma dw_t + \int_{\mathbb{R}} H(X_t - y) U_t(dy)\,dt \ , \text{ where } U_t(dy) \text{ is the law of } X_t\,, \\ \\ X_{t=0} = X_0\,, \quad \text{of law } U_0 \ . \end{cases} \tag{3}$$

In the stochastic differential equation (3), the interaction kernel is not Lipschitz: as a matter of fact, the existence and uniqueness of a weak solution cannot be derived from classical results, and the error analysis of the stochastic particles method is much more complex than in the Lipschitz case (see [4]).

In §2, we give a proof of the existence and uniqueness of a weak solution to (2). In §3, we show that the distribution function V_t of the law of X_t is the classical solution of the Burgers equation, i.e the solution given by the Cole-Hopf transformation [15]. In §4, we use the probabilistic interpretation of the solution of the Burgers equation, and the ideas developed in [4] to construct a stochastic particles method. Its rate of convergence is established in §5 and §6. The Appendix proves some intermediate results.

The results of numerical experiments can be found in [4] and overall in M. Bossy's thesis [3]: in particular, they show the excellent behaviour of the algorithm when the viscosity coefficient σ is small: in fact, the computation time is independent of σ; this property is not satisfied by the deterministic algorithms, which require very thin grids in the area where the gradient of the solution is very large whereas our stochastic particles have a dynamics which makes them naturally concentrate in this area.

Remark. If the initial condition of the Burgers equation is of the type

$$V_0(x) \;=\; 1 \;-\; \int_{-\infty}^{x} U_0(dy) \;=\; \int_{x}^{+\infty} U_0(dy)\,,$$

where U_0 is a probability law, we then consider the equation

$$\begin{cases} \dfrac{\partial U}{\partial t} \;=\; \dfrac{1}{2}\sigma^2 \dfrac{\partial^2 U}{\partial x^2} \;-\; \dfrac{\partial}{\partial x}\left(U \left(\int_{\mathbb{R}} (1 - H(x-y))\, U_t(dy) \right) \right) \\ U_{t=0} \;=\; U_0. \end{cases}$$

If U_t denotes the law of the corresponding process, with similar arguments as above, we obtain that the function $\tilde{V}(x,t)$ defined by

$$\tilde{V}(t,x) \;=\; 1 \;-\; \int_{-\infty}^{x} U_t(dy) \;=\; \int_{x}^{+\infty} U_t(dy)$$

is a weak solution to the Burgers equation; our algorithm and our convergence rate can easily be extended to that situation.

Contents

2 Existence and uniqueness of a weak solution to (3)

2.1 Link between (3) and the Burgers equation

Suppose that the existence in law of a solution to (3) holds. Then, applying Itô's formula, one gets that, for any function $f \in C^\infty([0,T] \times \mathbb{R})$, with a compact support in $]0,T[\times \mathbb{R}$:

$$\begin{aligned} 0 &= f(T, X_T) - f(0, X_0) \\ &= \int_0^T \left(\frac{\partial f}{\partial s}(s, X_s) + \frac{\partial f}{\partial x}(s, X_s) \times \left(\int_{\mathbb{R}} H(X_s - y) U_s(dy) \right) + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2}(s, X_s) \right) ds \\ &\quad + \int_0^T \frac{\partial f}{\partial x}(s, X_s)\, dw_s \ . \end{aligned}$$

We deduce that

$$\int_0^T \mathbb{E} \left(\frac{\partial f}{\partial s}(s, X_s) + \frac{\partial f}{\partial x}(s, X_s) \times \left(\int_{\mathbb{R}} H(X_s - y) U_s(dy) \right) + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2}(s, X_s) \right) ds = 0 \ .$$

Thus it holds that

$$\begin{aligned} \int_0^T \Bigg[\int_{\mathbb{R}} \frac{\partial f}{\partial s}(s,x) U_s(dx) + \int_{\mathbb{R}} \frac{\partial f}{\partial x}(s,x) \left(\int_{\mathbb{R}} H(x-y) U_s(dy) \right) U_s(dx) \\ + \frac{1}{2}\sigma^2 \int_{\mathbb{R}} \frac{\partial^2 f}{\partial x^2}(s,x) U_s(dx) \Bigg] = 0 \ . \end{aligned}$$

That shows that U_t is a weak solution to the McKean-Vlasov equation (2) in $]0,T[\times \mathbb{R}$.

Let $V(t,x)$ denote the distribution function of U_t:

$$V(t,x) = \int_{-\infty}^x U_t(dy)\, , \quad \forall\, (t,x) \in [0,T] \times \mathbb{R}.$$

We now show that V is a weak solution to the Burgers equation, following arguments developed by Sznitman [26].

As $\dfrac{\partial V}{\partial x} = U$ in the sense of distributions, (2) implies that

$$\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial t} \right) = \frac{\partial}{\partial x}\left(\frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial x^2} - V \frac{\partial V}{\partial x} \right).$$

The distributions $\dfrac{\partial V}{\partial t}$ and $\dfrac{1}{2}\sigma^2 \dfrac{\partial^2 V}{\partial x^2} - V\dfrac{\partial V}{\partial x}$ have the same spatial derivates, thus their difference is a distribution invariant by a translation on the x-axis (cf. Schwartz [24]).

Thus, for any test function $f(t,x)$ and for any $z \in I\!R$, one has that

$$
\begin{aligned}
&< -\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2} - V\frac{\partial V}{\partial x}, f > \\
&\quad = \int_{[0,T]\times I\!R} V(t,x)\left(\frac{\partial f}{\partial t}(t,x+z) + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}(t,x+z)\right)dtdx \\
&\quad\quad + \int_{[0,T]\times I\!R} \frac{1}{2}V^2(t,x)\frac{\partial f}{\partial x}(t,x+z)dtdx \\
&\quad = \int V(t,x-z)\left(\frac{\partial f}{\partial t}(t,x) + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}(t,x)\right)dtdx + \int \frac{1}{2}V^2(t,x-z)\frac{\partial f}{\partial x}(t,x)dtdx \quad .
\end{aligned}
$$

As, for any t in $[0,T]$, $V(t,x)$ is bounded and tends to 0 when x tends to $-\infty$, the right handside term tends to 0 when z tends to $+\infty$ by the bounded convergence theorem. This implies that V solves the Burgers equation in the distribution sense.

To sum the preceding discussion up, if the equation (3) has a weak solution, the distribution function of the law U_t is a weak solution of the Burgers equation.

2.2 Characterization of the law of X_t

To get the uniqueness in the sense of probability law of a solution to (3), we adapt arguments used by Meléard and Roelly in [18] for a similar equation.

We first state a result which appears in the proof of the proposition 1.1 of Meléard & Roelly [18]:

Lemma 2.1 *On a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), I\!P)$, consider the real process defined by*

$$
Y_t = Y_0 + \sigma w_t + \int_0^t C_s ds \ , \quad 0 \le t \le T \ ,
$$

where Y_0 is a random variable independent of the Brownian motion (w_t) and (C_t) is a process $(\mathcal{F}_t)$-adapted and bounded. Then, for all t in $]0,T]$, the law of Y_t has a density u_t which belongs to $L^2(I\!R)$ and it holds that

$$
\|u_t\|_2 \le \frac{C}{t^{\frac{1}{4}}} \ . \tag{4}
$$

Suppose that the existence of a weak solution to (3) holds; we are going to show that then the law of each random variable X_t ($t \in]0,T]$) is fully characterized.

Thus, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), (w_t), (X_t))$ be a weak solution; let U_t be the law of X_t. Set

$$C_t = \int_{\mathbb{R}} H(X_t - y)\, U_t(dy)\,.$$

(C_t) being a bounded process, the preceding lemma shows that U_t has a density in $L^2(\mathbb{R})$; we denote it by u_t. We now give another characterization of u_t.

For any $t > 0$, we denote by g_t the density of the law of σw_t, and by S_t the heat semigroup: $S_t U \;=\; g_t * U$.

Suppose that there exists a density p_t such that

$$p_t \;=\; S_t U_0 \;-\; \int_0^t S_{t-s}\left(\frac{\partial}{\partial x}\left(p_s \cdot \int_{\mathbb{R}} H(x-y) p_s(y) dy\right)\right) ds, \quad \forall\, t\, \in]0, T]\;\;. \tag{5}$$

By a formal differentiation of (5), one obtains that $\dfrac{\partial p_t}{\partial t}$ verifies

$$\begin{aligned}
\frac{\partial p_t}{\partial t} \;&=\; \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2}\left(S_t U_0\right) - S_0\left(\frac{\partial}{\partial x}\left(p_t \int_{\mathbb{R}} H(x-y) p_t(y) dy\right)\right) \\
&\qquad\qquad - \int_0^t \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}\left(S_{t-s}\left(\frac{\partial}{\partial x}\left(p_s \int_{\mathbb{R}} H(x-y) p_s(y) dy\right)\right)\right) ds \\
&=\; \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2}\left[S_t U_0 \;-\; \int_0^t S_{t-s}\left(\frac{\partial}{\partial x}\left(p_s \int_{\mathbb{R}} H(x-y) p_s(y) dy\right)\right) ds\right] \\
&\qquad\qquad - \frac{\partial}{\partial x}\left(p_t \int_{\mathbb{R}} H(x-y) p_t(y) dy\right)\;.
\end{aligned}$$

Thus p_t is a weak solution to (2):

$$\frac{\partial p_t}{\partial t} = \frac{1}{2}\sigma^2 \frac{\partial^2 p_t}{\partial x^2} \;-\; \frac{\partial}{\partial x}\left(p_t \int_{\mathbb{R}} H(x-y) p_t(y) dy\right) \quad .$$

The following lemma characterizes the density of the law of X_t as the unique solution to the equation (5).

Lemma 2.2 **(i)** *For any weak solution (X_t) to (3), the density of X_t is a weak solution to (5).*

(ii) *For any $0 < t \leq T$, there exists at most one function p_t in $L^1(\mathbb{R})$ weak solution to (5) and such that*

$$\exists C > 0 \;\;,\;\; \sup_{t \in]0,T]} \|p_t\|_{L_1(\mathbb{R})} \;\leq\; C\,.$$

Proof. We first show (i). Fix t in $]0,T]$ and f in $C^\infty(\mathbb{R})$ with a compact support; set

$$G(s,x) = S_{t-s}f(x) \quad , \quad 0 \leq s < t\,.$$

$G(s,x)$ solves the heat equation in backward time

$$\begin{cases} \dfrac{\partial G}{\partial s} \;+\; \dfrac{1}{2}\sigma^2 \dfrac{\partial^2 G}{\partial x^2} \;=\; 0 \quad , \quad 0 \leq s < t \quad , \\ \\ G(t,x) \;=\; f(x) \quad . \end{cases}$$

The Itô formula implies that

$$G(t,X_t) = G(0,X_0) + \int_0^t \frac{\partial G}{\partial x}(s,X_s)dw_s + \int_0^t \frac{\partial G}{\partial x}(s,X_s)\left(\int_{\mathbb{R}} H(X_s-y)u_s(y)dy\right)ds \quad .$$

We deduce that

$$\begin{aligned} &\int_{\mathbb{R}} f(x)u_t(x)dx \\ &\quad = \int_{\mathbb{R}} G(0,x)U_0(dx) \;+ \int_0^t \int_{\mathbb{R}} \frac{\partial}{\partial x} G(s,x)\,\left(\int_{\mathbb{R}} H(x-y)u_s(y)dy\right) u_s(x)dx\,ds \\ &\quad = \int_{\mathbb{R}} \left(S_t U_0\right)(x)\; f(x)dx \;+ \int_0^t \int_{\mathbb{R}} \frac{\partial}{\partial x} S_{t-s}f(x)\,\left(\int_{\mathbb{R}} H(x-y)u_s(y)dy\right) u_s(x)dx\,ds \quad . \end{aligned}$$

An integration by parts shows that

$$\begin{aligned} &\int_{\mathbb{R}_x} \frac{\partial}{\partial x}\left(\int_{\mathbb{R}_z} g_{t-s}(x-z)f(z)dz\right)\left(\int_{\mathbb{R}_y} H(x-y)u_s(y)dy\right)u_s(x)dx \\ &\quad = -\int_{\mathbb{R}_x}\int_{\mathbb{R}_z} g_{t-s}(x-z)f(z)\frac{\partial}{\partial x}\left[u_s(x)\left(\int_{\mathbb{R}_y} H(x-y)u_s(y)dy\right)\right]dx\,dz \\ &\quad = -\int_{\mathbb{R}_z} f(z)\;S_{t-s}\left(\frac{\partial}{\partial x}\left[u_s(x)\left(\int_{\mathbb{R}_y} H(x-y)u_s(y)dy\right)\right]\right)(z)\;dz \quad , \end{aligned}$$

so that we conclude that u_t solves (5) in the weak sense.

Let us now show (ii). Let u_t and v_t be two weak solutions to (5) belonging to $L^1(\mathbb{R})$ and verifying

$$\exists\, C > 0,\ \sup_{t\in]0,T]}\ \left(\|u_t\|_{L^1(\mathbb{R})} + \|v_t\|_{L^1(\mathbb{R})}\right) \;\leq\; C \quad .$$

Then, for any $t \in]0,T]$, it holds that

$$\begin{aligned}
\|u_t - v_t\|_1 &= \left\| \int_0^t S_{t-s} \frac{\partial}{\partial x} \left(u_s(x) \int_{\mathbb{R}} H(x-y) u_s(y) dy - v_s(x) \int_{\mathbb{R}} H(x-y) v_s(y) dy \right) ds \right\|_{L^1(\mathbb{R})} \\
&\leq \int_0^t \left\| g_{t-s} * \frac{\partial}{\partial x} \left(u_s(x) \int_{-\infty}^x u_s(y) dy \; - \; v_s(x) \int_{-\infty}^x v_s(y) dy \right) \right\|_{L^1(\mathbb{R})} ds \\
&\leq \int_0^t \left\| \frac{\partial}{\partial x} g_{t-s} \right\|_1 \cdot \left\| u_s(x) \int_{-\infty}^x u_s(y) dy \; - \; v_s(x) \int_{-\infty}^x v_s(y) dy \right\|_{L^1(\mathbb{R})} ds \\
&\leq \int_0^t \frac{2}{\sqrt{2\pi(t-s)\sigma^2}} \left\| u_s(x) \int_{-\infty}^x u_s(y) dy \; - \; v_s(x) \int_{-\infty}^x v_s(y) dy \right\|_1 ds \quad .
\end{aligned}$$

But one has

$$\begin{aligned}
& u_s(x) \int_{-\infty}^x u_s(y) dy - v_s(x) \int_{-\infty}^x v_s(y) dy \\
& \qquad = u_s(x) \int_{-\infty}^x \left(u_s(y) - v_s(y) \right) dy - \left(v_s(x) - u_s(x) \right) \int_{-\infty}^x v_s(y) dy \\
& \qquad \leq |u_s(x)| \, \|u_s \; - \; v_s\|_1 \; + \; C \, |v_s(x) - u_s(x)| \; ,
\end{aligned}$$

where C is a constant uniform with respect to t; thus,

$$\|u_t \; - \; v_t\|_{L^1(\mathbb{R})} \quad \leq \quad \int_0^t \frac{4\,C}{\sqrt{2\pi(t-s)\sigma^2}} \, \|u_s \; - \; v_s\|_{L^1(\mathbb{R})} \, ds \quad .$$

As $s \to 1/\sqrt{t-s}$ is integrable on $[0,t]$, an application of Gronwall's lemma ends the proof. ∎

2.3 A nonlinear martingale problem

Thus, having supposed the existence of a weak solution to (3), we have fully characterized the law of each random variable X_t. Let us show the existence of a weak solution: a classical method is to pose the associated martingale problem.

We first fix some notation. For any space E, $\mathcal{M}(E)$ denotes the set of probability measures on E; $x(\cdot)$ is the canonical process on the space of continuous functions from $[0,T]$ to $\mathbb{R}$; for any measure $\mu \in \mathcal{M}(\mathbb{R})$, the differential operator $\mathcal{L}_{(\mu)}$ is defined by

$$\mathcal{L}_{(\mu)} f(x) \; = \; \frac{1}{2} \sigma^2 \frac{\partial^2 f}{\partial x^2}(x) \; + \; \left(\int_{\mathbb{R}} H(x-y) \mu(dy) \right) \frac{\partial f}{\partial x}(x) \, .$$

A solution to the hereafter nonlinear martingale problem (6) associated to the operator $\mathcal{L}_{(.)}$ and an initial distribution $U_0 \in \mathcal{M}(\mathbb{R})$, is an element $\mathbb{Q}$ of $\mathcal{M}(C([0,T];\mathbb{R}))$ (we denote by $\mathbb{Q}_t$, $t \in [0,T]$, its onedimensional distributions), such that

$$\left.\begin{array}{ll} (i) & \mathbb{Q}_0 \;=\; U_0 \quad , \\ (ii) & \forall f \in C_K^2(\mathbb{R})\,,\;\; f(x(t)) - f(x(0)) - \displaystyle\int_0^t \mathcal{L}_{(\mathbb{Q}_s)} f(x(s))\,ds,\;\; t \in [0,T] \\ & \text{is a } \mathbb{Q} \text{ martingale} \quad . \end{array}\right\} \qquad (6)$$

Suppose that there exists a solution $\mathbb{Q}$ to the *nonlinear* martingale problem (6). Set

$$\widehat{C}(t,x) \;:=\; \int_{\mathbb{R}} H(x-y)\mathbb{Q}_t(dy)\,.$$

$\mathbb{Q}$ also solves the *linear* martingale problem associated to the operator $\widehat{\mathcal{L}}$ defined by

$$\widehat{\mathcal{L}} f(x) \;=\; \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2}(x) \;+\; \widehat{C}(t,x)\,.\,\frac{\partial f}{\partial x}(x)\,.$$

Thus (cf. Karatzas & Shreve [16], e.g.) there exists a $\Big(C(0,T),\Big|_T,\mathbb{Q},(\mathcal{F}_t)-(w_t)\Big)$ Brownian motion such that

$$x(t) \;=\; X_0 \;+\; \int_0^t \widehat{C}(s,x(s))ds \;+\; \sigma w_t \quad , \quad \mathbb{Q}-\text{a.s}\,.$$

As the probability measure $\mathbb{Q}_t$ is the law of $x(t)$ under $\mathbb{Q}$, we deduce that, under $\mathbb{Q}$, $x(t)$ is a weak solution to (3). Conversely, if there exists a solution in the sense of probability law to (3), then $\mathbb{Q} = \mathbb{P} \circ X^{-1}$ is a solution to the martingale problem (6).

2.4 Uniqueness of the solution to the nonlinear martingale problem

Let $\mathbb{Q}$ be a solution to the nonlinear martingale problem (6). The lemma 2.2 characterizes the law of $x(t)$ under $\mathbb{Q}$, so that $\mathbb{Q}_t = p_t(x)dx$. This is not enough to characterize $\mathbb{Q}$, but set

$$\widetilde{C}(t,x) \;:=\; \int_{\mathbb{R}} H(x-y)\,p_t(y)dy\,;$$

note (cf. page 327 in Karatzas & Shreve [16], e.g.) that there exists a unique solution $\widetilde{\mathbb{Q}}$ to the linear martingale problem associated to the operator $\widetilde{\mathcal{L}}$ defined by

$$\widetilde{\mathcal{L}} f(x) \;=\; \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2}(x) \;+\; \widetilde{C}(t,x)\,\frac{\partial f}{\partial x}(x)\,.$$

As $\mathbb{Q}_t = p_t(x)dx$, $\mathbb{Q}$ is a solution to this linear martingale problem: thus $\mathbb{Q} = \widetilde{\mathbb{Q}}$.

2.5 Existence of a solution to the nonlinear martingale problem

We now construct a solution to the martingale problem (6) as the limit of a sequence of probability measures of $\mathcal{M}(C([0,T];\mathbb{R}))$.

Consider the functions $\left(H^k;\ k \in \mathbb{N}^*\right)$ defined by

$$H^k(x) \ = \begin{cases} 0\,, & \text{if } x < \ -\frac{1}{k}\ , \\ kx \ + \ 1\,, & \text{if } x \ \in]-\frac{1}{k},0[\ , \\ 1\,, & \text{if } x \geq \ 0 \ . \end{cases}$$

Then

$$\forall \ x \ \in \ \mathbb{R}\,, \ \lim_{k\to\infty} H^k(x) \ = \ H(x) \ \ ,$$

and, for any k,

$$\left|H^k(x) \ - \ H^k(y)\right| \ \leq \ k|x \ - \ y| \ \ .$$

Substituting H^k to H in (3), we introduce the differential equation

$$\begin{cases} dX_t^k = \sigma dw_t + \displaystyle\int_{\mathbb{R}} H^k(X_t^k - y)U_t^k(dy)\,dt \ \ , \\ \qquad\qquad\qquad\qquad \text{where } U_t^k \text{ is the law of } X_t^k \ \ , \\ \\ X_{t=0}^k = X_0 \quad \text{whose law is } U_0 \ \ . \end{cases}$$

The corresponding interaction kernel $\left(b(x,y) = H^k(x-y)\right)$ is Lipschitz, so that (cf Sznitman [27] e.g.) the above equation has a unique strong solution.

For a fixed measure $\mu \in \mathcal{M}(\mathbb{R})$ and for any $k > 1$, define the operator $\mathcal{L}^k_{(\mu)}$ by

$$\mathcal{L}^k_{(\mu)}f(x) \ = \ \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}(x) \ + \ \left(\int_{\mathbb{R}} H^k(x-y)\mu(dy)\right)\ \frac{\partial f}{\partial x}(x) \ \ .$$

The probability $\mathbb{Q}^k \ := \ \mathbb{P} \circ (X^k)^{-1}$ solves the martingale problem similar to (6), obtained by substituting $\mathcal{L}^k_{(.)}$ to $\mathcal{L}_{(.)}$.

Proposition 2.3 *The family $(\mathbb{Q}^k)$ is tight.*

Proof. As $\mathbb{Q}^k = \mathbb{P}\circ(X^k)^{-1}$, it is enough to check that there exist strictly positive constants C_T, α and β such that

$$\sup_k \mathbb{E} \mid X_t^k \ - \ X_s^k \mid^\alpha \leq \ C_T\,(t-s)^{1+\beta}\,, \ \forall \ 0 \leq s \leq t \leq T \ \ .$$

We choose $\alpha = 4$, $\beta = 1$ and readily conclude. ∎

Now we show that any limit point $\mathbb{Q}^\infty$ of a convergent subsequence (still denoted by $(\mathbb{Q}^k)$) of $(\mathbb{Q}^k)$ solves the martingale problem (6), i.e that, for any f in $C_K^2(\mathbb{R})$, one has

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}^\infty}\left[f(x(t)) - f(x(s)) - \int_s^t \mathcal{L}_{(\mathbb{Q}^\infty_\tau)} f(x(\tau))\, d\tau \ \mid \ x(\theta) \ , \ 0 < \theta \le s \ \right] \\ = 0 \ , \ 0 \le s \le t \le T \ . \end{aligned} \tag{7}$$

Set

$$M_t := f(x(t)) - f(x(0)) - \int_0^t \mathcal{L}_{(\mathbb{Q}^\infty_\tau)} f(x(\tau))\, d\tau \ .$$

Thus (7) is equivalent to

$$\mathbb{E}_{\mathbb{Q}^\infty}[(M_t - M_s)\phi(x(t_1),..,x(t_n))] = 0, \ \forall\, \phi \in C_b(R^n) \text{ and } 0 \le t_1 < .... < t_n < s \ .$$

In fact, we only need to prove that for all $\varepsilon > 0$, for all $\phi \in C_b(R^n)$ and $0 < \varepsilon \le t_1 < .... < t_n < s$,

$$\mathbb{E}_{\mathbb{Q}^\infty}[(M_t - M_s)\phi(x(t_1),..,x(t_n))] = 0 \ , \tag{8}$$

since then

$$\mathbb{E}_{\mathbb{Q}^\infty}[M_t|\mathcal{F}_\varepsilon] = M_\varepsilon \ , \ \forall\, \varepsilon > 0 \ ,$$

so that, as M_t is uniformly bounded on $\Omega \times [0,T]$, $\mathbb{E}_{\mathbb{Q}^\infty}[M_t|\mathcal{F}_0] = 0 = M_0$.

Set

$$M_t^k := f(x(t)) - f(x(0)) - \int_0^t \mathcal{L}_{(\mathbb{Q}^k_\tau)} f(x(\tau))\, d\tau \ .$$

As $\mathbb{Q}^k$ solves the martingale problem associated to $\mathcal{L}^k_{(.)}$, for all function $\phi \in C_b(R^n)$ and all $0 < \varepsilon \le t_1 < .... < t_n < s$, one has that

$$\begin{aligned} 0 &= \mathbb{E}_{\mathbb{Q}^k}\left[(M_t^k - M_s^k)\, \phi(x(t_1),..,x(t_n))\right] \\ &= \mathbb{E}_{\mathbb{Q}^k}\Big[(f(x(t)) - f(x(s)))\, \phi(x(t_1),..,x(t_n)) \\ &\qquad - \phi(x(t_1),..,x(t_n)) \int_s^t \mathcal{L}^k_{(\mathbb{Q}^k_\tau)} f(x(\tau))\, d\tau\Big] \ . \end{aligned} \tag{9}$$

From this equality and the weak convergence of $(\mathbb{Q}^k)$, one easily concludes that (8) is implied by

$$\begin{aligned} \lim_{k\to\infty} \mathbb{E}_{\mathbb{Q}^k}\left[\phi(x(t_1),..,x(t_n)) \int_s^t \int_{\mathbb{R}} f'(x(\tau)) H^k(x-y) U^k_\tau(dy)\, d\tau\right] \\ = \mathbb{E}_{\mathbb{Q}^\infty}\left[\phi(x(t_1),..,x(t_n)) \int_s^t \int_{\mathbb{R}} f'(x(\tau)) H(x-y) \mathbb{Q}^\infty_\tau(dy)\, d\tau\right] \ . \end{aligned} \tag{10}$$

In order to prove this latter equality, we observe:

- the lemma 2.1 shows that, for any $t \in]0,T]$, U_t^k has a density u_t^k in $L^2(\mathbb{R})$ verifying (cf. (4))
$$\|u_t^k\|_2 \leq C \frac{1}{t^{\frac{1}{4}}} \quad ; \tag{11}$$
thus we have
$$\begin{aligned}&\Big| \mathbb{E}_{\mathbb{Q}^k}\left[\phi(x(t_1),..,x(t_n)) \int_s^t f'(x(\tau)) \int_{\mathbb{R}} H^k(x(\tau)-y) U_\tau^k(dy)\, d\tau\right] \\ &\qquad - \mathbb{E}_{\mathbb{Q}^k}\left[\phi(x(t_1),..,x(t_n)) \int_s^t f'(x(\tau)) \int_{\mathbb{R}} H(x(\tau)-y) \mathbb{Q}_\tau^\infty(dy)\, d\tau\right]\Big| \\ &\quad \leq \int_s^t \frac{C\|\phi\|_\infty}{\tau^{1/4}} \sqrt{\int_{\mathbb{R}} f'^2(x) \left[\int_{\mathbb{R}} H^k(x-y) U_\tau^k(dy) - \int_{\mathbb{R}} H(x-y)\mathbb{Q}_\tau^\infty(dy)\right]^2 dx\, d\tau} \quad .\end{aligned}$$
We observe that
$$\begin{aligned} D_1 := &\left[\int_{\mathbb{R}} H^k(x-y) U_\tau^k(dy) \ - \ \int_{\mathbb{R}} H(x-y)\mathbb{Q}_\tau^\infty(dy)\right]^2 \\ &\leq \|\phi\|_\infty \frac{1}{\sqrt{k}} \left(\int_s^t \frac{C}{\sqrt{\tau}}\, d\tau\right) \|f'\|_{L^2(\mathbb{R})} \\ &\quad + C\|\phi\|_\infty \int_s^t \sqrt{\int_{\mathbb{R}} f'^2(x) \left[\int_{-\infty}^x U_\tau^k(dy) - \int_{-\infty}^x \mathbb{Q}_\tau^\infty(dy)\right]^2 dx} \ \frac{d\tau}{\tau^{1/4}} \quad ;\end{aligned}$$
from (11), we deduce that for all function $g \in C_K(\mathbb{R})$,
$$< \mathbb{Q}_t^\infty \ , \ g > \leq \frac{C}{t^{\frac{1}{4}}} \|g\|_2 \quad ;$$
therefore, for all $t > 0, \mathbb{Q}_t^\infty$ has a density q_t^∞ w.r.t. the Lebesgue measure belonging to $L^2(\mathbb{R})$; this implies that the distribution function $V_t^\infty(\cdot)$ of $\mathbb{Q}_t^\infty$ is continuous, so that $V_t^k(\cdot)$ converges to $V_t^\infty(\cdot)$ everywhere, and thus D_1 tends to 0 when k tends to infinity.

- Now we consider
$$\begin{aligned} D_2 \ := \ &\mathbb{E}_{\mathbb{Q}^k}\left[\phi(x(t_1),..,x(t_n)) \int_s^t f'(x(\tau)) \int_{\mathbb{R}} H(x(\tau)-y)\mathbb{Q}_\tau^\infty(dy)\, d\tau\right] \\ &- \mathbb{E}_{\mathbb{Q}^\infty}\left[\phi(x(t_1),..,x(t_n)) \int_s^t f'(x(\tau)) \int_{\mathbb{R}} H(x(\tau)-y)\mathbb{Q}_\tau^\infty(dy)\, d\tau\right] \quad .\end{aligned}$$
We again need to introduce a smoothing of the kernel:
$$D_2 \ = \ \mathbb{E}_{\mathbb{Q}^k}\left[\phi(x(t_1),..,x(t_n)) \int_s^t f'(x(\tau)) \int_{\mathbb{R}} H(x(\tau)-y)\mathbb{Q}_\tau^\infty(dy)\, d\tau\right]$$

$$
\begin{aligned}
&- \mathbb{E}_{\mathbb{Q}^k}\left[\phi(x(t_1),..,x(t_n))\int_s^t f'(x(\tau))\int_{\mathbb{R}} H^{k_0}(x(\tau)-y)\mathbb{Q}_\tau^\infty(dy)\,d\tau\right]\\
&+ \mathbb{E}_{\mathbb{Q}^k}\left[\phi(x(t_1),..,x(t_n))\int_s^t f'(x(\tau))\int_{\mathbb{R}} H^{k_0}(x(\tau)-y)\mathbb{Q}_\tau^\infty(dy)\,d\tau\right]\\
&- \mathbb{E}_{\mathbb{Q}^\infty}\left[\phi(x(t_1),..,x(t_n))\int_s^t f'(x(\tau))\int_{\mathbb{R}} H^{k_0}(x(\tau)-y)\mathbb{Q}_\tau^\infty(dy)\,d\tau\right]\\
&+ \mathbb{E}_{\mathbb{Q}^\infty}\left[\phi(x(t_1),..,x(t_n))\int_s^t f'(x(\tau))\int_{\mathbb{R}} H^{k_0}(x(\tau)-y)\mathbb{Q}_\tau^\infty(dy)\,d\tau\right]\\
&- \mathbb{E}_{\mathbb{Q}^\infty}\left[\phi(x(t_1),..,x(t_n))\int_s^t f'(x(\tau))\int_{\mathbb{R}} H(x(\tau)-y)\mathbb{Q}_\tau^\infty(dy)\,d\tau\right] \quad .
\end{aligned}
\tag{12}
$$

From (11), we readily obtain that

$$\|q_t^\infty\|_{L^2(\mathbb{R})} \leq \frac{C}{t^{\frac{1}{4}}} \quad ;$$

thus

$$\left[\int_{\mathbb{R}}\Big(H(x-y)-H^{k_0}(x-y)\Big)\mathbb{Q}_\tau^\infty(dy)\right]^2 \leq \frac{C}{k_0\sqrt{t}} \quad ,$$

so that, as f having a compact support, one can choose k_0 uniformly in k to make arbitrarily small the first and the last differences of (12); such a k_0 being fixed, the second difference tends to 0 when k goes to infinity as a consequence of the weak convergence of $(\mathbb{Q}^k)$, since the smoothness of H^{k_0} implies that the functional

$$C([0,T];\mathbb{R}) \ni x(\cdot) \longrightarrow \phi(x(t_1),..,x(t_n))\int_s^t f'(x(\tau))\int_{\mathbb{R}} H^{k_0}(x(\tau)-y)\mathbb{Q}_\tau^\infty(dy)d\tau \in \mathbb{R}$$

is continuous.

Consequently we have proven that $\mathbb{Q}^\infty$ solves the nonlinear martingale problem (6). ∎

3 The classical solution of the Burgers equation

Let (X_t) be a weak solution of the above nonlinear stochastic differential equation.

We have shown that the distribution function V_t of the law U_t of X_t is a weak solution of the Burgers equation. We now show that V_t is the "classical" solution, which can be explicited by the Cole-Hopf transformation (cf. Cole [7], Hopf [15]): if V is a solution of class $C^{1,2}((0,T] \times \mathbb{R})$, then W defined by

$$W(t,x) := \exp\left(-\frac{1}{\sigma^2}\int_{-\infty}^{x} V(t,y)dy\right)$$

for all $0 \leq t \leq T$, satifies the heat equation

$$\begin{cases} \dfrac{\partial W}{\partial t}(t,x) = \dfrac{1}{2}\sigma^2 \dfrac{\partial^2 W}{\partial x^2}(x,t) \quad , \quad (t,x) \in]0,T] \times \mathbb{R} \quad , \\ W(0,x) = W_0(x) \quad , \end{cases}$$

so that one has that

$$V(t,x) = \frac{\displaystyle\int_{\mathbb{R}} \frac{(x-y)}{t} \exp\left(-\frac{1}{\sigma^2}\left[\frac{(x-y)^2}{2t} + \int_{-\infty}^{y} V_0(z)dz\right]\right) dy}{\displaystyle\int_{\mathbb{R}} \exp\left(-\frac{1}{\sigma^2}\left[\frac{(x-y)^2}{2t} + \int_{-\infty}^{y} V_0(z)dz\right]\right) dy} \quad . \tag{13}$$

We suppose:

(H0) the initial law U_0 verifies:

(i) either U_0 is probability measure with a compact support,

(ii) or U_0 has a continuous and strictly positive density u_0, and there exist strictly positive constants M, η et α such that

$$\forall |x| > M \quad , \quad u_0(x) \leq \eta \exp\left(-\alpha \frac{x^2}{2}\right) \quad .$$

Proposition 3.1 *Under (H0), the distribution function $V(t,x)$ of the law of X_t is the classical solution of the Burgers equation obtained by the Cole-Hopf transformation.*

The proof is an adaption of the proof given in [26] for the case where the initial condition of the Burgers equation is a density. For the sake of completeness, we give it in the Appendix.

From this explicit representation we deduce an estimate concerning the first spatial derivative of V.

Lemma 3.2 *If U_0 satisfies H0-(ii), then*

$$\left\| \frac{\partial V}{\partial x}(t,x) \right\|_{L^\infty([0,T]\times\mathbb{R})} \leq L_0 ,$$

where L_0 depends on σ, u_0 and T.

If U_0 is the Dirac measure in zero, then for any $t \in]0,T]$ one has

$$\left\| \frac{\partial V}{\partial x}(t,.) \right\|_{L^\infty(\mathbb{R})} \leq \frac{L_0}{\sqrt{t}} \quad ,$$

where L_0 depends on σ and T.

Proof. The proof requires easy computations from the equality

$$V(t,x) = \frac{\displaystyle\int_{\mathbb{R}} V_0(x-y) \exp\left(-\frac{1}{\sigma^2} \left[\frac{y^2}{2t} + \int_{-\infty}^{x-y} V_0(z)dz \right] \right) dy}{\displaystyle\int_{\mathbb{R}} \exp\left(-\frac{1}{\sigma^2} \left[\frac{y^2}{2t} + \int_{-\infty}^{x-y} V_0(z)dz \right] \right) dy} \quad .$$

The details can be found in [3]. ∎

4 Algorithm and convergence rate

In all the sequel, we suppose:

(H1) the initial law U_0 satisfies:

- **(i)** either U_0 is the Dirac measure at 0,
- **(ii)** or U_0 has a smooth density u_0, satisfying one of the two conditions:
  - $u_0(\cdot)$ is a continuous, strictly positive and bounded function, and there exists strictly positive constants M, η et α such that

$$\forall |x| > M \quad , \quad u_0(x) \leq \eta \exp\left(-\alpha \frac{x^2}{2}\right) \quad ,$$

  - u_0 is a function with a compact support and is continuous on this support.

The existence in the sense of probability law of a solution to (3) implies the existence in the sense of probability law of a solution to

$$\begin{cases} dz_t = V(t, z_t)\, dt + \sigma\, dw_t \quad , \\ \\ z_{t=0} = z_0 \quad . \end{cases} \tag{14}$$

Under (H1), the lemma 3.2 shows that $V(t,\cdot)$ is a Lipschitz function in x, with a Lipschitz constant upper bounded by $\dfrac{L_0}{\sqrt{t}}$ for all $t \in]0,T]$, wich implies the pathwise uniqueness of the solution to (14): indeed, if (z_t^1) and (z_t^2) are two solutions, then

$$|z_t^1 - z_t^2| \leq \int_0^t \frac{L_0}{\sqrt{s}} |z_s^1 - z_s^2| ds \quad ,$$

so that $z_t^1 = z_t^2$ by Gronwall's lemma.

The Markov process (z_t) with the initial distribution U_0 coincides with (X_t) and

$$V(t,x) = \mathbb{E}_{U_0} H(x - z_t) \quad .$$

From that representation, we obtain our algorithm formula by successive approximations.

4.1 Approximation of the initial condition

Choose N points in $\mathbb{R}$, $(y_0^1, .., y_0^N)$, such that the piecewise constant function

$$\overline{V}_0(x) = \frac{1}{N}\sum_{i=1}^{N} H(x - y_0^i)$$

approximates V_0 and denote by $\overline{U}_0 = \frac{1}{N}\sum_{i=1}^{N}\delta_{y_0^i}$ the corresponding empirical measure: if U_0 is the Dirac measure in 0, set $y_0^i = 0$, $\overline{U}_0 = U_0$ and $\overline{V}_0 = V_0$; in the other situations, the locations are determined by inverting the initial distribution function:

$$y_0^i \;=\; \begin{cases} V_0^{-1}(\frac{i}{N}) \;\;,\;\; i = 1, .., N-1 \;\;, \\ \\ V_0^{-1}(1 - \frac{1}{2N}) \;\;,\;\; i = N \;\;. \end{cases}$$

A first approximation of $V(t, .)$ is

$$V(t,x) \;\simeq\; \mathbb{E}_{\overline{U}_0} H(x - z_t) \;=\; \frac{1}{N}\sum_{i=1}^{N} \mathbb{E} H(x - z_t(y_0^i)) \;\;.$$

4.2 Approximation of the expectation

Consider N independent copies $(w_t^i)_{i=1}^N$ of the Brownian motion (w_t), and the family of independent processes $(z_t^i)_{(i=1,..,N)}$ defined by

$$\begin{cases} dz_t^i \;=\; V(t, z_t^i)\, dt \;+\; \sigma\, dw_t^i \;\;, \\ \\ z_0^i \;=\; y_0^i \;\;. \end{cases} \tag{15}$$

We now approximate $V(t, .)$ by applying the strong law of large numbers:

$$V(t,x) \;\simeq\; \frac{1}{N}\sum_{i=1}^{N} H(x - z_t^i)\,.$$

4.3 Time discretization

T being fixed, define $\Delta t > 0$ and $K \in \mathbb{N}$ such that $T = K\Delta t$. The discretization times are denoted by $t_k = k\Delta t \;\; , 1 \leq k \leq K$. Applying the Euler scheme to the stochastic differential equations (15), one defines independent discrete time processes $(\overline{z}^i_{t_k})$:

$$\begin{cases} \overline{z}^i_{t_{k+1}} = \overline{z}^i_{t_k} + V(t_k, \overline{z}^i_{t_k})\Delta t + \sigma \left(w^i_{t_{k+1}} - w^i_{t_k}\right) \quad , \\ \overline{z}^i_0 = y^i_0 \quad . \end{cases} \tag{16}$$

Thus, at time t_k $(k = 1, .., K)$, $V(t_k, .)$ is approximated by

$$V(t_k, x) \simeq \frac{1}{N} \sum_{i=1}^{N} H(x - \overline{z}^i_{t_k}) \quad .$$

4.4 Approximation of the interaction kernel

The dynamics of the $\overline{z}^i$'s depend on the function V, which is our unknown; thus we are led to approximate $V(t_k, \cdot)$ by the empirical distribution function of the particles, that we denote by $\overline{V}_{t_k}(\cdot)$.

Let $Y^i_{t_k}$ be the position of the i^{th} particle at time t_k, and let $\overline{U}_{t_k}$ be the corresponding empirical measure. Then one has that

$$\overline{V}_{t_k}(x) = \int_{\mathbb{R}} H(x-y)\overline{U}_{t_k}(dy) = \frac{1}{N}\sum_{i=1}^{N} H(x - Y^i_{t_k}) \quad .$$

Substituting V in (16) for this approximation, one gets the algorithm of the particles motion: the motion is described by the trajectories of discrete time and dependent processes, $(Y^i_{t_k})^N_{i=1}$, defined by

$$\begin{cases} Y^i_{t_{k+1}} & = Y^i_{t_k} + \overline{V}_{t_k}(Y^i_{t_k})\,\Delta t + \sigma\,\Delta w^i_{k+1} \quad , i = 1, \dots, N \quad , \\ & = Y^i_{t_k} + \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} H(Y^i_{t_k} - Y^j_{t_k})\,\Delta t + \sigma\,\Delta w^i_{k+1} \quad , \\ Y^i_0 & = y^i_0 \quad , \end{cases}$$

where $\Delta w^i_{k+1} = w^i_{t_{k+1}} - w^i_{t_k}$.

4.5 Convergence rate

We now state our estimate on the convergence rate of the empirical distribution function:

Theorem 4.1 *T being fixed, let $\Delta t > 0$ be such that $T = \Delta t K$, $K \in \mathbb{N}$.*

Let $V(t_k, x)$ be the solution at time $t_k = k\Delta t$ of the Burgers equation (1) with the initial condition V_0.

Let $\overline{V}_{t_k}(x)$ be defined as above, N being the number of particles.

Under (H1), there exist strictly positive constants L_1, L_2 and L_3, depending on σ, U_0 and T, such that, for all $k \in \{1, .., K\}$:

$$\mathbb{E}\, \|V(t_k, .) - \overline{V}_{t_k}(.)\|_{L^1(\mathbb{R})} \leq L_1 \|V_0 - \overline{V}_0\|_{L^1(\mathbb{R})} + L_2 \frac{1}{\sqrt{N}} + L_3 \sqrt{\Delta t} \ .$$

Remark. If the initial law U_0 is a Dirac measure, then $\|V_0 - \overline{V}_0\|_{L^1(\mathbb{R})} = 0$. In the other case, one can prove (see [4]) that $\|V_0(\cdot) - \overline{V}_0(\cdot)\|_{L^1(\mathbb{R})}$ converges with the order $\mathcal{O}\left(\frac{1}{N}\sqrt{\log(N)}\right)$. Thus the convergence rate of our algorithm is of order $\mathcal{O}\left(\frac{1}{\sqrt{N}} + \frac{1}{\sqrt{\Delta t}}\right)$.

4.6 Propagation of chaos

Consider N particles which at time 0 are independent with law U_0 and follow the dynamics

$$dX_t^{i,N} = \frac{1}{N} \sum_{j=1}^{N} H(X_t^{i,N} - X_t^{j,N})dt + \sigma dW_t^i \ .$$

In this section, we readily prove the propagation of chaos for this system of particles. The propagation of chaos property explains the convergence of the algorithm: when N goes to infinity, the particles system (with no time discretization) tends to behave like a system of independent particles, each one having the law $\mathbb{Q}$ defined in the section 2.4.

Theorem 4.2 *Let $\mathbb{P}^N$ be the joint law on $(C(0,T;\mathbb{R})^N)$ of the particles system $(X_\cdot^{1,N}, \ldots, X_\cdot^{N,N})$. For any $k \in \mathbb{N}^*$, for any continuous and bounded functions $f_1, \ldots, f_k : C(0,T;\mathbb{R}) \to \mathbb{R}$, one has that*

$$\lim_{N \to +\infty} < \mathbb{P}^N, f_1 \otimes \ldots \otimes f_k \otimes 1 \ldots \otimes 1 >= \prod_{i=1}^{k} < \mathbb{Q}, f_i > \ ,$$

where $\mathbb{Q}$ is the solution of the nonlinear martingale problem of the preceding subsection (the sequence $(\mathbb{P}^N)$ is said "$\mathbb{Q}$-chaotic").

Proof. To our knowledge, our context does not satisfy the hypotheses of the systems studied in the literature but, as we will see, the proof is straightforward; we simply adapt now classical arguments appearing in Meléard & Roelly [18] or Sznitman [27] e.g.

The $\mathbb{Q}$-chaoticity is equivalent to the convergence of the laws of the empirical measures $\mu^N := \frac{1}{N}\sum_{i=1}^{N}\delta_{X_\cdot^{i,N}}$ to $\delta_{\mathbb{Q}}$ (cf. Aldous [1] or Tanaka [28]).

First, we note that the sequence of the laws of the μ^N's is tight; indeed, a sufficient criterion due to Sznitman [25] is the tightness of the sequence of the intensity measures I^N defined by $< I^N, f >= \mathbb{E} < \mu^N, f >$, which by symmetry reduces here to the tightness of the laws $\mathbb{P}_{X_\cdot^{1,N}}$: this latter fact is implied by

$$\mathbb{E}\sup_{t\leq T}|X_t^{1,N}|^2 \leq CT \quad .$$

Second, any limit point of a convergent subsequence of $\mathcal{L}aw(\mu^N)$ is equal to $\delta_{\mathbb{Q}}$: indeed, let Π be a limit point and set

$$F(m) \ := < m, \left(f(x(t)) - f(x(s)) - \int_s^t \mathcal{L}_{(m)}f(x(\theta))d\theta\right) \ g(x(s_1),\ldots,x(s_k)) > \quad ,$$

where $f \in C_b^2(\mathbb{R})$, $g \in C_b(\mathbb{R}^k)$, $0 < s_1 < \ldots < s_k \leq T$ and m is a probability on $C(0,T;\mathbb{R})$; we note:

$$\begin{aligned}
\int F(m)^2\Pi(dm) &= \lim_{N\to+\infty}\mathbb{E}[F(\mu^N)]^2 \\
&\leq \lim_{N\to+\infty}\frac{C}{N^2}\mathbb{E}\left(\sum_{i=1}^{N}\left\{f(X_t^{i,N}) - f(X_s^{i,N}) - \int_s^t \mathcal{L}_{(\mu^N)}f(X_\theta^{i,N})d\theta\right\}\right)^2 \\
&= \lim_{N\to+\infty}\frac{C}{N^2}\sum_{i=1}^{N}\mathbb{E}\left(\int_s^t \sigma dW_\theta^i\right)^2 \\
&= 0 \ ,
\end{aligned}$$

and we conclude by using the uniqueness of the solution to the nonlinear martingale problem.

5 Proof of the theorem 4.1

5.1 Notation

In all the sequel, C will denote any strictly positive real number independent of N and Δt; typically it will depend on σ, T and U_0.

We also will denote by $\mathbb{E}_\mu f(z_t)$ the expectation of $f(z_t)$ when (z_t) has the initial distribution μ.

5.2 Preliminaries

As in the case of smooth kernels (cf. [4]), we decompose the error at time t_k, $\left(V(t_k,.) - \overline{V}_{t_k}(.)\right)$, in three terms:

$$
\begin{aligned}
\mathbb{E}\left\|V(t_k,x) \;-\; \overline{V}_{t_k}(x)\right\|_{L^1(\mathbb{R})} \;\leq\;\; & \left\|\mathbb{E}_{U_0} H(x - z_{t_k}) \;-\; \mathbb{E}_{\overline{U}_0} H(x - z_{t_k})\right\|_{L^1(\mathbb{R})} \\
& + \mathbb{E}\left\|\mathbb{E}_{\overline{U}_0} H(x - z_{t_k}) \;-\; \frac{1}{N}\sum_{i=1}^{N} H(x - z_{t_k}^i)\right\|_{L^1(\mathbb{R})} \\
& + \mathbb{E}\left\|\frac{1}{N}\sum_{i=1}^{N} H(x - z_{t_k}^i) \;-\; \frac{1}{N}\sum_{i=1}^{N} H(x - Y_{t_k}^i)\right\|_{L^1(\mathbb{R})} .
\end{aligned}
\tag{17}
$$

In the right handside, the first term corresponds to the approximation of the initial condition V_0 by the piecewise constant function $\overline{V}_0$; the second term corresponds to the introduction of the independent processes (z_t^i) and is a statistical error; estimates on these two terms can be obtained by using arguments similar to those used in the proofs of the lemmae 3.1 and 3.2 of Bossy & Talay [4] for the case of smooth interaction kernels, combining them with the following remark: if U_0 is the Dirac measure at 0, the first term is zero, whereas if U_0 satisfies (H1-(ii)), the lemma 3.2 shows that V is uniformly Lipschitz in (t,x), so that one has the following estimates (cf. Friedman [9], p.139-150, or the chapter 1 of [8]) concerning the transition probability $\gamma(t,x,y)$ of the process $(z_t(x))$: for any T, there exist strictly positive constants C_0 and C_1 such that, $\forall t \in [0,T]$, $\forall x,y$, $\forall \bar{\sigma} > \sigma$,

$$\begin{aligned} \mid \gamma_t(x,y) \mid &\leq \frac{C_0}{\sqrt{t}}\, exp\left(-\frac{(x-y)^2}{2\bar{\sigma}^2 t}\right) , \\ \mid \frac{\partial}{\partial y}\gamma_t(x,y) \mid &\leq \frac{C_1}{t}\, exp\left(-\frac{(x-y)^2}{2\bar{\sigma}^2 t}\right) . \end{aligned} \tag{18}$$

Thus, it holds that

$$\left\| \mathbb{E}_{U_0} H(x - z_{t_k}) - \mathbb{E}_{\overline{U}_0} H(x - z_{t_k}) \right\|_{L^1(\mathbb{R})} \leq C \left\| V_0 - \overline{V}_0 \right\|_{L^1(\mathbb{R})} ,$$

and

$$\mathbb{E} \left\| \mathbb{E}_{\overline{U}_0} H(x - z_{t_k}) - \frac{1}{N}\sum_{i=1}^N H(x - z^i_{t_k}) \right\|_{L^1(\mathbb{R})} \leq \frac{C}{\sqrt{N}} .$$

Thus, it remains to treat the third term of the right handside of (17), namely

$$\mathbb{E} \left\| \frac{1}{N}\sum_{i=1}^N H(x - z^i_{t_k}) \ - \ \frac{1}{N}\sum_{i=1}^N H(x - Y^i_{t_k}) \right\|_{L^1(\mathbb{R})} . \tag{19}$$

When the interaction kernel is smooth (cf. [4], one can separately treat

$$\mathbb{E} \left\| \frac{1}{N}\sum_{i=1}^N H(x - z^i_{t_k}) \ - \ \frac{1}{N}\sum_{i=1}^N H(x - \overline{z}^i_{t_k}) \right\|_{L^1(\mathbb{R})}$$

and

$$\mathbb{E} \left\| \frac{1}{N}\sum_{i=1}^N H(x - \overline{z}^i_{t_k}) \ - \ \frac{1}{N}\sum_{i=1}^N H(x - Y^i_{t_k}) \right\|_{L^1(\mathbb{R})} .$$

Here, as the kernel is equal to the Heaviside function, this method does not work and a more complex analysis must be developed. The rest of this section is devoted to the proof of the

Lemma 5.1 *There exists a constant $L > 0$, only depending on V_0, σ and T such that*

$$\forall\, k = 1,..,K, \quad \mathbb{E} \left\| \frac{1}{N}\sum_{i=1}^N H(x - z^i_{t_k}) - \frac{1}{N}\sum_{i=1}^N H(x - Y^i_{t_k}) \right\|_{L^1(\mathbb{R})} \leq L\left(\sqrt{\Delta t} + \frac{1}{\sqrt{N}}\right) .$$

In the proof of this estimate, we use that for any $t \in]0,T]$, $V(t,\cdot)$ is Lipschitz in x, with a Lipschitz constant upper bounded by $\frac{L_0}{\sqrt{t}}$, which is true under (H1): cf. lemma 3.2. In the case where U_0 is smooth, some steps of the proof can be simplified, but the convergence rate is not improved.

Besides, we use estimates similar to (18); but Friedman's hypotheses to get (18) are not satisfied when the initial distribution of (z_t) is a Dirac measure (the drift coefficient is V, which is not smooth enough); nevertheless, we can prove:

Lemma 5.2 *Under (H1), if $\gamma_t(x,y)$ denotes the density of the law of $z_t(x)$ $(t \in]0,T])$, then there exists a constant C_0 only depending on T and σ such that*

$$\gamma_t(x,y) \;\leq\; \frac{C_0}{\sqrt{2\pi t\sigma^2}} \exp\left(-\frac{(y-x)^2}{4t\sigma^2}\right) \quad .$$

The proof of this lemma, postponed to the Appendix, uses a representation formula for $\gamma_t(x,y)$ given in I.I. Gihman and A.V. Skorohod [10].

5.3 Proof of Lemma 5.1

Remarking that

$$\forall\, a,\; b \in \mathbb{R}, \qquad \int_{\mathbb{R}} \mid H(x-a) - H(x-b) \mid dx \;= \mid a-b \mid \;\; , \tag{20}$$

one gets

$$\mathbb{E} \left\| \frac{1}{N}\sum_{i=1}^{N} H(x - z^i_{t_k}) \;-\; \frac{1}{N}\sum_{i=1}^{N} H(x - Y^i_{t_k}) \right\|_{L^1(\mathbb{R})} \leq \frac{1}{N}\sum_{i=1}^{N} \mathbb{E} \mid z^i_{t_k} \;-\; Y^i_{t_k} \mid \quad .$$

Our objective is to upper bound $\left(\frac{1}{N}\sum_{i=1}^{N} \mathbb{E} \mid z^i_{t_k} \;-\; Y^i_{t_k} \mid\right)_{k=0,..,K}$.

We mention that the strong approximation of a family of nonlinear diffusion processes has been studied by Ogawa ([19], [20]), but in a different context than ours: Ogawa supposes that one can construct *independent* estimators of the density of the law U_t on another probability space, which permits to construct *independent* individual trajectories of the process (X_t).

Note that

$$
\begin{aligned}
\mathbb{E} \mid z^i_{t_k} - Y^i_{t_k} \mid \;&\leq\; \mathbb{E}|z^i_{t_{k-1}} - Y^i_{t_{k-1}}| \;+\; \mathbb{E}\left|\int_{t_{k-1}}^{t_k} V(s,z^i_s)ds \;-\; \Delta t \overline{V}_{t_{k-1}}(Y^i_{t_{k-1}})\right| \\
&\leq\; \mathbb{E}\left|z^i_{t_{k-1}} - Y^i_{t_{k-1}}\right| \;+\; \mathbb{E}\int_{t_{k-1}}^{t_k} \left|V(s,z^i_s) \;-\; V(t_{k-1},z^i_{t_{k-1}})\right| ds \qquad (21) \\
&\qquad + \Delta t\; \mathbb{E}\left|V(t_{k-1},z^i_{t_{k-1}}) \;-\; \overline{V}_{t_{k-1}}(Y^i_{t_{k-1}})\right| \;.
\end{aligned}
$$

For all $t > 0$, $V(t,\cdot)$ is Lipschitz with a Lipschitz constant upper bounded by $\frac{L_0}{\sqrt{t}}$; therefore,

$$
\begin{aligned}
&\mathbb{E}\int_{t_{k-1}}^{t_k} \left|V(s,z^i_s) - V(t_{k-1},z^i_{t_{k-1}})\right| ds \\
&\quad\leq \mathbb{E}\int_{t_{k-1}}^{t_k} \left|V(s,z^i_s) - V(t_{k-1},z^i_s)\right| ds \;+\; \mathbb{E}\int_{t_{k-1}}^{t_k} \left|V(t_{k-1},z^i_s) - V(t_{k-1},z^i_{t_{k-1}})\right| ds \\
&\quad\leq \mathbb{E}\int_{t_{k-1}}^{t_k} \left|V(s,z^i_s) - V(t_{k-1},z^i_s)\right| ds \;+\; \int_{t_{k-1}}^{t_k} \frac{L_0}{\sqrt{t_{k-1}}}\mathbb{E}|z^i_s \;-\; z^i_{t_{k-1}}|ds \;. \qquad (22)
\end{aligned}
$$

When u_0 is smooth, one can upper bound $\mathbb{E}\int_{t_{k-1}}^{t_k} \left|V(s,z^i_s) - V(t_{k-1},z^i_s)\right| ds$ using the Holder property of $t \to V(t,x)$; in any case, under (H1-i) or (H1-ii), one can apply the lemma 5.2 and get

$$
\begin{aligned}
\mathbb{E}\int_{t_{k-1}}^{t_k} \left|V(s,z^i_s) - V(t_{k-1},z^i_s)\right| ds \;&\leq\; \int_{t_{k-1}}^{t_k}\int_{\mathbb{R}} |V(s,x) - V(t_{k-1},x)|\, \frac{C_0}{\sqrt{2\pi\sigma^2 s}} dx\, ds \\
&\leq\; \int_{t_{k-1}}^{t_k} \mathbb{E}_{U_0}\int_{\mathbb{R}} \left|H(x - z_s) - H(x - z_{t_{k-1}})\right| \frac{C_0}{\sqrt{2\pi\sigma^2 s}} dx\, ds\;,
\end{aligned}
$$

from which, by (20), one gets that

$$
\mathbb{E}\int_{t_{k-1}}^{t_k} \left|V(s,z^i_s) - V(t_{k-1},z^i_s)\right| ds \;\leq\; \frac{C_0}{\sqrt{2\pi\sigma^2 t_{k-1}}}\int_{t_{k-1}}^{t_k} \mathbb{E}_{U_0}\left|z_s - z_{t_{k-1}}\right| ds \;.
$$

As

$$
\begin{aligned}
\mathbb{E} \mid z^i_s \;-\; z^i_{t_{k-1}} \mid \;&\leq\; \Delta t \;+\; \sigma \mathbb{E} \mid w^i_s \;-\; w^i_{t_{k-1}} \mid \;, \\
\text{and} \quad \mathbb{E}_{U_0} \mid z_s \;-\; z_{t_{k-1}} \mid \;&\leq\; \Delta t \;+\; \sigma \mathbb{E} \mid w_s \;-\; w_{t_{k-1}} \mid \;,
\end{aligned}
$$

the inequality (22) becomes:

$$
\begin{aligned}
\mathbb{E}\int_{t_{k-1}}^{t_k} \left|V(s,z^i_s) \;-\; V(t_{k-1},z^i_{t_{k-1}})\right| ds \;&\leq\; \Delta t\left(\frac{C_0}{\sqrt{2\pi\sigma^2 t_{k-1}}} \;+\; \frac{L_0}{\sqrt{t_{k-1}}}\right)(\Delta t + \sigma\sqrt{\Delta t}) \\
&\leq\; \frac{C}{\sqrt{t_{k-1}}}\,\Delta t^{\frac{3}{2}} \;,
\end{aligned}
$$

where C is a constant depending only on T, σ and V_0. Coming back to (21) and using again that $V(t_{k-1},\cdot)$ is Lipschitz, one gets:

$$\begin{aligned}\mathbb{E}\left|z^i_{t_k}-Y^i_{t_k}\right| \;&\leq\; \left(1+\frac{L_0}{\sqrt{t_{k-1}}}\Delta t\right)\mathbb{E}\left|z^i_{t_{k-1}}-Y^i_{t_{k-1}}\right| + \frac{C}{\sqrt{t_{k-1}}}\,\Delta t^{\frac{3}{2}}\\ &\qquad + \Delta t\mathbb{E}\left|V(t_{k-1},Y^i_{t_{k-1}})-\overline{V}_{t_{k-1}}(Y^i_{t_{k-1}})\right| \quad ,\end{aligned}$$

from which it comes:

$$\begin{aligned}\mathbb{E}\left|z^i_{t_k}-Y^i_{t_k}\right| \;&\leq\; \prod_{l=1}^{k-2}\left(1+\frac{L_0}{\sqrt{t_{k-l}}}\Delta t\right)\mathbb{E}\left|z^i_{\Delta t}-Y^i_{\Delta t}\right| + \frac{C}{\sqrt{t_{k-1}}}\,\Delta t^{\frac{3}{2}}\\ &\quad + \Delta t\mathbb{E}\left|V(t_{k-1},Y^i_{t_{k-1}})-\overline{V}_{t_{k-1}}(Y^i_{t_{k-1}})\right|\\ &\quad + \sum_{l=2}^{k-2}\left(\frac{C}{\sqrt{t_{k-l}}}\,\Delta t^{\frac{3}{2}} \;+\; \Delta t\mathbb{E}\left|V(t_{k-l},Y^i_{t_{k-l}})-\overline{V}_{t_{k-l}}(Y^i_{t_{k-l}})\right|\right)\\ &\qquad\qquad \prod_{j=1}^{l-1}\left(1+\frac{L_0}{\sqrt{t_{k-j}}}\Delta t\right) \quad .\end{aligned}$$

For all $l\in\{2,..,k-2\}$,

$$\prod_{j=1}^{l-1}(1+\frac{L_0}{\sqrt{t_{k-j}}}\Delta t)\leq \exp\left(\sum_{j=k-l+1}^{k-1}\frac{L_0\Delta t}{\sqrt{j\Delta t}}\right)\leq \exp\left(\int_{t_{k-l+1}}^{t_k}\frac{L_0}{\sqrt{s}}ds\right)\;\leq\;\exp\left(2L_0\sqrt{T}\right) \quad .$$

Therefore,

$$\begin{aligned}\mathbb{E}\left|z^i_{t_k}-Y^i_{t_k}\right| \;&\leq\; \exp\left(2L_0\sqrt{T}\right)\\ &\left(\mathbb{E}\left|z^i_{\Delta t}-Y^i_{\Delta t}\right| \;+\sum_{l=1}^{k-1}\Delta t\mathbb{E}\left|V(t_l,Y^i_{t_l})-\overline{V}_{t_l}(Y^i_{t_l})\right| \;+\sum_{l=1}^{k-1}\frac{C}{\sqrt{t_{k-l}}}\Delta t^{\frac{3}{2}}\right).\end{aligned}$$

As $z^i_0=Y^i_0$, one has that $\mathbb{E}|z^i_{\Delta t}-Y^i_{\Delta t}|\leq\Delta t$, so that

$$\mathbb{E}\left|z^i_{t_k}-Y^i_{t_k}\right| \;\leq\; \exp(L_0T)\left(\Delta t \;+\;\sum_{l=1}^{k-1}\Delta t\mathbb{E}\left|V(t_l,Y^i_{t_l})-\overline{V}_{t_l}(Y^i_{t_l})\right| \;+\; 2C\,\sqrt{t_k}\,\sqrt{\Delta t}\right) \quad .$$

For $k=0,..,K$, set

$$E_k \;:=\; \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left|V(t_k,Y^i_{t_k})-\overline{V}_{t_k}(Y^i_{t_k})\right| \quad . \tag{23}$$

Then

$$\left\{\begin{array}{l} \displaystyle \frac{1}{N}\sum_{i=1}^{N} \mathbb{E}\left|z^i_{t_k} - Y^i_{t_k}\right| \;\leq\; C\left(\sum_{l=1}^{k-1}\Delta t\, E_l \;+\; \sqrt{t_k}\,\sqrt{\Delta t}\right) \quad , \quad k = 2,..,K\,, \\ \\ \displaystyle \frac{1}{N}\sum_{i=1}^{N} \mathbb{E}\left|z^i_{\Delta t} - Y^i_{\Delta t}\right| \;\leq\; \Delta t \quad , \end{array}\right. \tag{24}$$

where C is a constant depending only on T, σ and V_0.

Below (lemma 5.3) we will prove that there exists a constant C depending only on V_0, T and σ such that, for any $k = 0,..,K$,

$$E_k \;\leq\; C\left(\sqrt{\Delta t} \;+\; \frac{1}{\sqrt{N}}\right) \quad . \tag{25}$$

Assuming this result, (24) becomes

$$\frac{1}{N}\sum_{i=1}^{N} \mathbb{E}\left|z^i_{t_k} - Y^i_{t_k}\right| \;\leq\; C\left(\sqrt{\Delta t} \;+\; \frac{1}{\sqrt{N}}\right) \quad . \tag{26}$$

Thus the lemma 5.1 is proved. ∎

Lemma 5.3 *There exists a constant C depending only on V_0, T and σ such that, for any $k = 0,..,K$,*

$$E_k \;=\; \frac{1}{N}\sum_{i=1}^{N} \mathbb{E}\left|V(t_k, Y^i_{t_k}) - \overline{V}_{t_k}(Y^i_{t_k})\right| \;\leq\; C\left(\sqrt{\Delta t} \;+\; \frac{1}{\sqrt{N}}\right) \quad . \tag{27}$$

Proof. First note that, if U_0 is a Dirac measure, then $V_0 = \overline{V}_0$ and thus $E_0 = 0$; if not, by definition of the (y^i_0)'s, one has

$$\begin{aligned} E_0 \;&=\; \frac{1}{N}\sum_{i=1}^{N}\left|V(0, z^i_0) - \overline{V}_0(Y^i_0)\right| \\ &=\; \frac{1}{N}\sum_{i=1}^{N-1}\left|V_0(V_0^{-1}(\frac{i}{N})) - \frac{i}{N}\right| \;+\; \left|V_0(V_0^{-1}(1 - \frac{1}{2N})) - 1\right| \;=\; \frac{1}{2N} \quad . \end{aligned}$$

Now fix $k \in \{1,..,K\}$ and decompose E_k into three terms:

$$
\begin{aligned}
E_k \;=\; &= \frac{1}{N}\sum_{i=1}^{N} \mathbb{E}\left|V(t_k,Y^i_{t_k}) - \frac{1}{N}\sum_{j=1}^{N} H(Y^i_{t_k}-Y^j_{t_k})\right| \\
&\leq \frac{1}{N}\sum_{i=1}^{N} \mathbb{E}\left|V(t_k,Y^i_{t_k}) - V(t_k,z^i_{t_k})\right| + \frac{1}{N}\sum_{i=1}^{N} \mathbb{E}\left|V(t_k,z^i_{t_k}) - \frac{1}{N}\sum_{j=1}^{N} H(z^i_{t_k}-z^j_{t_k})\right| \\
&\qquad + \frac{1}{N}\sum_{i=1}^{N} \mathbb{E}\left|\frac{1}{N}\sum_{j=1}^{N} H(z^i_{t_k}-z^j_{t_k}) - \frac{1}{N}\sum_{j=1}^{N} H(Y^i_{t_k}-Y^j_{t_k})\right| \\
&\leq \frac{L_0}{\sqrt{t_k}}\,\frac{1}{N}\sum_{i=1}^{N} \mathbb{E}\left|z^i_{t_k}-Y^i_{t_k}\right| \;+\; \frac{1}{N}\sum_{i=1}^{N} \mathbb{E}\left|V(t_k,z^i_{t_k}) - \frac{1}{N}\sum_{j=1}^{N} H(z^i_{t_k}-z^j_{t_k})\right| \\
&\qquad + \frac{1}{N^2}\sum_{i,j=1}^{N} \mathbb{E}\left|H(z^i_{t_k}-z^j_{t_k}) - H(Y^i_{t_k}-Y^j_{t_k})\right| \quad .
\end{aligned}
\tag{28}
$$

We now use the following arguments:

- we have just seen (cf. (24)) how we can upper bound $\frac{1}{N}\sum_{i=1}^{N} \mathbb{E}\left|z^i_{t_k}-Y^i_{t_k}\right|$ in terms of the E_l's ($l=0,\ldots,k-1$); note that we cannot use (26) since we use (27) to get it;
- in the next subsection (lemma 5.4), we will prove the following upper bound for the second term of the right handside: there exists a constant L, depending only on T and σ such that, for all $t \in [0,T]$ and any $i=1,..,N$, one has

$$
\mathbb{E}\left|V(t,z^i_t) - \frac{1}{N}\sum_{j=1}^{N} H(z^i_t-z^j_t)\right| \leq \frac{C}{\sqrt{N}} \quad . \tag{29}
$$

- in the next subsection (lemma 5.5), we will prove that there exists a constant C, depending only on T, σ and V_0 such that, for all $k=1,\ldots,K$, one has

$$
\begin{aligned}
&\frac{1}{N^2}\sum_{i,j=1}^{N} \mathbb{E}\left|H(z^i_{t_k}-z^j_{t_k}) - \frac{1}{N}\sum_{j=1}^{N} H(Y^i_{t_k}-Y^j_{t_k})\right| \\
&\qquad \leq C\left(\sqrt{\Delta t} + \frac{1}{N} + \sum_{l=0}^{k-1}\frac{\Delta t E_l}{\sqrt{t_k-t_l}}\right) \quad .
\end{aligned}
\tag{30}
$$

Assume the above estimates; for $k \geq 3$, one then has:

$$E_k \;\leq\; \frac{C}{\sqrt{t_k}} \left(\sum_{l=1}^{k-1} \Delta t \, E_l \;+\; \sqrt{t_k}\,\sqrt{\Delta t} \right) \;+\; \frac{C}{\sqrt{N}}$$
$$+ \; C \left[\sqrt{\Delta t} \;+\; \frac{1}{N} \;+\; \sum_{l=2}^{k-1} \frac{\Delta t}{\sqrt{t_k - t_l}} \left(E_l \;+\; \frac{\Delta t}{\sqrt{t_l}} \sum_{q=1}^{l-1} E_q \right) \right] \quad .$$

Besides, as $I\!\!E \left| z^i_{\Delta t} - Y^i_{\Delta t} \right| \leq \Delta t$ and $I\!\!E \left| z^i_{2\Delta t} - Y^i_{2\Delta t} \right| \leq 2\Delta t$, the inequalities (30) and (29) imply that $E_1 \leq C \left(\sqrt{\Delta t} + \frac{1}{\sqrt{N}} \right)$ and $E_2 \leq C \left(\sqrt{\Delta t} + \frac{1}{\sqrt{N}} \right)$. Thus:

$$\left\{ \begin{array}{l} E_k \leq \; C \left[\displaystyle\sum_{l=1}^{k-1} \frac{\Delta t}{\sqrt{t_k}} \, E_l \;+\; \sum_{l=2}^{k-1} \frac{\Delta t}{\sqrt{t_k - t_l}} \, E_l \;+\; \sum_{l=2}^{k-1} \frac{\Delta t}{\sqrt{t_k - t_l}\sqrt{t_l}} \, \sum_{q=1}^{l-1} \Delta t E_q \right] \\[2ex] \qquad\qquad + \, C \left(\sqrt{\Delta t} \;+\; \dfrac{1}{\sqrt{N}} \right) \quad , \quad k = 3, .., K \; , \\[2ex] E_0 = \dfrac{1}{2N} \quad , \quad E_1 \;\leq\; C \left(\sqrt{\Delta t} + \dfrac{1}{N} \right) \quad , \quad : E_2 \;\leq\; C \left(\sqrt{\Delta t} + \dfrac{1}{N} \right) \quad , \end{array} \right.$$

where C is a constant depending only on T, σ and V_0.

We are now in position to prove (27).

For all $t \in [0, T]$, define the function $\varepsilon(t)$ by

$$\varepsilon(t) \;:=\; \sum_{k=0}^{K-1} \; 1\!\!1_{[t_k, t_{k+1}[}(t) \, E_k \quad , \quad \varepsilon(T) \;:=\; E_K \quad .$$

This function is measurable, positive and bounded by 1 (remember the definition (23)).

The function $s \to \frac{1}{\sqrt{t_k - s}}$ being increasing on $]0, t_k[$, one has that

$$\sum_{l=2}^{k-1} \frac{\Delta t}{\sqrt{t_k - t_l}} E_l \;\leq\; \int_0^{t_k} \frac{\varepsilon(s)}{\sqrt{t_k - s}} ds \quad .$$

The function $s \to \frac{1}{\sqrt{s}\sqrt{t_k - s}}$ being decreasing on $]0, t_k/2[$ and increasing on $]t_k/2, t_k[$, one has that

$$\sum_{l=2}^{k-1} \frac{\Delta t}{\sqrt{t_k - t_l}\sqrt{t_l}} \sum_{q=1}^{l-1} \Delta t E_q \quad \leq \quad \int_0^{t_k} \epsilon(s) ds \; \sum_{l=2}^{k-1} \frac{\Delta t}{\sqrt{t_k - t_l}\sqrt{t_l}}$$

$$\leq \int_0^{t_k} \epsilon(s)ds \int_0^{t_k} \frac{1}{\sqrt{t_k - s}\sqrt{s}} ds \leq 4 \int_{\Delta t}^{t_k} \varepsilon(s)ds \quad .$$

Thus, the function $\varepsilon(t)$ satisfies:

$$\varepsilon(t_k) \leq C\left(\sqrt{\Delta t} + \frac{1}{\sqrt{N}}\right) + \int_0^{t_k} C\left(\frac{1}{\sqrt{t_k}} + \frac{1}{\sqrt{t_k - s}} + 1\right)\varepsilon(s)\, ds \quad .$$

We conclude by applying Gronwall's lemma:

$$\varepsilon(T) \leq C' \exp\left(C\,(3\sqrt{T} + T)\right)\left(\sqrt{\Delta t} + \frac{1}{\sqrt{N}}\right) \quad .$$

∎

5.4 Technical lemmae

We now prove the estimates (29) and (30).

Lemma 5.4 (Proof of (29)) *There exists a constant L, depending only on T and σ such that, for all $t \in [0,T]$ and any $i = 1,..,N$, one has*

$$\mathbb{E}\left|V(t, z_t^i) - \frac{1}{N}\sum_{j=1}^{N} H(z_t^i - z_t^j)\right| \leq \frac{L}{\sqrt{N}} \quad .$$

Proof. For $i \in 1,..,N$ and $t \in [0,T]$ fixed, consider

$$\begin{aligned}
&\mathbb{E}\left|V(t, z_t^i) - \frac{1}{N}\sum_{j=1}^{N} H(z_t^i - z_t^j)\right| \\
\leq\ & \mathbb{E}\left|V(t, z_t^i) - \mathbb{E}_{\overline{U}_0} H(x - z_t)\Big|_{x=z_t^i}\right| \\
&+\mathbb{E}\left|\frac{1}{N}\sum_{j=1}^{N} \mathbb{E} H(x - z_t^j)\Big|_{x=z_t^i} - \frac{1}{N}\sum_{j=1}^{N} H(z_t^i - z_t^j)\right| \\
:=\ & A + B \quad .
\end{aligned}$$

Let us first treat A.

If U_0 is a Dirac measure, then $U_0 = \overline{U}_0$ and A is 0; if (H1-ii) holds, then we can easily use the arguments of the proof of the lemma 3.1 in [4] to prove

$$\left|V(t,x) - \mathbb{E}_{\overline{U}_0} H(x - z_t)\right| \leq C \left\|V_0(\cdot) - \overline{V}_0(\cdot)\right\|_{L^\infty} \quad , \quad \forall x \in \mathbb{R} \ .$$

By definition of $\overline{V}_0$, it comes that

$$A \leq \frac{1}{N} \ .$$

Now consider B:

$$\begin{aligned}
&\mathbb{E}\left(\frac{1}{N}\sum_{j=1}^{N}\left(\mathbb{E}H(x - z_t^j)\Big|_{x=z_t^i} - H(z_t^i - z_t^j)\right)\right)^2 \\
&\quad = \frac{1}{N^2}\sum_{j=1}^{N} \mathbb{E}\left(\mathbb{E}H(x - z_t^j)\Big|_{x=z_t^i} - H(z_t^i - z_t^j)\right)^2 \\
&\qquad + \frac{1}{N^2}\sum_{\substack{j,k=1\\ j\neq k}}^{N} \mathbb{E}\left[\left(\mathbb{E}H(x - z_t^j)\Big|_{z_t^i} - H(z_t^i - z_t^j)\right)\right. \\
&\qquad\qquad \left.\cdot\left(\mathbb{E}H(x - z_t^k)\Big|_{z_t^i} - H(z_t^i - z_t^k)\right)\right] \ .
\end{aligned}$$

The $(z_t^j, j = 1, .., N)$ being independent, one gets that

$$\mathbb{E}\left(\frac{1}{N}\sum_{j=1}^{N}\left(\mathbb{E}H(x - z_t^j)\Big|_{x=z_t^i} - H(z_t^i - z_t^j)\right)\right)^2 \leq \frac{1}{N} ,$$

which ends the proof of the lemma. ∎

Lemma 5.5 (Proof of (30)) *There exists a constant C, depending only on T, σ and V_0 such that, for any $k \in 1, .., K$, one has:*

$$\frac{1}{N^2}\sum_{i,j=1}^{N} \mathbb{E}\left|H(z_{t_k}^i - z_{t_k}^j) - \frac{1}{N}\sum_{j=1}^{N} H(Y_{t_k}^i - Y_{t_k}^j)\right| \leq C\left(\sqrt{\Delta t} + \frac{1}{N} + \sum_{l=0}^{k-1}\frac{\Delta t}{\sqrt{t_k - t_l}} E_l\right) \ .$$

Proof. As $z_0^i = Y_0^i$, it is clear that the left handside is 0 when $k = 0$.

When $k=1$, $Y^i_{\Delta t} = \overline{z}^i_{\Delta t}$, thus for $i \neq j$,

$$\begin{aligned}
&\mathbb{E}\left|H(z^i_{\Delta t} - z^j_{\Delta t}) - H(\overline{z}^i_{\Delta t} - \overline{z}^j_{\Delta t})\right| \\
&\quad \leq \mathbb{E}\left|H(z^i_{\Delta t} - z^j_{\Delta t}) - H(\overline{z}^i_{\Delta t} - z^j_{\Delta t})\right| + \mathbb{E}\left|H(\overline{z}^i_{\Delta t} - z^j_{\Delta t}) - H(\overline{z}^i_{\Delta t} - \overline{z}^j_{\Delta t})\right| .
\end{aligned}$$

Integrate w.r.t. the law of $z^j_{\Delta t}$ and apply the lemma 5.2 and (20):

$$\begin{aligned}
&\mathbb{E}\left|H(z^i_{\Delta t} - z^j_{\Delta t}) - H(\overline{z}^i_{\Delta t} - z^j_{\Delta t})\right| \\
&\quad \leq \mathbb{E}\int_{\mathbb{R}} \frac{C}{\sqrt{\Delta t}}\left|H(z^i_{\Delta t} - x) - H(\overline{z}^i_{\Delta t} - x)\right| dx \leq \frac{C}{\sqrt{\Delta t}}\mathbb{E}\left|z^i_{\Delta t} - \overline{z}^i_{\Delta t}\right| \\
&\quad \leq C\sqrt{\Delta t} .
\end{aligned}$$

Besides, for $i \neq j$:

$$\begin{aligned}
&\mathbb{E}\left|H(\overline{z}^i_{\Delta t} - z^j_{\Delta t}) - H(\overline{z}^i_{\Delta t} - \overline{z}^j_{\Delta t})\right| \\
&\quad \leq \mathbb{E}\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2\Delta t}}\left|H(x - z^j_{\Delta t}) - H(x - \overline{z}^j_{\Delta t})\right| dx \leq C\sqrt{\Delta t} .
\end{aligned}$$

Thus (30) holds for $k=1$ and, similarly, for $k=2$.

Now fix $k \in \{3,\ldots,K\}$. The difficulty is to find the appropriate relation between the left handside of (30) and the E_j's.

For all $x \in \mathbb{R}$, $i = 1,..,N$ and $l = 0,..,K$, define the process $(z^{i,l}_t(x))$ by

$$z^{i,l}_t(x) := x + \int_0^t V\left(t_l + s, z^{i,l}_s(x)\right) ds + \sigma\left(w^i_{t+t_l} - w^i_{t_l}\right) \quad , \quad t \in [0, T - t_l] . \tag{31}$$

Then

$$\mathbb{E}\left|H(Y^i_{t_k} - Y^j_{t_k}) - H(z^i_{t_k} - z^j_{t_k})\right| = \mathbb{E}\left|H(Y^i_{t_k} - Y^j_{t_k}) - H(z^{i,0}_{t_k}(y^i_0) - z^{j,0}_{t_k}(y^j_0))\right| ,$$

from which

$$\begin{aligned}
&\mathbb{E}\left|H(Y^i_{t_k} - Y^j_{t_k}) - H(z^i_{t_k} - z^j_{t_k})\right| \leq \qquad (32) \\
&\sum_{l=0}^{k-1} \mathbb{E}\left|H\left(z^{i,k-l}_{l\Delta t}(Y^i_{t_{k-l}}) - z^{j,k-l}_{l\Delta t}(Y^j_{t_{k-l}})\right) - H\left(z^{i,k-(l+1)}_{(l+1)\Delta t}(Y^i_{t_{k-(l+1)}}) - z^{j,k-(l+1)}_{(l+1)\Delta t}(Y^j_{t_{k-(l+1)}})\right)\right| .
\end{aligned}$$

Fix l and $i \neq j$.

$$\left|H\left(z^{i,k-l}_{l\Delta t}(Y^i_{t_{k-l}}) - z^{j,k-l}_{l\Delta t}(Y^j_{t_{k-l}})\right) - H\left(z^{i,k-(l+1)}_{(l+1)\Delta t}(Y^i_{t_{k-(l+1)}}) - z^{j,k-(l+1)}_{(l+1)\Delta t}(Y^j_{t_{k-(l+1)}})\right)\right| \leq$$

$$\left| H\left(z^{i,k-l}_{l\Delta t}(Y^i_{t_{k-l}}) - z^{j,k-l}_{l\Delta t}(Y^j_{t_{k-l}})\right) - H\left(z^{i,k-l}_{l\Delta t}(Y^i_{t_{k-l}}) - z^{j,k-(l+1)}_{(l+1)\Delta t}(Y^j_{t_{k-(l+1)}})\right)\right| +$$

$$\left| H\left(z^{i,k-l}_{l\Delta t}(Y^i_{t_{k-l}}) - z^{j,k-(l+1)}_{(l+1)\Delta t}(Y^j_{t_{k-(l+1)}})\right) - H\left(z^{i,k-(l+1)}_{(l+1)\Delta t}(Y^i_{t_{k-(l+1)}}) - z^{j,k-(l+1)}_{(l+1)\Delta t}(Y^j_{t_{k-(l+1)}})\right)\right|$$
$$=: A + B \ .$$

We first upper bound $\mathbb{E}A$.

Let $(\mathcal{F}_t)$ the σ-field generated by $(w^i_t \ , \ 1 \le i \le N)$. Then, denoting by Δw^i_p the quantity $w^i_p - w^i_{p-1}$, one has:

$$\begin{aligned}
\mathbb{E}A &= \mathbb{E}\left| H\left(z^{i,k-l}_{l\Delta t}(Y^i_{t_{k-l}}) - z^{j,k-l}_{l\Delta t}(Y^j_{t_{k-l}})\right) - H\left(z^{i,k-l}_{l\Delta t}(Y^i_{t_{k-l}}) - z^{j,k-(l+1)}_{(l+1)\Delta t}(Y^j_{t_{k-(l+1)}})\right)\right| \\
&= \mathbb{E}\mathbb{E}^{\mathcal{F}_{t_{k-(l+1)}}}\left| H\left(z^{i,k-l}_{l\Delta t}(Y^i_{t_{k-l}}) - z^{j,k-l}_{l\Delta t}(Y^j_{t_{k-l}})\right) - \right. \\
&\qquad\qquad \left. - H\left(z^{i,k-l}_{l\Delta t}(Y^i_{t_{k-l}}) - z^{j,k-(l+1)}_{(l+1)\Delta t}(Y^j_{t_{k-(l+1)}})\right)\right| \\
&= \mathbb{E}\mathbb{E}^{\mathcal{F}_{t_{k-(l+1)}}}\left| H\left(z^{i,k-l}_{l\Delta t}(Y^i_{t_{k-(l+1)}} + \Delta t \overline{V}_{t_{k-(l+1)}}(Y^i_{t_{k-(l+1)}}) + \Delta w^i_{k-l}) - z^{j,k-l}_{l\Delta t}(Y^j_{t_{k-l}})\right)\right. \\
&\qquad \left. - H\left(z^{i,k-l}_{l\Delta t}(Y^i_{t_{k-(l+1)}} + \Delta t \overline{V}_{t_{k-(l+1)}}(Y^i_{t_{k-(l+1)}}) + \Delta w^i_{k-l}) - z^{j,k-(l+1)}_{(l+1)\Delta t}(Y^j_{t_{k-(l+1)}})\right)\right| \ .
\end{aligned}$$

Let $g_{\sigma^2\Delta t}(\cdot)$ denote the Gaussian density of mean 0 and variance $\sigma^2\Delta t$.

The random variables $z^{j,k-l}_{l\Delta t}(Y^j_{t_{k-l}})$ and $z^{j,k-(l+1)}_{(l+1)\Delta t}(Y^j_{t_{k-(l+1)}})$ are independent of Δw^i_{k-l}; besides, $z^{i,k-l}_{l\Delta t}(x)$ is independent of Δw^i_{k-l}; therefore,

$$\begin{aligned}
\mathbb{E}A &= \mathbb{E}\int_{\mathbb{R}} g_{\sigma^2\Delta t}(z)\left| H\left(z^{i,k-l}_{l\Delta t}(Y^i_{t_{k-(l+1)}} + \Delta t \overline{V}_{t_{k-(l+1)}}(Y^i_{t_{k-(l+1)}}) + z) - z^{j,k-l}_{l\Delta t}(Y^j_{t_{k-l}})\right)\right. \\
&\qquad \left. - H\left(z^{i,k-l}_{l\Delta t}(Y^i_{t_{k-(l+1)}} + \Delta t \overline{V}_{t_{k-(l+1)}}(Y^i_{t_{k-(l+1)}}) + z) - z^{j,k-(l+1)}_{(l+1)\Delta t}(Y^j_{t_{k-(l+1)}})\right)\right| dz \ .
\end{aligned}$$

Remember that $\gamma^{i,k}_t(x,\cdot)$ denotes the density of the law of $z^{i,k}_t(x)$ defined in (31); as $i \neq j$, it comes:

$$\begin{aligned}
\mathbb{E}A &= \mathbb{E}\int_{\mathbb{R}}\left| H\left(y - z^{j,k-l}_{l\Delta t}(Y^j_{t_{k-l}})\right) - H\left(y - z^{j,k-(l+1)}_{(l+1)\Delta t}(Y^j_{t_{k-(l+1)}})\right)\right| \\
&\qquad \times \int_{\mathbb{R}} g_{\sigma^2\Delta t}(z)\, \gamma^{i,k-l}_{l\Delta t}\left(y, Y^i_{t_{k-(l+1)}} + \Delta t \overline{V}_{t_{k-(l+1)}}(Y^i_{t_{k-(l+1)}}) + z\right) dz\ dy \ .
\end{aligned}$$

We apply the lemma 5.2 and get

$$\gamma^{i,k-l}_t(x,y) \ \le \ \frac{C}{\sqrt{2\pi t\sigma^2}} \exp\left(-\frac{(x-y)^2}{4t\sigma^2}\right) \ . \tag{33}$$

Thus,

$$\int_{\mathbb{R}} g_{\sigma^2\Delta t}(z)\, \gamma_{l\Delta t}^{i,k-l}\left(y, Y^i_{t_{k-(l+1)}} + \Delta t \overline{V}_{t_{k-(l+1)}}(Y^i_{t_{k-(l+1)}}) + z\right)\, dz \ \leq\ \frac{C}{\sqrt{t_{l+1}}}\quad .$$

Finally using (20) once more, we get:

$$\mathbb{E}A \ \leq\ \frac{C}{\sqrt{t_{l+1}}}\ \mathbb{E}\left|z_{l\Delta t}^{j,k-l}(Y^j_{t_{k-l}})\ -\ z_{(l+1)\Delta t}^{j,k-(l+1)}(Y^j_{t_{k-(l+1)}})\right|\quad . \tag{34}$$

To upper bound $\mathbb{E}B$, we follow the same way and obtain

$$\begin{aligned}
\mathbb{E}B \ &\leq\ \mathbb{E}\int_{\mathbb{R}} \frac{C}{\sqrt{t_{l+1}}}\left|H\left(z_{l\Delta t}^{i,k-l}(Y^i_{t_{k-l}}) - y\right) - H\left(z_{(l+1)\Delta t}^{i,k-(l+1)}(Y^i_{t_{k-(l+1)}}) - y\right)\right| dy \\
&\leq\ \frac{C}{\sqrt{t_{l+1}}}\ \mathbb{E}\left|z_{l\Delta t}^{i,k-l}(Y^i_{t_{k-l}}) - z_{(l+1)\Delta t}^{i,k-(l+1)}(Y^i_{t_{k-(l+1)}})\right|\quad . \qquad (35)
\end{aligned}$$

From(34) and (35), it comes:

$$\begin{aligned}
&\mathbb{E}\left|H\left(z_{l\Delta t}^{i,k-l}(Y^i_{t_{k-l}}) - z_{l\Delta t}^{j,k-l}(Y^j_{t_{k-l}})\right) - H\left(z_{(l+1)\Delta t}^{i,k-(l+1)}(Y^i_{t_{k-(l+1)}}) - z_{(l+1)\Delta t}^{j,k-(l+1)}(Y^j_{t_{k-(l+1)}})\right)\right| \\
&\quad\leq\ \frac{C}{\sqrt{t_{l+1}}}\left(\mathbb{E}\left|z_{l\Delta t}^{j,k-l}(Y^j_{t_{k-l}}) - z_{(l+1)\Delta t}^{j,k-(l+1)}(Y^j_{t_{k-(l+1)}})\right|\right. \qquad (36) \\
&\qquad\qquad\left. + \ \mathbb{E}\left|z_{l\Delta t}^{i,k-l}(Y^i_{t_{k-l}}) - z_{(l+1)\Delta t}^{i,k-(l+1)}(Y^i_{t_{k-(l+1)}})\right|\right)\quad .
\end{aligned}$$

Below, we will prove: for any $i = 1,..,N$ and $l = 0,..,k-3$,

$$\begin{aligned}
&\mathbb{E}\left|z_{l\Delta t}^{i,k-l}(Y^i_{t_{k-l}}) - z_{(l+1)\Delta t}^{i,k-(l+1)}(Y^i_{t_{k-(l+1)}})\right| \\
&\quad\leq\ \Delta t\, C\, \mathbb{E}\left|\overline{V}_{t_{k-(l+1)}}\left(Y^i_{t_{k-(l+1)}}\right) - V\left(t_{k-(l+1)}, Y^i_{t_{k-(l+1)}}\right)\right| \\
&\qquad + \Delta t\ \frac{C}{\sqrt{t_{k-(l+1)}}}\left(\sum_{q=1}^{k-(l+2)} \Delta t \mathbb{E}\left|V(t_q, Y^i_{t_q}) - \overline{V}_{t_q}(Y^i_{t_q})\right| \ +\ \sqrt{\Delta t}\right)\ ; \qquad (37)
\end{aligned}$$

using this estimate in (36), we get, for all $l \in \{0,..,k-3\}$:

$$\begin{aligned}
&\mathbb{E}\left|H\left(z_{l\Delta t}^{i,k-l}(Y^i_{t_{k-l}}) - z_{l\Delta t}^{j,k-l}(Y^j_{t_{k-l}})\right) - H\left(z_{(l+1)\Delta t}^{i,k-(l+1)}(Y^i_{t_{k-(l+1)}}) - z_{(l+1)\Delta t}^{j,k-(l+1)}(Y^j_{t_{k-(l+1)}})\right)\right| \\
&\quad\leq\ \frac{C\,\Delta t}{\sqrt{t_{l+1}}}\left\{\mathbb{E}\left|\overline{V}_{t_{k-(l+1)}}\left(Y^j_{t_{k-(l+1)}}\right)\ -\ V\left(t_{k-(l+1)}, Y^j_{t_{k-(l+1)}}\right)\right|\right. \\
&\qquad\qquad + \mathbb{E}\left|\overline{V}_{t_{k-(l+1)}}\left(Y^i_{t_{k-(l+1)}}\right)\ -\ V\left(t_{k-(l+1)}, Y^i_{t_{k-(l+1)}}\right)\right|
\end{aligned}$$

$$+ \frac{\Delta t}{\sqrt{t_{k-(l+1)}}} \left(\sum_{q=1}^{k-(l+2)} \mathbb{E} \left| V(t_q, Y_{t_q}^j) - \overline{V}_{t_q}(Y_{t_q}^j) \right| \right)$$
$$+ \frac{\Delta t}{\sqrt{t_{k-(l+1)}}} \left(\sum_{q=1}^{k-(l+2)} \mathbb{E} \left| V(t_q, Y_{t_q}^i) - \overline{V}_{t_q}(Y_{t_q}^i) \right| \right) + \frac{\sqrt{\Delta t}}{\sqrt{t_{k-(l+1)}}} \Bigg\}.$$

Thus, for $k \geq 3$, using the definition of E_k (cf. (23)), we get

$$\frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbb{E} \left| H(z_{t_k}^i - z_{t_k}^j) - \frac{1}{N} \sum_{j=1}^{N} H(Y_{t_k}^i - Y_{t_k}^j) \right|$$
$$\leq C \sqrt{\Delta t}$$
$$+ \sum_{l=0}^{k-3} \frac{C \, \Delta t}{\sqrt{t_{l+1}}} \left\{ E_{k-(l+1)} + \frac{\Delta t}{\sqrt{t_{k-(l+1)}}} \sum_{q=1}^{k-(l+2)} E_q + \frac{\sqrt{\Delta t}}{\sqrt{t_{k-(l+1)}}} \right\} + \frac{1}{N} \quad .$$

Remark that

$$\sum_{l=0}^{k-3} \frac{\Delta t^{\frac{3}{2}}}{\sqrt{\Delta t(l+1)} \sqrt{\Delta t(k-(l+1))}} = \sum_{l=1}^{k-2} \frac{\Delta t^{\frac{3}{2}}}{\sqrt{l \Delta t} \sqrt{\Delta t(k-l)}}$$
$$\leq \sum_{l=1}^{[k/2]} \frac{\Delta t^{\frac{3}{2}}}{\sqrt{l \Delta t} \sqrt{\Delta t(k-[k/2])}} + \sum_{[k/2]+1}^{k-2} \frac{\Delta t^{\frac{3}{2}}}{\sqrt{\Delta t([k/2]+1)} \sqrt{\Delta t(k-l)}}$$
$$\leq \Delta t \left(\frac{1}{\sqrt{\Delta t(k-[k/2])}} + \frac{1}{\sqrt{\Delta t([k/2]+1)}} \right) \int_0^{[k/2]} \frac{1}{\sqrt{x}} dx \leq 4 \sqrt{\Delta t} \quad .$$

We deduce that, for $k \geq 3$, we have that

$$\frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbb{E} \left| H(z_{t_k}^i - z_{t_k}^j) - \frac{1}{N} \sum_{j=1}^{N} H(Y_{t_k}^i - Y_{t_k}^j) \right|$$
$$\leq \sum_{l=0}^{k-3} \frac{C \, \Delta t}{\sqrt{t_{l+1}}} \left\{ E_{k-(l+1)} + \frac{\Delta t}{\sqrt{t_{k-(l+1)}}} \sum_{q=1}^{k-(l+2)} E_q \right\} + C \sqrt{\Delta t} + \frac{1}{N} \quad .$$

The inequality 30 is proved. ∎

Lemma 5.6 (Proof of (37)) *For all $i = 1, .., N$ and for all $l = 0, .., k-3$, one has that*

$$\mathbb{E} \left| z_{l \Delta t}^{i,k-l}(Y_{t_{k-l}}^i) - z_{(l+1)\Delta t}^{i,k-(l+1)}(Y_{t_{k-(l+1)}}^i) \right|$$

$$\leq C\Delta t\, \mathbb{E}\left|\overline{V}_{t_{k-(l+1)}}\left(Y^i_{t_{k-(l+1)}}\right) - V\left(t_{k-(l+1)}, Y^i_{t_{k-(l+1)}}\right)\right|$$
$$+ \frac{C\Delta t}{\sqrt{t_{k-(l+1)}}}\left(\sum_{q=1}^{k-(l+2)} \Delta t \mathbb{E}\left|V(t_q, Y^i_{t_q}) - \overline{V}_{t_q}(Y^i_{t_q})\right| + \sqrt{\Delta t}\right) \quad .$$

For $l = k-2$ and $l = k-1$, it holds that

$$\left(\mathbb{E}\left|z^{i,2}_{t_{k-2}}(Y^i_{2\Delta t}) - z^{i,1}_{t_{k-1}}(Y^i_{\Delta t})\right| + \mathbb{E}\left|z^{i,1}_{t_{k-1}}(Y^i_{\Delta t}) - z^{i,0}_{t_k}(y^i_0)\right|\right) \leq C\Delta t\,,$$

where C is a positive constant depending only on T, σ and V_0.

Proof. We already noticed the strong uniqueness of the solution to (14): for all $k = 0,..,K$ and $i = 1,..,N$,

$$z^{i,k}_{t+\Delta t}(x) = z^{i,k+1}_t\left(z^{i,k}_{\Delta t}(x)\right) \quad . \tag{38}$$

An easy computation also shows that

$$\mathbb{E}\left|z^{i,k}_t(x) - z^{i,k}_t(y)\right| \leq \exp(L_0\, 2\sqrt{T})\ |x - y| \quad . \tag{39}$$

Thus,

$$\begin{aligned}\mathbb{E}\left|z^{i,k-l}_{l\Delta t}(Y^i_{t_{k-l}}) - z^{i,k-(l+1)}_{(l+1)\Delta t}(Y^i_{t_{k-(l+1)}})\right| &= \mathbb{E}\left|z^{i,k-l}_{l\Delta t}(Y^i_{t_{k-l}}) - z^{i,k-l}_{l\Delta t}\left(z^{i,k-(l+1)}_{\Delta t}(Y^i_{t_{k-(l+1)}})\right)\right| \\ &= \mathbb{E}\mathbb{E}^{\mathcal{F}_{t_{k-l}}}\left|z^{i,k-l}_{l\Delta t}(Y^i_{t_{k-l}}) - z^{i,k-l}_{l\Delta t}\left(z^{i,k-(l+1)}_{\Delta t}(Y^i_{t_{k-(l+1)}})\right)\right| \\ &\leq \exp(L_0\, 2\sqrt{T})\ \mathbb{E}\left|Y^i_{t_{k-l}} - z^{i,k-(l+1)}_{\Delta t}(Y^i_{t_{k-(l+1)}})\right|.\end{aligned} \tag{40}$$

This shows that

$$\mathbb{E}\left|Y^i_{\Delta t} - z^{i,0}_{\Delta t}(y^i_0)\right| = \mathbb{E}\left|\Delta t\overline{V}_0\left(y^i_0\right) - \int_0^{\Delta t} V\left(s, z^0_s(y^i_0)\right) ds\right| \leq \Delta t\,,$$

and

$$\mathbb{E}\left|Y^i_{2\Delta t} - z^{i,1}_{\Delta t}(Y^i_{\Delta t})\right| = \mathbb{E}\left|\Delta t\overline{V}_{\Delta t}\left(Y^i_{\Delta t}\right) - \int_0^{\Delta t} V\left(\Delta t + s, z^{i,1}_s(Y^i_{\Delta t})\right) ds\right| \leq \Delta t \quad .$$

For $l \in \{0,..,k-3\}$:

$$\begin{aligned}&\mathbb{E}\left|Y^i_{t_{k-l}} - z^{i,k-(l+1)}_{\Delta t}(Y^i_{t_{k-(l+1)}})\right| \\ &= \mathbb{E}\left|\Delta t\overline{V}_{t_{k-(l+1)}}\left(Y^i_{t_{k-(l+1)}}\right) - \int_0^{\Delta t} V\left(t_{k-(l+1)} + s, z^{i,k-(l+1)}_s(Y^i_{t_{k-(l+1)}})\right) ds\right| \\ &\leq \Delta t\mathbb{E}\left|\overline{V}_{t_{k-(l+1)}}\left(Y^i_{t_{k-(l+1)}}\right) - V\left(t_{k-(l+1)}, Y^i_{t_{k-(l+1)}}\right)\right| \\ &\quad + \mathbb{E}\int_0^{\Delta t}\left|V\left(t_{k-(l+1)} + s, z^{i,k-(l+1)}_s(Y^i_{t_{k-(l+1)}})\right) - V\left(t_{k-(l+1)}, Y^i_{t_{k-(l+1)}}\right)\right| ds \quad .\end{aligned} \tag{41}$$

As V is Lipschitz, it comes that

$$
\begin{aligned}
& \mathbb{E}\int_0^{\Delta t}\left|V\left(t_{k-(l+1)}+s, z_s^{i,k-(l+1)}(Y^i_{t_{k-(l+1)}})\right)-V\left(t_{k-(l+1)},Y^i_{t_{k-(l+1)}}\right)\right|ds \\
& \quad \leq \mathbb{E}\int_0^{\Delta t}\left|V\left(t_{k-(l+1)}+s, z_s^{i,k-(l+1)}(Y^i_{t_{k-(l+1)}})\right)-V\left(t_{k-(l+1)},z_s^{i,k-(l+1)}(Y^i_{t_{k-(l+1)}})\right)\right|ds \\
& \qquad +\int_0^{\Delta t}\frac{L_0}{\sqrt{t_{k-(l+1)}}}\mathbb{E}\left|z_s^{i,k-(l+1)}(Y^i_{t_{k-(l+1)}})-Y^i_{t_{k-(l+1)}}\right|ds \\
& \quad \leq \mathbb{E}\int_0^{\Delta t}\left|V\left(t_{k-(l+1)}+s, z_s^{i,k-(l+1)}(Y^i_{t_{k-(l+1)}})\right)-V\left(t_{k-(l+1)},z_s^{i,k-(l+1)}(Y^i_{t_{k-(l+1)}})\right)\right|ds \\
& \qquad +\frac{L_0}{\sqrt{t_{k-(l+1)}}}\Delta t\left(\Delta t+\sigma\sqrt{\Delta t}\right) .
\end{aligned}
$$

We introduce the random variable $z_s^{i,k-(l+1)}(z^i_{t_{k-(l+1)}}) := z^i_{t_{k-(l+1)}+s}$:

$$
\begin{aligned}
& \mathbb{E}\int_0^{\Delta t}\left|V\left(t_{k-(l+1)}+s, z_s^{i,k-(l+1)}(Y^i_{t_{k-(l+1)}})\right)-V\left(t_{k-(l+1)},Y^i_{t_{k-(l+1)}}\right)\right|ds \\
& \leq \mathbb{E}\int_0^{\Delta t}\left|V\left(t_{k-(l+1)}+s, z_s^{i,k-(l+1)}(Y^i_{t_{k-(l+1)}})\right)-V\left(t_{k-(l+1)}+s,z_s^{i,k-(l+1)}(z^i_{t_{k-(l+1)}})\right)\right|ds \\
& \quad +\mathbb{E}\int_0^{\Delta t}\left|V\left(t_{k-(l+1)}+s, z^i_{t_{k-(l+1)}+s}\right)-V\left(t_{k-(l+1)},z^i_{t_{k-(l+1)}+s}\right)\right|ds \\
& \quad +\mathbb{E}\int_0^{\Delta t}\left|V\left(t_{k-(l+1)}, z_s^{i,k-(l+1)}(z^i_{t_{k-(l+1)}})\right)-V\left(t_{k-(l+1)},z_s^{i,k-(l+1)}(Y^i_{t_{k-(l+1)}})\right)\right|ds \\
& \quad +\frac{L_0}{\sqrt{t_{k-(l+1)}}}\Delta t\left(\Delta t+\sigma\sqrt{\Delta t}\right) .
\end{aligned}
$$

Using that $V(t,\cdot)$ is Lipschitz, (39) and (33), we get:

$$
\begin{aligned}
& \mathbb{E}\int_0^{\Delta t}\left|V\left(t_{k-(l+1)}+s, z_s^{i,k-(l+1)}(Y^i_{t_{k-(l+1)}})\right)-V\left(t_{k-(l+1)},Y^i_{t_{k-(l+1)}}\right)\right|ds \\
& \quad \leq 2\,\frac{L_0}{\sqrt{t_{k-(l+1)}}}\,\exp(L_0 2\sqrt{T})\,\Delta t\,\mathbb{E}\left|z^i_{t_{k-(l+1)}}-Y^i_{t_{k-(l+1)}}\right| \\
& \qquad +\int_0^{\Delta t}\int_{\mathbb{R}}\left|\mathbb{E}_{U_0}H(x-z_{t_{k-(l+1)}+s})-\mathbb{E}_{U_0}H(x-z_{t_{k-(l+1)}})\right|\frac{C}{\sqrt{t_{k-(l+1)}}}dx\,ds \\
& \qquad +\frac{L_0}{\sqrt{t_{k-(l+1)}}}\Delta t\left(\Delta t+\sigma\sqrt{\Delta t}\right) .
\end{aligned}
$$

Applying (20), it comes that

$$\begin{aligned}
&\mathbb{E}\int_0^{\Delta t}\left|V\left(t_{k-(l+1)}+s, z_s^{i,k-(l+1)}(Y^i_{t_{k-(l+1)}})\right)-V\left(t_{k-(l+1)},Y^i_{t_{k-(l+1)}}\right)\right|ds\\
&\leq 2\,\frac{L_0}{\sqrt{t_{k-(l+1)}}}\,\exp(L_0 2\sqrt{T})\,\Delta t\,\mathbb{E}\left|z^i_{t_{k-(l+1)}}-Y^i_{t_{k-(l+1)}}\right|\\
&+\frac{(L_0+C)}{\sqrt{t_{k-(l+1)}}}\Delta t\left(\Delta t+\sigma\sqrt{\Delta t}\right) .
\end{aligned}$$

Finally, we upper bound $\mathbb{E}\left|z^i_{t_{k-(l+1)}}-Y^i_{t_{k-(l+1)}}\right|$ as in (24) and get

$$\begin{aligned}
&\mathbb{E}\int_0^{\Delta t}\left|V\left(t_{k-(l+1)}+s, z_s^{i,k-(l+1)}(Y^i_{t_{k-(l+1)}})\right)-V\left(t_{k-(l+1)},Y^i_{t_{k-(l+1)}}\right)\right|ds\\
\leq\;& 2\,\frac{L_0}{\sqrt{t_{k-(l+1)}}}\,\exp(L_0 2\sqrt{T})\,\Delta t\,C\left(\sum_{q=1}^{k-(l+2)}\Delta t\mathbb{E}\left|V(t_q,Y^i_{t_q})-\overline{V}_{t_l}(Y^i_{t_q})\right|+\sqrt{\Delta t}\right)\\
&+\frac{C\Delta t^{\frac{3}{2}}}{\sqrt{t_{k-(l+1)}}}\\
\leq\;&\frac{C\Delta t}{\sqrt{t_{k-(l+1)}}}\left(\sum_{q=1}^{k-(l+2)}\Delta t\mathbb{E}\left|V(t_q,Y^i_{t_q})-\overline{V}_{t_q}(Y^i_{t_q})\right|+\sqrt{\Delta t}\right) .
\end{aligned}$$

We conclude by using this estimate in (41), and then considering (40). ∎

6 Conclusion

We have constructed a stochastic particles method for the one-dimensional Burgers equation and given its convergence rate for the $L^1(\mathbb{R} \times \Omega)$-norm of the error.

Here, the initial condition is supposed equal to a distribution function. It is not too hard to extend the method and the theoretical estimate of the convergence rate to non monotonic initial conditions: this is done in Bossy & Talay [4] and in Bossy [3].

Our next objective is to extend the algorithm and our error analysis to treat the 2-D inviscid Navier-Stokes equation, which would permit to give new error estimates for the Chorin's random vortex methods. The additional difficulty is due to the fact that the corresponding interaction kernel is singular.

A Appendix

A.1 Proof of the proposition 3.1

We again stress that this proof is adapted from [26]; we give the essential arguments, the details of the computations can be found in M. Bossy's thesis [3].

We start with an easy lemma.

Lemma A.1 *Under (H0), the function $V(t,x) = \mathbb{E}H(x - X_t)$ is integrable in $-\infty$; more precisely, it satisfies: there exist strictly positive constants C, γ, δ such that, for all $t \in [0,T]$,*

$$\forall x < -M \quad , V(t,x) \ \leq \ C \ \exp\left(-\frac{(x+\delta)^2}{\gamma}\right) \quad .$$

Proof. The proof only requires easy computations from

$$\begin{aligned} V(t,x) \ &= \ \mathbb{P}(X_t \leq x) \ = \ \mathbb{P}\left(X_0 + \int_0^t \left(\int_{\mathbb{R}} H(X_s - y) U_s(dy)\right) ds + \sigma w_t \ \leq x\right) \\ &\leq \ \mathbb{P}(\sigma w_t + X_0 \ \leq x) \quad , \end{aligned}$$

and the estimate

$$\forall x \in \mathbb{R} \quad , \quad \int_{|x|}^{+\infty} \exp\left(-\frac{y^2}{2}\right) dy \ \leq \ C \exp\left(-\frac{x^2}{2}\right) \quad . \ \blacksquare$$

We are now in position to prove the proposition 3.1.

For $(t,x) \in [0,T]$, set

$$F(t,x) \ = \ \int_{-\infty}^{x} V(t,y)dy$$

(which is well defined in view of the above lemma) and

$$W(t,x) \ = \ \exp\left(-\frac{1}{\sigma^2} \int_{-\infty}^{x} V(t,y)dy\right) \quad .$$

As V is a weak solution of the Burgers equation in $]0,T[\times\mathbb{R}$, F satisfies the following equality in the distribution sense:

$$\frac{\partial}{\partial x}\left(-\frac{\partial F}{\partial t} \ + \ \frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial x^2} \ \right) \ = \ \frac{1}{2}\frac{\partial}{\partial x}\left(\ V^2 \ \right) \quad \text{in }]0,T[\times\mathbb{R} \quad .$$

The distributions $(-\dfrac{\partial F}{\partial t}+\dfrac{1}{2}\sigma^2\dfrac{\partial^2 F}{\partial x^2})$ et $\frac{1}{2}V^2$ have the same spatial derivatives, therefore their difference is a distribution invariant by translations along the x-axis. Then, for any test function Φ and any $z \in I\!\!R$, one has

$$
\begin{aligned}
&< -\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 F}{\partial x^2} - \frac{1}{2}V^2 , \Phi > \\
&\quad = \int F(t,x)\left(\frac{\partial \Phi}{\partial t}(t,x+z) + \frac{1}{2}\sigma^2\frac{\partial^2\Phi}{\partial x^2}(t,x+z)\right)dtdx - \int \frac{1}{2}V^2(t,x)\,\Phi(t,x+z)dtdx \\
&\quad = \int F(t,x-z)\left(\frac{\partial \Phi}{\partial t}(t,x) + \frac{\sigma^2}{2}\frac{\partial^2\Phi}{\partial x^2}(t,x)\right)dtdx - \int \frac{1}{2}V^2(t,x-z)\,\Phi(t,x)dtdx \quad .
\end{aligned}
$$

Using the preceding lemma, the bounded convergence theorem and the fact that $V(t,.)$ is a distribution function, one can check that the right handside tends to 0 when z tends to $+\infty$.

Denote by (Φ_k) a sequence of smoothing functions in $I\!\!R^2$, define $\bar{F}$ and $\bar{V}$ in $I\!\!R^2$ by

$$
\bar{F}(t,x) = \begin{cases} F(t,x), & \text{if } (t,x) \in (0,T] \times I\!\!R \\ 0, & \text{if } (t,x) \in I\!\!R^2 \backslash (0,T] \times I\!\!R\,, \end{cases}
$$

and

$$
\bar{V}(t,x) = \begin{cases} V(t,x), & \text{if } (t,x) \in (0,T] \times I\!\!R \\ 0, & \text{if } (t,x) \in I\!\!R^2 \backslash (0,T] \times I\!\!R \quad . \end{cases}
$$

Define the functions F_k, V_k and W_k on $(0,T] \times I\!\!R$ by

$$
\begin{aligned}
F_k(t,x) &:= \left(\Phi_k * \bar{F}\right)(t,x) \\
V_k(t,x) &:= \left(\Phi_k * \bar{V}\right)(t,x) \\
W_k(t,x) &:= \exp\left(-\frac{1}{\sigma^2}\,F_k(t,x)\right) \quad .
\end{aligned}
$$

First, we note that (W_k) converges to W in the distribution sense: let ϕ be a test function and let K be such that $Supp\phi \subset (0,T)\times]-K,K[$; for any k such that $Supp\Phi_k \subset (]-K,K[)^2$, one has

$$
\begin{aligned}
&\sigma^2 \int_{(0,T]\times I\!\!R} |W_k(t,x) - W(t,x)| \cdot |\phi(t,x)|\, dt\, dx \\
&\qquad \le \int_{(0,T]\times I\!\!R} |F_k(t,x) - F(t,x)|\ |\phi(t,x)|\, dt\, dx
\end{aligned}
$$

$$
\begin{aligned}
&\leq \int_{Supp\phi} \left| F(t,x) - \mathbb{1}_{]-2K,2K[}(x) F(t,x) \right| \, |\phi(t,x)| \, dt \, dx \\
&\quad + \int_{Supp\phi} \left| \mathbb{1}_{]-2K,2K[}(x) F(t,x) - \left(\mathbb{1}_{]-2K,2K[} F \right)_k (t,x) \right| \, |\phi(t,x)| \, dt \, dx \\
&\quad + \int_{Supp\phi} \left| \Phi_k * \left(\bar{F} - \mathbb{1}_{]-2K,2K[} \bar{F} \right) (t,x) \right| \, |\phi(t,x)| \, dt \, dx \\
&= \int_{Supp\phi} \left| \mathbb{1}_{]-2K,2K[}(x) F(t,x) - \left(\mathbb{1}_{]-2K,2K[} F \right)_k (t,x) \right| \, |\phi(t,x)| \, dt \, dx \, ;
\end{aligned}
$$

the lemma A.1 shows that the function $\mathbb{1}_{]-2K,2K[}F$ belongs to $L^1((0,T) \times \mathbb{R})$, which implies that the sequence $(\mathbb{1}_{]-2K,2K[}F)_k$ converges to $\mathbb{1}_{]-2K,2K[}F$ in $L^1((0,T) \times \mathbb{R})$.

Besides, denoting $(V^2)_k := \Phi_k * V^2$, one can check that

$$
\frac{\partial W_k}{\partial t} - \frac{1}{2}\sigma^2 \frac{\partial^2 W_k}{\partial x^2} = \frac{1}{2\sigma^2}\left[(V^2)_k - (\frac{\partial F_k}{\partial x})^2 \right] W_k = \frac{1}{2\sigma^2}\left[(V^2)_k - (V_k)^2 \right] W_k \quad .
$$

Then, letting k go to infinity, easy computations show that W satisfies the heat equation, so that, for $0 < s < t \leq T$,

$$
\begin{aligned}
W(t,x) &= \frac{1}{\sqrt{2\pi\sigma^2(t-s)}} \int_{\mathbb{R}} W(s,y) \, \exp\left(-\frac{(x-y)^2}{2\sigma^2(t-s)} \right) dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} W(s, \sigma\sqrt{t-s}z + x) \, \exp\left(-\frac{z^2}{2} \right) dz \quad .
\end{aligned}
$$

We now make s tend to zero; the lemma A.1 and the bounded convergence theorem imply that $F(s,x)$ converges to $F(0,x)$ when s tends to 0; consequently, we get that

$$
W(t,x) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \int_{\mathbb{R}} \exp\left(-\frac{1}{\sigma^2} \left[\frac{(x-y)^2}{2t} + \int_{-\infty}^{y} V_0(z) dz \right] \right) dy \quad . \blacksquare
$$

A.2 Proof of the lemma 5.2

In the chapter 13 of [10], I. I. Gihman and A. V. Skorohod give a representation of the transition density of a process (X_t) solution to

$$
dX_t = b(t, z_t) + \sigma dw_t \quad ,
$$

under the condition that the derivatives $\partial_x b(t,x)$ and $\partial_t b(t,x)$ are well defined, and that the function B defined by

$$
B(t,x) = -\frac{1}{2\sigma^2} b^2(t,\sigma x) - \frac{1}{2}\frac{\partial b}{\partial x}(t,\sigma x) - \int_0^x \frac{1}{\sigma}\frac{\partial b}{\partial t}(t,\sigma z) dz
$$

satisfies:

$$\overline{\lim_{x\to\infty}} \frac{\sup_{0\le t\le T} B(t,x)}{1+x^2} = 0 \quad . \tag{42}$$

The formula for the density $\gamma_t(x,y)$ of the law of $X_t(x)$ is as follows:

$$\begin{aligned}\gamma_t(x,y) &= \frac{1}{\sqrt{2\pi t\sigma^2}} \exp\left(-\frac{(y-x)^2}{2t\sigma^2}\right) \exp\left\{\frac{1}{\sigma^2}\left(\int_0^y b(t,z)dz - \int_0^x b(0,z)dz\right)\right\} \\ &\times \mathbb{E}\exp\left\{t\int_0^1 B\left(ut, \frac{y}{\sigma} + (w(tu)-w(t)) + \frac{u}{\sigma}(x-y)\right)du\right\} \quad .\end{aligned}$$

In our case b is equal to V and the condition (42) seems difficult to check for the process $(z_t(x))$ because of the discontinuity of the derivatives of V at $t=0$ when U_0 is a Dirac measure. Thus we introduce an intermediate process $(z_t^\varepsilon(x))$ with $\varepsilon>0$, and we will get the desired result by making ε decrease to 0.

Let $(z_t^\varepsilon(x))$ be defined by

$$z_t^\varepsilon(x) = x + \int_0^t V(s+\varepsilon, z_s^\varepsilon(x))\, ds + \sigma w_t \quad .$$

Set

$$B^\varepsilon(t,x) = -\frac{1}{2\sigma^2}V^2(t+\varepsilon,\sigma x) - \frac{1}{2}\frac{\partial V}{\partial x}(t+\varepsilon,\sigma x) - \int_0^x \frac{1}{\sigma}\frac{\partial V}{\partial t}(t+\varepsilon,\sigma z)dz \quad .$$

As V is the solution of the Burgers equation,

$$B^\varepsilon(t,x) = \frac{1}{2}\frac{\partial V}{\partial x}(t+\varepsilon,0) - \frac{\partial V}{\partial x}(t+\varepsilon,\sigma x) - \frac{1}{2\sigma^2}V^2(t+\varepsilon,0) \quad .$$

We already proved that, for all $t\in]0,T]$, $\left\|\frac{\partial V}{\partial x}(t,.)\right\|_{L^\infty(\mathbb{R})} \le \frac{C}{\sqrt{t}}$. Thus, for all (t,x) in $]0,T]\times\mathbb{R}$, one has

$$|B^\varepsilon(t,x)| \le \frac{C}{\sqrt{\varepsilon}} + C \quad .$$

The condition (42) is satisfied, so that, $\gamma_t^\varepsilon(x,y)$ denoting the law of $z_t^\varepsilon(x)$,

$$\gamma_t^\varepsilon(x,y) = \frac{C}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{(y-x)^2}{2t\sigma^2}\right) \exp\left\{\frac{1}{\sigma^2}\left(\int_0^y V(t+\varepsilon,z)dz - \int_0^x V(s,z)dz\right)\right\} \quad .$$

But $V(t+\varepsilon,x) = \mathbb{E}_{U_0} H(x - z_{t+\varepsilon})$, thus

$$\gamma_t^\varepsilon(x,y) \;\leq\; \frac{C}{\sqrt{2\pi\sigma^2 t}} \exp\left(- \frac{(y-x)^2 \;-\; 2t|y-x|}{2t\sigma^2} \right) \quad .$$

For all $\gamma > \sigma$, an easy computation shows that

$$\exp\left(- \frac{(|y-x|-t)^2}{2t\sigma^2} \right) \;\leq\; \exp\left(- \frac{(y-x)^2}{2t\gamma^2} \right) \; \exp\left(\frac{t^2}{2(\gamma^2-\sigma^2)} \right) \quad .$$

Choose $\gamma = \sqrt{2}\,\sigma$:

$$\gamma_t^\varepsilon(x,y) \;\leq\; \frac{C}{\sqrt{2\pi t\sigma^2}} \exp\left(- \frac{(y-x)^2}{4t\sigma^2} \right) \quad .$$

Using the lemma (3.2), one easy obtains that $(z_t^\varepsilon(x))$ converges to $(z_t(x))$ in $L^1(\Omega)$ when $\varepsilon \to 0$. Thus, for any continuous and bounded function f, it holds that

$$\int_{\mathbb{R}} f(y)\gamma_t^\varepsilon(x,y)dy \stackrel{\varepsilon\to 0}{\longrightarrow} \int_{\mathbb{R}} f(y)\gamma_t(x,y) \quad .$$

Set

$$g_t(x,y) \;:=\; \frac{1}{\sqrt{2\pi t\sigma^2}} \exp\left(- \frac{(y-x)^2}{4t\sigma^2} \right) \quad .$$

As

$$\int_{\mathbb{R}} f(y)\gamma_t^\varepsilon(x,y)dy \;\leq\; \int_{\mathbb{R}} f(y)g_t(x,y)dy\,,$$

it comes that

$$\int_{\mathbb{R}} f(y)\gamma_t(x,y)dy \;\leq\; \int_{\mathbb{R}} f(y)g_t(x,y)dy\,,$$

which implies that $\gamma_t(x,y) \leq g_t(x,y)$. ∎

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Éditeur
INRIA, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)
ISSN 0249-6399