Large scale direct optimal control applied to the re-entry problem

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Large scale direct optimal control applied to the re-entry problem

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Abstract: We present the numerical solution of an atmospheric reentry problem for a space shuttle. We discretize the control and state with the same grid, and use a large-scale successive quadratic programming technique. With the help of sliding horizon and successive refinement of the discretization, we can solve on a workstation a problem with 1600 time step, an unusually large figure for this kind of real world optimal control problem.

Key-words: Optimal control, Discretization, Path following, Differential equations, Newton's method, Successive quadratic programming.

(Résumé : tsvp)

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Une méthode directe de contrôle optimal de grande taille appliquée au problème de rentrée

Résumé : Nous présentons la résolution numérique du problème de la rentrée dans l'atmosphère d'une navette spatiale. Le contrôle et l'état sont discrétisés de manière identique et on utilise une méthode de programmation quadratique successive de grande taille. Grâce à une technique d'horizon fuyant et de raffinement de la discrétisation, on peut résoudre sur une station de travail un problème avec 1600 pas de temps, nombre élevé pour ce type de problème de contrôle optimal appliqué.

Mots-clé : Contrôle optimal, Discrétisation, Suivi de trajectoire, Equations différentielles, Méthode de Newton, Programmation quadratique successive

1. Introduction. The technique called *direct optimal control* consists in discretizing an optimal control problem, and then solving the resulting nonlinear programming problem. It is often opposed to the techniques based on Pontryagin's principle, in which the control is expressed as a function of the state and costate, reducing the optimality system to a two points (in the simplest case) boundary value problem (TPBVP), which can be solved by a multiple shooting algorithm (Stoer and Burlirsch [15]).

The advantages of each method have been discussed thoroughly by many authors, among them Pesch [11]. It is recognized that multiple shooting is most effective when the starting point (for the state and costate) is good. In terms of complexity, this algorithm is optimal in the sense that the computational effort is (in the case of an integration scheme of order 1) proportional to the number of points used when integrating the differential system. In addition, the integration can be done using a device for controlling the precision. The drawbacks are that the method may have difficulties in converging if the starting point is poor, which may occur often as it is no easy to gives good initial values for the costate. In addition, any structural change in the constraints implies a modification of the system of equations to be solved.

The advantage of a priori discretizing an optimal control problem is that it is a general method, that is not so sensitive to an initial guess for the costate, and which allows to use the software already available for solving nonlinear programming problems. In the past, this kind of technique has often been combined with a low-dimension parametrization of the control (see e.g. Kraft [8]). In that case, the nonlinear programming problem has a small number of variables, and a large number of constraints: the distributed control and state constraints. There exists effective algorithms for dealing with this kind of structure, the so-called active set methods (Gill, Murray, Wright [7], p.). However, parametrizing the control destroys the local structure of optimal control problems. It is difficult to evaluate how far is the solution of the parametrized problem from the solution of original problem.

Another possibility is to discretize the control using the same time intervals as for the state. The aim of this paper is to explore such a possibility. The disadvantage we have to face up is the difficulty of solving the resulting large scale nonlinear programming problem. In particular, it seems difficult to obtain the same computational complexity as for multiple shooting. Rather, we may hope to obtain a less precise estimate of the optimal control, but it will be easier to obtain due to the generality of the method. Some results along this line were obtained recently by Betts and Huffman [1]. [2]

In this paper we study the application of large scale direct optimal control algorithm to the problem of atmospheric reentry of a space shuttle. In section 2, we explain how our model optimal control problem is discretized, and how the nonlinear programming problem is solved. In particular, we give a path algorithm that takes in account a poor initial guess for the optimal control, and a method of refinement of discretization that allows us to compute a more precise solution. In section 3, we expose the reentry problem, which is a highly nonlinear and state-constrained problem. Then in section 4, we give the numerical results. These results tend to show that the resolution of problems by a direct method and with a precise discretization, is possible at least in some realistic optimal control problems.

2. Formulation and resolution of the discrete optimal control problem. We consider the following family of optimal control problems (see e.g. Bryson and Ho [5]):

$$\begin{array}{l}
\text{minimize } V(y(T), u_c);\\
\dfrac{dy}{dt} = F(y(t), u, v(t), t) \quad t \in [0,T],\ y(0) = y_0,\\
\underset{\sim}{c}(t) \leq c(y(t), u, v(t)) \leq \tilde{c}(t) \quad t \in [0,T],\\
\underset{\sim}{c}_f \leq c_f(y(T), u) \leq \tilde{c}_f,\\
\underset{\sim}{y}(t) \leq y(t) \leq \tilde{y}(t),\\
\underset{\sim}{u} \leq u \leq \tilde{u},\\
\underset{\sim}{v}(t) \leq v(t) \leq \tilde{v}(t),
\end{array}$$

in which:

$T > 0$ is the final time,
$y(t) \in \mathbb{R}^{n_y}$ is the value of the state at time t,
$v(t) \in \mathbb{R}^{n_v}$ is the control,
$u \in \mathbb{R}^{n_c}$ is a set of parameters non depending on time to be optimized,
y_0 is a given value of the state at time 0,
V is the value function,
F is the dynamics of the problem,
c are the distributed constraints,
c_f are the final constraints.

We discretize the time interval as

$$0 = t_0 < t_1 < \cdots < t_{n_t} = T.$$

We discretize the control variables by taking functions that are of constant value u^k on each time step $[t_{k-1}, t_k[$, $1 \leq k \leq n_t$. Then we discretize the differential equation using an explicit one-step method (the classical fourth order Runge-Kutta scheme in our implementation). The discrete problem can be formulated as

$$\begin{array}{l}
\min V(y^{n_t}, u_c);\\
y^{k+1} = \Phi(y^{k-1}, u^k) \quad k = 1, \ldots, n_t,\ y^0 = y_0,\\
\underset{\sim}{c}^k \leq c(y^k, u, v^k) \leq \tilde{c}^k,\\
\underset{\sim}{c}_f \leq c_f(y^{n_t}, u) \leq \tilde{c}^f,\\
\underset{\sim}{y}^k \leq y^k \leq \tilde{y}^k,\\
\underset{\sim}{u} \leq u \leq \tilde{u},\\
\underset{\sim}{v}^k \leq v^k \leq \tilde{v}^k,
\end{array}$$

where of course the discrete bounds are discretization of the continuous bounds.

The discrete problem is a nonlinear programming problem of the following form:

$$(NLP) \qquad \min_{x \in \mathbb{R}^n} f(x)\ ;\ \underset{\sim}{x} \leq x \leq \tilde{x}\ ;\ \underset{\sim}{g} \leq g(x) \leq \tilde{g},$$

with $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^p$; indeed, x is here the vector composed by discrete state and control variables as well as the parameters. More precisely x is the concatenation of the vectors $(y^k)_{0 \leq k \leq n_t}, u_c, (u^k)_{1 \leq k \leq n_t}$ whereas $g(x)$ include the state equations as well as the distributed and final constraints of the discrete problem.

Problem (NLP) is, if the time discretization is fine, a large scale problem. An old idea for solving such problems is successive linear programming (SLP) which consists in, given a current point x^k, computing d^k solution of

$$(LP_k) \qquad \min_{d \in \mathbb{R}^n} f'(x^k)d\ ;\ \underset{\sim}{x} \leq x^k + d^k \leq \tilde{x}\ ;\ \underset{\sim}{g} \leq g(x^k) + g'(x^k)d^k \leq \tilde{g}.$$

The new point x^{k+1} may be $x^k + d^k$, or a point in the segment $[x^k, x^k + d^k]$ if a linesearch is used.

Another class of methods is sequential quadratic programming (SQP) in which, at each iteration, a direction d^k is computed as a local solution of

$$(QP_k) \qquad \min_{d \in \mathbb{R}^n} f'(x^k)d + \frac{1}{2} d^t H^k d \; ; \; \underset{\sim}{x} \leq x^k + d^k \leq \tilde{x} \; ; \; \underset{\sim}{g} \leq g(x^k) + g'(x^k)d^k \leq \tilde{g}.$$

Here H^k is an $n \times n$ matrix which is an approximation of the Hessian with respect to x of the Lagrangian associated with (NLP), i.e.

$$\mathcal{L}(x, \lambda, \mu) := f(x) + \lambda^t g(x) + \mu^t x$$

whose Hessian can be written as $\nabla^2_x \mathcal{L}(x, \lambda, \mu) = H(x, \lambda)$ with

$$H(x, \lambda) = \nabla^2 f(x) + \sum_{i=1}^{p} \lambda_i \nabla^2 g_i(x).$$

For the study of local convergence of successive quadratic programming, we refer to [4]. An early reference for successive quadratic programming applied to optimal control is Mitter [9].

We have compared (SQP) and (SLP) without linesearches. For (SLP) the solution of the linear program to be solved at each iteration is computed using a primal simplex algorithm. For (SQP) we use a conjugate reduced-gradient algorithm. Details of the implementation of these optimization algorithms may be found in [3]. For other implementation of large scale nonlinear programming algorithms we quote Murthagh and Saunders [10] and Shanno and Marsten [14].

For future reference we note that the first-order optimality system of (NLP) can be written in the following compact form

$$\begin{cases} \nabla f(x) + g'(x)^t \lambda + \mu = 0, \\ \mu \in \partial I_{[\underset{\sim}{x}, \tilde{x}]}(x), \\ \lambda \in \partial I_{[\underset{\sim}{g}, \tilde{g}]}(g(x)), \end{cases}$$

where I_K denotes the indicator of set K:

$$I_K(z) := \begin{cases} 0 \; if \; z \in K, \\ +\infty \; if \; not, \end{cases}$$

and ∂ is the subdifferential in the sense of convex analysis. The subdifferential of the indicatrix coincides with the set of outward normals in the sense of convex analysis (see Hiriart-Urruty and Lemaréchal [6]).

We consider the problem at atmospheric reentry of a space shuttle, the function to be minimized being the integral of thermal flux, subject to some state constraints.

3. The shuttle reentry problem. The dynamic equations are those of flight dynamics, without thrust:

$$\begin{aligned}
dV/dt &= -g_r \sin\gamma - g_\lambda \cos\gamma \cos\chi - \tfrac{\rho}{2m} V^2 S C_x \\
&\quad +\Omega^2 r(\cos\lambda \sin\gamma - \sin\lambda \cos\gamma \cos\chi)\cos\lambda \\
d\gamma/dt &= -g_r \cos\gamma/V + g_\lambda \sin\gamma \cos\chi/V + V(\cos\gamma)/r \\
&\quad +\tfrac{\rho}{2m} V S C_z \cos\mu + 2\Omega \sin\chi \cos\lambda \\
&\quad +\Omega^2 r \cos\lambda(\cos\lambda \cos\gamma - \sin\lambda \sin\gamma \cos\chi)/V \\
d\chi/dt &= g_\lambda \sin\chi/V(\cos\gamma) + V(\cos\gamma \sin\chi \tan\lambda)/r \\
&\quad +\tfrac{\rho}{2m} V S C_z \sin\mu/cos\gamma \\
&\quad +\Omega^2 r \cos\lambda \sin\lambda \sin\chi/(V \cos\gamma) \\
&\quad +2\Omega(\sin\lambda - \cos\lambda \cos\chi \tan\gamma) \\
dr/dt &= V \sin\gamma \\
d\lambda/dt &= \tfrac{V}{r} \cos\gamma \cos\chi
\end{aligned}$$

with the state variables

V the modulus of velocity,
γ flight path angle,
χ azimuth,
r distance from center of earth to shuttle,
λ latitude.

The equations include the following fixed parameters :
S reference surface,

Ω velocity of rotation of earth,
m mass of the vehicle,
g_r, g_λ taken equal to $g = 9.81$.

The intermediate functions
$\rho = \rho(r)$ atmospheric density,
$Mach(V, \rho) = V/V(\rho)$ with $V(\rho)$ the velocity of sound,
$C_x(\alpha, Mach)$ and $C_z(\alpha, Mach)$ drag and lift coefficients.

Two variables appear a priori as controls:
α the angle of attack,
μ the bank angle.

However, their derivatives are subject to bounds, so that we include them as state variables and consider their derivatives as the actual control:

$$\begin{aligned} \frac{d\alpha}{dt} &= \beta, \\ \frac{d\mu}{dt} &= \eta. \end{aligned}$$

There is an integral cost which is the total thermal flux modelized as

$$\int_0^T C_q \sqrt{\rho} V^3 dt.$$

Here $C_q > 0$ is a given constant. In order to comply with the general formulation, we write

$$\begin{aligned} \frac{dc}{dt} &= C_q \sqrt{\rho} V^3, \\ c(0) &= 0, \end{aligned}$$

so that the cost can be written as $c(T)$. There are some distributed constraints on

$$\begin{array}{ll} n_z := \rho S V^2 (C_x \sin\alpha + C_z \cos\alpha)/2mg & \text{normal acceleration,} \\ \varphi := C_q \sqrt{\rho} V^3 & \text{thermal flux,} \end{array}$$

which are

$$n_z \leq 2.5 \text{ and } \varphi \leq 4.10^5.$$

In addition there are some bound constraints on the state:

$$\gamma \leq 0 \ ; \ 0 \leq \alpha \leq 40^o \ ; \ 1^o \leq \mu(t) \leq 90^o$$

and the control

$$-1^o \leq \beta \leq 1^o \ ; \ -6^o \leq \eta \leq 6^o.$$

Also we take in account a final state constraint on the velocity

$$V(T) = V_T.$$

The final time is free. Through a change of variable on time we have a formulation of the problem with final time equal to 1.

4. Numerical experiments. We have built a fourth order Runge-Kutta integrator, and we compute the exact gradients for the discretized system. We also compute the Hessian of the Lagrangian, using a first-order discretization formula, so that our numerical Hessian is not far from the exact one. We have linked this piece of Fortran 77 code to SOS-OPSYC.[1] The overall software is called DOC (Direct Optimal Control). It uses the sparse LU factorization of Reid [13].

An essential difficulty in this kind of study consists in finding a reasonable starting point for the optimal control. This may require a high expertise level and a lot of time, whereas the aim of optimization techniques is precisely to speed up the design of the trajectory. In order to tackle with this difficulty we decided to optimize first over a small time interval, choosing a target (the final velocity) close to the initial value, and then to decrease the value of the final velocity; the solution computed for a given target is used as the initial point for the new problem with a lower target. In this case the length T of the time interval, being a result of the optimization process, increases automatically. Of course fixing these values of the

[1] An INRIA software for solving nonlinear programming problems by a large-scale successive linear or quadratic programming. SOS stands for Sparse Optimization Solver.

final velocity needs itself some tuning. Computation made with 50 times intervals used the following values for the final velocity:

Iteration	1	2	3	4	5	6	7	8	9	10	11
Final velocity	6.6	6.3	6.0	5.5	5.0	4.5	4.0	3.5	3.0	2.5	2.0

Successive values of final velocity (km/sec)

A more accurate discretization is desirable, but would lead to prohibitive computing times. We prefer to perform the above path-following method with a poor discretization, and then refine discretization. The difficulty is to be able to predict a reasonably good value of the set of active constraints for the refined problem. More than that –and here we have to describe a little more in detail the algorithms– reduced gradient methods use at each iteration of the algorithm a basis. This is a subset B of the components of the variable x, of cardinality $|B| = p$, where p is the dimension of the image space, i.e. $g(x) \in \mathbb{R}^p$. Writing $x = (x_B, x_N)$, where $N := \{1, \ldots, n\} \backslash B$, B is choosed in such a way that $\partial g(x)/\partial x_B$ is invertible. This allows to compute displacements of x such that the linearized constraints of (QP_k) are satisfied. The difficulty is to compute a reasonable basis for the refined problem. This prevents us from choosing an arbitrary refinement. In our experiments we always divided each step by half, so that each variable splits into two variables. Now in our application, the numerical solution has the property that the only variable not distributed on time, the final time, is basic whereas there is exactly one active final constraint. It follows that if each variable of the refined problem inherits from the status of the one from which it was created, i.e. basic, non basic, binding, then we have exactly the right number of basic variables for the refined problem.

In Table 1 below we give computing time for each refinement of the grid (computations were made on an IBM R6000/350 work station). In order to have an idea of the effectiveness of refinement, we ran the sliding horizon with 100 time step (instead of 50 as before). We compare in Table 2 the resulting computing times. The interest of refinement is clear from this result.

n			User-time		
50	$\rightarrow$	100	418 sec	=	6 min 58 sec
100	$\rightarrow$	200	923 sec	=	15 min 23 sec
200	$\rightarrow$	400	19,762 sec	=	5 h 29 min 22 sec
400	$\rightarrow$	800	12,078 sec	=	3 h 21 min 18 sec
800	$\rightarrow$	1600	21,655 sec	=	6 h 55 sec

Table 1
User time for doubling

Computation	User-time		
path with $nt = 100$	4,302 sec	=	1 h 11 min 42 sec
path with $nt = 50$	593 sec	=	9 min 53 sec
path with $nt = 50$ plus one doubling	593 sec + 418 sec = 1011 sec	=	16 min 51 sec

Table 2
Effectiveness of doubling

In Figures 1 to 4, we represent the bank angle and its derivative, the nonlinear state constraints, the velocity, flight, path angle and the altitude (comment).

-

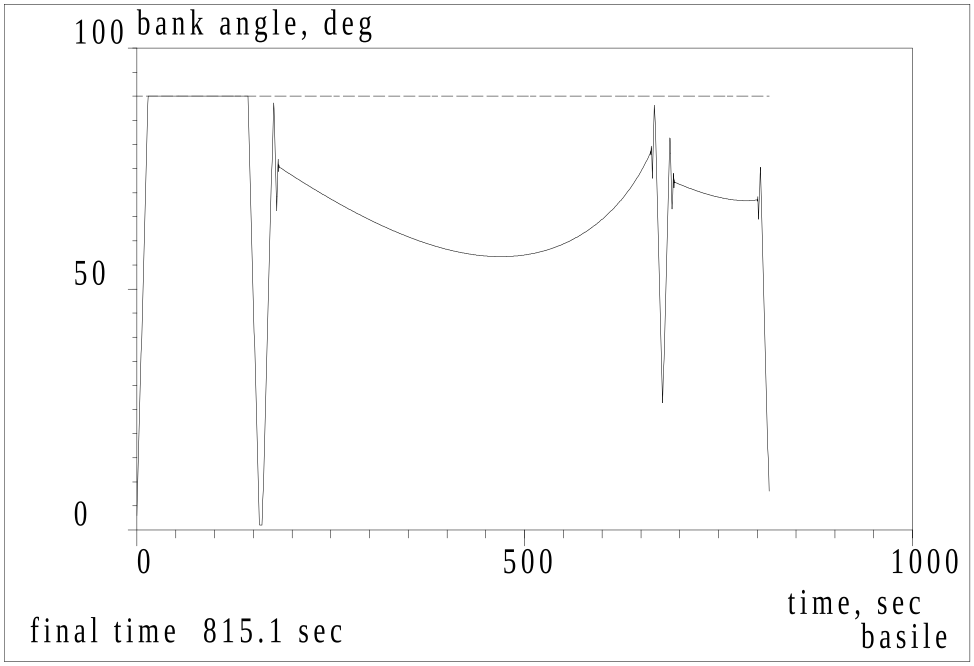

Figure 1(i): Bank angle

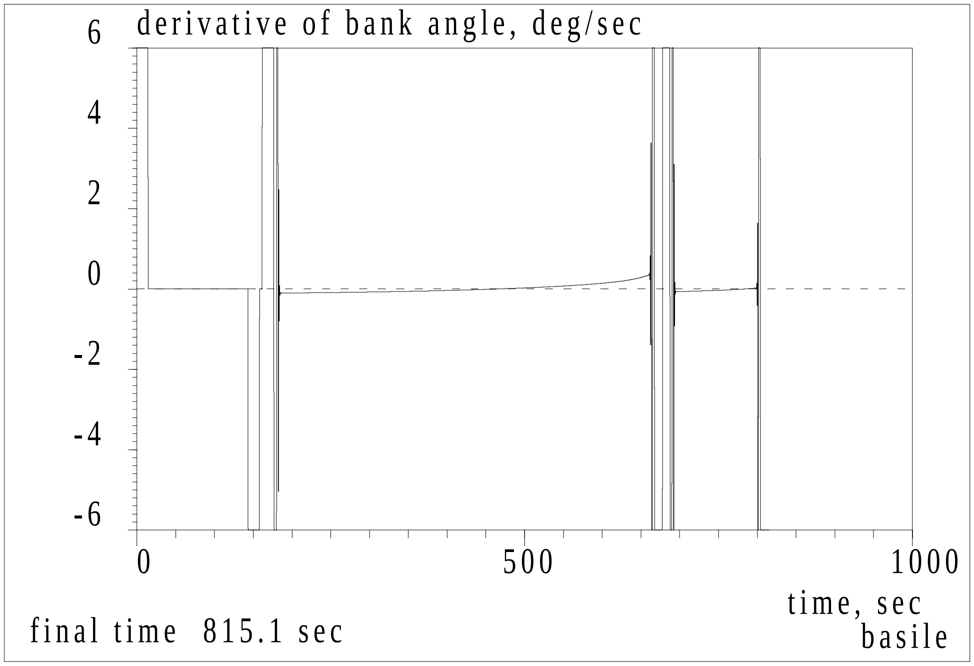

Figure 1(ii): Derivative of bank angle

-

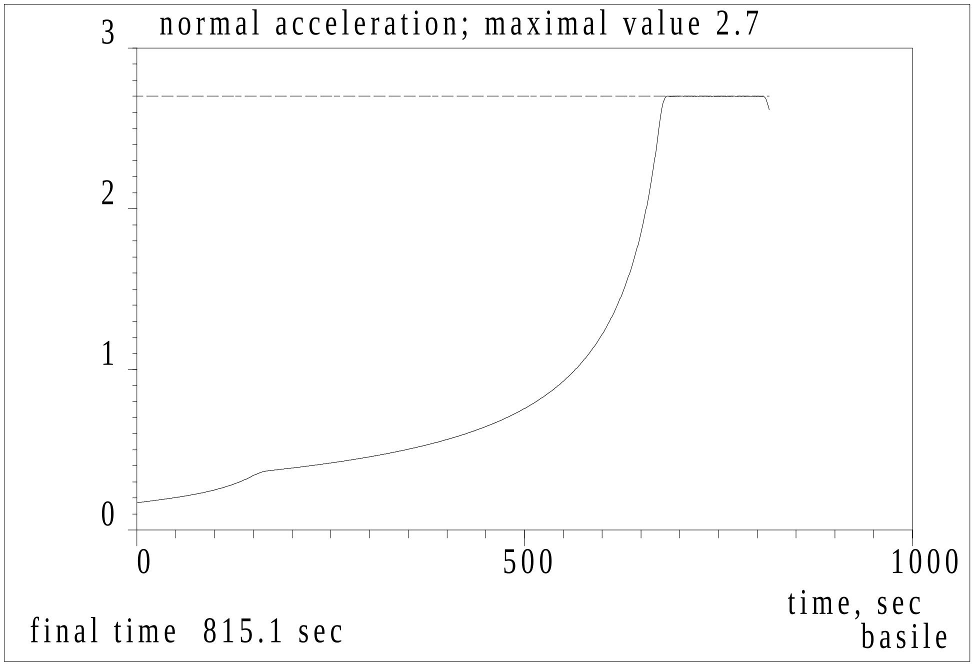

Figure 2(i) Normal acceleration

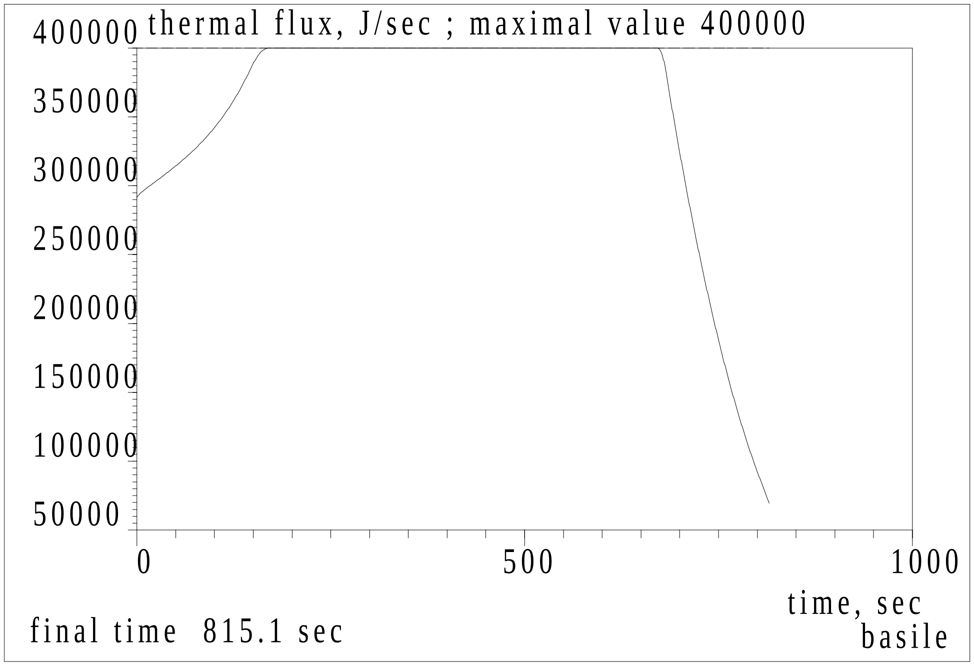

Figure 2(ii) Thermal flux

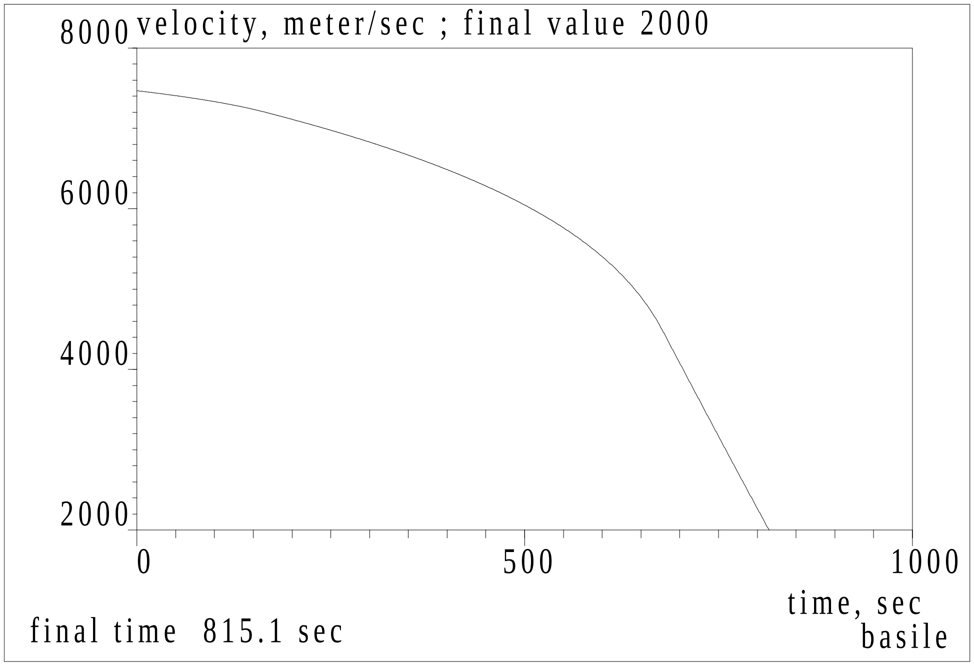

Figure 3(i): Velocity

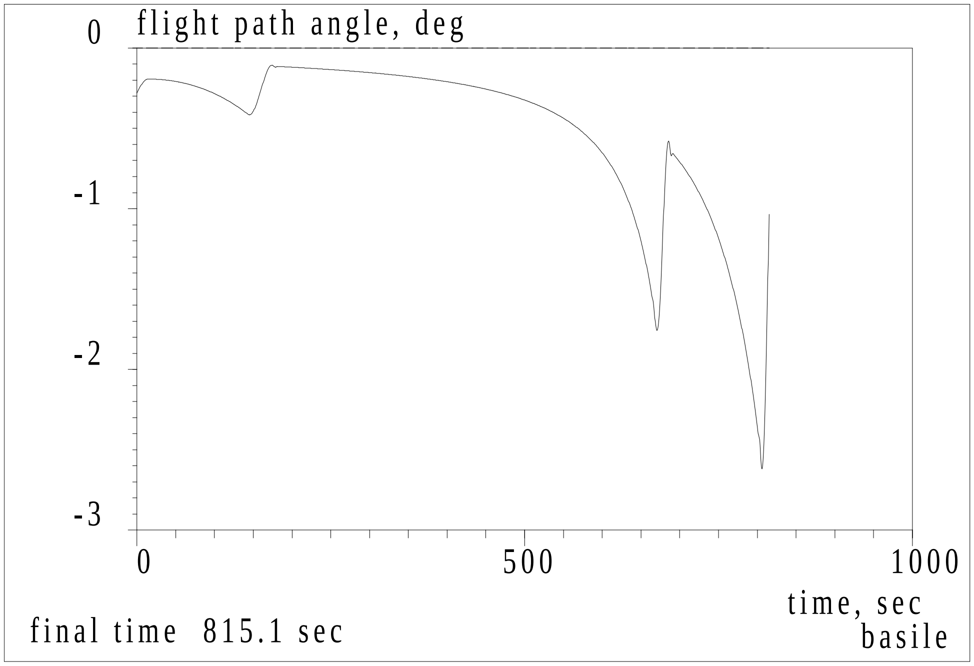

Figure 3(ii): Flight path angle

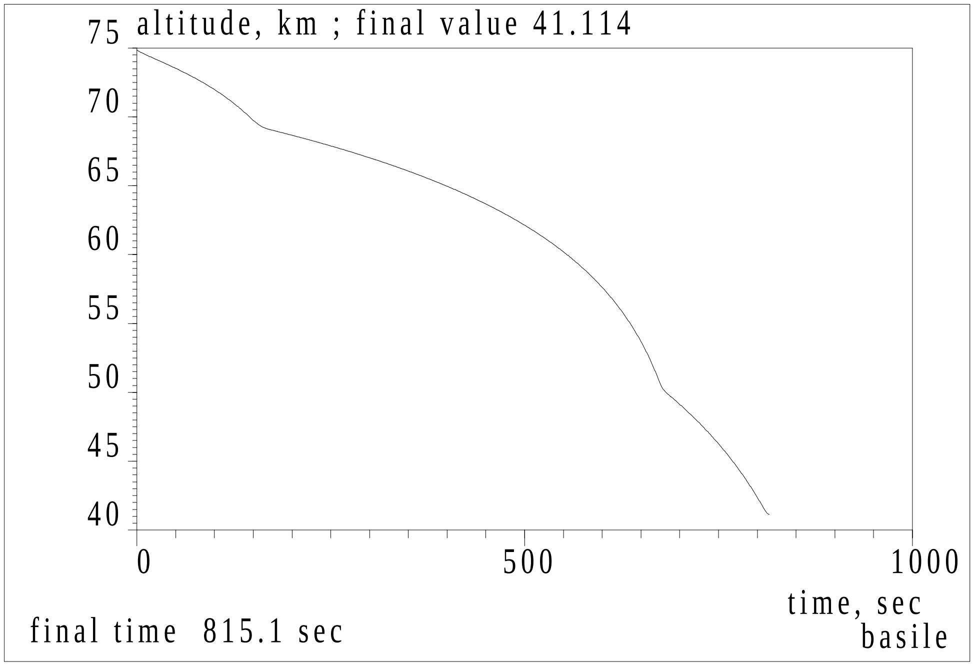

Figure 4: Altitude

We note that two state constraints are active: the thermal flux during the first stage of the trajectory, and then the normal acceleration. The optimal control can be described in the following way. The angle of attack remains binded to its upper bound. The trajectory is divided in three phases. In the first the bank angle is equal to its upper bound. Then the constraint on the thermal flux becomes active. Finally the constraint on normal acceleration is active. In addition there are four very short manoeuvres, in order to reach the upper bound of the bank angle, and to switch successively to the thermal flux and normal acceleration constraints, and then to reach the final velocity.

These results show that using large-scale SQP algorithm and an a priori discretization of the optimal control problem, we are able, at least on this example, to reach a reasonably good accuracy, for the computation of the optimal control and state and, may be more important, the optimal strategy has a simple physical interpretation.

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