# Predictive maintenance of an unreliable two-unit system

Chengbin Chu, Jean-Marie Proth, Philippe Wolff

INRIA

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

# *Predictive maintenance of an unreliable two-unit system*

Chengbin CHU - Jean-Marie PROTH
Philippe WOLFF

N° 2401
Novembre 1994

PROGRAMME 5

Rapport de recherche

# PREDICTIVE MAINTENANCE OF AN UNRELIABLE TWO-UNIT SYSTEM

Chengbin CHU[1], Jean-Marie PROTH[1,2], Philippe WOLFF[1]

**Abstract.** - We consider a system composed of two independent and non-identical units. This system is under periodic inspection. Each unit is characterized by a variable whose value evolves from one inspection to the next one according to an exponential distribution. The parameters of this distribution can be different for the two units.

The system breaks down if at least one of the two states exceeds a given value. These limits are referred to as breakdown limits.

A preventive maintenance is applied on one unit if its state is less than the breakdown limit but exceeds a given threshold, called maintenance limit. The cost of a maintenance is much less than the cost of a repair.

If a repair or a maintenance is applied on one unit, while the state of the second unit is less than the maintenance limit, a maintenance is performed on this second unit.

The goal of this study is to adjust the maintenance limits to minimize the long-run average cost per unit of time. We study the probability density for the system to be in a given state. We then give the analytical expression of the cost function according to the maintenance limits. The values of these limits that minimize this function are then calculated using gradient estimation.

**Keywords :** Ageing Process, Multi-Unit Maintenance Policy, Stochastic System, Unreliable System.

---

1. INRIA-Lorraine, Technopôle METZ 2000, 4 rue Marconi 57070 METZ, FRANCE.

Tel : (33) 87 20 35 00 Fax : (33) 87 76 39 77 Email : Wolffp@loria.fr

2. Institute for System Research, University of Maryland, College Park, MD 20742, USA.

# MAINTENANCE PREDICTIVE D'UN SYSTEME NON FIABLE A DEUX UNITES

Chengbin CHU[1], Jean-Marie PROTH[1,2], Philippe WOLFF[1]

**Résumé.** - Nous nous intéressons à un système composé de deux unités indépendantes et non identiques. Chaque unité est caractérisée par une variable d'état dont l'évolution suit une loi de probabilité exponentielle.

Le système tombe en panne si l'état d'au moins une des deux unités dépasse une limite donnée, appelée limite de panne.

Une maintenance préventive est effectuée sur une unité si son état devient supérieur à un seuil, appelé limite de maintenance, tout en restant inférieur à la limite de panne.

Nous supposons que le coût de maintenance est beaucoup plus faible que celui de réparation. Si une opération de maintenance ou une réparation doit être effectuée sur une unité tandis que l'état de la seconde est inférieur à sa limite de maintenance, on réalise aussi une opération de maintenance sur cette deuxième unité.

Le but de cette étude est d'ajuster les limites de maintenance afin de minimiser le coût moyen de cette politique. Nous donnons l'expression analytique de la fonction coût puis nous calculons, par une méthode d'estimation du gradient, la valeur des limites de maintenance qui la minimisent. L'intérêt de cette politique de maintenance prédictive est illustré par une comparaison avec des politiques classiques de maintenance.

**Mots clés :** Politique de Maintenance pour Système à Plusieurs Unités, Processus de Vieillissement, Systèmes Non-Fiables, Processus Stochastiques.

---

1. INRIA-Lorraine, Technopôle METZ 2000, 4 rue Marconi 57070 METZ, FRANCE.

Tel : (33) 87 20 35 00 Fax : (33) 87 76 39 77 Email : Wolffp@loria.fr

2. Institute for Systems Research, University of Maryland, College Park, MD 20742, USA.

## 1. INTRODUCTION

Many maintenance models for multi-unit systems have been studied in the literature. These models usually combine periodic replacement with a policy such that maintenance can only be performed when a unit of the system fails. The decision to perform a preventive maintenance is usually based on the age of the components and/or the number of failed units in the system (ASSAF et al. [1], HAURIE et al. [6], RITCHKEN et al. [8], VANNESTE et al. [10]). BERG [2][3] consider a two-unit system and investigate the preventive replacement of a unit upon the failure of the other one. A complete survey of these models can be found in CHO and PARLAR [4].

The starting point of all these studies is the probability density of failure of the elements of the system. For single-unit system, work has been done based on the transition probability of the state variable of the system (TAYLOR [9], CHU et al. [5]). In these models, a preventive maintenance can be performed to avoid breakdowns. This intervention is called predictive maintenance as it is based on the prediction of the evolution of the system. Avoiding breakdowns result in a drastic reduction of the cost since corrective maintenance cost is very high compared with the one of a preventive maintenance.

The goal of this work is to extend this approach to two-unit systems.

We consider a system composed of two elements stochastically independent (i.e. the probability of transition of the state variable of a unit does not depend on the state of the other unit). Furthermore, the possibility to combine interventions (maintenance or repair) on both units generates an important cost reduction.

Indeed, the cost of the maintenance can be divided into two parts :

- the cost of the preventive maintenance or repair of a unit,
- the cost of the intervention on the system (i.e. cost incurred by the loss of productivity if the system has to be stopped, maintenance staff, ...).

If the cost of the units are very small compared to the cost of intervention, it could be interesting to performed a maintenance on the whole system when a maintenance on one part is needed.

In section 2 we present the model of the system and define our predictive maintenance policy. In section 3, we study the probability density for the system to be at a given state. The analytical expression of the mean value of the cost function with regards to the maintenance limits is given in section 4. In section 5, we propose a method based on gradient estimation to compute the values of the maintenance limits that minimize this cost. This policy is then compared with corrective maintenance and "classical" preventive maintenance policies.

## 2. MODEL DESCRIPTION AND MAINTENANCE POLICY

### 2.1. Model of the system:

We consider an unreliable system composed of two components in series. The transition probability from one value of the state variable to the next one, i.e. the one reached at the next inspection, follows an exponential probability law with parameter $\mu_x$ for the first unit and $\mu_y$ for the second one ($\mu_x$>0 and $\mu_y$>0).

We denote by $S_t = \begin{pmatrix} x_t \\ y_t \end{pmatrix}$ the vector representing the state of the system after inspection t.

The first (respectively second) component of $S_t$ describes the state of the first (respectively second) unit.

The transition probability from state $S_{t-1}$ to state $S_t$ depends on state $S_{t-1}$. This density is denoted by $f(S_t / S_{t-1})$, and is expressed as :

$$f(S_t / S_{t-1}) = \begin{cases} \mu_x \mu_y e^{-\mu_x(x_t - x_{t-1}) - \mu_y(y_t - y_{t-1})} & \text{if } x_t > x_{t-1} \text{ and } y_t > y_{t-1} \\ 0 \quad \text{otherwise} \end{cases}$$

Since $S_t^-$ never decreases (i.e. $\forall t > 0$, $x_{t-1} < x_t$ and $y_{t-1} < y_t$), this system follows an ageing process.

We assume that the system is in working order if $x_t \in [0, L_x)$ and $y_t \in [0, L_y)$ ($L_x$ and $L_y$ are the strictly positive breakdown limits). The system breaks down as soon as either $x_t$ becomes greater than or equal to $L_x$ or $y_t$ becomes greater than or equal to $L_y$.

## 2.2. Predictive maintenance policy :

The maintenance policy is defined as follows :

- if the state of the first unit exceeds $L_x$ or the state of the second unit exceeds $L_y$, the corresponding unit is repaired,
- if the state of the first (respectively second) unit exceeds a maintenance limit X ($X<L_x$) (respectively Y ($Y<L_y$)) while remaining less than the breakdown limit, a preventive maintenance is performed on it.

These interventions are instantaneous and put the unit in its original state.

As we have already seen in the introduction, we consider a system for which the cost of the elements are very small compared to the cost incurred when stopping the system to perform a repair or a maintenance. Thus, we decide to maintain a unit which is subject to neither a repair nor a maintenance when the other unit must be repaired or maintained. We can then add the following rule to the definition of the maintenance policy :

- if a repair or a maintenance is performed on one unit, and if the other unit is in normal functioning, a maintenance is also done on it. These interventions are instantaneous and put the unit in its original state, i.e. $S_t = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Figure 1 shows the different domains of intervention and their cost in the state-plan.

The cost associated with a predictive maintenance is, of course, smaller than the cost of a repair after a breakdown.

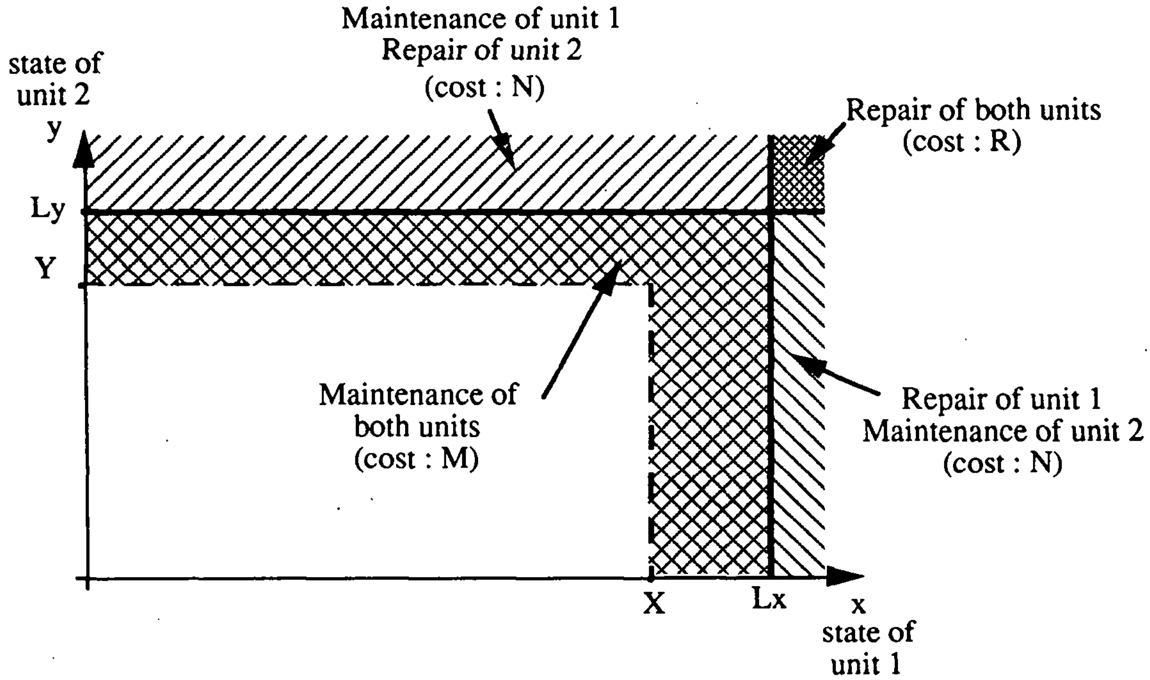


Fig. 1 : Domains of maintenance/repair and associated costs

## 3. PROBABILITY DENSITY OF THE STATE OF THE SYSTEM

The cost function depends on the probability density of the state of the system.

Let $A = \begin{pmatrix} u \\ v \end{pmatrix}$ and $B = \begin{pmatrix} x \\ y \end{pmatrix}$ be two vectors representing the state of the system.

The density of probability for the state to be in (x,y) is denoted p(x,y).

There are two types of transitions from state A to state B :

- A was in the normal functionning domain, i.e. $u \leq X$ and $v \leq Y$. Due to the fact that the process we are studying is an ageing process, the previous state can not be greater than B, i.e. $u \leq x$ and $v \leq y$. The transition probability from A to B is then :

$$\mu_x \mu_y p(u,v) e^{-\mu_x (x-u) - \mu_y (y-v)} du dv$$

- A was a repair or maintenance state of one or both units of the system ($u > X$ or/and $v > Y$). As we have seen in the definition of the maintenance policy, the state of the system is immediately brought back to the original state, i.e. $A = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. The transition probability from A to B is then :

$$\mu_x\mu_y p(u,v)e^{-\mu_x x-\mu_y y}dudv$$

In steady state, the density p of the probability to be in state (x,y) is then given by the following integral equation :

$$p(x,y)=\mu_x\mu_y\Big(\int_{0^+}^{Min(x,X)}\int_{0^+}^{Min(y,Y)} p(u,v)e^{-\mu_x(x-u)-\mu_y(y-v)}dvdu + \int_X^{+\infty}\int_Y^{+\infty} p(u,v)e^{-\mu_x x-\mu_y y}dvdu + \int_0^X\int_Y^{+\infty} p(u,v)e^{-\mu_x x-\mu_y y}dvdu + \int_X^{+\infty}\int_0^Y p(u,v)e^{-\mu_x x-\mu_y y}dvdu\Big) \quad (1)$$

p(x,y) is a probability density. We can then add the following constraints :

$$\int_{0^+}^{+\infty}\int_{0^+}^{+\infty} p(u,v)dudv = 1 \quad (2)$$

and $p(u,v)\geq 0 \ \forall(u,v)\in R^+ x R^+$

Due to the Min operator in the first term of equation (1), we have to consider four cases :

**3.1. For $0<x\leq X$ and $0<y\leq Y$,** the system is in the normal functionning domain. Equation (1) can be written as :

$$p(x,y)=\mu_x\mu_y\Big(\int_{0^+}^{x}\int_{0^+}^{y} p(u,v)e^{-\mu_x(x-u)-\mu_y(y-v)}dvdu + \int_X^{+\infty}\int_Y^{+\infty} p(u,v)e^{-\mu_x x-\mu_y y}dvdu + \int_0^X\int_Y^{+\infty} p(u,v)e^{-\mu_x x-\mu_y y}dvdu + \int_X^{+\infty}\int_0^Y p(u,v)e^{-\mu_x x-\mu_y y}dvdu\Big)$$

If we note by $C(X,Y)$ the constant part of the right side of this equation, i.e.:

$$C(X,Y)=\mu_x\mu_y\left(\int_X^{+\infty}\int_Y^{+\infty}p(u,v)dvdu+\int_0^X\int_Y^{+\infty}p(u,v)dvdu+\int_X^{+\infty}\int_0^Y p(u,v)dvdu\right),$$

the equality can be rewritten as :

$$p(x,y)=\mu_x\mu_y\int_{0^+}^x\int_{0^+}^y p(u,v)e^{-\mu_x(x-u)-\mu_y(y-v)}dvdu+C(X,Y)e^{-\mu_x x-\mu_y y}$$

i.e :

$$p(x,y)e^{\mu_x x+\mu_y y}=\mu_x\mu_y\int_{0^+}^x\int_{0^+}^y p(u,v)e^{\mu_x u+\mu_y v}dvdu+C(X,Y)$$

if we set $q(x,y)=p(x,y)e^{\mu_x x+\mu_Y y}$, we obtain :

$$q(x,y)=\mu_x\mu_y\int_{0^+}^x\int_{0^+}^y q(u,v)\,dvdu+C(X,Y)$$

On the compact $(0,X)x(0,Y)$, this integral equation admits a unique solution in $C^1((0,X)x(0,Y))$. This solution is obtained by using the Picard-Lindelöf iteration method :

$$q_0=C(X,Y)$$
$$q_1=(\mu_x\mu_y xy+1)C(X,Y)$$
$$q_2=\left(\frac{(\mu_x\mu_y xy)^2}{4}+\mu_x\mu_y xy+1\right)C(X,Y)$$
$$\ldots$$
$$q_n=\left(\sum_{i=1}^{n}\frac{(\mu_x\mu_y xy)^i}{(i!)^2}+1\right)C(X,Y)=\left(\sum_{i=0}^{n}\frac{(\mu_x\mu_y xy)^i}{(i!)^2}\right)C(X,Y)$$

$q_n$ converges uniformly to the solution $q(x,y)$ in $C^1((0,X)x(0,Y))$ when $n\to+\infty$.

Thus for $0<x\leq X$ and $0<y\leq Y$, we have :

$$p(x,y)=\left(\sum_{i=0}^{+\infty}\frac{(\mu_x\mu_y xy)^i}{(i!)^2}\right)C(X,Y)e^{-\mu_x x-\mu_y y}\qquad(3)$$

**3.2. For x>X and y>Y**, the system has to be completely repaired due to failures on both units. The probability density is then given by the following equation :

$$p(x,y)=\mu_x\mu_y\Big(\int_{0^+}^{X}\int_{0^+}^{Y}p(u,v)e^{-\mu_x(x-u)-\mu_y(y-v)}dvdu$$
$$+\int_{0}^{X}\int_{Y}^{+\infty}p(u,v)e^{-\mu_x x-\mu_y y}dvdu+\int_{X}^{+\infty}\int_{0}^{Y}p(u,v)e^{-\mu_x x-\mu_y y}dvdu\Big)$$

Using the notations introduced in 3.1, we obtain :

$$p(x,y)=\mu_x\mu_y\int_{0^+}^{X}\int_{0^+}^{Y}p(u,v)e^{-\mu_x(x-u)-\mu_y(y-v)}dvdu+C(X,Y)e^{-\mu_x x-\mu_y y}$$

In the domain 0<x≤X and 0<y≤Y, the expression of the density p(x,y) is given by equation (3). Then :

$$p(x,y)=\left(\mu_x\mu_y\int_{0^+}^{X}\int_{0^+}^{Y}\left(\sum_{i=0}^{+\infty}\frac{(\mu_x\mu_y uv)^i}{(i!)^2}\right)dvdu\right)Ce^{-\mu_x x-\mu_y y}$$

The solution of (1) for x>X and y>Y is thus :

$$p(x,y)=\left(\sum_{i=0}^{+\infty}\frac{(\mu_x\mu_y XY)^i}{(i!)^2}\right)Ce^{-\mu_x x-\mu_y y} \qquad (4)$$

**3.3. For 0<x≤X and Y>Y**, one unit has failed and is repaired, while a maintenance is performed on the other one. We have :

$$p(x,y)=\mu_x\mu_y\Big(\int_{0^+}^{x}\int_{0^+}^{Y}p(u,v)e^{-\mu_x(x-u)-\mu_y(y-v)}dvdu+\int_{X}^{+\infty}\int_{Y}^{+\infty}p(u,v)e^{-\mu_x x-\mu_y y}dvdu$$
$$+\int_{0}^{X}\int_{Y}^{+\infty}p(u,v)e^{-\mu_x x-\mu_y y}dvdu+\int_{X}^{+\infty}\int_{0}^{Y}p(u,v)e^{-\mu_x x-\mu_y y}dvdu\Big)$$

Using the same notations as in 3.1, we obtain :

$$p(x,y) = \mu_x\mu_y \int_{0^+}^{x}\int_{0^+}^{Y} p(u,v)e^{-\mu_x(x-u)-\mu_y(y-v)}dvdu + C(X,Y)e^{-\mu_x x-\mu_y y}$$

The solution of (1) for $0<x\leq X$ and $y\geq Y$ is then :

$$p(x,y) = \left(\sum_{i=0}^{+\infty}\frac{(\mu_x\mu_y Yx)^i}{(i!)^2}\right)C(X,Y)e^{-\mu_x x-\mu_y y} \qquad (5)$$

**3.4. For x>X and 0<y≤Y,** the problem is the symmetric in y of the previous one, i.e. :

$$p(x,y) = \left(\sum_{i=0}^{+\infty}\frac{(\mu_x\mu_y Xy)^i}{(i!)^2}\right)C(X,Y)e^{-\mu_x x-\mu_y y} \qquad (6)$$

To obtain the exact function p(x,y), we need the expression of C(X,Y). This is done in the next subsection.

**3.5. Constant C(X,Y) :**

C(X,Y) is defined in 1.1 by the following equality :

$$C(X,Y) = \mu_x\mu_y\left(\int_X^{+\infty}\int_Y^{+\infty} p(u,v)dvdu + \int_0^{X}\int_Y^{+\infty} p(u,v)dvdu + \int_X^{+\infty}\int_0^{Y} p(u,v)dvdu\right) \quad (7)$$

Thus, C(X,Y) represents the probability to be either in breakdown or in preventive maintenance for at least one unit.

By using constraint (2) with equality (7), we obtain :

$$\int_{0^+}^{X}\int_{0^+}^{Y} p(u,v)du\,dv + \frac{C(X,Y)}{\mu_x\mu_y} = 1$$

If we replace p(x,y) by its expression for $0 \le x < X$ and $0 \le y < Y$, we get :

$$\int_{0^+}^{X}\int_{0^+}^{Y}\left(\sum_{i=0}^{+\infty}\frac{(\mu_x\mu_y uv)^i}{(i!)^2}\right)C(X,Y)e^{-\mu_x u-\mu_y v}du\,dv+\frac{C(X,Y)}{\mu_x\mu_y}=1$$

that is :

$$\frac{\mu_x\mu_y}{C(X,Y)}=1+\sum_{i=0}^{+\infty}\frac{1}{(i!)^2}\left(\int_0^{\mu_x X}u^i e^{-u}du\cdot\int_0^{\mu_Y Y}v^i e^{-v}dv\right)$$

If we introduce the Euler Gamma function $\Gamma(a)=\int_0^{+\infty}t^{a-1}e^{-t}dt$ and the incomplete Euler Gamma function $\Gamma(a,z)=\int_z^{+\infty}t^{a-1}e^{-t}dt$, the previous equality leads to :

$$\frac{\mu_x\mu_y}{C(X,Y)}=1+\sum_{i=0}^{+\infty}\frac{1}{(i!)^2}\left(\Gamma(i+1)-\Gamma(i+1,\mu_x X)\right)\left(\Gamma(i+1)-\Gamma(i+1,\mu_y Y)\right)$$

We know that if the parameter a is a positive integer, $\Gamma(a,z)=(a-1)!e^{-z}\sum_{r=0}^{a-1}\frac{z^r}{r!}$.

Then :

$$\frac{\mu_x\mu_y}{C(X,Y)}=1+\sum_{i=0}^{+\infty}\frac{1}{i!}\left(1-e^{-\mu_x X}\sum_{r=0}^{i}\frac{(\mu_x X)^r}{r!}\right)\left(1-e^{-\mu_y Y}\sum_{r=0}^{i}\frac{(\mu_y Y)^r}{r!}\right)$$

Finally :

$$C(X,Y)=\frac{\mu_x\mu_y}{1+\sum_{i=0}^{+\infty}\frac{1}{i!}\left(1-e^{-\mu_x X}\sum_{r=0}^{i}\frac{(\mu_x X)^r}{r!}\right)\left(1-e^{-\mu_y Y}\sum_{r=0}^{i}\frac{(\mu_y Y)^r}{r!}\right)} \qquad (8)$$

## 4. EXPRESSION OF THE COST FUNCTION

We define the following cost associated with the different types of interventions on the system :

- M : cost of the preventive maintenance on both units.
- N : cost of the preventive maintenance on one unit and the repair of the other.
- R : cost of the repair of both units.

As we notice before, M<N<R.

The long-run average cost per unit of time of the maintenance policy is equal to the sum of the cost for each type of intervention multiplied by the probability to perform it. It is given by the following function :

$$g(X,Y) = M\left(\int_0^X\int_Y^{L_y} p(u,v)dvdu + \int_X^{L_x}\int_0^{Y} p(u,v)dvdu + \int_X^{L_x}\int_Y^{L_y} p(u,v)dvdu\right)$$
$$+N\left(\int_X^{L_x}\int_{L_y}^{+\infty} p(u,v)dvdu + \int_{L_x}^{+\infty}\int_Y^{L_y} p(u,v)dvdu + \int_0^{X}\int_{L_y}^{+\infty} p(u,v)dvdu + \int_{L_x}^{+\infty}\int_0^{Y} p(u,v)dvdu\right)$$
$$+R\int_{L_x}^{+\infty}\int_{L_y}^{+\infty} p(u,v)dvdu$$

Using results (3) to (6) and (8), the previous expression can be rewritten as :

$$g(X,Y) = M\left(\int_0^X\int_Y^{L_y}\left(\sum_{i=0}^{+\infty}\frac{(\mu_x\mu_y Yu)^i}{(i!)^2}\right)C(X,Y)e^{-\mu_x u-\mu_y v}dvdu\right.$$
$$+\int_X^{L_x}\int_0^{Y}\left(\sum_{i=0}^{+\infty}\frac{(\mu_x\mu_y Xv)^i}{(i!)^2}\right)C(X,Y)e^{-\mu_x u-\mu_y v}dvdu$$
$$\left.+\int_X^{L_x}\int_Y^{L_y}\left(\sum_{i=0}^{+\infty}\frac{(\mu_x\mu_y XY)^i}{(i!)^2}\right)C(X,Y)e^{-\mu_x u-\mu_y v}dvdu\right)$$
$$+N\left(\int_X^{L_x}\int_{L_y}^{+\infty}\left(\sum_{i=0}^{+\infty}\frac{(\mu_x\mu_y XY)^i}{(i!)^2}\right)C(X,Y)e^{-\mu_x u-\mu_y v}dvdu\right.$$

$$+\int_{L_X}^{+\infty}\int_{Y}^{L_Y}\left(\sum_{i=0}^{+\infty}\frac{(\mu_x\mu_y XY)^i}{(i!)^2}\right)C(X,Y)e^{-\mu_X u-\mu_Y v}dvdu$$

$$+\int_{0}^{X+\infty}\int_{L_y}\left(\sum_{i=0}^{+\infty}\frac{(\mu_x\mu_y Yu)^i}{(i!)^2}\right)C(X,Y)e^{-\mu_x u-\mu_y v}dvdu$$

$$\left.+\int_{L_x}^{+\infty}\int_{0}^{Y}\left(\sum_{i=0}^{+\infty}\frac{(\mu_x\mu_y Xv)^i}{(i!)^2}\right)C(X,Y)e^{-\mu_x u-\mu_y v}dvdu\right)$$

$$+R\int_{L_x}^{+\infty}\int_{L_y}^{+\infty}\left(\sum_{i=0}^{+\infty}\frac{(\mu_x\mu_y XY)^i}{(i!)^2}\right)C(X,Y)e^{-\mu_x u-\mu_y v}dvdu$$

The resolution of this integral equation gives the analytical expression of the cost function :

$$g(X,Y)=\frac{1}{1+\sum_{i=0}^{+\infty}\frac{1}{i!}\left(1-e^{-\mu_x X}\sum_{r=0}^{i}\frac{(\mu_x X)^r}{r!}\right)\left(1-e^{-\mu_y Y}\sum_{r=0}^{i}\frac{(\mu_y Y)^r}{r!}\right)}$$

$$\left\{\left(\sum_{i=0}^{+\infty}\frac{(\mu_x\mu_y XY)^i}{(i!)^2}\right)\left[M\left(e^{-\mu_x X}-e^{-\mu_x L_x}\right)\left(e^{-\mu_y Y}-e^{-\mu_y L_y}\right)\right.\right.$$

$$+N\left(\left(e^{-\mu_x X}-e^{-\mu_x L_x}\right)e^{-\mu_y L_y}+\left(e^{-\mu_y Y}-e^{-\mu_y L_y}\right)e^{-\mu_x L_x}\right)$$

$$\left.+R\left(e^{-\mu_x L_x-\mu_y L_y}\right)\right]$$

$$+\left(\sum_{i=0}^{+\infty}\frac{(\mu_y Y)^i}{i!}\left(1-e^{-\mu_x X}\sum_{r=0}^{i}\frac{(\mu_x X)^r}{r!}\right)\right)\left[Ne^{-\mu_y L_y}+M\left(e^{-\mu_y Y}-e^{-\mu_y L_y}\right)\right]$$

$$\left.+\left(\sum_{i=0}^{+\infty}\frac{(\mu_x X)^i}{i!}\left(1-e^{-\mu_y Y}\sum_{r=0}^{i}\frac{(\mu_y Y)^r}{r!}\right)\right)\left[Ne^{-\mu_x L_x}+M\left(e^{-\mu_x X}-e^{-\mu_x L_x}\right)\right]\right\} \quad (9)$$

## 5. MINIMIZATION OF THE COST FUNCTION

we use a gradient approach to find an optimal solution to equation (9). Since we did not succeed to prove that g(X,Y) is convex, the result of this approach could be a local optimum. To evaluate our result, we compare it with some classical approaches in section 6.

We use an iterative approach to find the desirable solution: at each step of the computation, we try to evaluate the gradient $\nabla g(X_k,Y_k)$, where $X_k$ and $Y_k$ are the values of the parameters at the beginning of the k-th iteration. Let us denote by $\hat{\nabla} g(X_k,Y_k)$ the evaluation of $\nabla g(X_k,Y_k)$.

The general formulation of the algorithm is :

$$\begin{pmatrix} X_{k+1} \\ Y_{k+1} \end{pmatrix} = P\left(\begin{pmatrix} X_k \\ Y_k \end{pmatrix} - \alpha_k \hat{\nabla} g(X_k, Y_k)\right)$$

where: - $\begin{pmatrix} X_{k+1} \\ Y_{k+1} \end{pmatrix}$ is the value of the parameters at the beginning of the (k+1)-th iteration.

- P is the projection of the parameter values on the set of feasible values of the parameters.

- $\{\alpha_k\}$ is a series of parameters such that $\sum_{k=1}^{+\infty} \alpha_k \rightarrow +\infty$ and $\sum_{k=1}^{+\infty} \alpha_k^2 < +\infty$.

In practice, we decrease the value of $\alpha_k$ only when the sign of the gradient estimated through the simulation is different from the one used to estimate the criterion knowing the values $\begin{pmatrix} X_n \\ Y_n \end{pmatrix}$ of the parameters.

## 6. COMPARISON BETWEEN DIFFERENT MAINTENANCE POLICIES ON A NUMERICAL EXAMPLE

In this section, we compare the above policy with two other classical maintenance policies.

We define the following cost associated with the different types of intervention on the system :

- $M_1$ : cost of the preventive maintenance on one unit.
- M : cost of the preventive maintenance on both units.
- N : cost of the preventive maintenance on one unit and repair of the other.

- $R_1$ : cost of the repair of one unit.
- R : cost of the repair of both units.

Note : for the sake of simplicity, the cost of repair or maintenance has been considered equal for both units. All the previous and following calculus could also be done without additional difficulty with different costs for each unit.

### 6.1. Corrective maintenance :

This policy consists in repairing only failed units without doing any preventive maintenance.

Thus, the two units can be considered as independent of each other.

In CHU et al. [5], we have already studied a single-unit system whose state follows a exponential distribution of parameter $\mu$. If L is the breakdown limit, the distribution of probability of the state is :

$$p(x) = \begin{cases} \dfrac{\mu}{1+\mu L} & \text{for } x \le L \\ \dfrac{\mu e^{-\mu(x-L)}}{1+\mu L} & \text{for } x > L \end{cases}$$

For a two unit system, if we use the same notation as in 2), we obtain :

$$p(x,y) = \begin{cases} \dfrac{\mu_x \mu_y e^{-\mu_x(x-L_x)}}{(1+\mu_x L_x)(1+\mu_y L_y)} & \text{for } x > L_x \text{ and } y \le L_y \\ \dfrac{\mu_x \mu_y e^{-\mu_y(y-L_y)}}{(1+\mu_x L_x)(1+\mu_y L_y)} & \text{for } x \le L_x \text{ and } y > L_y \\ \dfrac{\mu_x \mu_y e^{-\mu_x(x-L_x)-\mu_y(y-L_y)}}{(1+\mu_x L_x)(1+\mu_y L_y)} & \text{for } x > L_x \text{ and } y > L_y \end{cases}$$

The first (respectively second) line corresponds to a state where the unit represented by the state variable x (respectively y) has to be repaired. The last line give the density of probability to be in a state where both units fail.

The cost of this policy depends only on the breakdown limits and is given by the following equation :

$$g_1 = R_1\left(\int_{L_x}^{+\infty}\int_{0^+}^{L_y} p(x,y)dydx + \int_{0^+}^{L_x}\int_{L_y}^{+\infty} p(x,y)dydx\right) + R\int_{L_x}^{+\infty}\int_{L_y}^{+\infty} p(x,y)dydx$$

The average cost per unit of time for the corrective maintenance policy is then :

$$g_1 = \frac{R_1\left(\mu_x L_x + \mu_y L_y\right) + R}{(1+\mu_x L_x)(1+\mu_y L_y)}$$

**6.2. "Classical" preventive maintenance :**

In this policy, a maintenance limit exists for each unit. As in the previous case, a unit fails if its state exceeds its breakdown limit. A maintenance is performed only if the state of a unit exceeds the maintenance limit and is less than its breakdown limit.

The two units can then be considered as independent.

From CHU et al. [5], for a single-unit system whose state x follow a exponential distribution of parameter $\mu$, with a breakdown limit L and a maintenance limit X, we have the following probabilities :

$$\Pr(x \le X) = \frac{\mu X}{(1+\mu X)},\quad \Pr(X < x \le L) = \frac{1 - e^{-\mu(L-X)}}{1+\mu X},\quad \Pr(x > L) = \frac{e^{-\mu(L-X)}}{1+\mu X}$$

For a two-unit system, the cost of this policy is :

$$\begin{aligned} g_2(X,Y) = {} & M_1\left(\Pr(X < x \le L_x)\Pr(y \le Y) + \Pr(Y < y \le L_y)\Pr(x \le X)\right) \\ & + M\left(\Pr(X < x \le L_x)\Pr(Y < y \le L_y)\right) \\ & + R_1\left(\Pr(x > L_x)\Pr(y \le Y) + \Pr(y > L_y)\Pr(x \le X)\right) \\ & + N\left(\Pr(X < x \le L_x)\Pr(y > L_y) + \Pr(Y < y \le L_y)\Pr(x > L_x)\right) \\ & + R\left(\Pr(x > L_x)\Pr(y > L_y)\right) \end{aligned}$$

The average cost per unit of time for the "classical" preventive maintenance policy is then :

$$g_2(X,Y)=\frac{1}{(1+\mu_x L_x)(1+\mu_y L_y)}\Bigg(M_1\Big(\mu_y Y\big(1-e^{-\mu_x(L_x-X)}\big)+\mu_x X\big(1-e^{-\mu_y(L_y-Y)}\big)\Big)$$
$$+M\Big(\big(1-e^{-\mu_x(L_x-X)}\big)\big(1-e^{-\mu_y(L_y-Y)}\big)\Big)$$
$$+R_1\Big(\mu_y Y e^{-\mu_x(L_x-X)}+\mu_x X e^{-\mu_y(L_y-Y)}\Big)$$
$$+N\Big(\big(1-e^{-\mu_y(L_y-Y)}\big)e^{-\mu_x(L_x-X)}+\big(1-e^{-\mu_x(L_x-X)}\big)e^{-\mu_y(L_y-Y)}\Big)$$
$$+R\Big(e^{-\mu_x(L_x-X)-\mu_y(L_y-Y)}\Big)\Bigg)$$

### 6.3. Comparison of the maintenance policies :

For this numerical example, we use the following values of the parameters :

$\mu_x=1$ $\quad\mu_y=1$ $\quad L_x=5$ $\quad L_y=7$

Note : it is always possible to modify the variables in order to have $\mu_x=\mu_y=1$.

The mean number of inspections before a breakdown is then 5 for the unit represented by x and 7 for the other.

The cost of intervention on the system is 20 u.c. (unity of cost). The cost of the maintenance of a unit is 0.5 u.c. and the cost of the repair of a unit is 100 u.c.. The global costs of maintenance and repair are then :

$M_1$=20.5 u.c. M=21 u.c. N=120.5 u.c. $R_1$=120 u.c. R=220 u.c.

| | Corrective | Classical | Predictive |
|---|---|---|---|
| X* | ($L_x$=5) | 2.34 | 2.21 |
| Y* | ($L_y$=7) | 3.75 | 3.61 |
| Minimum Cost | 34.58 | 12.09 | 10.46 |

Tab. I : Comparison of the different maintenance policies

The values of the maintenance limits X* and Y* that minimize the cost function were computed using perturbation analysis. The results are given in Table I. Predictive maintenance reduces the cost of the classical maintenance by more than 13%.

## 7. CONCLUDING REMARKS

In this work, we propose a predictive maintenance policy for an unreliable two-unit system. We assume that the repair cost is important compared to the cost of a maintenance. Based on this assumption, we observe via an numerical example that the predictive maintenance policy results in a smaller cost than the corrective and classical preventive maintenance policies.

Although this assumption is not restrictive, we plan to extend this study to cases where the repair cost is not very important.

We also continue to work on the theoretical properties of the cost function and particularly on its convexity.

## REFERENCES

[1] D. ASSAF, G. SHANTHIKUMAR, "Optimal Group Maintenance Policy with Continuous and Periodic Inspections", *Management Science*, vol. 33, no. 11, pp. 1440-1452, 1987.

[2] M. BERG, "Optimal Replacement Policies for Two-Unit Machines with Running Cost - I", *Stochastic Processes and their Applications*, vol. 4, pp. 89-106, 1976.

[3] M. BERG, "Optimal Replacement Policies for Two-Unit Machines with Running Cost - II", *Stochastic Processes and their Applications*, vol. 5, pp. 315-322, 1977.

[4] D.I. CHO, M. PARLAR, "A Survey of Maintenance Models for Multi-Unit Systems", *European Journal of Operational Research*, vol. 51, pp. 1-23, 1991.

[5] C. CHU, J-M. PROTH, Ph. WOLFF, "Prediction of the Behaviour of an Unreliable System. Application to the Choice of the Optimal Maintenance Policy", *INRIA Research Report* , RR-2207, 1994.

[6] A. HAURIE, P. L'ECUYER, "A Stochastic Control Approach to Group Preventive Replacement in a Multicomponent System", *IEEE Transactions on Automatic Control*, vol. 27, no. 2, pp.387-393, 1982.

[7] Y.C. HO, X.R. CAO, *Perturbation Analysis of Discrete Events Dynamic Systems*, Kluwer Academic Publisher, 1991.

[8] P. RITCHKEN, J.G. WILSON, "($m$,T) Group Maintenance Policies", *Management Science*, vol. 36, no. 5, pp. 632-639, 1990.

[9] H.M. TAYLOR, "Optimal Replacement Under Additive Damage and Other Failure Models", *Naval Research Logistics Quarterly*, vol. 22, pp.1-18, 1975.

[10] F.A. VAN DER DUYN SCHOUTEN, S.G. VANNESTE, "Two Simple Control Policies for a Multicomponent Maintenance System", *Operations Research*, vol.41, no. 6, pp. 1125-1135, 1993.

Unité de recherche INRIA Lorraine
Technopôle de Nancy-Brabois - Campus scientifique
615, rue du Jardin Botanique - B.P. 101 - 54602 Villers lès Nancy Cedex (France)
Unité de recherche INRIA Rennes - IRISA, Campus universitaire de Beaulieu 35042 Rennes Cedex (France)
Unité de recherche INRIA Rhône-Alpes - 46, avenue Félix Viallet - 38031 Grenoble Cedex 1 (France)
Unité de recherche INRIA Rocquencourt - Domaine de Voluceau - Rocquencourt - B.P. 105 - 78153 Le Chesnay Cedex (France)
Unité de recherche INRIA Sophia Antipolis - 2004, route des Lucioles - B.P. 93 - 06902 Sophia Antipolis Cedex (France)

Éditeur
INRIA - Domaine de Voluceau - Rocquencourt - B.P. 105 - 78153 Le Chesnay Cedex (France)

ISSN 0249 - 6399

*RR.2481*