Lyapunov exponents of controlled SDE's and stabilizability property : Some examples

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N° 2397

Novembre 1994

PROGRAMME 5

Traitement du signal,

automatique

et productique

Rapport de recherche

1994

ISSN 0249-6399

Lyapunov exponents of controlled SDE's and stabilizability property : Some examples

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Programme 5 — Traitement du signal, automatique et productique
Projet Mefisto

Rapport de recherche n° 2397 — Novembre 1994 — 15 pages

Abstract: We consider a stochastic differential equation with linear feedback control :

$$dX_t = (A + B\,K)\,X_t\,dt + \sum_{k=1}^{r} (A_k + B_k\,K)\,X_t \circ dW_k(t)$$

where K is the feedback gain matrix. For each value of K, let λ_K be the Lyapunov exponent associated with the solution of the SDE. The set of λ_K, as K describe the set of matrices, is a connected interval of $\mathbb{R}$. We present some examples where $-\infty$ is the lower bound of this set. For these cases, we say that the corresponding EDS is stabilizable.

Key-words: Stochastic differential equation, stabilizability, Lyapunov exponent.

(Résumé : tsvp)

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Exposants de Lyapunov d'EDS contrôlées et propriété de stabilisabilité : Quelques exemples

Résumé : On considère une équation différentielle stochastique avec contrôle linéaire en boucle fermée :

$$dX_t = (A + B\,K)\,X_t\,dt + \sum_{k=1}^{r}(A_k + B_k\,K)\,X_t\,\circ dW_k(t)$$

où K est une matrice de gain. Pour chaque valeur de K, soit λ_K l'exposant de Lyapunov associé à la solution de cette EDS. L'ensemble de valeurs de λ_K lorsque K parcours l'espace des matrices est un intervalle connexe de $\mathbb{R}$. On présente quelques exemples où $-\infty$ est la borne inférieure de cet ensemble. Pour ces exemples, nous dirons que l'EDS correspondante est stabilizable.

Mots-clé : Équation différentielle stochastique, stabilisabilité, exposant de Lyapunov.

1 Preliminaries

We consider the following linear stochastic differential equation in $\mathbb{R}^d$

$$dX_t = A\, X_t\, dt + \sum_{k=1}^{r} A_k\, X_t \circ dW_k(t)\ ,\ X_0 = x_0 \in \mathbb{R}^d\ ,\ x_0 \neq 0\ , \tag{1}$$

where $A, A_1, \ldots, A_k$ are $d \times d$ matrices, $W_1, \ldots, W_r$ are independent standard Wiener processes. Here "$\circ dW$" (resp. "dW") refer to the Stratonovich (resp. Itô) stochastic integral.

We define the Lyapunov exponent of the solution of (1) starting at x_0

$$\lambda(x_0) \triangleq \lim_{t\to\infty} \frac{1}{t} \log \|X_t\|\ . \tag{2}$$

Oseledec's multiplicative ergodic theorem states that the limit (2) exists with probability one and that there are d fixed numbers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_d$ – called the Lyapunov exponents of (1) – such that the random variable $\lambda(x_0)$ takes on only these values (see [1] for a review). Moreover, (1) is exponentially stable with probability one if and only if $\lambda_1 < 0$.

Let $S^{d-1} = \{x \in \mathbb{R}^d; \|x\| = 1\}$ denote the unit sphere of $\mathbb{R}^d$. We can define the projection of X_t onto the sphere by

$$U_t \triangleq \|X_t\|^{-1}\, X_t\ .$$

U_t is the solution of the following SDE on S^{d-1}

$$dU_t = h(A, U_t)\, dt + \sum_{k=1}^{r} h(A_k, U_t) \circ dW_k(t)\ , \quad U_0 = u_0 \triangleq \|x_0\|^{-1}\, x_0 \tag{3}$$

where

$$h(C, u) \triangleq C\, u - (C\, u, u)\, u\ . \tag{4}$$

Here (x, y) is the scalar product on $\mathbb{R}^d$ and $\|x\|^2 = (x, x)$.

Moreover,

$$\|X_t\| = \|x_0\|\, \exp\left\{\int_0^t q(U_s)\, ds + \sum_{k=1}^{r} \int_0^t p_k(U_s)\, dW_k(s)\right\} \tag{5}$$

with

$$\begin{aligned} q_0(u) &\stackrel{\triangle}{=} (A\,u,u)\ , \quad p_k(u) \stackrel{\triangle}{=} (A_k\,u,u)\ , \quad k=1,\dots,r\ , \\ q_1(u) &\stackrel{\triangle}{=} \frac{1}{2}\sum_{k=1}^{r}[(A_k^2\,u,u)+\|A_k\,u\|^2-2(A_k\,u,u)^2]\ , \\ q(u) &\stackrel{\triangle}{=} q_0(u)+q_1(u)\ . \end{aligned}$$

For each matrix M, $h(M,-u)=-h(M,u)$, so that $h(M,\cdot)$ can be viewed as a vector field on the projective space P^{d-1} (obtained from S^{d-1} by identifying u and $-u$). Therefore (3) can be considered as a stochastic differential equation on P^{d-1} and (5) is still valid with this definition. We make the following

Hypothesis 1.1 *For all $u \in P^{d-1}$*

$$\dim \mathsf{Lie\ Algebra}\{h(A,.),h(A_k,.),k=1,\dots,r\}(u)=d-1\ .$$

Theorem 1.2 *Under Hypothesis 1.1*

(i) *The diffusion process U_t admits a unique invariant probability measure μ. Moreover μ has a C^∞ density p with respect to the Lebesgue measure on P^{d-1} which solves the Fokker–Planck equation $L^*p=0$, where L is the infinitesimal generator associated with equation (3).*

(ii) *The number*
$$\lambda \stackrel{\triangle}{=} \int_{P^{d-1}} q(u)\,\mu(du)$$
is equal to the top Lyapunov exponent λ_1.

(iii) *For all $x_0 \in \mathbb{R}^d$, $x_0 \neq 0$, $\lambda(x_0)=\lambda$ with probability one. When $\lambda<0$, the system (1) is exponentially stable with probability one.*

2 Main result

We consider the controlled SDE

$$dX_t=(A\,X_t+B\,u_t)\,dt+\sum_{k=1}^{r}(A_k\,X_t+B_k\,u_t)\circ dW_k(t)$$

where A, A_k, $k = 1, \ldots, r$ are $d \times d$ matrices, B, B_k, $k = 1, \ldots, r$ are $d \times p$ matrices. We suppose that these matrices are given.

We restrict ourselves to feedback controls, i.e. $u_t = K X_t$, where K is a $p \times d$ matrix. The resulting SDE is

$$dX_t = (A + B\,K)\,X_t\,dt + \sum_{k=1}^{r} (A_k + B_k\,K)\,X_t \circ dW_k(t)\ ,\ X_0 = x_0 \in \mathbb{R}^d\ ,\ x_0 \neq 0\ . \tag{6}$$

The problem is to choose the feedback gain matrix K so as to stabilize the system (6).

The projection of X_t onto P^{d-1}, which now depends on K, satisfies

$$dU_t = h(A + B\,K, U_t)\,dt + \sum_{k=1}^{r} h(A_k + B_k\,K, U_t) \circ dW_k(t)\ , \tag{7}$$

with $U_0 = u_0 \triangleq \|x_0\|^{-1}\,x_0$, and

$$\|X_t\| = \|x_0\|\ \exp\left\{\int_0^t q(K, U_s)\,ds + \sum_{k=1}^{r} \int_0^t p_k(K, U_s)\,dW_k(s)\right\} \tag{8}$$

with

$$\begin{aligned}
q_0(K,u) &\triangleq ((A + B\,K)\,u, u)\ ,\\
q_1(K,u) &\triangleq \frac{1}{2}\sum_{k=1}^{r}\Big\{((A_k + B_k\,K)^2 u, u) + |(A_k + B_k\,K)\,u|^2 \\
&\qquad\qquad -2((A_k + B_k\,K)\,u, u)^2\Big\}\ ,\\
q(K,u) &\triangleq q_0(K,u) + q_1(K,u)\ ,\\
p_k(K,u) &\triangleq ((A_k + B_k\,K)\,u, u)\ , \quad k = 1, \ldots, r\ .
\end{aligned}$$

2.1 First case : $B_k = 0$, $k = 1, \ldots, r$

Here we suppose that only the drift coefficient is controlled, i.e. $B_k = 0$, $k = 1, \ldots, r$. The equation for U_t reduce to U_t is solution of the following equation

$$dU_t = h(A + B\,K, U_t)\,dt + \sum_{k=1}^{r} h(A_k, U_t) \circ dW_k(t)\ . \tag{9}$$

We make the following

Hypothesis 2.1 *For all $u \in P^{d-1}$ and $K \in M(p \times d)$*

$$\dim \mathsf{Lie\ Algebra}\{h(A+BK,.), h(A_k,.),\, k=1,\dots,r\}(u) = d-1 \ ,$$

where $M(p \times d)$ is the set of $p \times d$ matrices.

Under Hypothesis 2.1, the Theorem 1.2 states that, for all $K \in M(p \times d)$ the diffusion process U_t admits a unique invariant probability measure μ_K. Let λ_K denote the Lyapunov exponent associated with Equation (6)

$$\lambda_K \triangleq \int_{P^{d-1}} q(K,u)\, \mu_K(du) \ .$$

Proposition 2.2 *Under Hypothesis 2.1, $K \mapsto \lambda_K$ define a continuous function defined on $M(p \times d)$.*

Since $M(p \times d)$ is a connected set, we have the following

Corollary 2.3 *Under Hypothesis 2.1,*

$$\mathcal{D} \triangleq \{\lambda_K; K \in M(p \times d)\}$$

is a connected interval of $\mathbb{R}$.

In order to prove Proposition 2.2 we need the following

Lemma 2.4 *For all $t \geq 0$, there exist $C_t < \infty$, such that for all K_1, $K_2 \in M(p \times d)$*

$$\sup_{u \in S^{d-1}} E \left\| U_t^{1,u} - U_t^{2,u} \right\|^2 \leq C_t \, \|K_1 - K_2\| \ ,$$

where $U_t^{i,u}$ denote the solution of (9) with control matrix K_i and starting at point u.

Proof Let U_t^u denote the solution of Equation (9) with control matrix K and starting at point u. U_t^u is solution of the following Itô equation

$$dU_t^u = h(A{+}B\,K, U_t^u)\,dt + \frac{1}{2}\sum_{k=1}^{r} h'(A_k, U_t^u)\,h(A_k, U_t^u)\,dt + \sum_{k=1}^{r} h(A_k, U_t^u)\,dW_k(t)\ ,$$

with $h'(C,u)v = C\,v - (C\,u,u)\,v - (C\,u,v)\,u - (C\,v,u)\,u$. We get

$$dU_t^u = b(K, U_t^u)\,dt + \sum_{k=1}^{r} \sigma_k(U_t^u)\,dW_k(t)\ , \quad U_0^u = u\ ,$$

where

$$b(M,u) \triangleq h(A + B\,M, u) + \frac{1}{2}\sum_{k=1}^{r} h'(A_k,u)\,h(A_k,u)\ , \quad \sigma_k(u) \triangleq h(A_k,u)\ .$$

The drift coefficient $b(M,\cdot)$ and diffusion coefficients $\sigma_k(\cdot)$ are polynomial functions of u, so they are locally Lipschitz. But they are also globally Lipschitz because S^{d-1} is a compact set. Hence, there exist $L > 0$ such that for all $u, v \in S^{d-1}$ and $k = 1, \ldots, r$

$$\|b(M,u) - b(M,v)\| + \|\sigma_k(u) - \sigma_k(v)\| \le L\ \|u - v\|\ .$$

Also, for all $K, K' \in M(p \times d)$ and $u \in S^{d-1}$

$$\|b(K,u) - b(K',u)\| \le \|B\|\ \|K - K'\|\ .$$

Now we go back to the proof of the lemma :

$$\begin{aligned}
&\left\|U_t^{1,u} - U_t^{2,u}\right\|^2 \\
&\le 2\left\|\int_0^t [b(K_1, U_s^{1,u}) - b(K_2, U_s^{2,u})]\,ds\right\|^2 \\
&\qquad + 2\left\|\sum_{k=1}^{r}\int_0^t [\sigma_k(U_s^{1,u}) - \sigma_k(U_s^{2,u})]\,dW_k(s)\right\|^2 \\
&\le 2\,t\int_0^t \left\|b(K_1, U_s^{1,u}) - b(K_2, U_s^{2,u})\right\|^2 ds \\
&\qquad + 2\,r\sum_{k=1}^{r}\left\|\int_0^t [\sigma_k(U_s^{1,u}) - \sigma_k(U_s^{2,u})]\,dW_k(s)\right\|^2\ .
\end{aligned}$$

So

$$\begin{aligned} E\left\|U_t^{1,u}-U_t^{2,u}\right\|^2 &\leq 4\,t\int_0^t E\left\|b(K_1,U_s^{1,u})-b(K_2,U_s^{1,u})\right\|^2\,ds \\ &\quad +4\,t\int_0^t E\left\|b(K_2,U_s^{1,u})-b(K_2,U_s^{2,u})\right\|^2\,ds \\ &\quad +2\,r\sum_{k=1}^r\int_0^t E\left\|\sigma_k(U_s^{1,u})-\sigma_k(U_s^{2,u})\right\|^2\,ds\ . \end{aligned}$$

So we get

$$\begin{aligned} E\left\|U_t^{1,u}-U_t^{2,u}\right\|^2 &\leq (4\,t+2\,r)\,L^2\int_0^t E\left\|U_s^{1,u}-U_s^{2,u}\right\|^2\,ds \\ &\quad +4\,t^2\,\|B\|^2\,\|K_1-K_2\|^2\ . \end{aligned}$$

From this last inequality and Gronwall's inequality we prove the lemma. □

Proof of Proposition 2.2 Let $K_n \to K$ as $n\to\infty$. We want to prove that $\lambda_{K_n}\to\lambda_K$ as $n\to\infty$. Let U_t^u (resp. $U_t^{n,u}$) denote the solution of Equation (9) with control matrix K (resp. K_n) and starting at point u.

From Lemma 2.4,

$$\lim_{n\to\infty}\sup_{u\in S^{d-1}} E\,\|U_t^{n,u}-U_t^u\|^2=0\ . \tag{10}$$

Now we show that the sequence $\mu_n\stackrel{\triangle}{=}\mu_{K_n}$ admits a weak limit μ and that $\mu=\mu_K$. First, it is clear that the sequence $\{\mu_n\}$ is tight because S^{d-1} is a compact set. So there exist a sub–sequence, denoted $\{\mu_{n'}\}$, and a probability measure μ defined on S^{d-1} such that $\mu_{n'}\Rightarrow\mu$.

Now we prove that $\mu=\mu_K$. Let f be Lipschitz on S^{d-1}, then there exists $\alpha>0$ such that $|f(u)-f(v)|\leq\alpha\,\|u-v\|$ for all $u,v\in S^{d-1}$, and from (10), we have

$$\begin{aligned} \sup_{u\in S^{d-1}}|Ef(U_t^{n',u})-Ef(U_t^u)|^2 &\leq \sup_{u\in S^{d-1}} E|f(U_t^{n',u})-f(U_t^u)|^2 \\ &\leq \alpha^2\sup_{u\in S^{d-1}} E\left\|U_t^{n',u}-U_t^u\right\|^2\to 0\ . \end{aligned}\tag{11}$$

From the fact that $\mu_{n'} \Rightarrow \mu$ and that $\mu_{n'}$ is an invariant probability measure for $U_t^{n',u}$, we have

$$\int_{P^{d-1}} Ef(U_t^{n',u})\,\mu_{n'}(du) = \int_{P^{d-1}} f(u)\mu_{n'}(du) \to \int_{P^{d-1}} f(u)\,\mu(du)\ .$$

Furthermore

$$\begin{aligned}
&\left|\int_{P^{d-1}} Ef(U_t^{n',u})\,\mu_{n'}(du) - \int_{P^{d-1}} Ef(U_t^{u})\,\mu(du)\right| \\
&\quad \le \int_{P^{d-1}} |Ef(U_t^{n',u}) - Ef(U_t^{u})|\,\mu_{n'}(du) \\
&\qquad + \left|\int_{P^{d-1}} Ef(U_t^{u})\,[\mu_{n'}(du) - \mu(du)]\right| \\
&\quad \le \sup_{u\in S^{d-1}} |Ef(U_t^{n',u}) - Ef(U_t^{u})| + \left|\int_{P^{d-1}} Ef(U_t^{u})\,[\mu_{n'}(du) - \mu(du)]\right| \\
&\quad \to 0\ .
\end{aligned}$$

Indeed, the first term tends to 0 because of (11) and the second term tend to 0 because the function $u \mapsto Ef(U_t^u)$ is continuous and $\mu_{n'} \Rightarrow \mu$.

At last, we get

$$\int_{P^{d-1}} Ef(U_t^{u})\,\mu(du) = \int_{P^{d-1}} f(u)\,\mu(du)\ , \quad \forall t \ge 0$$

that is $\mu = \mu_K$.

The invariant measure μ_K is unique, so the whole sequence $\{\mu_n\}$ converge to μ_K.

Finally

$$\begin{aligned}
|\lambda_{K_n} - \lambda_K| &\le \left|\int_{P^{d-1}} q(K_n,u)\,\mu_n(du) - \int_{P^{d-1}} q(K,u)\,\mu(du)\right| \\
&\le \sup_{u\in S^{d-1}} |q(K_n,u) - q(K,u)| + \left|\int_{P^{d-1}} q(K,u)\,[\mu_n(du) - \mu(du)]\right|
\end{aligned}$$

which tends to 0. □

2.2 General case

We go back to the general set up (6)–(8). We make the following

Hypothesis 2.5 *For all $u \in P^{d-1}$ and $K \in M(p \times d)$*

$$\dim \mathsf{Lie\ Algebra}\,\{h(A + B\,K, .),\, h(A_k + B_k\,K, .)\,;\; k = 1, \ldots, r\}\,(u) = d - 1\ .$$

Under this hypothesis, U_t admit a unique invariant measure μ_K and the Lyapunov exponent is given by

$$\lambda_K = \int_{P^{d-1}} q(K, u)\, \mu_K(du)\ .$$

Proposition 2.6 *Under Hypothesis 2.5, $K \mapsto \lambda_K$ defines a continuous function defined on $M(p \times d)$.*

Since $M(p \times d)$ is a connected set, we have the following

Corollary 2.7 *Under Hypothesis 2.5,*

$$\mathcal{D} \stackrel{\triangle}{=} \{\lambda_K; K \in M(p \times d)\}$$

is a connected interval of $\mathbb{R}$.

Proof of Proposition 2.6 The proof is equivalent to Proposition 2.2. □

3 Examples

In these examples, we consider feedback gain matrices $K(\alpha)$ parametrized by a one dimensional parameter $\alpha \in \mathbb{R}$. We suppose that $\{K(\alpha); \alpha \in \mathbb{R}\}$ is a connected subset of $M(p \times d)$. Let λ_α (resp. μ_α) denote the Lyapunov exponent (resp. the invariant measure) associated with $K(\alpha)$ and

$$\lambda_\star = \inf_{\alpha \in \mathbb{R}} \lambda_\alpha$$

We provide in this section two examples where $\lambda_\star = -\infty$ and one where $\lambda_\star = 1$.

In all the examples, $d = 2$ and $r = 1$. For any 2×2 matrix C, $h(C, \cdot)$ defined in (4) is a vector field on P^1, and

$$h(C, \theta) = (-c_{11} + c_{22})\, \cos\theta\, \sin\theta - c_{12}\, \sin^2\theta + c_{21}\, \cos^2\theta$$

for all $\theta \in P^1$.

3.1 Example 1

We identify P^1 to $[-\pi/2, \pi/2]$. Let

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \quad A_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \quad B_1 = 0 .$$

We consider feedback gain matrices K of the form

$$K(\alpha) = \begin{pmatrix} \alpha & -1 \\ -1 & 2\alpha \end{pmatrix}$$

parametrized by $\alpha \in \mathbb{R}$.

The Hypothesis 2.1 is satisfied, since

$$h(A + BK(\alpha), \theta) = \alpha \sin\theta \cos\theta , \quad h(A_1, \theta) = 1 , \quad \forall \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}] .$$

We have

$$q_0(K(\alpha), \theta) = \alpha + \alpha \sin^2\theta , \quad q_1(\theta) = 0 , \quad \forall \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}] .$$

Finally we get

$$\lambda_\alpha = \alpha + \alpha \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2(\theta)\, \mu_\alpha(d\theta) \leq \alpha .$$

So the system is stabilizable because

$$\lim_{\alpha \to -\infty} \lambda_\alpha = -\infty .$$

3.2 Example 2

We identify P^1 and $[0, \pi]$, and we take

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} , \quad B = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} , \quad A_1 = 0 , \quad B_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ,$$

and

$$K(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} .$$

So

$$h(A+BK(\alpha),\theta)=\alpha\,\sin\theta\,\cos\theta\;,\quad h(A_1+B_1K(\alpha),\theta)=\alpha\;,\quad \forall\theta\in[0,\pi]\;,$$

and

$$q_0(K(\alpha),\theta)=-1+\alpha\,\sin^2\theta\;,\quad q_1(K(\alpha),\theta)=0\;,\quad \forall\theta\in[0,\pi]\;.$$

The Lyapunov exponent is

$$\lambda_\alpha=-1+\alpha\int_0^\pi \sin^2(\theta)\,\mu_\alpha(d\theta)\;. \tag{12}$$

We want to compute the limit of λ_α as $\alpha\to-\infty$. We can check that the projected process U_t is solution of

$$dU_t=h(A,U_t)\,dt+\alpha\,h(A',U_t)dt+\alpha\,h(A'',U_t)\circ dW(t)$$

with

$$A'=\begin{pmatrix}0&0\\0&1\end{pmatrix}\;,\quad A''=\begin{pmatrix}0&-1\\1&0\end{pmatrix}\;.$$

We make the following time scale transformation $\tilde U^\alpha_t\triangleq U_{t/\alpha^2}$. $\tilde U_t$ is solution of

$$\tilde U^\alpha_t=U_0+\frac{1}{\alpha^2}\int_0^t h(A,\tilde U^\alpha_s)\,ds+\frac{1}{\alpha}\int_0^t h(A',\tilde U^\alpha_s)\,ds+\int_0^t h(A'',\tilde U^\alpha_s)\circ d\tilde W^\alpha_s \tag{13}$$

where $\tilde W^\alpha_t\triangleq\alpha W_{t/\alpha^2}$ is a standard Wiener process.

When $\alpha\to-\infty$, we get the following limit equation

$$\tilde U_t=U_0+\int_0^t h(A'',\tilde U_s)\circ d\tilde W_s$$

where $\tilde W_t$ is a standard Wiener process. Let $\tilde U_t=(\cos\tilde\theta_t,\sin\tilde\theta_t)$. Because $h(A'',\theta)=1$, we get

$$\tilde\theta_t=\theta_0+\tilde W_t\;. \tag{14}$$

Proposition 3.1

$$\mu_\alpha\Rightarrow\mathcal U[0,\pi]\quad \textit{as}\quad \alpha\to-\infty$$

where $\mathcal U[0,\pi]$ *is the uniform law on* $[0,\pi]$.

Proof It is clear that $\mu_\alpha \Rightarrow \mu_{-\infty}$, where $\mu_{-\infty}$ satisfies the following Fokker–Planck equation $L^*\mu_{-\infty} = 0$ and L is the infinitesimal generator associated with equation (14). Moreover, $\mu_{-\infty}$ is the uniform law on $[0,\pi]$. $\square$

Then using this proposition and (12) we have

Corollary 3.2

$$\lambda_\alpha \to -\infty \quad as \quad \alpha \to -\infty \; .$$

3.3 Example 3

Now we present an example which is not stabilizable. We consider the same coefficients as in the previous section except for matrices A, B_1 :

$$A = I \; , \quad B_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \; .$$

Let $X_t = (X_t^1, X_t^2)$, we get the following system

$$\begin{aligned} dX_t^1 &= X_t^1\, dt - \alpha\, X_t^2 \circ dW_t \; , \quad X_0^1 = x_0^1 \; , \\ dX_t^2 &= (1+\alpha)\, X_t^2\, dt \; , \quad X_0^2 = x_0^2 \; , \quad x_0 \neq 0 \; , \end{aligned}$$

whose solution is $X_t^2 = e^{(1+\alpha)t}\, x_0^2$, and

$$\begin{aligned} X_t^1 &= e^t\, x_0^1 - \alpha \int_0^t e^{t-s}\, X_s^2 \circ dW_s \\ &= e^t\, x_0^1 - \alpha\, x_0^2 \int_0^t e^{t+\alpha\, s}\, dW_s \\ &= e^t\,(x_0^1 - Y_t) \end{aligned}$$

with

$$Y_t \triangleq \alpha\, x_0^2 \int_0^t e^{\alpha s}\, dW_s \; .$$

If $x_0^2 \neq 0$, we deduce from

$$\|X_t\| \geq |X_t^2| = e^{(1+\alpha)t}\, |x_0^2|$$

that $\lambda_\alpha \geq 1+\alpha$. If $x_0^2=0$ then $x_0^1 \neq 0$ (because, $x_0 \neq 0$) and $\lambda_\alpha = 1$. So, for $\alpha \geq 0$, $\lambda_\alpha \geq 1$.

Let us consider the case $\alpha < 0$. By the theorem of convergence of martingales, $Y_t \to Y_\infty \stackrel{\triangle}{=} \alpha\, X_0^2 \int_0^\infty e^{\alpha s}\, dW_s$ as $t \to \infty$ and this convergence holds a.s. and in L^2.

We deduce from

$$||X_t||^2 = e^{2t}\left\{e^{2\alpha t}|x_0^2|^2 + |x_0^1 - Y_t|^2\right\}$$

and from (2) that

$$\lambda_\alpha = 1 + \lim_{t\to+\infty} \frac{1}{2t} \log\left[e^{2\alpha t}|x_0^2|^2 + |x_0^1 - Y_t|^2\right] = 1 \quad a.s.$$

(this limit is valid whether $x_0^2 = 0$ or not). Which proves that the system is not stabilizable.

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ISSN 0249-6399