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INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

***Lyapunov exponents of controlled SDE's  
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# Lyapunov exponents of controlled SDE's and stabilizability property : Some examples

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**Abstract:** We consider a stochastic differential equation with linear feedback control :

$$dX_t = (A + B K) X_t dt + \sum_{k=1}^r (A_k + B_k K) X_t \circ dW_k(t)$$

where  $K$  is the feedback gain matrix. For each value of  $K$ , let  $\lambda_K$  be the Lyapunov exponent associated with the solution of the SDE. The set of  $\lambda_K$ , as  $K$  describe the set of matrices, is a connected interval of  $\mathbb{R}$ . We present some examples where  $-\infty$  is the lower bound of this set. For these cases, we say that the corresponding EDS is stabilizable.

**Key-words:** Stochastic differential equation, stabilizability, Lyapunov exponent.

*(Résumé : tsvp)*

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# Exposants de Lyapunov d'EDS contrôlées et propriété de stabilisabilité : Quelques exemples

**Résumé :** On considère une équation différentielle stochastique avec contrôle linéaire en boucle fermée :

$$dX_t = (A + B K) X_t dt + \sum_{k=1}^r (A_k + B_k K) X_t \circ dW_k(t)$$

où  $K$  est une matrice de gain. Pour chaque valeur de  $K$ , soit  $\lambda_K$  l'exposant de Lyapunov associé à la solution de cette EDS. L'ensemble de valeurs de  $\lambda_K$  lorsque  $K$  parcourt l'espace des matrices est un intervalle connexe de  $\mathbb{R}$ . On présente quelques exemples où  $-\infty$  est la borne inférieure de cet ensemble. Pour ces exemples, nous dirons que l'EDS correspondante est stabilizable.

**Mots-clé :** Équation différentielle stochastique, stabilisabilité, exposant de Lyapunov.

# 1 Preliminaries

We consider the following linear stochastic differential equation in  $\mathbb{R}^d$

$$dX_t = A X_t dt + \sum_{k=1}^r A_k X_t \circ dW_k(t) , \quad X_0 = x_0 \in \mathbb{R}^d , \quad x_0 \neq 0 , \quad (1)$$

where  $A, A_1, \dots, A_k$  are  $d \times d$  matrices,  $W_1, \dots, W_r$  are independent standard Wiener processes. Here “ $\circ dW$ ” (resp. “ $dW$ ”) refer to the Stratonovich (resp. Itô) stochastic integral.

We define the Lyapunov exponent of the solution of (1) starting at  $x_0$

$$\lambda(x_0) \triangleq \lim_{t \rightarrow \infty} \frac{1}{t} \log \|X_t\| . \quad (2)$$

Oseledec's multiplicative ergodic theorem states that the limit (2) exists with probability one and that there are  $d$  fixed numbers  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_d$  – called the Lyapunov exponents of (1) – such that the random variable  $\lambda(x_0)$  takes on only these values (see [1] for a review). Moreover, (1) is exponentially stable with probability one if and only if  $\lambda_1 < 0$ .

Let  $S^{d-1} = \{x \in \mathbb{R}^d; \|x\| = 1\}$  denote the unit sphere of  $\mathbb{R}^d$ . We can define the projection of  $X_t$  onto the sphere by

$$U_t \triangleq \|X_t\|^{-1} X_t .$$

$U_t$  is the solution of the following SDE on  $S^{d-1}$

$$dU_t = h(A, U_t) dt + \sum_{k=1}^r h(A_k, U_t) \circ dW_k(t) , \quad U_0 = u_0 \triangleq \|x_0\|^{-1} x_0 \quad (3)$$

where

$$h(C, u) \triangleq C u - (C u, u) u . \quad (4)$$

Here  $(x, y)$  is the scalar product on  $\mathbb{R}^d$  and  $\|x\|^2 = (x, x)$ .

Moreover,

$$\|X_t\| = \|x_0\| \exp \left\{ \int_0^t q(U_s) ds + \sum_{k=1}^r \int_0^t p_k(U_s) dW_k(s) \right\} \quad (5)$$

with

$$\begin{aligned} q_0(u) &\triangleq (A u, u), \quad p_k(u) \triangleq (A_k u, u), \quad k = 1, \dots, r, \\ q_1(u) &\triangleq \frac{1}{2} \sum_{k=1}^r [(A_k^2 u, u) + \|A_k u\|^2 - 2(A_k u, u)^2], \\ q(u) &\triangleq q_0(u) + q_1(u). \end{aligned}$$

For each matrix  $M$ ,  $h(M, -u) = -h(M, u)$ , so that  $h(M, \cdot)$  can be viewed as a vector field on the projective space  $P^{d-1}$  (obtained from  $S^{d-1}$  by identifying  $u$  and  $-u$ ). Therefore (3) can be considered as a stochastic differential equation on  $P^{d-1}$  and (5) is still valid with this definition. We make the following

**Hypothesis 1.1** For all  $u \in P^{d-1}$

$$\dim \text{Lie Algebra}\{h(A, \cdot), h(A_k, \cdot), k = 1, \dots, r\}(u) = d - 1.$$

**Theorem 1.2** Under Hypothesis 1.1

(i) The diffusion process  $U_t$  admits a unique invariant probability measure  $\mu$ . Moreover  $\mu$  has a  $C^\infty$  density  $p$  with respect to the Lebesgue measure on  $P^{d-1}$  which solves the Fokker–Planck equation  $L^*p = 0$ , where  $L$  is the infinitesimal generator associated with equation (3).

(ii) The number

$$\lambda \triangleq \int_{P^{d-1}} q(u) \mu(du)$$

is equal to the top Lyapunov exponent  $\lambda_1$ .

(iii) For all  $x_0 \in \mathbb{R}^d$ ,  $x_0 \neq 0$ ,  $\lambda(x_0) = \lambda$  with probability one. When  $\lambda < 0$ , the system (1) is exponentially stable with probability one.

## 2 Main result

We consider the controlled SDE

$$dX_t = (A X_t + B u_t) dt + \sum_{k=1}^r (A_k X_t + B_k u_t) \circ dW_k(t)$$

where  $A, A_k, k = 1, \dots, r$  are  $d \times d$  matrices,  $B, B_k, k = 1, \dots, r$  are  $d \times p$  matrices. We suppose that these matrices are given.

We restrict ourselves to feedback controls, i.e.  $u_t = K X_t$ , where  $K$  is a  $p \times d$  matrix. The resulting SDE is

$$dX_t = (A + B K) X_t dt + \sum_{k=1}^r (A_k + B_k K) X_t \circ dW_k(t), \quad X_0 = x_0 \in \mathbb{R}^d, \quad x_0 \neq 0. \quad (6)$$

The problem is to choose the feedback gain matrix  $K$  so as to stabilize the system (6).

The projection of  $X_t$  onto  $P^{d-1}$ , which now depends on  $K$ , satisfies

$$dU_t = h(A + B K, U_t) dt + \sum_{k=1}^r h(A_k + B_k K, U_t) \circ dW_k(t), \quad (7)$$

with  $U_0 = u_0 \triangleq \|x_0\|^{-1} x_0$ , and

$$\|X_t\| = \|x_0\| \exp \left\{ \int_0^t q(K, U_s) ds + \sum_{k=1}^r \int_0^t p_k(K, U_s) dW_k(s) \right\} \quad (8)$$

with

$$\begin{aligned} q_0(K, u) &\triangleq ((A + B K) u, u), \\ q_1(K, u) &\triangleq \frac{1}{2} \sum_{k=1}^r \left\{ ((A_k + B_k K)^2 u, u) + |(A_k + B_k K) u|^2 \right. \\ &\quad \left. - 2((A_k + B_k K) u, u)^2 \right\}, \\ q(K, u) &\triangleq q_0(K, u) + q_1(K, u), \\ p_k(K, u) &\triangleq ((A_k + B_k K) u, u), \quad k = 1, \dots, r. \end{aligned}$$

## 2.1 First case : $B_k = 0, k = 1, \dots, r$

Here we suppose that only the drift coefficient is controlled, i.e.  $B_k = 0, k = 1, \dots, r$ . The equation for  $U_t$  reduce to  $U_t$  is solution of the following equation

$$dU_t = h(A + B K, U_t) dt + \sum_{k=1}^r h(A_k, U_t) \circ dW_k(t). \quad (9)$$

We make the following



**Hypothesis 2.1** For all  $u \in P^{d-1}$  and  $K \in M(p \times d)$

$$\dim \text{Lie Algebra}\{h(A + BK, \cdot), h(A_k, \cdot), k = 1, \dots, r\}(u) = d - 1 ,$$

where  $M(p \times d)$  is the set of  $p \times d$  matrices.

Under Hypothesis 2.1, the Theorem 1.2 states that, for all  $K \in M(p \times d)$  the diffusion process  $U_t$  admits a unique invariant probability measure  $\mu_K$ . Let  $\lambda_K$  denote the Lyapunov exponent associated with Equation (6)

$$\lambda_K \triangleq \int_{P^{d-1}} q(K, u) \mu_K(du) .$$

**Proposition 2.2** Under Hypothesis 2.1,  $K \mapsto \lambda_K$  define a continuous function defined on  $M(p \times d)$ .

Since  $M(p \times d)$  is a connected set, we have the following

**Corollary 2.3** Under Hypothesis 2.1,

$$\mathcal{D} \triangleq \{\lambda_K; K \in M(p \times d)\}$$

is a connected interval of  $\mathbb{R}$ .

In order to prove Proposition 2.2 we need the following

**Lemma 2.4** For all  $t \geq 0$ , there exist  $C_t < \infty$ , such that for all  $K_1, K_2 \in M(p \times d)$

$$\sup_{u \in S^{d-1}} E \left\| U_t^{1,u} - U_t^{2,u} \right\|^2 \leq C_t \|K_1 - K_2\| ,$$

where  $U_t^{i,u}$  denote the solution of (9) with control matrix  $K_i$  and starting at point  $u$ .

**Proof** Let  $U_t^u$  denote the solution of Equation (9) with control matrix  $K$  and starting at point  $u$ .  $U_t^u$  is solution of the following Itô equation

$$dU_t^u = h(A+B K, U_t^u) dt + \frac{1}{2} \sum_{k=1}^r h'(A_k, U_t^u) h(A_k, U_t^u) dt + \sum_{k=1}^r h(A_k, U_t^u) dW_k(t),$$

with  $h'(C, u)v = C v - (C u, u)v - (C u, v)u - (C v, u)u$ . We get

$$dU_t^u = b(K, U_t^u) dt + \sum_{k=1}^r \sigma_k(U_t^u) dW_k(t), \quad U_0^u = u,$$

where

$$b(M, u) \triangleq h(A + B M, u) + \frac{1}{2} \sum_{k=1}^r h'(A_k, u) h(A_k, u), \quad \sigma_k(u) \triangleq h(A_k, u).$$

The drift coefficient  $b(M, \cdot)$  and diffusion coefficients  $\sigma_k(\cdot)$  are polynomial functions of  $u$ , so they are locally Lipschitz. But they are also globally Lipschitz because  $S^{d-1}$  is a compact set. Hence, there exist  $L > 0$  such that for all  $u, v \in S^{d-1}$  and  $k = 1, \dots, r$

$$\|b(M, u) - b(M, v)\| + \|\sigma_k(u) - \sigma_k(v)\| \leq L \|u - v\|.$$

Also, for all  $K, K' \in M(p \times d)$  and  $u \in S^{d-1}$

$$\|b(K, u) - b(K', u)\| \leq \|B\| \|K - K'\|.$$

Now we go back to the proof of the lemma :

$$\begin{aligned} & \|U_t^{1,u} - U_t^{2,u}\|^2 \\ & \leq 2 \left\| \int_0^t [b(K_1, U_s^{1,u}) - b(K_2, U_s^{2,u})] ds \right\|^2 \\ & \quad + 2 \left\| \sum_{k=1}^r \int_0^t [\sigma_k(U_s^{1,u}) - \sigma_k(U_s^{2,u})] dW_k(s) \right\|^2 \\ & \leq 2t \int_0^t \|b(K_1, U_s^{1,u}) - b(K_2, U_s^{2,u})\|^2 ds \\ & \quad + 2r \sum_{k=1}^r \left\| \int_0^t [\sigma_k(U_s^{1,u}) - \sigma_k(U_s^{2,u})] dW_k(s) \right\|^2. \end{aligned}$$

So

$$\begin{aligned} E \left\| U_t^{1,u} - U_t^{2,u} \right\|^2 &\leq 4t \int_0^t E \left\| b(K_1, U_s^{1,u}) - b(K_2, U_s^{1,u}) \right\|^2 ds \\ &\quad + 4t \int_0^t E \left\| b(K_2, U_s^{1,u}) - b(K_2, U_s^{2,u}) \right\|^2 ds \\ &\quad + 2r \sum_{k=1}^r \int_0^t E \left\| \sigma_k(U_s^{1,u}) - \sigma_k(U_s^{2,u}) \right\|^2 ds . \end{aligned}$$

So we get

$$\begin{aligned} E \left\| U_t^{1,u} - U_t^{2,u} \right\|^2 &\leq (4t + 2r) L^2 \int_0^t E \left\| U_s^{1,u} - U_s^{2,u} \right\|^2 ds \\ &\quad + 4t^2 \|B\|^2 \|K_1 - K_2\|^2 . \end{aligned}$$

From this last inequality and Gronwall's inequality we prove the lemma.  $\square$

**Proof of Proposition 2.2** Let  $K_n \rightarrow K$  as  $n \rightarrow \infty$ . We want to prove that  $\lambda_{K_n} \rightarrow \lambda_K$  as  $n \rightarrow \infty$ . Let  $U_t^u$  (resp.  $U_t^{n,u}$ ) denote the solution of Equation (9) with control matrix  $K$  (resp.  $K_n$ ) and starting at point  $u$ .

From Lemma 2.4,

$$\lim_{n \rightarrow \infty} \sup_{u \in S^{d-1}} E \left\| U_t^{n,u} - U_t^u \right\|^2 = 0 . \quad (10)$$

Now we show that the sequence  $\mu_n \triangleq \mu_{K_n}$  admits a weak limit  $\mu$  and that  $\mu = \mu_K$ . First, it is clear that the sequence  $\{\mu_n\}$  is tight because  $S^{d-1}$  is a compact set. So there exist a sub-sequence, denoted  $\{\mu_{n'}\}$ , and a probability measure  $\mu$  defined on  $S^{d-1}$  such that  $\mu_{n'} \Rightarrow \mu$ .

Now we prove that  $\mu = \mu_K$ . Let  $f$  be Lipschitz on  $S^{d-1}$ , then there exists  $\alpha > 0$  such that  $|f(u) - f(v)| \leq \alpha \|u - v\|$  for all  $u, v \in S^{d-1}$ , and from (10), we have

$$\begin{aligned} \sup_{u \in S^{d-1}} |E f(U_t^{n',u}) - E f(U_t^u)|^2 &\leq \sup_{u \in S^{d-1}} E |f(U_t^{n',u}) - f(U_t^u)|^2 \\ &\leq \alpha^2 \sup_{u \in S^{d-1}} E \left\| U_t^{n',u} - U_t^u \right\|^2 \rightarrow 0 . \quad (11) \end{aligned}$$

From the fact that  $\mu_{n'} \Rightarrow \mu$  and that  $\mu_{n'}$  is an invariant probability measure for  $U_t^{n',u}$ , we have

$$\int_{P^{d-1}} E f(U_t^{n',u}) \mu_{n'}(du) = \int_{P^{d-1}} f(u) \mu_{n'}(du) \rightarrow \int_{P^{d-1}} f(u) \mu(du) .$$

Furthermore

$$\begin{aligned} & \left| \int_{P^{d-1}} E f(U_t^{n',u}) \mu_{n'}(du) - \int_{P^{d-1}} E f(U_t^u) \mu(du) \right| \\ & \leq \int_{P^{d-1}} |E f(U_t^{n',u}) - E f(U_t^u)| \mu_{n'}(du) \\ & \quad + \left| \int_{P^{d-1}} E f(U_t^u) [\mu_{n'}(du) - \mu(du)] \right| \\ & \leq \sup_{u \in S^{d-1}} |E f(U_t^{n',u}) - E f(U_t^u)| + \left| \int_{P^{d-1}} E f(U_t^u) [\mu_{n'}(du) - \mu(du)] \right| \\ & \rightarrow 0 . \end{aligned}$$

Indeed, the first term tends to 0 because of (11) and the second term tend to 0 because the function  $u \mapsto E f(U_t^u)$  is continuous and  $\mu_{n'} \Rightarrow \mu$ .

At last, we get

$$\int_{P^{d-1}} E f(U_t^u) \mu(du) = \int_{P^{d-1}} f(u) \mu(du) , \quad \forall t \geq 0$$

that is  $\mu = \mu_K$ .

The invariant measure  $\mu_K$  is unique, so the whole sequence  $\{\mu_n\}$  converge to  $\mu_K$ .

Finally

$$\begin{aligned} |\lambda_{K_n} - \lambda_K| & \leq \left| \int_{P^{d-1}} q(K_n, u) \mu_n(du) - \int_{P^{d-1}} q(K, u) \mu(du) \right| \\ & \leq \sup_{u \in S^{d-1}} |q(K_n, u) - q(K, u)| + \left| \int_{P^{d-1}} q(K, u) [\mu_n(du) - \mu(du)] \right| \end{aligned}$$

which tends to 0. □

## 2.2 General case

We go back to the general set up (6)–(8). We make the following

**Hypothesis 2.5** For all  $u \in P^{d-1}$  and  $K \in M(p \times d)$

$$\dim \text{Lie Algebra } \{h(A + B K, \cdot), h(A_k + B_k K, \cdot); k = 1, \dots, r\} (u) = d - 1 .$$

Under this hypothesis,  $U_t$  admit a unique invariant measure  $\mu_K$  and the Lyapunov exponent is given by

$$\lambda_K = \int_{P^{d-1}} q(K, u) \mu_K(du) .$$

**Proposition 2.6** Under Hypothesis 2.5,  $K \mapsto \lambda_K$  defines a continuous function defined on  $M(p \times d)$ .

Since  $M(p \times d)$  is a connected set, we have the following

**Corollary 2.7** Under Hypothesis 2.5,

$$\mathcal{D} \triangleq \{\lambda_K; K \in M(p \times d)\}$$

is a connected interval of  $\mathbb{R}$ .

**Proof of Proposition 2.6** The proof is equivalent to Proposition 2.2.  $\square$

### 3 Examples

In these examples, we consider feedback gain matrices  $K(\alpha)$  parametrized by a one dimensional parameter  $\alpha \in \mathbb{R}$ . We suppose that  $\{K(\alpha); \alpha \in \mathbb{R}\}$  is a connected subset of  $M(p \times d)$ . Let  $\lambda_\alpha$  (resp.  $\mu_\alpha$ ) denote the Lyapunov exponent (resp. the invariant measure) associated with  $K(\alpha)$  and

$$\lambda_\star = \inf_{\alpha \in \mathbb{R}} \lambda_\alpha$$

We provide in this section two examples where  $\lambda_\star = -\infty$  and one where  $\lambda_\star = 1$ .

In all the examples,  $d = 2$  and  $r = 1$ . For any  $2 \times 2$  matrix  $C$ ,  $h(C, \cdot)$  defined in (4) is a vector field on  $P^1$ , and

$$h(C, \theta) = (-c_{11} + c_{22}) \cos \theta \sin \theta - c_{12} \sin^2 \theta + c_{21} \cos^2 \theta$$

for all  $\theta \in P^1$ .

### 3.1 Example 1

We identify  $P^1$  to  $[-\pi/2, \pi/2]$ . Let

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad B_1 = 0.$$

We consider feedback gain matrices  $K$  of the form

$$K(\alpha) = \begin{pmatrix} \alpha & -1 \\ -1 & 2\alpha \end{pmatrix}$$

parametrized by  $\alpha \in \mathbb{R}$ .

The Hypothesis 2.1 is satisfied, since

$$h(A + BK(\alpha), \theta) = \alpha \sin \theta \cos \theta, \quad h(A_1, \theta) = 1, \quad \forall \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}].$$

We have

$$q_0(K(\alpha), \theta) = \alpha + \alpha \sin^2 \theta, \quad q_1(\theta) = 0, \quad \forall \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}].$$

Finally we get

$$\lambda_\alpha = \alpha + \alpha \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2(\theta) \mu_\alpha(d\theta) \leq \alpha.$$

So the system is stabilizable because

$$\lim_{\alpha \rightarrow -\infty} \lambda_\alpha = -\infty.$$

### 3.2 Example 2

We identify  $P^1$  and  $[0, \pi]$ , and we take

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_1 = 0, \quad B_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and

$$K(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}.$$

So

$$h(A + BK(\alpha), \theta) = \alpha \sin \theta \cos \theta, \quad h(A_1 + B_1K(\alpha), \theta) = \alpha, \quad \forall \theta \in [0, \pi],$$

and

$$q_0(K(\alpha), \theta) = -1 + \alpha \sin^2 \theta, \quad q_1(K(\alpha), \theta) = 0, \quad \forall \theta \in [0, \pi].$$

The Lyapunov exponent is

$$\lambda_\alpha = -1 + \alpha \int_0^\pi \sin^2(\theta) \mu_\alpha(d\theta). \quad (12)$$

We want to compute the limit of  $\lambda_\alpha$  as  $\alpha \rightarrow -\infty$ . We can check that the projected process  $U_t$  is solution of

$$dU_t = h(A, U_t) dt + \alpha h(A', U_t) dt + \alpha h(A'', U_t) \circ dW(t)$$

with

$$A' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad A'' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We make the following time scale transformation  $\tilde{U}_t^\alpha \triangleq U_{t/\alpha^2}$ .  $\tilde{U}_t$  is solution of

$$\tilde{U}_t^\alpha = U_0 + \frac{1}{\alpha^2} \int_0^t h(A, \tilde{U}_s^\alpha) ds + \frac{1}{\alpha} \int_0^t h(A', \tilde{U}_s^\alpha) ds + \int_0^t h(A'', \tilde{U}_s^\alpha) \circ d\tilde{W}_s^\alpha \quad (13)$$

where  $\tilde{W}_t^\alpha \triangleq \alpha W_{t/\alpha^2}$  is a standard Wiener process.

When  $\alpha \rightarrow -\infty$ , we get the following limit equation

$$\tilde{U}_t = U_0 + \int_0^t h(A'', \tilde{U}_s) \circ d\tilde{W}_s$$

where  $\tilde{W}_t$  is a standard Wiener process. Let  $\tilde{U}_t = (\cos \tilde{\theta}_t, \sin \tilde{\theta}_t)$ . Because  $h(A'', \theta) = 1$ , we get

$$\tilde{\theta}_t = \theta_0 + \tilde{W}_t. \quad (14)$$

### Proposition 3.1

$$\mu_\alpha \Rightarrow \mathcal{U}[0, \pi] \quad \text{as } \alpha \rightarrow -\infty$$

where  $\mathcal{U}[0, \pi]$  is the uniform law on  $[0, \pi]$ .

**Proof** It is clear that  $\mu_\alpha \Rightarrow \mu_{-\infty}$ , where  $\mu_{-\infty}$  satisfies the following Fokker-Planck equation  $L^*\mu_{-\infty} = 0$  and  $L$  is the infinitesimal generator associated with equation (14). Moreover,  $\mu_{-\infty}$  is the uniform law on  $[0, \pi]$ .  $\square$

Then using this proposition and (12) we have

**Corollary 3.2**

$$\lambda_\alpha \rightarrow -\infty \quad \text{as} \quad \alpha \rightarrow -\infty .$$

**3.3 Example 3**

Now we present an example which is not stabilizable. We consider the same coefficients as in the previous section except for matrices  $A, B_1$  :

$$A = I , \quad B_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} .$$

Let  $X_t = (X_t^1, X_t^2)$ , we get the following system

$$\begin{aligned} dX_t^1 &= X_t^1 dt - \alpha X_t^2 \circ dW_t , & X_0^1 &= x_0^1 , \\ dX_t^2 &= (1 + \alpha) X_t^2 dt , & X_0^2 &= x_0^2 , \quad x_0 \neq 0 , \end{aligned}$$

whose solution is  $X_t^2 = e^{(1+\alpha)t} x_0^2$ , and

$$\begin{aligned} X_t^1 &= e^t x_0^1 - \alpha \int_0^t e^{t-s} X_s^2 \circ dW_s \\ &= e^t x_0^1 - \alpha x_0^2 \int_0^t e^{t+\alpha s} dW_s \\ &= e^t (x_0^1 - Y_t) \end{aligned}$$

with

$$Y_t \triangleq \alpha x_0^2 \int_0^t e^{\alpha s} dW_s .$$

If  $x_0^2 \neq 0$ , we deduce from

$$\|X_t\| \geq |X_t^2| = e^{(1+\alpha)t} |x_0^2|$$



that  $\lambda_\alpha \geq 1 + \alpha$ . If  $x_0^2 = 0$  then  $x_0^1 \neq 0$  (because,  $x_0 \neq 0$ ) and  $\lambda_\alpha = 1$ . So, for  $\alpha \geq 0$ ,  $\lambda_\alpha \geq 1$ .

Let us consider the case  $\alpha < 0$ . By the theorem of convergence of martingales,  $Y_t \rightarrow Y_\infty \triangleq \alpha X_0^2 \int_0^\infty e^{\alpha s} dW_s$ , as  $t \rightarrow \infty$  and this convergence holds a.s. and in  $L^2$ .

We deduce from

$$\|X_t\|^2 = e^{2t} \left\{ e^{2\alpha t} |x_0^2|^2 + |x_0^1 - Y_t|^2 \right\}$$

and from (2) that

$$\lambda_\alpha = 1 + \lim_{t \rightarrow +\infty} \frac{1}{2t} \log \left[ e^{2\alpha t} |x_0^2|^2 + |x_0^1 - Y_t|^2 \right] = 1 \quad a.s.$$

(this limit is valid whether  $x_0^2 = 0$  or not). Which proves that the system is not stabilizable.

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