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Abstract: We consider approximation of L^p functions by Hardy functions on subsets of the circle. We first derive some properties of traces of Hardy classes on such subsets, and then turn to a generalization of classical extremal problems involving norm constraints on the complementary subset.

Key-words: Hardy spaces, dual extremal problems, Nehari extension.

(Résumé : tsvp)

MSC: 30D55, 30E99, 93B30, 93C80.

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Approximation dans l'espace de Hardy de fonctions L^p sur un sous–ensemble du cercle

Résumé : Nous étudions l'approximation de fonctions L^p par des fonctions de l'espace de Hardy H^p sur des sous–ensembles du cercle. Nous établissons tout d'abord des propriétés de densité des classes de Hardy sur ces sous–ensembles, puis nous considérons une généralisation des problèmes extrêmaux classiques en présence de contraintes en norme sur le sous–ensemble complémentaire.

Mots-clé : espaces de Hardy, problèmes extrêmaux duaux, extension de Nehari.

1 Introduction

The issue under investigation here may be considered as a generalization to arbitrary subsets of the unit circle $\mathbb{T}$ of classical extremal problems in Hardy spaces of the unit disk. More precisely, if K is a subset of $\mathbb{T}$ and if we denote its complement by $\mathbb{T} \setminus K$, the question we address can be stated as follows.

For $p \geq 1$, let us be given $f \in L^p(K)$, $h \in L^p(\mathbb{T} \setminus K)$, and $M > 0$. We want to find $g \in H^p$ whose distance to h in $L^p(\mathbb{T} \setminus K)$ does not exceed M and which is as close as possible to f in the $L^p(K)$ metric under this constraint.

When $K = \mathbb{T}$, then h and M play no role and we recognize a standard (dual) extremal problem (see e.g. [6, chap.8], [8, chap.IV], [11, chap.VII]). We shall refer to the above generalization as a bounded extremal problem. If K is not of full measure and $p < \infty$, traces of Hardy functions turn out to be dense in $L^p(K)$. Hence, unless f is already the trace of a Hardy function, the constraint $\|g - h\|_{L^p(\mathbb{T}\setminus K)} \leq M$ is needed for the problem to be well–posed ; the case where $p = \infty$ is slightly more complex, as we shall see, but behaves similarly provided f and K are regular enough.

We shall begin in section 2 with a somewhat detailed analysis of traces of Hardy functions on subsets of $\mathbb{T}$. Then, in section 3, we show that well–known existence and uniqueness properties of solutions to standard extremal problems do generalize to bounded extremal problems, and that a bounded extremal problem reduces to a standard one when $p = \infty$. We shall be working entirely on the disk, but it is clear how to derive half–plane versions of our results, either by direct inspection or using conformal mapping (see e.g. [10, chap.8]).

Part of the authors motivations for studying such problems originates with some questions in identification of linear dynamical control systems that we would now like to describe. A linear time–invariant dynamical control system, or simply a system, is a convolution operator associating to an input function

$u : [0, +\infty) \rightarrow \mathbb{R}$ an output function $y : [0, +\infty) \rightarrow \mathbb{R}$ given by

$$y(t) = \int_0^{+\infty} r(t-\tau)u(\tau)\, d\tau,$$

where $r : [0, \infty) \rightarrow \mathbb{R}$ is called the impulse response of the system. As a general rule, restrictions on the growth of the impulse response express various boundedness properties of the input–output map $u \mapsto y$, and a connection with complex analysis can be made when r is of exponential type, so that we may consider its Laplace transform :

$$\mathcal{F}(s) = \int_0^{\infty} e^{-st} r(t)\, dt,$$

which is analytic for $\operatorname{Re} s$ large enough and is called the transfer function of the system. This is where Hardy classes arise in connection with two important kinds of stability. The first is the boundedness of the input–output map $L^2[0, \infty) \rightarrow L^2[0, \infty)$ which occurs if and only if $\mathcal{F}$ belongs to $\mathcal{H}^\infty$ of the right half–plane ; the corresponding operator norm, that is the maximum rate of energy transmission, is then precisely $\|\mathcal{F}\|_\infty$. The second, appearing in a stochastic context, is the variance of the process generated by the system when the input is a white noise ; this variance is finite if and only if $\mathcal{F}$ belongs to $\mathcal{H}^2$ of the right half–plane, and it is then equal to $\|\mathcal{F}\|_2$. If $1 \leq p \leq 2$ and q is the conjugate exponent, we get more generally from the Hausdorff–Young inequalities that $\mathcal{F} \in \mathcal{H}^q$ when $r \in L^p[0, \infty)$ but there is no longer such a nice interpretation for the norm.

Now, if $r \in L^1(\mathbb{R}^+)$, a common way of identifying a system from experiments is to plug a periodic signal with frequency ω into it as this produces asymptotically an output which is also periodic and whose phase and modulus are those of the complex number $\mathcal{F}(i\omega)$. However, due to the bandwidth limitation of the system and also to the restricted domain of validity of the linear model, the points $i\omega_k$ at which the transfer function can be evaluated are confined within a certain range of frequencies Ω, say, and only some rough information on the roll–off rate at higher frequencies is available. To recap, we have a description of the $\mathcal{H}^p$ function $\mathcal{F}$ on Ω which is accurate up to experimental errors, and an expected behaviour of the same function outside

Ω which need not be exceedingly precise. When trying to recover a plausible model from these data, we are naturally led to consider the bounded extremal problem stated before except that the setting is now the half–plane, and particular instances of this question were studied for $p = 2$ in [12] and [1] both of which emphasize the connections with system–theory. Note that the standard extremal problem with $p = \infty$ (also known as the Nehari extension) already plays an important role in robust identification and has been extensively used in control for several years (see e.g. [7]). In addition, it should be observed that the above identification scheme can be turned into a design procedure helping one to decide how feasible a given frequency behaviour may be.

There is of course much more to approximation in this identification scheme because $\mathcal{F}$ has to be reconstructed on Ω from pointwise values. For details on how this merges into the framework of robust identification discussed in [9] and [15], we refer the reader to [4].

As a final remark in this section, observe that if the subset K is invariant under conjugation and f and h are real–symmetric, namely if $\overline{f(z)} = f(\bar{z})$ and similarly for h, a best approximant g in our problem can always be chosen so as to be real–symmetric since $\frac{1}{2}\left(g(z) + \overline{g(\bar{z})}\right)$ does the job as well ; if moreover we know that such a g is unique, we conclude that the best approximant is in fact real–symmetric. Since transfer–functions are real–symmetric, such a property is mandatory for system–theoretic applications.

2 Some density results

We first fix a few notations. If E is a set and $F \subset E$, we write $E \setminus F$ for the set–theoretic complement of F in E. When E is a topological space, we let as usual $\overline{F}$ and $\mathring{F}$ be respectively the closure and the interior of F in E ; further, $C(E)$ will denote the space of continuous complex–valued functions and $\mathcal{M}_E$ the space of complex Borel measures on E. Call $\mathbb{T}$ the unit circle and $\mathbb{D}$ the unit disk. We let λ be the Lebesgue measure on $\mathbb{T}$ and, for $1 \leq p \leq \infty$, we designate by H^p the Hardy space with exponent p of the unit disk while $\mathcal{A}$ stands for the disk algebra. For $E \subset \mathbb{T}$, we denote by $L^p(E)$ the familiar Lebesgue space and by $\|.\|_{L^p(E)}$ the corresponding norm. If f is a function defined on some set containing E, the symbol $f_{|_E}$ indicates the restriction of f to E, and we write without ambiguity $\|f\|_{L^p(E)}$ for the norm of this restriction. Accordingly, we set $H^p_{|_E}$ to mean the space of traces on E of H^p functions and we define $\mathcal{A}_{|_E}$ similarly. Whenever f is defined on E and h is defined on $\mathbb{T} \setminus E$, the notation $f \vee h$ is used for the concatenated function which is defined on all of $\mathbb{T}$.

Since H^p is a closed subspace of $L^p(\mathbb{T})$, the distance from $f \in L^p(\mathbb{T})$ to H^p is positive unless $f \in H^p$. The corresponding approximation problem on a subset of $\mathbb{T}$ which is *not* of full measure behaves quite differently :

Theorem 1 *Let K be a subset of $\mathbb{T}$ such that $\lambda(\mathbb{T}\setminus K) > 0$, and let $1 \leq p < \infty$.*
(i) $H^p_{|_K}$ is dense in $L^p(K)$.
(ii) If K is closed, $\mathcal{A}_{|_K}$ is dense in $C(K)$.
(iii) If $\lambda(K) > 0$, $\mathcal{A}_{|_K} \neq C(K)$ and $H^p_{|_K} \neq L^p(K)$.

Proof : let q be conjugate to p, and assume that $f \in L^q(K)$ is orthogonal to $H^p_{|_K}$. This may be expressed as

$$\int_{\mathbb{T}} (f \vee 0)\, g \, d\lambda = 0, \quad \text{for all } g \in H^p.$$

In particular, the Fourier coefficients of $f \vee 0$ of negative index are all zero so $f \vee 0 \in H^q$. Since it vanishes on a set of positive measure, namely $\mathbb{T} \setminus K$, it must vanish identically. This proves (i).
Since K does not separate $\mathbb{C}$, (ii) is immediate from Mergelyan's theorem [19, thm.20.5].
To prove (iii), we need only construct $f \in C(K)$ which is not the trace on K of a function in H^1. This will certainly be the case if f is nonzero but vanishes on some subset of K of positive measure. Since we assume now that $\lambda(K)$ is positive, we can construct by dichotomy two disjoint subsets K_1, K_2 of K, closed with respect to K, such that $\lambda(K_1) > 0$ and $\lambda(K_2) > 0$. Being a metric space, K is certainly normal hence, by the Tietze–Urysohn extension theorem [19, thm.20.4], there exists $f \in C(K)$ which is identically 1 on K_1 and zero on K_2. ■

Remarks

(a) The assumption that K is closed is necessary in (ii) : if K is dense in $\mathbb{T}$, then the distance from $f \in C(\mathbb{T})$ to $\mathcal{A}$ is equal to the distance from $f_{|_K}$ to $\mathcal{A}_{|_K}$.

(b) If $\lambda(K) = 0$, then $L^p(K)$ reduces to the zero space and we have $L^p(K) = H^p_{|_K} = \{0\}$, so the second half of (iii) becomes trivially false. More subtle is the fact that the first half of (iii) also becomes false. Indeed, if K is closed and $\lambda(K) = 0$, a theorem of Rudin asserts that $C(K) = \mathcal{A}_{|_K}$. A proof of Rudin's result can be found in [10, ch.6], where three other ways of establishing (ii) are also given, one resting on Runge's theorem, another on the F. and M. Riesz theorem, and a third on Wermer's maximality theorem. Quoting [10], (ii) "is a very old theorem which has a variety of proofs". The short argument above, based on Mergelyan's theorem, was suggested to the authors by E. B. Saff. One may also appeal to another deep result due to Levinson on sums of exponentials [20, III.2,thm.II].

(c) When $\bar{K} \neq \mathbb{T}$, note that (i) follows readily from (ii) as applied to $\bar{K}$ and from the density of $C(\bar{K})$ in $L^p(\bar{K})$. For K an interval, assertions (i) and (ii) are stated in [12] for the half–plane rather than the disk with reference to some work by P.Ya. Nudel'man.

In [16], an approximation procedure is given which is strongly related to theorem 1 : if $1 < p < \infty$ and $P_{H^p} \; : \; L^p(\mathbb{T}) \to H^p$ denotes the natural projection, one can associate to every function $g \in H^p$ a family of H^p functions depending on a real parameter $\rho \geq 0$ by setting

$$T_\rho(g) := \rho \, h_\rho \, P_{H^p} \left(\chi_K \, g \, \bar{h}_\rho\right), \tag{1}$$

where χ_K denotes the characteristic function of K and $h_\rho \in H^\infty$ is the outer factor

$$h_\rho(z) := \exp \left\{ -\frac{1}{4\,\pi} \log(1+\rho) \int_K \frac{e^{it}+z}{e^{it}-z} \, dt \right\} . \tag{2}$$

It follows from [16, thm.1] that $T_\rho(g)$ converges to g in H^p when ρ goes to infinity. Now, formula (1) makes perfectly good sense for any g in $L^p(K)$ – the values of g outside K being irrelevant – and we elaborate on this result to obtain a constructive version of part (i) of theorem 1 in the case where $p > 1$:

Proposition 1 *Let K be a subset of $\mathbb{T}$ such that $\lambda(\mathbb{T}\setminus K)>0$ and let $1<p<\infty$. For $f\in L^p(K)$, define $T_\rho(f)\in H^p$ by formula (1). Then $T_\rho(f)_{|K}$ converges to f in $L^p(K)$ as ρ goes to infinity.*

Proof : observe that $|h_\rho|^2 = 1\,/\,(1+\rho\,\chi_K)$ on $\mathbb{T}$, hence

$$\|T_\rho(f)\|_{L^p(K)} = \frac{\rho}{\sqrt{1+\rho}}\,\|P_{H^p}\,(\chi_K\,f\,\bar h_\rho)\|_{L^p(K)} \leq \frac{\rho}{\sqrt{1+\rho}}\,\|P_{H^p}\,(\chi_K\,f\,\bar h_\rho)\|_{L^p(\mathbb{T})}$$

$$\leq \frac{\rho\,C_p}{\sqrt{1+\rho}}\,\|\chi_K\,f\,\bar h_\rho\|_{L^p(\mathbb{T})} = \frac{\rho\,C_p}{1+\rho}\,\|f\|_{L^p(K)} \leq C_p\,\|f\|_{L^p(K)}\,, \tag{3}$$

where the constant C_p is given by the M. Riesz theorem on projections from $L^p(\mathbb{T})$ to H^p (see e.g. [10, chap. 9]). The family T_ρ is thus a uniformly bounded family of operators on $L^p(K)$, converging pointwise to the identity on $H^p_{|K}$ by [16, thm.1]. Since $H^p_{|K}$ is dense in $L^p(K)$ by theorem 1, the conclusion follows from Ascoli's theorem. ■

When $p=2$, a more elaborate process was derived by Van Winter in [21] to recover Hardy functions on the half–plane from their values on a semi–axis. This theory, based on Mellin transforms, has been generalized by Rosenblum and Rovnyak in [18] so as to deal with values on arbitrary Borel subsets of the real line and we shall briefly account for their results while carrying them over to the disk.
Assume that $K\subset\mathbb{T}$ is such that $\lambda(K)$ and $\lambda(\mathbb{T}\setminus K)$ are positive. We first introduce an auxiliary function :

$$\gamma(z) := -\exp\left\{\frac{1}{2i}\int_K \frac{e^{it}+z}{e^{it}-z}\,dt\right\}, \tag{4}$$

which is clearly analytic off $\mathbb{T}$. For instance, when K is the arc $(e^{i\theta_1},e^{i\theta_2})$, (4) defines the function

$$-\frac{z-e^{i\theta_1}}{z-e^{i\theta_2}}\,\exp\frac{i(\theta_2-\theta_1)}{2}\,.$$

By a classical theorem of Fatou and well–known properties of conjugate functions (see e.g. [10, chap.3,6]), the argument of the exponential in (4) has non-tangential limits from inside the disk almost everywhere on $\mathbb{T}$, and these limits have imaginary part $-\pi$ on K and 0 on $\mathbb{T} \setminus K$. Hence γ itself has non tangential limits a.e. on $\mathbb{T}$, that are positive on K and negative on $\mathbb{T} \setminus K$. Since $\gamma(1/\bar{z}) = \overline{\gamma(z)}$, the same limits exist from outside the disk as well, and will serve as a definition of γ on $\mathbb{T}$. For $\rho > 0$, we shall further partition K into

$$K_{\rho_+} := \{e^{i\theta} \in K; \quad \gamma(e^{i\theta}) > \rho\} \quad \text{and} \quad K_{\rho_-} := \{e^{i\theta} \in K; \quad \gamma(e^{i\theta}) \leq \rho\}.$$

Now, a summary of [18] is contained in the following two assertions.

Assertion (a). If $F \in L^2(\mathbb{T})$ is such that $e^{\pi\gamma}F$ also belongs to $L^2(\mathbb{T})$, the function $S(F)$ given by

$$S(F)(z) := \frac{i}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \frac{\gamma(e^{it}) - \gamma(z)}{e^{it} - z} \gamma(z)^{-1/2-i\gamma(e^{it})} \, F(e^{it}) e^{it} \, dt \tag{5}$$

belongs to H^2. Moreover, we have :

$$\|S(F)\|_{L^2(K)} = \|F\|_{L^2(\mathbb{T})} \quad \text{and} \quad \|S(F)\|_{L^2(\mathbb{T}\setminus K)} = \|e^{\pi\gamma} F\|_{L^2(\mathbb{T})}. \tag{6}$$

Assertion (b). Conversely, for every $f \in L^2(K)$, we can define a function $R(f) \in L^2(\mathbb{T})$ such that $R(f)(e^{i\theta})$ is equal to :

$$\frac{i}{\sqrt{2\pi}} \left(\lim_{r\to 1^-} \int_{K_{1_+}} + \lim_{r \to 1^+} \int_{K_{1_-}} \right) \frac{\gamma(e^{it}) - \gamma(re^{i\theta})}{e^{it} - re^{i\theta}} \, \gamma(e^{it})^{-1/2+i\gamma(re^{i\theta})} \, f(e^{it}) \, e^{it} \, dt \, , \tag{7}$$

where the limit exists in $L^2(\mathbb{T})$. We have $e^{\pi\gamma}R(f) \in L^2(\mathbb{T})$ if and only if $f \in H^2_{|K}$, and in this case $f = S(R(f))$ where S is given by (5).

Using the conformal map $z(s) = (i-s)/(i+s)$ which induces an isometry

$$g(z) \mapsto \frac{g(z(s))}{\sqrt{\pi}(1-is)}$$

from H^2 onto the corresponding Hardy space of the upper half–plane, we leave it to the reader to check that assertions (a) and (b) above are merely a rephrasing of [18, thm. 2]. In the same vein than before, though less explicitly, one can derive from what precedes an approximation procedure in $L^2(K)$:

Proposition 2 *Let K be a subset of $\mathbb{T}$ such that $\lambda(K) > 0$ and $\lambda(\mathbb{T} \setminus K) > 0$. For $f \in L^2(K)$ and $\rho > 0$, define $S_\rho(f) \in H^2$ by*

$$S_\rho(f) := S(\chi_{\mathbb{T}\setminus K_{\rho+}} R(f)), \tag{8}$$

where the meaning of R and S is given by (7) and (5). Then $S_\rho(f)_{|K}$ converges to f in $L^2(K)$ as ρ goes to infinity. If, moreover, $f = g_{|K}$ with $g \in H^2$, then $S_\rho(f)$ converges to g in H^2.

Proof : since $R(f) \in L^2(\mathbb{T})$ by assertion (b) and γ is negative on $\mathbb{T} \setminus K$, we see that $e^{\pi\gamma}\chi_{\mathbb{T}\setminus K_{\rho+}} R(f)$ belongs to $L^2(\mathbb{T})$ by the definition of $K_{\rho+}$. Therefore $S_\rho : L^2(K) \to H^2$ is well–defined by assertion (a). Assume first that $f = g_{|K}$ where $g \in H^2$. Then $e^{\pi\gamma}R(f)$ belongs to $L^2(\mathbb{T})$ and $g = S(R(f))$ by assertion (b). Thus, it follows from (8) and (6) that

$$\|g - S_\rho(f)\|^2_{L^2(\mathbb{T})} = \|S(\chi_{K_{\rho+}} R(f))\|^2_{L^2(\mathbb{T})} = \|\chi_{K_{\rho+}} R(f)\|^2_{L^2(\mathbb{T})} + \|e^{\pi\gamma}\chi_{K_{\rho+}} R(f)\|^2_{L^2(\mathbb{T})}$$

which tends to zero when ρ tends to ∞ by monotone convergence. This proves the last assertion of the proposition. Next, observe that (7) defines R as a pointwise limit of bounded operators from $L^2(K)$ to $L^2(\mathbb{T})$. Thus, the very existence of the limit for every $f \in L^2(K)$ implies that R is bounded by the Banach–Steinhaus principle. When $f \in H^2_{|K}$, then $f = S(R(f))_{|K}$ by assertion (b) and we get from (6) that $\|R(f)\|_{L^2(\mathbb{T})} = \|f\|_{L^2(K)}$. Since $H^2_{|K}$ is dense in $L^2(K)$ by theorem 1, we deduce that R is an isometry. Therefore, using (8) and (6) again, we obtain for every $f \in L^2(K)$:

$$\|S_\rho(f)\|_{L^2(K)} = \|\chi_{\mathbb{T}\setminus K_{\rho+}} R(f)\|_{L^2(\mathbb{T})} \leq \|R(f)\|_{L^2(\mathbb{T})} = \|f\|_{L^2(K)}. \tag{9}$$

Consider the operator

$$S_{\rho_{|K}} : L^2(K) \mapsto L^2(K)$$

manufactured from S_ρ followed by the natural restriction. Equation (9) shows that it is a contraction. The family $S_{\rho_{|K}}$ is thus a uniformly bounded family of operators converging pointwise to the identity on $H^2_{|K}$ by the first part of the proof. As in proposition 1, the conclusion now follows from Ascoli's theorem. ∎

Observe that theorem 1 leaves out the case $p = \infty$, and this is no accident as the next result shows.

Theorem 2 *Let K be a subset of $\mathbb{T}$ such that $\lambda(K) > 0$.*
(i) $H^\infty_{|K}$ is not dense in $L^\infty(K)$.
(ii) If K is open, the closure of $H^\infty_{|K}$ in $L^\infty(K)$ is contained in $H^\infty_{|K} + C(K)$.
(iii) If K is a proper closed subset of $\mathbb{T}$, the closure of $H^\infty_{|K}$ in $L^\infty(K)$ contains $(H^\infty + C(\mathbb{T}))_{|K}$.

Proof : we will denote by $P_r(\theta)$ the familiar Poisson kernel :

$$P_r(\theta) := \frac{1 - r^2}{1 - 2\,r\,\cos\theta + r^2}\ , \text{ for } 0 \le r < 1 \text{ and } 0 \le \theta \le 2\pi\,.$$

Let us also define two positive measures ν_1 and ν_2 on $\mathbb{T}$ by

$$d\nu_1 := \chi_{_K}\,d\lambda \qquad \text{and} \qquad d\nu_2 := (1 - \chi_{_K})\,d\lambda\,.$$

Next, we consider a Blaschke product :

$$b(z) := z^k \prod_{n=1}^{\infty} \frac{\bar{\alpha}_n}{|\alpha_n|}\,\frac{\alpha_n - z}{1 - \bar{\alpha}_n z}, \quad \text{with } \alpha_n = r_n\,e^{i\theta_n} \text{ and } \sum_n (1 - r_n) < \infty. \tag{10}$$

Since $|b| = 1$ almost everywhere on $\mathbb{T}$, the distance from $\bar{b}_{|K}$ to $H^\infty_{|K}$ is at most one. Denote this distance by $d(\bar{b}, H^\infty_{|K})$, and assume it is less than one. Then, there exists $h \in H^\infty$ and $\eta < 1$ such that $|\bar{b} - h| < \eta$ almost everywhere on K or equivalently such that

$$|1 - b\,h| < \eta \ \text{ a.e. on } K\,. \tag{11}$$

Since bh lies in H^∞, we get from the Poisson representation formula

$$0 = bh(\alpha_n) = \frac{1}{2\,\pi}\int_{-\pi}^{\pi} P_{r_n}(\theta_n - t)\,bh(e^{it})\,dt,$$

so that

$$\left|\frac{1}{2\,\pi}\int_{-\pi}^{\pi} P_{r_n}(\theta_n - t)\,bh(e^{it})\,d\nu_1(t)\right| = \left|\frac{1}{2\,\pi}\int_{-\pi}^{\pi} P_{r_n}(\theta_n - t)\,bh(e^{it})\,d\nu_2(t)\right|. \tag{12}$$

Now, we have on the one hand

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} P_{r_n}(\theta_n - t)\, bh(e^{it})\, d\nu_2(t)\right| \leq \|h\|_{L^\infty(\mathbb{T}\setminus\mathrm{K})} \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{r_n}(\theta_n - t)\, d\nu_2(t), \tag{13}$$

while on the other hand (11) implies

$$\begin{aligned}\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} P_{r_n}(\theta_n - t)\, bh(e^{it})\, d\nu_1(t)\right| &\geq \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{r_n}(\theta_n - t)\, \mathrm{Re}[bh(e^{it})]\, d\nu_1(t) \\ &\geq (1-\eta)\frac{1}{2\pi}\int_{-\pi}^{\pi} P_{r_n}(\theta_n - t)\, d\nu_1(t).\end{aligned} \tag{14}$$

Because of (12), (13), and (14), we get

$$(1-\eta)\frac{1}{2\pi}\int_{-\pi}^{\pi} P_{r_n}(\theta_n - t)\, d\nu_1(t) \leq \|h\|_{L^\infty(\mathbb{T}\setminus\mathrm{K})} \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{r_n}(\theta_n - t)\, d\nu_2(t). \tag{15}$$

To prove (i), we will design our Blaschke product b in such a way that (15) becomes absurd, thereby contradicting (11).
By Lebesgue's theorem, ν_1 has derivative χ_K almost everywhere with respect to λ. Since $\lambda(K) > 0$, there exists a point $e^{i\theta_0}$ of K where ν_1 has a derivative which is equal to 1. At such a point, ν_2 also has a derivative which is equal to 0. In formula (10), let us choose the α_n's so that a subsequence (α_{n_k}) converges nontangentially to $e^{i\theta_0}$. If we set $n = n_k$ in (15) and invoke a well–known theorem of Fatou (see e.g. [10, chap.3]), the integral in the left hand–side goes to 2π if k goes to infinity whereas the integral in the right hand–side goes to 0, yielding $1 - \eta \leq 0$. This contradicts the definition of η, thereby showing that $d(\bar{b}, H^\infty_{|K}) = 1$. We have thus established (i) in the following more precise form : each time the zeroes of a Blaschke product b accumulate nontangentially at some density point of K, the function $\bar{b}_{|K}$ is badly approximable from $H^\infty_{|K}$.

Note that we used no special property of the Blaschke product b beyond the fact that it is invertible in $L^\infty(K)$ and has many zeroes. Actually, a closer look at the proof shows that a slightly more general statement holds true, namely :

if $f \in H^\infty$ is such that $\|f\|_{L^\infty(\mathbb{T})} \leq 1$ and $f^{-1}_{|K} \in L^\infty(K)$, and if moreover the zeroes of f accumulate nontangentially at some density point of K, then

$d(f^{-1}, H^\infty_{|K}) \geq 1$.

As a first step in the proof of the remaining assertions, we show that (15) again becomes absurd as soon as the α_n's accumulate at a point $e^{i\theta_0}$ which is *interior* to K. This is not a consequence of what precedes because we do not assume this time that the convergence is nontangential, and this is the reason why we need the stronger hypothesis that $e^{i\theta_0}$ is an interior point and not merely a density point. Using the previous sequence of inequalities, we shall give a short proof of this fact. However, it should be noticed that it also follows from a classical result, namely that every H^p function with non–negative real part on an arc has an inner factor which is analytic across this arc [8, II, ex.14] (in view of (11), all we have to do is to apply this to $b\,h \in H^\infty$).
Assume therefore that some subsequence (α_{n_k}) converges to $e^{i\theta_0} \in \overset{\circ}{K}$; thus, there exists $\sigma > 0$ such that

$$\{e^{it};\ \theta_0 - \sigma < t < \theta_0 + \sigma\} \subset K,$$

and we have that $\theta_{n_k} \in [\theta_0 - \frac{\sigma}{2}, \theta_0 + \frac{\sigma}{2}]$ when k is large enough. For such a k, we get from the monotonicity of the Poisson kernel with respect to θ that

$$\frac{1}{2\pi}\int_{-\pi}^{\theta_0-\sigma} P_{r_{n_k}}(\theta_{n_k} - t)\,dt + \frac{1}{2\pi}\int_{\theta_0+\sigma}^{\pi} P_{r_{n_k}}(\theta_{n_k} - t)\,dt \leq P_{r_{n_k}}(\sigma/2) \tag{16}$$

hence also

$$\frac{1}{2\pi}\int_{\theta_0-\sigma}^{\theta_0+\sigma} P_{r_{n_k}}(\theta_{n_k} - t)\,dt \geq 1 - P_{r_{n_k}}(\sigma/2). \tag{17}$$

If we set $n = n_k$ in (15) and use (17) and (16) to minorize and majorize the left and right hand–side respectively, we get

$$(1-\eta)\,(1 - P_{r_{n_k}}(\sigma/2)) \;\leq\; \|h\|_{L^\infty(\mathbb{T}\setminus K)}\, P_{r_{n_k}}(\sigma/2),$$

which again contradicts the definition of η, for $P_{r_{n_k}}(\sigma/2)$ tends to 0 when k goes to infinity.

Recall $j \in H^\infty$ is said to be inner if $|j(e^{it})| = 1$ almost everywhere on $\mathbb{T}$. Our second step toward the proof of (ii) will be to show that if j is inner and $d(\bar{j}, H^\infty_{|K}) < 1$, then j is *continuous* at every interior point $e^{i\theta_0}$ of K. When j

is a Blaschke product, this follows from the fact that the zeroes of j cannot accumulate at $e^{i\theta_0}$ by what we have just seen in the first step, and from the well–known analytic continuation properties of Blaschke products (see e.g. [10, chap.5]). For the general case, we use Frostman's theorem (see e.g. [8, II, cor. 6.5]) that Blaschke products are dense in the set of inner functions : if b is sufficiently close to j, then $d(\bar{b}, H^\infty_{|K}) < 1$ and b is continuous at $e^{i\theta_0}$. Since j is the uniform limit of such b's, we are done.

We are now in position to demonstrate (ii). Let $\mathcal{C}$ be the closure of $H^\infty_{|K}$ in $L^\infty(K)$ which is a closed subalgebra of $L^\infty(K)$. Since the restriction map $\Pi_K : L^\infty(\mathbb{T}) \to L^\infty(K)$ is continuous, $\mathcal{D} := \Pi_K^{-1}(\mathcal{C})$ is a closed subalgebra of $L^\infty(\mathbb{T})$ containing H^∞ and, by the Chang–Marshall theorem [8, IX, thm 3.1], there is a set $\mathcal{J}$ of inner functions such that

$$\mathcal{D} = [H^\infty, \bar{\mathcal{J}}]. \tag{18}$$

Let f be a function in $\mathcal{C}$. From (18), using the classical trick $h_1\bar{b}_1 + h_2\bar{b}_2 = (h_1b_2 + h_2b_1)\,\bar{b}_1\bar{b}_2$ for inner b_1, b_2, there exist a sequence (h_n) of H^∞ functions and a sequence (b_n) of functions of $\mathcal{J}$ such that

$$\lim_{n\to\infty} h_n\,\bar{b}_n = f \text{ in } L^\infty(K). \tag{19}$$

Clearly, any element b_n of $\mathcal{J}$ satisfies $d(\bar{b}_n, H^\infty_{|K}) = 0$. Because K is now assumed to be open, it consists entirely of interior points so that b_n, hence also $\bar{b}_n$, are continuous on K by what we saw in the second step above. This proves in particular that $\mathcal{C} \subset H^\infty_{|K} + C(K)$, which is (ii).

Finally, it is clear that (iii) is a consequence of part (ii) of theorem 1. ∎

We end up this section by showing that the approximants provided by theorems 1, 2 and propositions 1, 2 are bound to have wild behaviour outside K. Recall that H^p is reflexive for $1 < p < \infty$ whence balls are weakly compact, and that H^∞ is weak–* closed in $L^\infty(\mathbb{T})$ whence balls are weak–* compact. Moreover, balls are also weak–* compact in H^1 when imbedded in the space $\mathcal{M}_\mathbb{T}$ of complex measures, as this is essentially the content of the F. and M. Riesz theorem (see e.g.[11, VII.A]).

Proposition 3 *Let K be a subset of $\mathbb{T}$ such that $\lambda(K) > 0$, and f be in $L^p(K)$, with $1 \leq p \leq \infty$. Suppose (g_n) is a sequence of H^p functions such that $g_{n|_K}$ converges to f in $L^p(K)$. If f is not the trace of an H^p function, then $\lim_{n\to\infty} \|g_n\|_{L^p(\mathbb{T}\setminus K)} = \infty$.*

Proof : suppose the conclusion does not hold. Extracting a subsequence if necessary, we may assume that $(\|g_n\|_{L^p(\mathbb{T}\setminus K)})$ is bounded, hence that (g_n) is bounded in H^p. Then there exists a subsequence (g_{n_k}) converging weakly (weakly–* if $p = 1$ or $p = \infty$) toward some $g \in H^p$. Consider first the case where $p \neq 1$. Then it is clear that $(g_{n_k|_K})$ converges weakly (weakly–* if $p = \infty$) to $g_{|_K}$ in $L^p(K)$, and since it also converges strongly to f by hypothesis, we get $f = g$ a.e. on K so that $f \in H^p_{|_K}$. Suppose now that $p = 1$. We still want to prove that $f = g$ a.e. on K in this case. The weak–* convergence of (g_{n_k}) means here that

$$\lim_{k\to\infty} \int_{\mathbb{T}} g_{n_k} w \, d\lambda = \int_{\mathbb{T}} g w \, d\lambda \qquad \text{for all } w \in C(\mathbb{T}). \tag{20}$$

Upon multiplying f, g, and g_{n_k} by $e^{i\theta}$, we may as well assume that $g(0) = g_{n_k}(0) = 0$, for weak–* convergence is preserved by this multiplication since $e^{i\theta} \in C(\mathbb{T})$. In this case, g and g_{n_k} are orthogonal to H^∞ because the Cauchy formula implies

$$\int_{\mathbb{T}} g v \, d\lambda = 2\pi g(0) v(0) = 0 \qquad \text{whenever } v \in H^\infty,$$

and similarly if g is replaced by g_{n_k}. Therefore, (20) remains valid for $w \in H^\infty + C(\mathbb{T})$. Now, every measurable subset of $\mathbb{T}$ is almost everywhere the zero set of some function in $H^\infty + C(\mathbb{T})$ [2], so there exists $w_0 \in H^\infty + C(\mathbb{T})$ with the property that $w_0(e^{i\theta}) = 0$ a.e. on $\mathbb{T} \setminus K$ and $w_0(e^{i\theta}) \neq 0$ a.e. on K. Note also that $w_0 w \in H^\infty + C(\mathbb{T})$ whenever $w \in H^\infty + C(\mathbb{T})$ for the latter is an algebra (see e.g. [8, IX,thm 2.2]). Since the support of $w_0 w$ lies in K, we conclude that

$$\int_{\mathbb{T}} g w_0 w \, d\lambda = \lim_{k\to\infty} \int_{\mathbb{T}} g_{n_k} w_0 w \, d\lambda = \lim_{k\to\infty} \int_K g_{n_k} w_0 w \, d\lambda = \int_K f w_0 w \, d\lambda \tag{21}$$

for any $w \in H^\infty + C(\mathbb{T})$. We obtain in particular

$$\int_{\mathbb{T}} (g - f \vee 0) \, w_0 \, w \, d\lambda = 0 \qquad \text{for all } w \in C(\mathbb{T}).$$

This implies that $(g - f \vee 0)\, w_0 = 0$ a.e. on $\mathbb{T}$, and since w_0 is a.e. non zero on K, we get $f = g_{|K}$ as desired. ∎

Proposition 3 formalizes a basic principle to characterize traces on K of H^p functions that can be illustrated at several places in the literature : assuming that K and $\mathbb{T}\setminus K$ have positive measure, construct for each $f \in L^p(K)$ a family (g_ρ) of H^p functions such that :
(i) $g_{\rho\,|K}$ converges to f in $L^p(K)$ when ρ tends to some value ρ_0,
(ii) g_ρ converges in H^p as $\rho \to \rho_0$ whenever $f \in H^p_{|K}$.
Then, $f \in H^p_{|K}$ if and only if $\|g_\rho\|_{L^p(\mathbb{T})}$, or equivalently $\|g_\rho\|_{L^p(\mathbb{T}\setminus K)}$, remains bounded.

Though not explicitly stated, this approach is taken in [16, thm.2] where it is observed that the boundedness of the family $T_\rho(f)$ as $\rho \to \infty$ is a necessary and sufficient condition for f to belong to $H^p_{|K}$. The fact that (i) is satisfied in this case is just proposition 1.

A further example lies with the work in [18] where the condition $e^{\pi\gamma}R(f) \in L^2(\mathbb{T})$, which is necessary and sufficient for f to belong to $H^2_{|K}$, is equivalent to the boundedness of $\|S_\rho(f)\|_{L^2(\mathbb{T})}$ when ρ increases indefinitely. Proposition 2 asserts that (i) and (ii) are fulfilled here as well.

Yet another instance of this kind of characterization is implicit in [12, thm.4.1]. The problem considered by these authors on a half–plane translates to the disk as follows : given $f \in L^2(K)$, where K is an arc of $\mathbb{T}$, and some constant $\rho > 0$, find g_ρ of minimum $L^2(\mathbb{T} \setminus K)$–norm among those $g \in H^2$ such that $\|f - g\|_{L^2(K)} \leq \rho$. By construction, the family (g_ρ) satisfies (i) and (ii) with $\rho_0 = 0$ and the above mentioned theorem asserts that $f \in H^2_{|K}$ if and only if $\|g_\rho\|_{L^2(\mathbb{T})}$ remains bounded as ρ goes to zero.

A closely related and somewhat dual problem was studied in [1] where we are still given $f \in L^2(K)$ and some constant $\rho \geq 0$, but we want to find this time g_ρ minimizing $\|f - g\|_{L^2(K)}$ among those $g \in H^2$ whose $L^2(\mathbb{T} \setminus K)$–norm is at most ρ. Again by construction, the family (g_ρ) satisfies (i) and (ii) with $\rho_0 = \infty$. If one expresses that g_ρ remains bounded using the formulae given in [1], he will recover a characterization of $H^2_{|K}$ which was established in [17] in a different manner. It is interesting to note that these formulae are asymptotically equivalent to (1) where $p = 2$, though the latter was introduced in [16] without reference to best approximation.

The two questions above, namely those considered in [12] and [1] respectively, can be recast as particular cases of a generalized dual extremal problem which will be our object of study in the remaining of the paper. As was stressed in the introduction, this issue also comes up naturally in identification of linear dynamical control systems.

3 Bounded extremal problems

The question we address is the following :

Problem 1 *Let K be a subset of $\mathbb{T}$ of positive measure and p be such that $1 \leq p \leq \infty$. For $f \in L^p(K)$ and $h \in L^p(\mathbb{T} \setminus K)$, define a subset $\mathcal{B}^p_{M,h}$ of H^p by :*

$$\mathcal{B}^p_{M,h} := \{g \in H^p\,,\ \|h - g\|_{L^p(\mathbb{T}\setminus K)} \leq M\}\,,$$

where M is a positive real number. We seek $g_0 \in \mathcal{B}^p_{M,h}$ such that

$$\|f - g_0\|_{L^p(K)} = \min_{g \in \mathcal{B}^p_{M,h}} \|f - g\|_{L^p(K)}\,. \tag{22}$$

Such a problem we call a *bounded extremal problem.* When $K = \mathbb{T}$, we see that h and M play no role and that $\mathcal{B}^p_{M,h} = H^p$, so we recognize a classical (dual) extremal problem for which the following facts (see e.g. [8, IV]) are well–known :

(a) a solution does exist ;
(b) the solution is unique for $1 \leq p < \infty$;
(c) when $p = \infty$, the solution need not be unique but is unique at least if $f \in H^\infty + C(\mathbb{T})$, and the error function $|f - g_0|$ is then constant a.e. on $\mathbb{T}$;
(d) when $p = \infty$, g_0 is continuous as soon as f is Dini–continuous (the Carleson–Jacobs theorem).

Our objective, in this section, is to carry properties (a), (b), and (c) over to general bounded extremal problems, namely when $K \neq \mathbb{T}$. As we shall see,

property (d) has no straightforward generalization.
Standard proofs in the case where $K = \mathbb{T}$ rely on duality in Hardy spaces, interpreting the infimum in (22) as an operator norm (see e.g. [6], [8], [11]). Although such an interpretation is not available here, it can be replaced by strong convexity properties if $1 < p < \infty$ and, if $p = 1$, by combining differentiation with more refined results on Douglas algebras due to Axler. When $p = \infty$ – and this may not be too surprising since the L^∞ norm decouples the behaviour on K and $\mathbb{T} \setminus K$ respectively – problem 1 reduces, though in an implicit manner, to a classical extremal problem that we introduce now.

Problem 2 *Notations being as in problem 1 with $p = \infty$, let w_ρ be the outer factor whose modulus is 1 on $\mathbb{T} \setminus K$ and ρ on K :*

$$w_\rho(z) := \exp\left\{\frac{1}{2\pi} \log\rho \int_K \frac{e^{it}+z}{e^{it}-z}\, dt\right\}, \tag{23}$$

where ρ is a positive constant. Assume that the infimum in the right hand–side of (22) is positive and call it β_∞. We seek $v_0 \in H^\infty$ such that

$$\|(f \vee h) w_{\frac{M}{\beta_\infty}} - v_0\|_{L^\infty(\mathbb{T})} = \min_{v \in H^\infty} \|(f \vee h) w_{\frac{M}{\beta_\infty}} - v\|_{L^\infty(\mathbb{T})}. \tag{24}$$

Note that problem 1 is trivial if there exists $g \in \mathcal{B}^p_{M,h}$ such that $f = g_{|_K}$, since g is then the unique solution. Also, when $\lambda(\mathbb{T} \setminus K) = 0$, our results reduce to (a), (b), and (c) above which are already well–known. Observe that, for $1 \leq p < \infty$, it follows from theorem 1 that $\mathcal{B}^p_{M,h} \neq \emptyset$. However, when $p = \infty$, it may happen that $\mathcal{B}^\infty_{M,h} = \emptyset$, in which case problem 1 obviously has no solution. This will not occur if h belongs to the closure of $H^\infty_{|_K}$ in $L^\infty(K)$ (see theorem 2 for a partial characterization of this closure). We shall rule all these cases out when stating our theorems. In particular, the number β_∞ in problem 2 will now be positive.

Theorem 3 *If $p = \infty$, any solution v_0 to problem 2 is such that $g_0 = v_0 / w_{\frac{M}{\beta_\infty}}$ is a solution to problem 1 and, conversely, any solution g_0 to problem 1 gives rise to a solution $v_0 = g_0 w_{\frac{M}{\beta_\infty}}$ of problem 2.*

To state our next result, let us denote by $\mathcal{C}^p_{M,h}$ the trace of $\mathcal{B}^p_{M,h}$ on K :

$$\mathcal{C}^p_{M,h} := \{g_{|_K}\ ,\ g \in \mathcal{B}^p_{M,h}\}.$$

Theorem 4 *Concerning the bounded extremal problem stated above as problem 1 where it is assumed that $f \notin \mathcal{C}^p_{M,h}$ and $\lambda(\mathbb{T} \setminus K) > 0$, the following assertions hold true :*
(i) When $p < \infty$, a solution g_0 does exist ; when $p = \infty$, a solution g_0 does still exist, provided $\mathcal{B}^\infty_{M,h} \neq \emptyset$.
(ii) When $p < \infty$, the solution is unique and satisfies

$$\|h - g_0\|_{L^p(\mathbb{T}\setminus K)} = M. \tag{25}$$

(iii) If $p = \infty$, the solution need not be unique nor meet (25) ; however, any solution satisfies (25) when f belongs to the closure of $H^\infty_{|_K}$ in $L^\infty(K)$.
(iv) If $p = \infty$ the solution g_0 is unique at least when $f \vee h$ lies in $H^\infty + C(\mathbb{T})$ and, in this case, the functions $f - g_0$ and $h - g_0$ have constant modulus β_∞ and M a.e. on K and $\mathbb{T} \setminus K$ respectively.

Remark
Property (d) of ordinary extremal problems does not generalize to bounded extremal problems : there is no chance that g_0 be continuous in general, even if $f \vee h$ is very smooth. To see this, assume for instance that $f \vee h$ is Lipschitz of order α on $\mathbb{T}$, $0 < \alpha \leq 1$, and that $K = (-\theta_0, \theta_0)$ is an arc. There exists a polynomial P such that $(f \vee h - P)(\pm\theta_0) = 0$, and it is not hard to see that $(f \vee h - P)\, w_{\frac{M}{\beta_\infty}}$ is again Lipschitz of order α. In particular, it is Dini–continuous and, by the Carleson–Jacobs theorem, its best H^∞ approximant in $L^\infty(\mathbb{T})$, say u_0, is continuous. But we have, by theorem 3, that $g_0 = P + u_0\, w^{-1}_{\frac{M}{\beta_\infty}}$ which is not continuous at $\pm\theta_0$ unless $M = \beta_\infty$, that is when the bounded extremal problem is in fact a Nehari problem. The question as to when this occurs is an interesting one, to which the authors do not know the answer.

The proofs of theorems 4 and 3 will occupy the remainder of this section. As another piece of notation, for $1 \leq p \leq \infty$, we introduce the map

$$\begin{array}{rccc} \psi_p : & L^p(K) & \longrightarrow & \mathbb{R} \\ \text{given by} & \xi & \longmapsto & \|f - \xi\|_{L^p(K)}. \end{array}$$

3.1 Existence

In this subsection, we prove (i) of theorem 4.
The set $\mathcal{C}^p_{M,h}$ is clearly a convex subset of $L^p(K)$. When $1 < p < \infty$, it will be enough to show that it is closed to get at *the same time* existence and uniqueness of g_0 from the fact that there is a best approximation projection on any closed convex subset of a uniformly convex Banach space (see e.g. [5, 3,II,1,prop. 5 and 8]).

When $p = \infty$, it will also be sufficient to show that $\mathcal{C}^\infty_{M,h}$ is weak–* closed. Indeed, $\psi_\infty(\xi)$ goes to infinity with $\|\xi\|_{L^\infty(K)}$ so that we may restrict our attention to $\mathcal{C}^\infty_{M,h} \cap B(0,\varrho)$ where $B(0,\varrho)$ is the closed ball of radius ϱ centered at 0 in $L^\infty(K)$ and where ϱ is large enough. If $\mathcal{C}^\infty_{M,h}$ is weak–* closed, then $\mathcal{C}^\infty_{M,h} \cap B(0,\varrho)$ is weak–* compact because $B(0,\varrho)$ is. Now, ψ_∞ is lower semi–continuous on $L^\infty(K)$ equipped with the weak–* topology because it is the supremum of the family of continuous functions $\xi \to |\int_K (f-\xi) w \, d\lambda|$ when w ranges over the unit ball of $L^1(K)$. Therefore, ψ_∞ reaches its minimum on $\mathcal{C}^\infty_{M,h} \cap B(0,\varrho)$ hence on $\mathcal{C}^p_{M,h}$ by the choice of ϱ.

We now prove the desired closedness properties for $1 < p < \infty$ and $p = \infty$ simultaneously. Denote by q be the conjugate exponent and let $H^q_0 := e^{i\theta} H^q$ be the subspace of H^q consisting of functions vanishing at the origin. Since $1 \leq q < \infty$, it follows from theorem 1 that $H^q_{|\mathbb{T}\setminus K}$ is dense in $L^q(\mathbb{T}\setminus K)$ and it is then easily seen that the same is true of $H^q_{0|_{\mathbb{T}\setminus K}}$.
Now, take $w \in \mathcal{C}^p_{M,h}$ and write $w = g_{|K}$ with $g \in \mathcal{B}^p_{M,h}$. By duality and using the Cauchy formula :

$$M \geq \|h-g\|_{L^p(\mathbb{T}\setminus K)} = \sup_{\substack{v \in H^q_0 \\ \|v\|_{L^q(\mathbb{T}\setminus K) \leq 1}}} \left| \int_{\mathbb{T}\setminus K} (h-g) v \, d\lambda \right| = \sup_{\substack{v \in H^q_0 \\ \|v\|_{L^q(\mathbb{T}\setminus K) \leq 1}}} \left| \int_{\mathbb{T}} (w \vee h) v \, d\lambda \right| .$$

Conversely, let $w \in L^p(K)$ be such that :

$$\sup_{\substack{v \in H^q_0 \\ \|v\|_{L^q(\mathbb{T}\setminus K) \leq 1}}} \left| \int_{\mathbb{T}} (w \vee h) v \, d\lambda \right| \leq M .$$

Then the map $L_w : H^q_{0|_{\mathbb{T}\setminus K}} \to \mathbb{C}$ defined by

$$L_w(v_{|_{\mathbb{T}\setminus K}}) := \int_{\mathbb{T}} (w \vee h)\, v\, d\lambda \tag{26}$$

extends to a continuous linear form on $L^q(\mathbb{T}\setminus K)$ whose norm is at most M. Thus, there exists $w' \in L^p(\mathbb{T}\setminus K)$ such that $\|w'\|_{L^p(\mathbb{T}\setminus K)} \leq M$ and

$$L_w(v_{|_{\mathbb{T}\setminus K}}) = \int_{\mathbb{T}\setminus K} w' v\, d\lambda \quad \text{ for } v \in H^q_0. \tag{27}$$

Substracting (27) from (26), we see that the function $w \vee (h - w')$ belongs to H^p hence to $\mathcal{B}^p_{M,h}$. Consequently, w is in $\mathcal{C}^p_{M,h}$ and we have obtained for $1 < p \leq \infty$ the following characterization :

$$\mathcal{C}^p_{M,h} = \bigcap_{\substack{v \,\in\, H^q_0 \\ \|v\|_{L^q(\mathbb{T}\setminus K)\leq 1}}} \left\{ w \in L^p(K)\,;\ \left| \int_K wv\, d\lambda + \int_{\mathbb{T}\setminus K} hv\, d\lambda \right| \leq M \right\}. \tag{28}$$

When $h = 0$, this characterization is in the spirit of those obtained on the line for $p = 2$ and $p = \infty$ in [17, thms.1,4]. It shows that $\mathcal{C}^p_{M,h}$ is an intersection of closed sets (of weak–* closed sets if $p = \infty$), and settles the question of the existence of g_0 if $1 < p \leq \infty$.

When $p = 1$, we can still imbed $L^1(K)$ into $\mathcal{M}_K$ but there is a slight difficulty due to the fact that K is not necessarily locally compact so the duality between $C(K)$ and $\mathcal{M}_K$ may fail. However, there exists a sequence (K_m) of compact subsets of K such that $\lambda(K \setminus K_m) \to 0$ (see e.g. [19, thm.2.18]). We begin with a preliminary result.

Lemma 1 *Let (g_n) be a sequence of functions converging weakly–* to g in H^1. For E a compact subset of $\mathbb{T}$ and f an arbitrary function in $L^1(E)$, we have that*

$$\|f - g\|_{L^1(E)} \leq \liminf_n \|f - g_n\|_{L^1(E)}. \tag{29}$$

Proof : upon multiplying all functions by $e^{i\theta}$, which does not affect the norm, we may assume that $g_n(0) = g(0) = 0$. Then, as in the proof of proposition 3, weak–* convergence sharpens to

$$\lim_{n\to\infty} \int_{\mathbb{T}} g_n w\, d\lambda = \int_{\mathbb{T}} gw\, d\lambda \quad \text{ for all } w \in H^\infty + C(\mathbb{T}). \tag{30}$$

Let $\Lambda(E)$ denote the closed unit ball of $C(E)$ endowed with the sup norm. By the Tietze–Urysohn theorem, and since E is closed, we have $\Lambda(E) = \Lambda(\mathbb{T})_{|_E}$. Defining

$$\widetilde{\Lambda(E)} := \{v \in L^\infty(\mathbb{T})\ ;\ v_{|_E} \in \Lambda(E) \ \text{ and } \ v_{|_{\mathbb{T}\setminus E}} = 0\},$$

this can be rephrased as

$$\widetilde{\Lambda(E)} = \chi_E \Lambda(\mathbb{T}). \tag{31}$$

Also, since any function in $L^\infty(\mathbb{T})$ is the quotient of a member of $H^\infty + C(\mathbb{T})$ by a Blaschke product [2], we can write

$$\chi_E = \bar{b} w_E, \quad w_E \in H^\infty + C(\mathbb{T}), \quad b \text{ a Blaschke product.} \tag{32}$$

Now, we have by duality

$$\|f-g\|_{L^1(E)} = \|b(f-g)\|_{L^1(E)} = \sup_{v \in \widetilde{\Lambda(E)}} \int_{\mathbb{T}} b(f-g)v\, d\lambda = \sup_{w \in \Lambda(\mathbb{T})} \int_{\mathbb{T}} (f-g) w_E w\, d\lambda,$$

where the last equality uses (31) and (32), and we see from (30) that the argument of the last supremum is just

$$\lim_{n\to\infty} \int_{\mathbb{T}} (f - g_n) w_E w\, d\lambda. \tag{33}$$

To establish (29), it remains to prove that (33) is majorized by $\liminf \|f - g_n\|_{L^1(E)}$, but this follows at once from the inequality

$$\int_{\mathbb{T}} (f - g_n) w_E w\, d\lambda \le \|f - g_n\|_{L^1(E)},$$

which is plain if we recall that $w \in \Lambda(\mathbb{T})$ and that $|w_E|$ is a.e. 1 on E and 0 on $\mathbb{T} \setminus E$. ■

We return to the proof of the existence of g_0, and we denote by β_1 the infimum in the right hand–side of (22) where p is set to 1. Let $(g_{n_{|_K}})$ be a minimizing sequence for ψ_1 on $\mathcal{C}^1_{M,h}$ where $g_n \in \mathcal{B}^1_{M,h}$. Then $\|g_n\|_{L^1(K)}$ is bounded and, since $\|h - g_n\|_{L^1(\mathbb{T}\setminus K)} \le M$, (g_n) is bounded in $L^1(\mathbb{T})$. Hence, we may extract a subsequence (g_{n_k}) converging weakly–* in $\mathcal{M}_{\mathbb{T}}$ toward $g \in H^1$. By the definition of (g_n), we have

$$\lim_{n\to\infty} \|f - g_n\|_{L^1(K)} = \beta_1. \tag{34}$$

and *a fortiori*

$$\liminf_n \|f - g_n\|_{L^1(K_m)} \leq \beta_1, \quad \text{for each } m. \tag{35}$$

Applying lemma 1 with $E = K_m$, we get in view of (35)

$$\|f - g\|_{L^1(K_m)} \leq \beta_1, \quad \text{for each } m. \tag{36}$$

By monotone convergence, it follows now that

$$\|f - g\|_{L^1(K)} \leq \beta_1. \tag{37}$$

Because $\|h - g_n\|_{L^1(\mathbb{T}\setminus K)} \leq M$, a similar argument when changing K into $\mathbb{T} \setminus K$ and f into h shows that $\|h - g\|_{L^1(\mathbb{T}\setminus K)} \leq M$, in other words that $g \in \mathcal{B}^1_{M,h}$. Thus, equality must hold in (37) and we can take $g_0 = g$. ∎

3.2 Uniqueness and saturation of the constraint

In this subsection, we prove (ii) and the second half of (iii) of theorem 4.

We first show that (25) holds for $p < \infty$ and also for $p = \infty$ if f belongs to the closure of $H^\infty_{|_K}$ in $L^\infty(K)$.
Assume indeed that $\|h - g_0\|_{L^p(\mathbb{T}\setminus K)} < M$. There is a neighborhood $\mathcal{V}$ of g_0 in H^p such that $\mathcal{V} \subset \mathcal{B}^p_{M,h}$. Denoting by $R_K : H^p \to L^p(K)$ the natural restriction, this means that the map $\psi_p R_K$ attains a local minimum at g_0 hence a global minimum, for this map is convex. But thanks to theorem 1 when $1 \leq p < \infty$ and by our very assumption when $p = \infty$, we know that f belongs to the closure of $H^p_{|_K}$ in $L^p(K)$ so this global minimum can only be zero. This would entail $f = g_{0|_K}$ hence $f \in \mathcal{C}^p_{M,h}$, contrary to the hypothesis. Thus, (25) holds under the stated assumptions.

We now turn to uniqueness. The case where $1 < p < \infty$ has been settled in section 3.1 so we need only consider $p = 1$. We shall make use of the following lemma :

Lemma 2 *For any subset E of $\mathbb{T}$, the map :*

$$\begin{array}{rccl} \mathcal{N}_E : & L^1(E) & \to & \mathbb{R} \\ & v & \mapsto & \|v\|_{L^1(E)} \end{array}$$

is Gâteaux–differentiable at every $v \in L^1(E)$ such that v is a.e. nonzero on E ; its derivative at v in the direction $u \in L^1(E)$ is then given by :

$$D\mathcal{N}_E(v).[u] = \mathrm{Re} \int_E \frac{\overline{v}}{|v|} u \, d\lambda \,. \tag{38}$$

Proof : since v is a.e. nonzero on E, it has a unique norming functional which is given by (38). The result now follows from [5, 3,I,2,rmk.6]. ■

Let g_0 be a solution to problem 1 with $p = 1$. Our first task will be to derive a critical point equation for g_0 under the extra–assumption that

$$(f \vee h) \, - \, g_0 \neq 0 \text{ a.e. on } \mathbb{T}. \tag{39}$$

Thanks to (39), we may thus define a subspace of H^1 by

$$T = \{u \in H^1; \qquad \mathrm{Re} \int_{\mathbb{T}\setminus K} \frac{\overline{h - g_0}}{|h - g_0|} u \, d\lambda = 0\} \,.$$

Next, since $H^1_{|_{\mathbb{T}\setminus K}}$ is dense in $L^1(\mathbb{T} \setminus K)$ by theorem 1, we can find $v \in H^1$ such that

$$c_v := \mathrm{Re} \int_{\mathbb{T}\setminus K} \frac{\overline{h - g_0}}{|h - g_0|} (v - g_0) \, d\lambda > 0 \,, \tag{40}$$

for if $v_{|_{\mathbb{T}\setminus K}}$ tends to h in $L^1(\mathbb{T} \setminus K)$ the integral in (40) tends to $\|h - g_0\|_{L^1(\mathbb{T}\setminus K)}$ by dominated convergence. Fix such a v, choose $u \in T$ and some $t > 0$, and consider the map

$$\begin{array}{rccl} G : & \mathbb{R} & \to & \mathbb{R} \\ & \varepsilon & \mapsto & \|h - g_0 + \varepsilon \, (u + t(g_0 - v))\|_{L^1(\mathbb{T}\setminus K)}. \end{array}$$

Since we assumed that $h - g_0$ is a.e. non–zero on $\mathbb{T} \setminus K$, G is differentiable at $\varepsilon = 0$ by lemma 2 and we have the Taylor expansion :

$$G(\varepsilon) = G(0) + \varepsilon \, D\mathcal{N}_{\mathbb{T}\setminus K}(h - g_0).[u + t(g_0 - v)] + \varepsilon \, \eta(\varepsilon) \,, \tag{41}$$

where η depends of course on u, t, and v but in any case satisfies $\lim_{\varepsilon\to 0}\eta(\varepsilon) = 0$. Recalling that $G(0) = M$ by (25), which was established before, and using (38), we can rewrite (41) as

$$G(\varepsilon) = M + \varepsilon \,\mathrm{Re} \int_{\mathbb{T}\setminus K} \frac{\overline{h-g_0}}{|h-g_0|}(u + t(g_0 - v))\,d\lambda \, + \varepsilon\,\eta(\varepsilon).$$

Since $u \in T$, this reduces to

$$G(\varepsilon) = M - \varepsilon\, t\, c_v + \varepsilon\,\eta(\varepsilon). \tag{42}$$

From (42) and the definition of G, we see that

$$g_0 - \varepsilon\,(u + t(g_0 - v)) \quad \in \mathcal{B}^1_{M,h}$$

as soon as ε is positive and so small that $\eta(\varepsilon) < t\,c_v$. By the definition of g_0, this implies

$$\psi_1(g_{0|_K}) \le \psi_1\left(g_{0|_K} - \varepsilon\,(u + t(g_0 - v))_{|_K}\right)$$

as soon as $\varepsilon > 0$ is small enough. If we consider now the map

$$\begin{array}{rccl} P: & \mathbb{R} & \to & \mathbb{R} \\ & \varepsilon & \mapsto & \|f - g_0 + \varepsilon\,(u + t(g_0 - v))\|_{L^1(K)}, \end{array}$$

this means that $P(\varepsilon)$ attains a local minimum on $\mathbb{R}^+$ at $\varepsilon = 0$ and since it is differentiable at this point, thanks to lemma 2 and (39), we get from (38) again :

$$\frac{dP}{d\varepsilon}(0) = D\mathcal{N}_K(f - g_0).[u + t(g_0 - v)] = \mathrm{Re} \int_K \frac{\overline{f-g_0}}{|f-g_0|}(u + t(g_0 - v))\,d\lambda \ge 0.$$

By letting t go to zero, we obtain in the limit

$$\mathrm{Re} \int_K \frac{\overline{f-g_0}}{|f-g_0|} u\,d\lambda \ge 0,$$

which is valid for any $u \in T$. Changing u into $-u$ yields

$$\mathrm{Re} \int_K \frac{\overline{f-g_0}}{|f-g_0|} u\,d\lambda = 0 \quad \text{for all } u \in T. \tag{43}$$

This is the critical point equation we were looking for.

Now, define two linear forms ϕ_h and ϕ_f on $L^1(\mathbb{T})$by the formulae :

$$\phi_h(\xi) := \int_{\mathbb{T}\setminus K} \frac{\overline{h-g_0}}{|h-g_0|}\, \xi \, d\lambda \tag{44}$$

and

$$\phi_f(\xi) := \int_{K} \frac{\overline{f-g_0}}{|f-g_0|}\, \xi \, d\lambda, \tag{45}$$

where it should be observed that they are well–defined thanks to hypothesis (39) and that none of them can be the zero form. We have that Re $\phi_h(T) = 0$ by definition of the space T and also that Re $\phi_f(T) = 0$ by (43). Choose $w \in H^1$ such that $w \notin T$. This is possible for otherwise $\mathrm{Re}\, \phi_h$ would vanish on H^1 and, since $\phi_h(\xi)$ depends on $\xi_{|_{\mathbb{T}\setminus K}}$ only, it would vanish identically by the density of $H^1_{|_{\mathbb{T}\setminus K}}$ in $L^1(\mathbb{T}\setminus K)$. Since a linear form is completely determined by its real part, ϕ_h itself would vanish and this cannot happen. Set

$$m := \frac{\mathrm{Re}\ \phi_f(w)}{\mathrm{Re}\ \phi_h(w)}, \tag{46}$$

and consider the composite linear form on $L^1(\mathbb{T})$

$$L_{f,h} := \phi_f - m\, \phi_h. \tag{47}$$

By restriction, $L_{f,h}$ induces a continuous linear form on H^1 whose real part vanishes on T and at w. Because any function $\varphi \in H^1$ can be written as

$$\varphi = \frac{\phi_h(\varphi)}{\phi_h(w)} w + u$$

where u belongs to T, we conclude that Re $L_{h,f}$ vanishes on H^1, hence $L_{h,f}$ itself vanishes on H^1. This implies that there exists a function $\rho \in H^\infty$ with $\rho(0) = 0$ such that

$$L_{h,f}(\xi) = \int_{\mathbb{T}} \rho\, \xi\, d\lambda\,,\ \forall \xi \in L^1(\mathbb{T})\,. \tag{48}$$

But using (47), (44), and (45), $L_{h,f}$ rewrites as

$$L_{h,f}(\xi) = \int_{\mathbb{T}} ((1+m)\,\chi_{K} - m)\, \frac{\overline{f \vee h - g_0}}{|f \vee h - g_0|}\, \xi \, d\lambda \,,$$

so that necessarily,

$$((1+m)\,\chi_{K} - m)\, \frac{\overline{f \vee h - g_0}}{|f \vee h - g_0|} = \rho \,, \text{ a.e. on } \mathbb{T}\,. \tag{49}$$

Here, it is essential to recognize that T, ϕ_h, ϕ_f, m, and thus also $L_{h,f}$ and ρ *depend only on the function* $\overline{f \vee h - g_0}/|f \vee h - g_0|$.

We are now in position to establish uniqueness. To this effect, let g_0 and g_0' be two solution to problem 1, *where we no longer assume* (39) *to hold.* So, let $Z_0 \subset \mathbb{T}$ be the zero set of $f \vee h - g_0$ and Z_0' be the zero set of $f \vee h - g_0'$. As before, we denote by β_1 the infimum in the right hand–side of (22). Since $\mathcal{C}^1_{M,h}$ and ψ_1 are convex, $l\, g_0 + (1-l)\, g_0'$ is again a solution for each $l \in [0,1]$:

$$\int_K |l\,(f-g_0) + (1-l)\,(f-g_0')|\, d\lambda = \int_K l\,|f-g_0| + (1-l)\,|f-g_0'|\, d\lambda = \beta_1 \,,$$

and also :

$$\int_{\mathbb{T}\setminus K} |l\,(h-g_0) + (1-l)\,(h-g_0')|\, d\lambda = \int_{\mathbb{T}\setminus K} l\,|h-g_0| + (1-l)\,|h-g_0'|\, d\lambda = M$$

since (25) holds as we have seen. In each of the above equations, the first integrand is majorized by the second so they must in fact be equal a.e. and this means that $f \vee h - g_0$ and $f \vee h - g_0'$ have the same argument modulo 2π a.e. on the subset where both are defined :

$$\frac{\overline{f \vee h - g_0}}{|f \vee h - g_0|} = \frac{\overline{f \vee h - g_0'}}{|f \vee h - g_0'|} \quad \text{a.e. on } \mathbb{T} \setminus (Z_0 \cup Z_0')\,. \tag{50}$$

Assume first that $\lambda(Z_0) = \lambda(Z_0') = 0$. This means (39) is satisfied both for g_0 and g_0' and (50) shows, by a previous remark, that equation (49) holds both for g_0 and g_0' *with the same m and the same function ρ.* Therefore we can write

$$((1+m)\,\chi_{K} - m)\, \frac{\overline{f \vee h - g_0}}{|f \vee h - g_0|} = ((1+m)\,\chi_{K} - m)\, \frac{\overline{f \vee h - g_0'}}{|f \vee h - g_0'|} = \rho \,, \text{ a.e. on } \mathbb{T}.$$

Upon multiplying by $f \vee h - g_0$ and $f \vee h - g_0'$ respectively, we get

$$\rho\,(f \vee h - g_0) = ((1+m)\,\chi_{K} - m)\,|f \vee h - g_0|\,,$$
$$\rho\,(f \vee h - g_0') = ((1+m)\,\chi_{K} - m)\,|f \vee h - g_0'|\,,$$

and, taking the difference, we see that the function $\rho\,(g_0 - g_0') \in H^1$ takes real values a.e. on $\mathbb{T}$. Using the Poisson representation formula, this implies that $\rho\,(g_0 - g_0')$ takes real values a.e. in $\mathbb{D}$ hence is constant. Since $\rho(0) = 0$, this constant has to be zero, and $g_0 = g_0'$. This was the main part of the proof.

Assume now, for instance, that $\lambda(Z_0) > 0$.
If $\lambda(Z_0 \cap Z_0') > 0$ then g_0 and g_0' are both equal to $f \vee h$ a.e. over some common set of positive measure, so they are equal.
Next, consider the case where $\lambda(Z_0 \cap Z_0') = 0$. For every $l \in (0,1)$,

$$g_l = l\,g_0 + (1-l)\,g_0'$$

is also a solution of (22). Let Z_l be the zero set of $f \vee h - g_l$ and assume that $\lambda(Z_l) > 0$.
If $\lambda(Z_l \cap Z_0) > 0$, we get $g_0 = g_l$ by the previous argument and it follows then from the definition of g_l that $g_0 = g_0'$ also. We argue similarly if $\lambda(Z_l \cap Z_0') > 0$. Finally, assume that $\lambda(Z_l \cap Z_0) = 0$ and $\lambda(Z_l \cap Z_0') = 0$ for each $l \in (0,1)$. Then, a.e. on Z_l, $f \vee h - g_0$ and $f \vee h - g_0'$ have the same argument by (50). But

$$f \vee h - g_l = l\,(f \vee h - g_0) + (1-l)\,(f \vee h - g_0') = 0 \quad \text{a.e. on } Z_l$$

implies

$$\frac{f \vee h - g_0'}{f \vee h - g_0} = -\frac{l}{1-l} < 0\,, \text{ a.e. on } Z_l\,,$$

which contradicts the equality of the arguments.
Hence, either $g_0 = g_0'$ or else our initial assumption that $\lambda(Z_l) > 0$ is false. But if we suppose that $\lambda(Z_l) = 0$ for each l we get $g_l = g_{l'}$ for any $l, l' \in (0,1)$ by the main part of the proof. Choosing $l \neq l'$ and solving for g_0 and g_0' again leads to $g_0 = g_0'$, thereby achieving the proof of uniqueness. ∎

3.3 An example of non–uniqueness when $p = \infty$

In this subsection, we prove the first half of (iii) of theorem 4.

Let $K = [-\frac{\pi}{2}, \frac{\pi}{2}]$, $M = 1$, and consider the $L^\infty(K)$ function

$$f = \begin{cases} -1 & \text{on } [0, \frac{\pi}{2}]\,, \\ 1 & \text{on } [-\frac{\pi}{2}, 0)\,. \end{cases}$$

Take $h \equiv 0$ on $\mathbb{T} \setminus K$. We now show that these data generate at least two solutions of (22). This may not be surprising since $f \vee 0$ is a classical example of a function having at least two best approximants from H^∞ in $L^\infty(\mathbb{T})$ (see e.g. [6, 8.3(iii)], [8, IV.1.5], [11, VII.A.2]), and we merely show that these are solutions of a bounded extremal problem as well. Any of the references we just mentioned establishes that

$$\inf_{g \in H^\infty} \|f \vee 0 - g\|_{L^\infty(\mathbb{T})} = 1\,. \tag{51}$$

From (51), it follows that :

$$1 \leq \inf_{\substack{g \in H^\infty \\ \|g\|_{L^\infty(\mathbb{T}\setminus K)} \leq 1}} \|f \vee 0 - g\|_{L^\infty(\mathbb{T})} \leq \|f \vee 0\|_{L^\infty(\mathbb{T})} = \|f\|_{L^\infty(K)} = 1\,,$$

so that equality holds througout and

$$\inf_{g \in \mathcal{B}^\infty_{1,0}} \psi_\infty(g_{|K}) = \inf_{\substack{g \in H^\infty \\ \|g\|_{L^\infty(\mathbb{T}\setminus K)} \leq 1}} \|f - g\|_{L^\infty(K)} \leq \inf_{\substack{g \in H^\infty \\ \|g\|_{L^\infty(\mathbb{T}\setminus K)} \leq 1}} \|f \vee 0 - g\|_{L^\infty(\mathbb{T})} = 1\,.$$

Let us prove that in fact :

$$\inf_{g \in \mathcal{B}^\infty_{1,0}} \psi_\infty(g_{|K}) = 1\,. \tag{52}$$

Indeed, by part (i) of theorem 4 which we already proved, there is a function $g_0 \in \mathcal{B}^\infty_{1,0}$ such that

$$\psi_\infty({g_0}_{|K}) = \inf_{g \in \mathcal{B}^\infty_{1,0}} \psi_\infty(g_{|K})\,.$$

Assume that $\psi_\infty(g_{0|_K}) < 1$; then there exists $\nu \in (0,1)$ such that $\psi_\infty(g_{0|_K}) < 1-\nu$ so that in particular $g_{0|_K} \neq 0$. For every $l \in (-1,+1)$,

$$\|l\, g_0\|_{L^\infty(\mathbb{T}\setminus\mathrm{K})} < 1 \tag{53}$$

hence $l\, g_0 \in \mathcal{B}^\infty_{1,0}$. Choose $l = 1-\nu \,/\, \|g_0\|_{L^\infty(K)}$ to obtain :

$$\psi_\infty(l\, g_{0|_K}) \leq \psi_\infty(g_{0|_K}) + (1-l)\, \|g_0\|_{L^\infty(K)} < 1\,.$$

This, together with (53), implies that $\|f \vee 0 - l\, g_0\|_{L^\infty(\mathbb{T})} < 1$, which contradicts (51), thereby establishing (52).

As a consequence of (52), the zero function is a best approximant to f from $\mathcal{B}^\infty_{1,0}$ for which (25) does not hold.

Yet another solution is given here by the conformal mapping g_0 from $\mathbb{D}$ onto the half disk $\{w \in \mathbb{D}\,,\ \mathrm{Im}\, w < 0\}$ such that (see also [6, 8.3(iii)]) :

$$\left\{ \begin{array}{lcl} g_0(-i) & = & 1\,, \\ g_0(1) & = & 0\,, \\ g_0(i) & = & -1\,. \end{array} \right.$$

Indeed, it maps $\mathbb{T} \setminus K$ onto the arc $\mathbb{T} \cap \{\mathrm{Im}\, w < 0\}$ so that $\|g_0\|_{L^\infty(\mathbb{T}\setminus\mathrm{K})} = 1$ hence $g_0 \in \mathcal{B}^\infty_{1,0}$, and one can easily verify that $\|f - g_0\|_{L^\infty(K)} = 1$. This shows that a solution to problem 1 need not be unique when $p = \infty$.

3.4 More on the case $p = \infty$

In this subsection, we prove theorem 3 and part (iv) of theorem 4.

Proof of theorem 3 : suppose that $v_0 \in H^\infty$ is a solution to problem 2. Define

$$\delta := \|(f \vee h)\, w_{\frac{M}{\beta_\infty}} - v_0\|_{L^\infty(\mathbb{T})}\,.$$

Then with $g_0 = v_0 w^{-1}_{\frac{M}{\beta_\infty}}$, where it should be observed that $w_{\frac{M}{\beta_\infty}}$ is indeed invertible in H^∞, we have

$$\|(f \vee h - g_0)\, w_{\frac{M}{\beta_\infty}}\|_{L^\infty(\mathbb{T})} = \delta\,,$$

which, from the definition of w, shows that

$$\max\left(\frac{M}{\beta_\infty}\|f-g_0\|_{L^\infty(K)}, \|h-g_0\|_{L^\infty(\mathbb{T}\setminus K)}\right)=\delta.$$

We assert now that $\delta = M$, which implies that g_0 is a solution to problem 1. For it is clearly impossible for δ to be less than M, otherwise this makes $\|f-g_0\|_{L^\infty(K)} < \beta_\infty$ and $\|h-g_0\|_{L^\infty(\mathbb{T}\setminus K)} < M$, which contradicts the definition of β_∞. Moreover δ cannot exceed M, because there is a function g_1 (i.e. a solution to problem 1 which has been shown to exist in section 3.1) with

$$\max\left(\frac{M}{\beta_\infty}\|f-g_1\|_{L^\infty(K)}, \|h-g_1\|_{L^\infty(\mathbb{T}\setminus K)}\right)=M\,.$$

But then replacing v_0 by $v_1 = g_1 w_{\frac{M}{\beta_\infty}}$ in problem 2 would contradict the optimality of v_0. Hence $\delta = M$ and g_0 is a solution to problem 1.

Conversely, suppose that g_0 is a solution to problem 1. Thus

$$\|f-g_0\|_{L^\infty(K)} = \beta_\infty \text{ and } \|h-g_0\|_{L^\infty(\mathbb{T}\setminus K)} \leq M.$$

With $v_0 = g_0 w_{\frac{M}{\beta_\infty}}$ we then have

$$\|(f \vee h)w_{\frac{M}{\beta_\infty}} - v_0\|_{L^\infty(\mathbb{T})} = M$$

and we assert that v_0 is a solution to problem 2. For if not, letting v_1 be such that

$$\|(f \vee h)w_{\frac{M}{\beta_\infty}} - v_1\|_{L^\infty(\mathbb{T})} < M,$$

and writing $g_1 = v_1 w^{-1}_{\frac{M}{\beta_\infty}}$, we see that $\|f-g_1\|_{L^\infty(K)} < \beta_\infty$ and $\|h-g_1\|_{L^\infty(\mathbb{T}\setminus K)} < M$, which contradicts the definition of g_0 as a solution to problem 1. ∎

This completes the proof of theorem 3. Note that we have been using the existence of a solution to problem 1 to establish the equivalence between this problem and problem 2 and therefore could not deduce it from the existence of a solution to problem 2.

We finally prove part (iv) of theorem 4. Having at our disposal the equivalence of problem 1 with the classical extremal problem 2 (which is theorem 3), all the assertions follow immediately from the known properties (a), (b) and (c) listed just after problem 1, once it is recognized that $f \vee h$ belongs to $H^\infty + C(\mathbb{T})$ if and only if $(f \vee h) w_{\frac{M}{\beta_\infty}}$ does. This, in turn, follows immediately from the fact that $H^\infty + C(\mathbb{T})$ is an algebra and $w_{\frac{M}{\beta_\infty}}$ is invertible in H^∞. The proof of theorem 4 is now complete. ∎

4 Concluding remarks

Having derived in section 2 certain properties of traces of Hardy functions on subsets of the circle, we were able to obtain in section 3 a full analog for bounded extremal problems to the well–known qualitative features of classical (dual) extremal problems. As explained in the introduction, an incentive for studying these questions arises naturally from issues in the identification of linear dynamical systems.

Now, having analysed existence and uniqueness of solutions to bounded extremal problems, the next thing is how to compute them. From the system–theoretic viewpoint, the interest lies mostly with the situation where K is a union of intervals, $f \vee h$ is continuous, and $p = \infty$ though $p = 2$ is also of some value in robust identification [4]. So far, this question has not been investigated in depth and we shall merely comment on a few directions suggested by the literature and by the above study.

In the terminology of problem 1, the case $p = 2$ was considered with $f = 0$ in [12] and with $h = 0$ [1]. In both references, the proposed solution proceeds along the classical lines of constrained optimization, namely rests on solving a critical point equation which contains an unknown Lagrange multiplier. This reduces to a spectral equation of the Toeplitz type, and monotonicity properties proved in [1] allow one, in principle, to numerically solve with respect to the multiplier by dichotomy. Let us mention also that an analoguous bounded extremal problem was studied in [3] for the Hardy–Sobolev class of exponent

2 on the disk. While this remains to be done, there is little doubt that the technique can be extended to the case where neither f nor h is zero. The situation where $1 < p < \infty$ is more complicated. Though it is still true that the problem amounts to find the unique critical point of the smooth function ψ_p on the (infinite–dimensional) manifold $\mathcal{C}^p_{M,h}$, the critical point equation does not come out so nicely.

When $p = 1$, the problem looks more difficult. Much of the complexity in the proofs of the present paper was due to the lack of smoothness in this case, which itself reflects the usual failure of duality. Of course, one would expect the Gâteaux differentiability and the critical point equation established in the proof of theorem 4 to be helpful when $f \vee h - g_0$ is a.e. non–zero. How to ensure that this condition holds is yet unclear.

If $p = \infty$, the situation is best understood because the question reduces to a classical best–approximation problem (Nehari extension), namely problem 2. Analysing the solution to the latter from a practical perspective leads us to recall briefly the operator–theoretic approach to the Nehari extension which has undeniable elegance. (see e.g. [14], [22]).

Write $F = (f \vee h)w_{\frac{M}{\beta_\infty}}$. Problem 2 then consists of finding $v_0 \in H^\infty$ to meet

$$\inf_{v \in H^\infty} \|F - v\|_{L^\infty(\mathbb{T})}. \tag{54}$$

Let $\Gamma : H^2(\mathbb{T}) \to H^2(\mathbb{T})^\perp$ be the Hankel operator defined by

$$\Gamma(u) = P_{H^2(\mathbb{T})^\perp}(Fu), \qquad u \in H^2(\mathbb{T}),$$

where $P_{H^2(\mathbb{T})^\perp} : L^2(\mathbb{T}) \to H^2(\mathbb{T})^\perp$ is the orthogonal projection. It is a theorem of Nehari that the infimum in (54) is equal to $\|\Gamma\|$. Suppose that Γ attains its norm at a nonzero vector $u_0 \in H^2(\mathbb{T})$, that is, $\|\Gamma(u_0)\| = \|\Gamma\|\|u_0\|$. This will happen for example if Γ is compact, that is, if $F \in H^\infty + C(\mathbb{T})$ by Hartman's theorem (see e.g. [14], [22]).

Then there is a unique best H^∞ approximant v_0 to F and it is given by

$$v_0(z) = F(z) - (\Gamma(u_0))(z)/u_0(z).$$

Moreover $|F(e^{i\theta}) - v_0(e^{i\theta})| = \|\Gamma\|$ almost everywhere.

Now in our setting, if we assume that $f \vee h$ lies in $H^\infty + C(\mathbb{T})$, then $F \in H^\infty + C(\mathbb{T})$ also, and the Hankel operator is compact. Thus, in the notation of problems 1 and 2, we get with $g_0 = v_0 w_{\frac{M}{\beta_\infty}}^{-1}$:

$$|f(e^{i\theta}) - g_0(e^{i\theta})| = \|\Gamma\|\beta_\infty/M \quad \text{a.e. on } K$$

and

$$|h(e^{i\theta}) - g_0(e^{i\theta})| = \|\Gamma\| \quad \text{a.e. on } \mathbb{T} \setminus K,$$

which forces $\|\Gamma\| = M$ since we know, from theorem 3, that g_0 is a solution to problem 1. It should be noted that problem 2 is an implicit one, since the data depend on the solution through β_∞ : this is the counterpart, for $p = \infty$, of the Lagrange multiplier. But here also, a dichotomy procedure can be used to iteratively solve for β_∞ : the right value of β_∞ is the one that makes $\|\Gamma\|$ equal to M.

A more serious difficulty, however, is that the above solution requires the knowledge of the singular values of Γ, and also of the singular vector associated with the largest singular value. Such a computation can be effective only when Γ is of finite rank, that is, when there exists a rational symbol. Here, we will face a problem even if $f \vee h$ is given in rational form because $w_{\frac{M}{\beta_\infty}}$ does not belong to the space $\mathcal{R}$ of rational functions. One way to construct suboptimal functions is of course to approximate $(f \vee g)w_{\frac{M}{\beta_\infty}}$ from $H^\infty + \mathcal{R}$, but some care is required, for the nonlinear operator taking a continuous function to its best analytic approximation is not norm continuous in general [13]. Therefore, there is *a priori* no reason why suboptimal approximants should be close to the optimal one.

From the practical point of view of robust identification of linear dynamical systems, it would be important to be able to compute rational approximations to the solutions to problems 1 and 2 in the case $p = \infty$ in order to produce a rational function in H^∞ whose boundary values are close to f on K while still being constrained sufficiently on $\mathbb{T} \setminus K$. To do this requires some systematic interpolation and approximation procedures, the details of which we hope to present in a later work.

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