Hierarchies of decidable extensions of bounded quantification

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Abstract:

The system $F_{\leq}$, the well-known second-order polymorphic typed λ-calculus with subtyping and bounded universal type quantification [CW85, BL90, CG92, Pie92, CMMS94], appears to be undecidable [Pie92] because of undecidability of its subtyping component. Attempts were made to obtain decidable type systems with subtyping by *weakening* $F_{\leq}$ [CP94, KS92], and also by *reinforcing* or *extending* it [Vor94a, Vor94b, Vor95]. However, for the moment, these extensions lack the important proof-theoretic *minimum type property*, which holds for $F_{\leq}$ and guarantees that each typable term has the minimum type, being a subtype of any other type of the term in the same context [CG92, Vor94c].

As a preparation step to introducing the extensions of $F_{\leq}$ with the minimum type property and the decidable term typing relation (which we do in [Vor94e]), we define and study here the hierarchies of decidable extensions of the $F_{\leq}$ subtyping relation. We demonstrate conditions providing that each theory in a hierarchy:

1. *extends* $F_{\leq}$, proving everything that is $F_{\leq}$-provable;

2. satisfies the *substitution property* for boundedly quantified universal types:

$$\Gamma \vdash (\forall \gamma \leq \sigma_1 . \sigma_2) \leq (\forall \gamma \leq \tau_1 . \tau_2) \text{ and } \Gamma \vdash \theta \leq \tau_1 \text{ imply } \Gamma \vdash \sigma_2[\theta/\gamma] \leq \tau_2[\theta/\gamma];$$

3. is *transitive*: $\Gamma \vdash \sigma_1 \leq \sigma_2$ *and* $\Gamma \vdash \sigma_2 \leq \sigma_3$ *imply* $\Gamma \vdash \sigma_1 \leq \sigma_3$.

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Transitivity and substitutivity are indispensable for the minimum type property and typing proof normalization [Vor94e].

We give conditions guaranteeing that a hierarchy *condenses*, i.e.,

$$Th_{\leq}(F_{\leq}^{(i)}(\mathcal{H})) \supseteq Th_{\leq}(F_{\leq}^{(i+1)}(\mathcal{H})) \quad \textit{for each } i \in \mathbb{N}$$

and *converges* to $F_{\leq}$, i.e.,

$$\lim_{n\to\infty} Th_{\leq}(F_{\leq}^{(n)}(\mathcal{H})) \;=\; \bigcap_{n=1}^{\infty} Th_{\leq}(F_{\leq}^{(n)}(\mathcal{H}\;)) = \; Th_{\leq}(F_{\leq}) \;\; .$$

We also study cases when a hierarchy *collapses*, i.e.,

$$Th_{\leq}(F_{\leq}^{(i)}(\mathcal{H})) = Th_{\leq}(F_{\leq}^{(i+1)}(\mathcal{H})) \quad \textit{starting from some } i \in \mathbb{N} \;\; .$$

It turns out, however, that normally the hierarchies *do not collapse.*

Key-words: second-order polymorphic typed λ-calculus, subtyping, system $F_{\leq}$, bounded universal type quantification, (un)decidability, polymorphism, proof normalization.

(Résumé : tsvp)

Hiérarchies des extensions décidables de la quantification bornée

Résumé :

Le système $F_{\leq}$, le λ-calcul polymorphe typé avec le sous-typage et la quantification universelle bornée de types [CW85, BL90, CG92, Pie92, CMMS94], s'avère indécidable [Pie92] à cause de l'indécidabilité de sa composante de sous-typage. Des essais ont été effectués pour obtenir des systèmes de types décidables en *affaiblissant* $F_{\leq}$ [CP94, KS92], et aussi en le *renforçant* [Vor94a, Vor94b, Vor95]. Néanmoins, pour l'instant ces extensions ne satisfont pas la *propriéte de typage minimum*, très importante de point de vue de la théorie de la démonstration. Cette propriété est vraie pour $F_{\leq}$ et garantit que chaque terme typable possède un type minimum, étant le sous-type de n'importe quel autre type du terme dans le même contexte [CG92, Vor94c].

Pour préparer l'introduction des extensions de $F_{\leq}$ avec la propriété de typage minimum et la relation de typage décidable (ce que nous faisons dans [Vor94e]), nous définissons et étudions ici des hiérarchies des extensions décidables de la relation de sous-typage dans $F_{\leq}$. Nous établissons des conditions garantissant que chaque théorie dans une hiérarchie :

1. *étend* $F_{\leq}$, démontrant tout ce qui est $F_{\leq}$-démontrable;
2. satisfait la *propriété de substitution* pour les types universellement quantifiés;
3. est *transitive*.

Transitivité et substitutivité sont indispensables pour la propriété de typage minimum et pour la normalisation des démonstrations [Vor94e].

Nous donnons aussi des conditions garantissant qu'une hiérarchie *se condense* et *converge* vers $F_{\leq}$, et étudions des cas où une hiérarchie *s'effondre*. Il se trouve que normalement des hiérarchies *ne s'effondrent pas*.

Mots-clé : λ-calcul polymorphe du second ordre, sous-typage, système $F_{\leq}$, quantification universelle bornée de types, polymorphisme, (in)décidabilité, normalisation de démonstrations.

Contents

1 Introduction

Polymorphism, inheritance, and static type checking are three important cornerstone paradigms of the most contemporary approaches to programming language design and methodology. Statically typed languages stipulate that all function calls typecheck at compile time. This makes programs more error-protected and efficient, making the run-time type control unnecessary. Almost all recently developed programming languages feature in one way or another combinations of ideas of type control in presence of polymorphism. During last decade a substantial progress has been achieved both on elucidating theoretical foundations of the underlying type systems and practical implementation of type-checkers for programming systems and environments [CW85, Car88, Mit88, Car89, BL90, Mit90, BTCCS91, CL91, Bar92, CG92, Pie92, KS92, CP94, CMMS94].

Among various existing notions of polymorphism the most important and influential are the following two:

The *parametric* (or horizontal) polymorphism characterizes a function uniformly applicable to objects of *any* type. This is best captured by universal quantification. For example, the polymorphic identity $\lambda x \, . \, x$ does not care what the type of its argument is. Its generic type is therefore $\forall \alpha \, . \, \alpha \to \alpha$. The contemporary study of parametric polymorphism is based on Girard-Reynold's polymorphic second-order λ-calculus [Gir71, Gir72, Rey74, GLT89].

The *inheritance* (or vertical) polymorphism is based on the idea of hierarchical data organization where each new descendant type possesses all the properties of its ancestor types. Therefore, a function applicable to an ancestor type should successfully work on all its offsprings, i.e., polymorphic downwards. This idea of programming over taxonomic data, first appearing in Simula-67, has been evolved into a new programming paradigm, now best known as the object-oriented programming. The systematic study of inheritance polymorphism and its combination with parametric polymorphism was started by Cardelli [Car88], Cardelli and Wegner [CW85].

Cardelli and Wegner [CW85] suggested to unify the parametric and inheritance polymorphism in a single type-theoretic framework. Their language *Fun*, an extension of the second-order typed λ-calculus with subtyping [CMMS94], includes both kinds of polymorphism, allowing simultaneously:

- the universal quantification over types $\forall \alpha \, . \, type[\alpha]$, expressing a polymorphic $type[\alpha]$ parametrized by an arbitrary type α;
- the inheritance subtype relation on types: $subtype \leq type$ meaning that a *subtype* inherits everything from a *type*;
- the most revolutionary feature of *Fun* is the ability to combine the both, achieved by means of the *bounded universal quantification*:

$$\forall \alpha \leq \textit{bound-type} \, . \, type[\alpha]$$

expressing a polymorphic $type[\alpha]$ parametrized by α that could be substituted by *any* subtype of a *bound-type*. Since *Fun* has the largest type $\top$, the bounded quantification subsumes the unbounded one as a particular case: $\forall\alpha$ is simply expressed by $\forall\alpha \leq \top$.

The ability to derive subtyping judgments between types $\Gamma \vdash \sigma \leq \tau$ (meaning: "σ is a subtype of τ in context Γ") in addition to usual typing judgments $\Gamma \vdash M : \rho$ (meaning: "M is of type ρ in context Γ") is crucial in the presence of subtyping. In fact, to correctly type an application of a function of type $\tau \rightarrow \chi$ to an argument of a subtype σ of a type τ, one needs rules like:

$$\frac{\Gamma \vdash t : \sigma \qquad\qquad \Gamma \vdash \sigma \leq \tau}{\Gamma \vdash t : \tau} \qquad (Subsumption)$$

which promotes a type of a term to all supertypes and brings together typing and subtyping.

To derive subtyping judgments about boundedly quantified types the following rule was introduced in *Fun*:

$$\frac{\Gamma, \alpha \leq \rho \vdash \sigma_2 \leq \tau_2}{\Gamma \vdash (\forall\alpha \leq \rho \,.\, \sigma_2) \leq (\forall\alpha \leq \rho \,.\, \tau_2)} \qquad (\mathit{All\text{-}Fun})$$

meaning that two boundedly quantified types are in the subtype relation if so are their bodies.

For theoretical purposes *Fun* was purified and simplified by Bruce and Longo [BL90], then by Curien and Ghelli [CG92]. The resulting type system is now known as the system $F_{\leq}$. The system $F_{\leq}$ includes more subtle and powerful subtyping rule for boundedly quantified types:

$$\frac{\Gamma \vdash \tau_1 \leq \sigma_1 \qquad\qquad \Gamma, \alpha \leq \tau_1 \vdash \sigma_2 \leq \tau_2}{\Gamma \vdash (\forall\alpha \leq \sigma_1 \,.\, \sigma_2) \leq (\forall\alpha \leq \tau_1 \,.\, \tau_2)} \qquad (\mathit{All})$$

allowing one to subtype types with *different* type bounds.

$F_{\leq}$ and its variations become a proving ground for experimentation and studying of different forms and notions of subtyping on boundedly quantified types and its algorithmic and proof-theoretic properties. Curien and Ghelli [CG92] proved that $F_{\leq}$ with powerful (*All*) rule is coherent, possessing the *minimal type property*, i.e., each $F_{\leq}$-typable term has a minimal type, being a subtype of any other type of the same term with respect to the subtyping relation generated by $F_{\leq}$. Unfortunately, as proved Pierce [Pie92], the $F_{\leq}$ subtyping relation is *undecidable*. Consequently, the $F_{\leq}$ typecheck problem is undecidable too. Trying to obtain decidability, attempts were made to *weaken* $F_{\leq}$. Katiyar and Sankar [KS92] proved decidability of restricted fragment of $F_{\leq}$ disabling the maximal type $\top$ in bounds of polymorphic types. Castagna and Pierce obtained decidability of $F_{\leq}$ replacing the powerful rule (*All*) above by its weaker version

$$\frac{\Gamma \vdash \tau_1 \leq \sigma_1 \qquad\qquad \Gamma, \alpha \leq \top \vdash \sigma_2 \leq \tau_2}{\Gamma \vdash (\forall\alpha \leq \sigma_1 . \sigma_2) \leq (\forall\alpha \leq \tau_1 . \tau_2)} \quad (\textit{All-Top})$$

However, the resulting system does not possess the minimal type property, as was noticed recently by Ghelli[1]. Other weaker variants of $F_{\leq}$ are mentioned in the introduction to [CP94] and in [Pie92].

At that point, as remarked Pierce, there exists several ways: either to return to original rule (*All-Fun*) with its unnatural restriction that bound of compared universally quantified types are the same, or stay with powerful but undecidable (*All*), or, better, look for other, simpler notions of subtyping.

In this paper we demonstrate that this last way is indeed possible. We show that there exist *infinitely many hierarchies* of different subtyping systems that are simultaneously:

- *simple* in the sense that they are all *decidable*;
- *more powerful than* $F_{\leq}$, i.e., they extend $F_{\leq}$ proving everything that $F_{\leq}$ does;
- *transitive*, i.e., $\Gamma \vdash \sigma_1 \leq \sigma_2$ and $\Gamma \vdash \sigma_2 \leq \sigma_3$ imply $\Gamma \vdash \sigma_1 \leq \sigma_3$;
- satisfy the *substitution property* for universal types, i.e.,
  $\Gamma \vdash (\forall\gamma \leq \sigma_1 . \sigma_2) \leq (\forall\gamma \leq \tau_1 . \tau_2)$ and $\Gamma \vdash \rho \leq \tau_1$ imply $\Gamma \vdash \sigma_2[\rho/\gamma] \leq \tau_2[\rho/\gamma]$.

We construct hierarchies of more and more tight decidable extensions of the $F_{\leq}$ subtyping relation satisfying the above properties and converging to $F_{\leq}$. Our ultimate goal consists in creating, on the basis of these hierarchies, the infinite family of decidable type systems with subtyping more powerful than $F_{\leq}$, which would type *more* terms than $F_{\leq}$ does, and possess the unique canonical proofs and the minimum type properties. The transitivity and substitution properties are necessary to achieve this. Using the hierarchies constructed in this paper, in [Vor94e] we introduce an infinite family of extensions of $F_{\leq}$ with *decidable typing*.

Infinite sets of decidable extensions of undecidable $F_{\leq}$-subtyping relation, based on interpreting $F_{\leq}$ in Rabin's second-order arithmetic of two successors, were first introduced and studied in [Vor94a]. It turns out that the $F_{\leq}$ types and the $F_{\leq}$ subtyping rules could be read in many ways just as **S2S**-formulas and admissible rules of **S2S** respectively. The **S2S**-interpretations were generalized to recursive types [Vor94b]. The main drawback of **S2S**-interpretations of $F_{\leq}$, as was noted by Cardelli, Pierce and others, was their *unstructuredness*. The **S2S**-interpretations were defined without any inference rules and thus proved too many undesirable subtyping judgments. This disadvantage was partially remedied in [Vor95], where the direct **SnS**-interpretations of $F_{\leq}$ were combined with conventional subtype inference rules. In this paper we go further: the structural decidable extensions of [Vor95]

[1] "types@dcs.gla.ac.uk" electronic forum, January 1994

form just the first levels of infinite hierarchies presented in this paper. These hierarchies is what is really needed to get decidable extensions of the $F_{\leq}$ typing, see [Vor94e].

The three main ideas of our approach are the following:

1. we *restrict* the structure of (possibly) infinite $F_{\leq}$-subtyping inferences, which lead to undecidability;

2. we *enrich* the set of *non-logical hypotheses* (which is empty in the case of $F_{\leq}$); instead of involving ourselves into exploring possibly infinite inferences, we prune the proof branches, which *may lead* to infinite loops and verify whether the resulting unproved judgments belong to the non-logical hypotheses sets;

3. we suggest a *large choice of non-logical hypotheses* sets, allowing us to extend $F_{\leq}$ and to get the transitivity and substitution properties; **SnS**-interpretations are the main tool[2] to construct such hypotheses sets; each **SnS**-interpretation gives rise to its own hierarchy; as there exists infinitely many different **SnS**-interpretations of $F_{\leq}$ [Vor94a], we get infinitely many different hierarchies, all converging to $F_{\leq}$.

After informally considering the example of the typing proof normalization in the presence of subtyping and elucidating the problems in Section 2, we proceed to basic definitions concerning polymorphic subtyping in Sections 3 and 4. Section 5 introduces the hierarchies of subtyping theories, and Sections 6–17 are devoted to proving the transitivity in the hierarchies and related problems. Sections 18–20 address the substitution property. To make the paper self-contained, we added the Appendix A on **SnS**-interpretations of $F_{\leq}$. All the proofs, which are quite difficult and subtle in the presence of non-logical hypotheses, are given in Appendix B.

[2] There exist others

2 Rapid Introduction to Typing Proof Normalization

Let us start by informally considering an example of a typing proof and a typing proof normalization in the presence of subtyping. This will allow us to introduce some jargon, to understand problems, and to sharpen the intuition.

In contrast to the simply typed λ-calculus or Girard's polymorphic system F [Bar92], the typing proofs in the presence of subtyping are no more unique. A term may possess several, or even infinitely many different typing proofs and, correspondingly, several or infinitely many types.

Here is a fragment of a typing proof in the system $F_{\leq}$:

$$\dfrac{\dfrac{\Gamma \vdash t : \overset{(*I1*)}{(\forall\alpha \leq \sigma_1 . \sigma_2)} \quad \Gamma \vdash \overset{(*I2*)}{(\forall\alpha \leq \sigma_1 . \sigma_2) \leq (\forall\alpha \leq \tau_1 . \tau_2)}}{\Gamma \vdash t : (\forall\alpha \leq \tau_1 . \tau_2)}(Sub) \qquad \Gamma \vdash \overset{(*I3*)}{\rho \leq \tau_1}}{\Gamma \vdash t\{\rho\} : \tau_2[\rho/\alpha]}(TApp) \tag{1}$$

where:

- the term t is assigned the boundedly quantified universal type $(\forall\gamma \leq \sigma_1 . \sigma_2)$ in context Γ by the typing proof $(*I1*)$, which is not depicted here; the term t is therefore proved to be a partial function accepting type parameters; when applied to any subtype ρ of the bound σ_1 this function produces the object $t\ \{\rho\}$ of the type $\sigma_2[\rho/\alpha]$, i.e., σ_2 with ρ substituted for all free occurrences of α;
- $(*I2*)$ is an example of a subtyping judgment: $(\forall\alpha \leq \sigma_1 . \sigma_2)$ is a subtype of $(\forall\alpha \leq \tau_1 . \tau_2)$ in context Γ; it is intended that any object of a subtype belongs also to a supertype;
- the judgments $(*I1*)$ and $(*I2*)$ allow us to infer by the so-called *Subsumption* rule (Sub) the judgment $\Gamma \vdash t : (\forall\alpha \leq \tau_1 . \tau_2)$; in general the rule (Sub) allows one to promote a type of a term to any supertype:

  $$\Gamma \vdash s : \tau \ \textit{and}\ \Gamma \vdash \tau \leq \theta \ \textit{imply}\ \Gamma \vdash s : \theta$$

  (it is the subsumption rule, which leads to the multitude of types of the same term);
- by $(*I3*)$, the type ρ is a subtype of the bound τ_1 of the type $(\forall\alpha \leq \tau_1 . \tau_2)$ assigned to the term t; therefore, the term t is good to be applied to the type ρ, and we get after applying the inference rule $(TApp)$ for the type application, the final judgment $\Gamma \vdash t\{\rho\} : \tau_2[\rho/\alpha]$.

The above $F_{\leq}$-typing proof is however not unique, and we could proceed otherwise:

$$\dfrac{\dfrac{\Gamma \vdash t \ : \ \overset{(*I1*)}{\forall\alpha \leq \sigma_1 \,.\, \sigma_2} \qquad \Gamma \vdash \overset{(*I4*)}{\rho \leq \sigma_1}}{\Gamma \vdash t\{\rho\} \ : \ \sigma_2[\rho/\alpha]}\,(TApp) \qquad \Gamma \vdash \overset{(*I5*)}{\sigma_2[\rho/\alpha] \leq \tau_2[\rho/\alpha]}}{\Gamma \vdash t\{\rho\} \ : \ \tau_2[\rho/\alpha]}\,(Sub) \tag{2}$$

where instead of applying the subsumption rule (Sub) first, as in (1), we immediately use the type application rule $(TApp)$, and only afterwards proceed to the subsumption (Sub).

There are several necessary conditions guaranteeing that the transformation of (1) into (2) could always be performed correctly:

1. the subtyping judgment $(*I3*)$ should always imply $(*I4*)$;

   - the latter is guaranteed by the properties of subtyping for bounded universal types (contravariance on bounds):

     $$\Gamma \vdash (\forall\alpha \leq \sigma_1 \,.\, \sigma_2) \ \leq \ (\forall\alpha \leq \tau_1 \,.\, \tau_2) \ \Rightarrow \ \Gamma \vdash \tau_1 \ \leq \ \sigma_1 \ ,$$

   - and by transitivity of the subtyping relation:

     $$\Gamma \vdash \rho \ \leq \ \tau_1 \ \ \& \ \ \Gamma \vdash \tau_1 \ \leq \ \sigma_1 \ \Rightarrow \ \Gamma \vdash \rho \ \leq \ \sigma_1$$

2. the subtyping judgments $(*I2*)$ and $(*I3*)$ should always imply $(*I5*)$, i.e.,

   $$\Gamma \vdash (\forall\alpha \leq \sigma_1 \,.\, \sigma_2) \ \leq \ (\forall\alpha \leq \tau_1 \,.\, \tau_2) \ \textit{and} \ \Gamma \vdash \rho \ \leq \ \tau_1 \ \textit{imply} \ \Gamma \vdash \sigma_2[\rho/\alpha] \leq \tau_2[\rho/\alpha] \tag{3}$$

   (which is exactly the *Substitution property* for bounded universal types).

Note that transforming (1) into (2) we *improve* the type of the term $t\{\rho\}$: whereas (1) yields the type $\tau_2[\rho/\alpha]$, the proof (2) produces a *better (smaller)* type $\sigma_2[\rho/\alpha]$ (see $(*I5*)$). Again, to be sure that during such transformations the types really get smaller, we need the *Substitution property* (3).

The example just considered is a particular case of the so-called *typing proof normalization*. The target of of this process consists in eliminating the abnormal patterns as above, where a subsumption (Sub) *immediately precedes* a type application $(TApp)$. As a result, the normalization process produces finally a *unique canonical proof* and, as a by-product, the *least possible type* for a term, being a subtype of all other possible types of the term in the same context.

The moral one should extract from this example is:

Transitivity and Substitutivity are two properties indispensable for typing proof normalization.

Accidentally, the subtyping relation of the system $F_\leq$ possesses both transitivity and substitutivity properties, and any $F_\leq$-typable term has a unique canonical typing proof and the least type [CG92]. Unfortunately, the $F_\leq$ subtyping, and hence, the $F_\leq$ typing relations are *undecidable* [Pie92].

Our aim in this paper consists in studying hierarchies of subtyping relations that:

1. possess the *transitivity* and *substitutivity* properties,
2. *extend* the $F_\leq$-subtyping relation,
3. are all *decidable*.

Our ultimate goal consists in creating, on the basis of these hierarchies of extensions of the $F_\leq$ subtyping, the infinite family of typing systems, which would type *more* terms than $F_\leq$ and possess the unique canonical proofs and the least type properties. We implement this program in [Vor94e].

With these ideas in mind we now turn to formalities.

3 Preliminaries

Definition 3.1 (Boundedly quantified types) *The set of* $F_{\leq}$-types *is defined by the following abstract grammar:*

$$\mathbb{T} \ \equiv_{df} \ \mathbb{V} \mid \top \mid \mathbb{T} \to \mathbb{T} \mid \forall \mathbb{V} \leq \mathbb{T} \,.\, \mathbb{T}$$

where:

1. $\mathbb{V}$ *is a set of* type variables *denoted by Greek letters* α, β, γ;
2. $\top$ *is the largest type superior to any other type,* $\sigma \leq \top$;
3. $\to$ *is the functional type constructor,* $\sigma \to \tau$ *is the type of functions with domain of type* σ *and codomain of type* τ;
4. $\forall\alpha \leq \rho \,.\, \tau$ *is a polymorphic* boundedly quantified *type, i.e., a function assigning to each subtype* σ *of* ρ, $\sigma \leq \rho$, *the type* $\tau[\sigma/\alpha]$ *obtained from* τ *by substituting* σ *instead of free occurrences of* α *(with usual non-clashing preconditions on free variables). In* $\forall\alpha \leq \rho \,.\, \tau$ *the bound* ρ *does not contain* α *free.*

The Greek letters σ, τ, ρ, θ, ϕ, ψ *denote arbitrary (variable or compound)* $F_{\leq}$*-types;* $\forall\beta \,.\, \tau$ *abbreviates* $\forall\beta \leq \top.\tau$*;* $FV(\sigma)$ *is the set of free variables in* σ. □

Definition 3.2 (Subtyping Contexts) *An* $F_{\leq}$-subtyping context *is an ordered sequence*

$$\alpha_1 \leq \sigma_1, \ldots, \alpha_n \leq \sigma_n$$

of $\leq$*-relations between type variables and* $F_{\leq}$*-types such that:*

1. *all* α_i *are different type variables, and*
2. *for each* i, $FV(\sigma_i) \subseteq \{\alpha_1, \ldots, \alpha_{i-1}\}$.

Subtyping contexts are denoted by capital Greek Γ. $Dom(\Gamma)$ *is the set of type variables appearing to the left of* $\leq$ *in* Γ. *We write* $\Gamma(\alpha) = \sigma$ *if* Γ *contains* $\alpha \leq \sigma$ *and call* σ *a* bound *of* α *in* Γ. *We define* $\Gamma^*(\alpha)$ *as* $\Gamma(\alpha)$ *if the latter is not a variable, and as* $\Gamma^*(\Gamma(\alpha))$ *otherwise.*

Definition 3.3 (Subtyping Judgments) *An* $F_{\leq}$-subtyping judgment *is a figure of the form:*

$$\Gamma \vdash \sigma \ \leq \ \tau,$$

where $FV(\sigma) \cup FV(\tau) \subseteq Dom(\Gamma)$. □

4 System $F_{\leq}$ (Curien-Ghelli's Variant [CG92])

Definition 4.1 (System $F_{\leq}$) *The subtyping relation of the system $F_{\leq}$ is generated by the following set of axioms and inference rules intended to be applied bottom-up (from conclusions to premises):*

$$\Gamma \vdash \tau \leq \top \qquad (Top)$$

$$\Gamma \vdash \alpha \leq \alpha \quad (\alpha \textit{ is a variable}) \qquad (Refl)$$

$$\frac{\Gamma \vdash \Gamma(\alpha) \leq \beta}{\Gamma \vdash \alpha \leq \beta} \qquad (AlgTrans\text{-}Var)$$

$$\frac{\Gamma \vdash \Gamma^*(\alpha) \leq \sigma \to \tau}{\Gamma \vdash \alpha \leq \sigma \to \tau} \qquad (AlgTrans\text{-} \to)$$

$$\frac{\Gamma \vdash \Gamma^*(\alpha) \leq (\forall \beta \leq \rho \,.\, \tau)}{\Gamma \vdash \alpha \leq (\forall \beta \leq \rho \,.\, \tau)} \qquad (AlgTrans\text{-}\forall)$$

$$\frac{\Gamma \vdash \tau_1 \leq \sigma_1 \qquad \Gamma \vdash \sigma_2 \leq \tau_2}{\Gamma \vdash \sigma_1 \to \sigma_2 \leq \tau_1 \to \tau_2} \qquad (Arrow)$$

$$\frac{\Gamma \vdash \tau_1 \leq \sigma_1 \qquad \Gamma, \alpha \leq \tau_1 \vdash \sigma_2 \leq \tau_2}{\Gamma \vdash (\forall \alpha \leq \sigma_1 \,.\, \sigma_2) \leq (\forall \alpha \leq \tau_1 \,.\, \tau_2)} \qquad (All)$$

Remark. For the purposes, which will become clear shortly we explicitly stated three distinct rules: $(AlgTrans\text{-}Var)$, $(AlgTrans\text{-} \to)(AlgTrans\text{-}\forall)$ instead of just one generic scheme:

$$\frac{\Gamma \vdash \Gamma(\alpha) \leq \tau}{\Gamma \vdash \alpha \leq \tau} \qquad (AlgTrans)$$

Definition 4.2 **($F_{\leq}$-inference trees)** *Call an* $\forall$-hypothesis *a judgment of the form*

$$\Gamma \vdash \alpha \leq (\forall\beta \leq \rho \,.\, \tau)$$

Fix an arbitrary (normally recursive) set of $\forall$*-hypotheses* $\mathcal{H}$*. An* $F_{\leq}$-proof tree *of a judgment* J *from the set of* $\forall$*-hypotheses* $\mathcal{H}$ *(*$F_{\leq}(\mathcal{H})$*-proof tree for short) is a (possibly infinite) tree* $\mathcal{T}$ *with the root* J *such that:*

1. *each node in* $\mathcal{T}$ *is a subtyping judgment with at most two successors;*
2. *each node in* $\mathcal{T}$ *without successors is an instance of either the axiom* (Top)*, or the axiom* $(Refl)$*, or belongs to the set of* $\forall$*-hypotheses* $\mathcal{H}$*;*
3. *for each node* J' *in* $\mathcal{T}$*, if* J'' *is a unique successor of* J'*, then* J' *is not an instance of the* $(Refl)$ *axiom, and*

   $$\frac{J''}{J'}$$

   is a particular case of either the rule $(AlgTrans\text{-}Var)$*, or the rule* $(AlgTrans\text{-}\rightarrow)$*, or the rule* $(AlgTrans\text{-}\forall)$*;*
4. *for each node* J' *in* $\mathcal{T}$*, if* J'' *and* J''' *are two successors of* J'*, then*

   $$\frac{J'' \qquad J'''}{J'}$$

   is a particular case of either the rule $(Arrow)$ *or the rule* (All)*.*

An $F_{\leq}$-proof tree *is a finite* $F_{\leq}$*-inference tree from the empty set of* $\forall$*-hypotheses, i.e., a finite* $F_{\leq}(\emptyset)$*-proof tree. A judgment* J *is* $F_{\leq}$*-provable iff there exists a* finite $F_{\leq}$*-proof tree with the root* J*.*

Let $Th_{\leq}(F_{\leq})$ *denote the* $F_{\leq}$-subtyping theory, *i.e., the set of all* $F_{\leq}$*-provable judgments.* □

As an immediate consequence of the above definitions we have:

Proposition 4.3 *The following subtyping judgments are unprovable in* $F_{\leq}$*:*

- $\Gamma \vdash \top \leq \tau$ *with* $\tau \neq \top$,
- $\Gamma \vdash \tau \leq \alpha$ *with* α *variable and* τ *non-variable,*
- $\Gamma \vdash \sigma_1 \rightarrow \sigma_2 \leq (\forall\gamma \leq \tau_1 \,.\, \tau_2)$,
- $\Gamma \vdash (\forall\gamma \leq \sigma_1 \,.\, \sigma_2) \leq \tau_1 \rightarrow \tau_2$. □

One may think that constructing $F_{\leq}$-proof trees he gets a decision procedure for $Th_{\leq}(F_{\leq})$. Unfortunately, it is not the case. Ghelli gave examples of infinite $F_{\leq}$-proof trees, and Pierce proved

Theorem 4.4 (Pierce, [Pie92]) *The theory $Th_{\leq}(F_{\leq})$ is undecidable.* □

This result was inspired by Ghelli's example of a diverging alternating sequence of the bottom-up applications of the rules (*All*) and (*AlgTrans*-$\forall$), see [Pie92], Section 4. Using Ghelli's divergence pattern, Pierce succeeded to encode the termination problem in $Th_{\leq}(F_{\leq})$.

5 How to Make Your Extension of $F_{\leq}$ Decidable: Hierarchies $\{ Th_{\leq}(F_{\leq}^{(i)}(\mathcal{H})) \}_{i=0}^{\infty}$

Our main idea of gaining decidability consists in pruning (possibly) infinite chains of alternating (*AlgTrans*-$\forall$) and (*All*) applications in the $F_{\leq}(\mathcal{H})$-proof trees (see [Pie92], Section 4), together with using suitable $\forall$-hypotheses sets and their decision by using external (non-inferential) means.

Definition 5.1 (Theories $Th_{\leq}(F_{\leq}^{(n)}(\mathcal{H}))$) *Let $n \in \mathbb{N}$ and $\mathcal{H}$ be a set of $\forall$-hypotheses. An* $F_{\leq}^{(n)}(\mathcal{H})$-proof tree *$\mathcal{T}$ for a judgment J is an $F_{\leq}(\mathcal{H})$-proof tree of J with the following additional properties:*

1. *every path in $\mathcal{T}$ starting from the root J contains* **at most** *n applications of the rule (AlgTrans-$\forall$);*
2. *every path in $\mathcal{T}$ starting from the root J and finishing by an $\forall$-hypothesis from $\mathcal{H}$ contains* **exactly** *n applications of the rule (AlgTrans-$\forall$).*

Let $Th_{\leq}(F_{\leq}^{(n)}(\mathcal{H})) \equiv_{df} \{ J \mid \text{there exists an } F_{\leq}^{(n)}(\mathcal{H})\text{-proof tree for } J \}$.

By writing $J \in Th_{\leq}(F_{\leq}^{(n)}(\mathcal{H}))[J_1, \ldots, J_k]$ we stress the fact that $\{ J_1, \ldots, J_k \} \subseteq \mathcal{H}$ is the finite set of all $\forall$-hypotheses actually used in an $F_{\leq}^{(n)}(\mathcal{H})$-proof of J. □

The following proposition is a direct consequence of the above definition.

Proposition 5.2

1. *For every $J \in Th_{\leq}(F_{\leq}^{(n)}(\mathcal{H}))$ its $F_{\leq}^{(n)}(\mathcal{H})$-proof tree is uniquely determined.*
2. *Proposition 4.3 holds for all $Th_{\leq}(F_{\leq}^{(n)}(\mathcal{H}))$.*

3. $Th_{\leq}(F_{\leq}) = \bigcup_{i=0}^{\infty} Th_{\leq}(F_{\leq}^{(i)}(\emptyset))$. □

In contrast to Theorem 4.4 above we have:

Proposition 5.3 *For every* $n \in \mathbb{N}$ *and every decidable set of* $\forall$*-hypotheses* $\mathcal{H}$ *the theory* $Th_{\leq}(F_{\leq}^{(n)}(\mathcal{H}))$ *is decidable.* □

Proof . Immediate from definitions. Given a judgment we try to construct its $F_{\leq}^{(n)}(\mathcal{H})$-proof tree by deterministically applying the rules of $F_{\leq}$, verifying in parallel at each step the two additional properties of Definition 5.1. Either we succeed in building an $F_{\leq}^{(n)}(\mathcal{H})$-proof tree or we are blocked with something, which could be called the $F_{\leq}$*-rejection tree.* □

Below we show how to choose decidable sets of $\forall$-hypotheses correctly, so as each theory $Th_{\leq}(F_{\leq}^{(n)}(\mathcal{H}))$ in the hierarchy be transitive, extend $Th_{\leq}(F_{\leq})$, and satisfy additional good properties, as substitutivity.

The theories $Th_{\leq}(F_{\leq}^{(n)}(\mathcal{H}))$ and $Th_{\leq}(F_{\leq}^{(n+1)}(\mathcal{H}))$ are linked by the following simple property:

Proposition 5.4 *For every* $n \in \mathbb{N}$*:*

1. *Let* $\Gamma^*(\alpha) \equiv (\forall\gamma \leq \sigma_1 . \sigma_2)$*. A judgment* $\Gamma \vdash \alpha \leq (\forall\gamma \leq \tau_1 . \tau_2)$ *is in* $Th_{\leq}(F_{\leq}^{(n+1)}(\mathcal{H}))$ *iff:*
   - *(a)* $\Gamma \vdash \tau_1 \leq \sigma_1 \in Th_{\leq}(F_{\leq}^{(n)}(\mathcal{H}))$*;*
   - *(b)* $\Gamma, \gamma \leq \tau_1 \vdash \sigma_2 \leq \tau_2 \in Th_{\leq}(F_{\leq}^{(n)}(\mathcal{H}))$*;*
   - *(c)* $\Gamma \vdash (\forall\gamma \leq \sigma_1 . \sigma_2) \leq (\forall\gamma \leq \tau_1 . \tau_2) \in Th_{\leq}(F_{\leq}^{(n)}(\mathcal{H}))$*.*
2. *The conclusion of the inference rule:*

$$\frac{\Gamma \vdash \tau_1 \leq \sigma_1 \qquad\qquad \Gamma, \gamma \leq \tau_1 \vdash \sigma_2 \leq \tau_2}{\Gamma \vdash (\forall\gamma \leq \sigma_1 . \sigma_2) \leq (\forall\gamma \leq \tau_1 . \tau_2)} \quad (All)$$

   is in $Th_{\leq}(F_{\leq}^{(n)}(\mathcal{H}))$ *iff its two premises are also in* $Th_{\leq}(F_{\leq}^{(n)}(\mathcal{H}))$ *(the same is true for all other rules of* $F_{\leq}$*, except (AlgTrans-*$\forall$*)).* □

Proof . Immediate by definition. To prove 1, it suffices to consider the inference figure

$$\dfrac{\dfrac{\Gamma \vdash \tau_1 \le \sigma_1 \qquad\qquad \Gamma,\, \gamma \le \tau_1 \vdash \sigma_2 \;\le\; \tau_2}{\Gamma \vdash \Gamma^*(\alpha) \equiv (\forall\gamma \le \sigma_1 \,.\, \sigma_2) \;\le\; (\forall\gamma \le \tau_1 \,.\, \tau_2)}\,(All)}{\Gamma \vdash \alpha \;\le\; (\forall\gamma \le \tau_1 \,.\, \tau_2)}\,(AlgTrans\text{-}\forall)$$

Remarks. 1) Proposition 5.4.1 shows that the theories $Th_{\le}(F_{\le}^{(n+1)}(\mathcal{H}))$ are *not closed* with respect to *backward applications* of the $(AlgTrans\text{-}\forall)$, which makes impossible the proof of transitivity for $Th_{\le}(F_{\le}^{(n)}(\mathcal{H}))$ in general. However, if $Th_{\le}(F_{\le}^{(0)}(\mathcal{H}))$ is transitive and $Th_{\le}(F_{\le}^{(n)}(\mathcal{H}))$ covers $Th_{\le}(F_{\le}^{(n+1)}(\mathcal{H}))$ (Lemmas 9.3, 10.1, and 11.1 give conditions to attain that), then all theories $Th_{\le}(F_{\le}^{(n)}(\mathcal{H}))$ are transitive (Lemma 12.1).

2) The theory $Th_{\le}(F_{\le}^{(0)}(\mathcal{H}))$ is defined without any applications of the rule $(AlgTrans\text{-}\forall)$ at all! As this rule is absent, normally one has to add to the set of $\forall$-hypotheses $\mathcal{H}$ all judgments of the form

$$\Gamma \vdash \alpha \;\le\; (\forall\gamma \le \tau_1 \,.\, \tau_2) \;\equiv\; \Gamma^*(\alpha)$$

unnecessary in the presence of the rule $(AlgTrans\text{-}\forall)$, see Lemma 10.1. □

6 Choices of ∀-Hypotheses Sets

In principle, the set of $\forall$-hypotheses in Definition 5.1 may be arbitrary. Decidability of $\mathcal{H}$ is of course crucial for decidability of $Th_{\le}(F_{\le}^{(i)}(\mathcal{H}))$.

We may choose, for example:

1. $\mathcal{H}_1 \equiv \{\ J \mid J$ *is of the form* $\Gamma \vdash \alpha \;\le\; (\forall\beta \le \rho \,.\, \tau)$ *with* $\Gamma^*(\alpha)$ *being an* $\forall$*-type* $\}$.

   With this choice, for every $i \in \mathbb{N}$ the theory $Th_{\le}(F_{\le}^{(i)}(\mathcal{H}))$ is a consistent decidable extension of $Th_{\le}(F_{\le})$, the undecidable set of $F_{\le}$-provable judgments.

2. $\mathcal{H}_2 \equiv \{\ J \mid J$ *is an* $F_{\le}$*-provable judgment of the form* $\Gamma \vdash \alpha \;\le\; (\forall\beta \le \rho \,.\, \tau)\ \}$.

   In this case for every $i \in \mathbb{N}$ the theory $Th_{\le}(F_{\le}^{(i)}(\mathcal{H}))$ is exactly $Th_{\le}(F_{\le})$. Note, however, that $\mathcal{H}_2$ is undecidable.

3. Take any decidable extension $\mathcal{H}$ of $\mathcal{H}_2$ (if you think that $\mathcal{H}_1$ above is the only one such extension, you are wrong, see the infinite class of $\mathcal{H}_{\mathcal{I}}$ right below). For every decidable $\mathcal{H}$ extending $\mathcal{H}_2$ and every $i \in \mathbb{N}$ the theory $Th_{\le}(F_{\le}^{(i)}(\mathcal{H}))$ is a decidable extension of $Th_{\le}(F_{\le})$.

4. Consider the **SnS**-interpretations of the $F_{\leq}$-subtyping relation introduced in [Vor94a], see also [Vor95] and Appendix A. Choose and fix any such interpretation $\mathcal{I}$ (there are infinitely many of them). Let

   $\mathcal{H}_{\mathcal{I}} \equiv \{\, J \mid J \textit{ has the form } \Gamma \vdash \alpha \leq (\forall\beta \leq \rho \,.\, \tau),\ \Gamma^*(\alpha) \textit{ is a } \forall\textit{-type},\ J \textit{ is valid in } \mathcal{I} \,\}.$

   We will call such sets of $\forall$-hypotheses the **SnS**-*based* $\forall$-*hypotheses*. In this paper we prove, in particular, that every theory $Th_{\leq}(F_{\leq}^{(i)}(\mathcal{H}_{\mathcal{I}}))$ with an **SnS**-based $\forall$-hypotheses set $\mathcal{H}_{\mathcal{I}}$ is a consistent decidable extension of $Th_{\leq}(F_{\leq})$, closed with respect to transitivity and satisfying the substitution property.

7 Three Characterizations of $F_{\leq}$

1. Let $\mathcal{H}_{F_{\leq}}$ be $Th_{\leq}(F_{\leq})$ restricted to the $\forall$-hypotheses, (coincides with undecidable $\mathcal{H}_2$ from the previous Section). Then the hierarchy collapses:

$$Th_{\leq}(F_{\leq}) \;=\; Th_{\leq}(F_{\leq}^{(i)}(\mathcal{H}_{F_{\leq}})) \quad \textit{for every } i \in \mathbb{N}$$

2. Choosing the empty set of of $\forall$-hypotheses, we get:

$$Th_{\leq}(F_{\leq}) \;=\; \bigcup_{i=0}^{\infty} Th_{\leq}({F_{\leq}}^{(i)}(\emptyset))$$

3. For every set of $\forall$-hypotheses $\mathcal{H} \supseteq \mathcal{H}_{F_{\leq}}$, if $Th_{\leq}(F_{\leq}^{(n+1)}(\mathcal{H})) \subseteq Th_{\leq}(F_{\leq}^{(n)}(\mathcal{H}))$ for all $i \in \mathbb{N}$ then:

$$Th_{\leq}(F_{\leq}) \;=\; \lim_{n\to\infty} Th_{\leq}(F_{\leq}^{(n)}(\mathcal{H})) \;=\; \bigcap_{n=0}^{\infty} Th_{\leq}(F_{\leq}^{(n)}(\mathcal{H}))$$

Proposition 7.1 *There are no decidable sets of $\forall$-hypotheses $\mathcal{H}$ extending $\mathcal{H}_{F_{\leq}}$ such that the hierarchy $\{\, Th_{\leq}(F_{\leq}^{(i)}(\mathcal{H}))\,\}_{i=0}^{\infty}$ collapses.* □

Proof . Immediate from the limit property 3) above. □

8 Transitivity in Hierarchies $\{\, Th_{\leq}(F_{\leq}^{(i)}(\mathcal{H}))\,\}_{i=0}^{\infty}$: Proof Plan

We now proceed to the proof that under certain conditions on the set of $\forall$-hypotheses, each theory $Th_{\leq}(F_{\leq}^{(i)}(\mathcal{H}))$ in a hierarchy is transitive. First we prove the transitivity of

$Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H}))$ (Lemma 9.3). Then we demonstrate the inclusions $Th_{\leq}(F_{\leq}^{(i+1)}(\mathcal{H})) \subseteq Th_{\leq}(F_{\leq}^{(i)}(\mathcal{H}))$ (Lemmas 10.1 and 11.1). Finally, we demonstrate the transitivity of all $Th_{\leq}(F_{\leq}^{(i)}(\mathcal{H}))$ by induction using these two facts (Lemma 12.1).

9 Transitivity of $Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H}))$

The theory $Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H}))$ is defined *without any applications* of the transitivity rule ($AlgTrans$-$\forall$). The next lemma shows how to remedy this drawback by correctly choosing the set of $\forall$-hypotheses $\mathcal{H}$, so as $Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H}))$ becomes transitive.

The $\forall$-hypotheses should satisfy the natural closure conditions, corresponding to the assumption strengthening and to the congruences:

Definition 9.1 (Natural closure conditions on $\forall$-hypotheses) *We say that a set of $\forall$-hypotheses $\mathcal{H}$ satisfies the* natural closure conditions *iff:*

1. $\Gamma,\ \gamma \leq \tau_1 \ \vdash\ \delta\ \leq\ \theta\ \in\ \mathcal{H}$ *and* $\Gamma\ \vdash\ \rho_1\ \leq\ \tau_1\ \in\ Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H}))$

   imply $\Gamma,\ \gamma \leq \rho_1\ \vdash\ \delta\ \leq\ \theta\ \in\ \mathcal{H}$.

2. $\Gamma\ \vdash\ \alpha\ \leq\ (\forall\gamma \leq \tau_1 \,.\, \tau_2)\ \in\ \mathcal{H}$ *and*

   $\Gamma\ \vdash\ (\forall\gamma \leq \tau_1 \,.\, \tau_2)\ \leq\ (\forall\gamma \leq \rho_1 \,.\, \rho_2)\ \in\ Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H}))$

   imply $\Gamma\ \vdash\ \alpha\ \leq\ (\forall\gamma \leq \rho_1 \,.\, \rho_2)\ \in\ \mathcal{H}$.

3. $\Gamma\ \vdash\ \beta\ \leq\ (\forall\gamma \leq \tau_1 \,.\, \tau_2)\ \in\ \mathcal{H}$ *and* $\Gamma\ \vdash\ \alpha\ \leq\ \beta\ \in\ Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H}))$

   imply $\Gamma\ \vdash\ \alpha\ \leq\ (\forall\gamma \leq \tau_1 \,.\, \tau_2)\ \in\ \mathcal{H}$. □

Proposition 9.2 *Every* **SnS**-*based set of $\forall$-hypotheses (see Section 6 and Appendix A) satisfies the natural closure conditions of Definition 9.1.* □

Proof . Immediate by definition, see Appendix A. □

Lemma 9.3 (Transitivity of $Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H}))$) *Let a set of $\forall$-hypotheses $\mathcal{H}$ satisfy the natural closure conditions of Definition 9.1. Then the theory $Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H}))$ is transitive, i.e.,*

$$\Gamma\ \vdash\ \theta_1 \leq \theta_2\ \in\ Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H}))\ \ \textit{and}\ \ \Gamma\ \vdash\ \theta_2 \leq \theta_3 \in\ Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H}))$$

$$\textit{imply}\ \ \Gamma\ \vdash\ \theta_1 \leq \theta_3 \in\ Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H}))$$ □

Proof . See Appendix B.1. □

Lemma 9.3 has infinitely many immediate applications, for example:

Corollary 9.4 *The conditions of Lemma 9.3 are satisfied, if one takes $\mathcal{H}$ to be any* **SnS***-interpretation $\mathcal{H}_{\mathcal{I}}$ defined in Section 6 and Appendix A. So, all $Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H}_{\mathcal{I}}))$ are transitive.*

The direct proof of this fact can be found in [Vor95].

10 First Embedding Lemma

Lemma 10.1 (First Embedding Lemma) *Let a set of $\forall$-hypotheses $\mathcal{H}$ include all judgments of the form*

$$\Gamma \vdash \alpha \ \leq \ (\forall\gamma \leq \tau_1 \,.\, \tau_2) \ \equiv \ \Gamma^*(\alpha)$$

and satisfy the natural closure conditions of Definition 9.1.

Then the theory $Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H}))$ extends $Th_{\leq}(F_{\leq}^{(1)}(\mathcal{H}))$, i.e.,

$$Th_{\leq}(F_{\leq}^{(1)}(\mathcal{H})) \ \subseteq \ Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H})).$$ □

Proof . See Appendix B.2. □

11 Second Embedding Lemma

Lemma 11.1 (Second Embedding Lemma) *Let all the conditions of Lemma 10.1 be satisfied.*

Then for all $n \in \mathbb{N}$ one has:

$$Th_{\leq}(F_{\leq}^{(n+1)}(\mathcal{H})) \ \subseteq \ Th_{\leq}(F_{\leq}^{(n)}(\mathcal{H}))$$

Proof . See Appendix B.3. □

12 All $Th_{\le}(F_{\le}^{(n)}(\mathcal{H}))$ are Transitive

Lemma 12.1 (Transitivity) *Let all the conditions of Lemma 10.1 be satisfied. Then for every $n \in \mathbb{N}$ the theory $Th_{\le}(F_{\le}^{(n)}(\mathcal{H}))$ is transitive, i.e.,*

$$\Gamma \vdash \theta_1 \le \theta_2 \in Th_{\le}(F_{\le}^{(n)}(\mathcal{H})) \text{ and } \Gamma \vdash \theta_2 \le \theta_3 \in Th_{\le}(F_{\le}^{(n)}(\mathcal{H}))$$

$$\text{imply } \Gamma \vdash \theta_1 \le \theta_3 \in Th_{\le}(F_{\le}^{(n)}(\mathcal{H})). \qquad \Box$$

Proof . See Appendix B.4. □

13 Cover Property

For every $n \in \mathbb{N}$, if a set of $\forall$-hypotheses is properly selected, the subtype theory $Th_{\le}(F_{\le}^{(n)}(\mathcal{H}))$ extends $Th_{\le}(F_{\le})$, the set of $F_{\le}$-provable judgments:

Proposition 13.1 *Let a set of $\forall$-hypotheses satisfies the natural closure conditions of Definition 9.1 and contain the set of $F_{\le}$-provable judgments of the form $\Gamma \vdash \alpha \le (\forall\gamma \le \tau_1 . \tau_2)$. Then*

1) $Th_{\le}(F_{\le}) \subseteq Th_{\le}(F_{\le}^{(n)}(\mathcal{H})) \;\; \forall\, n \in \mathbb{N}$.

2) $Th_{\le}(F_{\le}) = \lim_{i\to\infty} Th_{\le}(F_{\le}^{(i)}(\mathcal{H})) = \bigcap_{i=0}^{\infty} Th_{\le}(F_{\le}^{(i)}(\mathcal{H}))$. □

Proof . Immediate. □

Proposition 13.2 *The conditions of Proposition 13.1 are satisfied by each of the $\forall$-hypotheses sets $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_{\mathcal{I}}$ from Section 6.* □

14 Collapse Problem

The following problem arises naturally:

Under what conditions the hierarchy $\{\, Th_{\le}(F_{\le}^{(i)}(\mathcal{H}))\,\}_{i=0}^{\infty}$ collapses?

For example, if the set of $\forall$-hypotheses is

$$\mathcal{H}_{F_{\le}} \;\; \equiv_{df} \;\; \{\, \Gamma \vdash \alpha \le (\forall\gamma \le \tau_1 . \tau_2) \in Th_{\le}(F_{\le})\,\}$$

then for all $i, k \in \mathbb{N}$

$$Th_{\leq}(F_{\leq}^{(i)}(\mathcal{H}_{F_{\leq}})) \;=\; Th_{\leq}(F_{\leq}^{(k)}(\mathcal{H}_{F_{\leq}})) \;=\; Th_{\leq}(F_{\leq})$$

In the next three Sections we show:

1. how to construct a typing judgment, which belongs to $Th_{\leq}(F_{\leq}^{(i)}(\mathcal{H}))$ and does not belong to
$Th_{\leq}(F_{\leq}^{(i+1)}(\mathcal{H}))$, provided that $Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H}))$ and $Th_{\leq}(F_{\leq}^{(1)}(\mathcal{H}))$ are different;

2. that the hierarchy $\{\ Th_{\leq}(F_{\leq}^{(i)}(\mathcal{H}))\ \}_{i=0}^{\infty}$ collapses iff $Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H}))$ coincides with $Th_{\leq}(F_{\leq}^{(1)}(\mathcal{H}))$.

15 Distinguishing $Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H}))$ and $Th_{\leq}(F_{\leq}^{(1)}(\mathcal{H}))$

Proposition 15.1 *Consider the following two types and the judgment (Γ is arbitrary):*

$$\tau \equiv \forall\gamma\,.\,((\top \to \top) \to \top) \tag{4}$$
$$\rho \equiv \forall\gamma\,.\,(\top \to \top) \tag{5}$$
$$J \equiv \Gamma,\ \alpha_0\ \leq\ \tau \vdash \alpha_0\ \leq\ \rho \tag{6}$$

The judgment (6) belongs to every **SnS***-based set of* $\forall$*-hypotheses.* □

Proof . Immediate by definition, see Appendix A. □

Remark. Note that a judgment of the form $\Gamma \ \vdash\ \alpha\ \leq\ (\forall \ldots)$ is in $\mathcal{H}$ iff it is in $Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H}))$. □

Proposition 15.2 *Let $\mathcal{H}$ be an arbitrary* **SnS***-based set of* $\forall$*-hypotheses.*

Then the judgment (6) is in $Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H}))$, but not in $\in Th_{\leq}(F_{\leq}^{(1)}(\mathcal{H}))$. □

Proof . See Appendix B.5. □

16 Distinguishing $Th_{\leq}(F_{\leq}^{(i)}(\mathcal{H}))$ and $Th_{\leq}(F_{\leq}^{(i+1)}(\mathcal{H}))$

We demonstrate the "simplest" distinguishing judgment.

Notation. Denote by Γ_{n+1}^n the empty context $\emptyset$ and for $k \leq n$ define the context Γ_k^n as:

$$\Gamma_k^n \equiv_{df} \Gamma_{k+1}^n,\ \alpha_k \leq (\forall \alpha_{k-1} . \alpha_{k-1})$$

In other words,

$$\Gamma_k^n \equiv \alpha_n \leq (\forall \alpha_{n-1} . \alpha_{n-1}),\ \alpha_{n-1} \leq (\forall \alpha_{n-2} . \alpha_{n-2}),\ \alpha_{n-2} \leq (\forall \alpha_{n-3} . \alpha_{n-3}),\ \ldots,$$

$$\ldots,\ \alpha_{k+1} \leq (\forall \alpha_k . \alpha_k),\ \alpha_k \leq (\forall \alpha_{k-1} . \alpha_{k-1})$$

Define the types θ_i $(i \in \mathbb{N})$ by induction (where the types τ and ρ are defined by (4) and (5)):

$$\begin{array}{rcl} \theta_0 & \equiv & \forall \alpha_0 \leq \tau . \rho \\ \theta_{i+1} & \equiv & \forall \alpha_{i+1} \leq (\forall \alpha_i . \alpha_i) . \theta_i \end{array}$$

Proposition 16.1 *For every $n \in \mathbb{N}$ and an arbitrary* **SnS***-based set of $\forall$-hypotheses $\mathcal{H}$:*

1. *the judgment*

$$\Gamma_1^n \vdash (\forall \alpha_0 . \alpha_0) \leq \theta_0$$

is in $Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H}))$ and not in $Th_{\leq}(F_{\leq}^{(1)}(\mathcal{H}))$;

2. *the judgment*

$$\Gamma_{n+1}^n \vdash (\forall \alpha_n . \alpha_n) \leq \theta_n$$

is in $Th_{\leq}(F_{\leq}^{(n)}(\mathcal{H}))$ and not in $Th_{\leq}(F_{\leq}^{(n+1)}(\mathcal{H}))$. □

Proof . See Appendix B.6. □

17 Non-Collapse

As a corollary to the above propositions we have:

Corollary 17.1 *A hierarchy* $\{\ Th_{\leq}(F_{\leq}^{(i)}(\mathcal{H}))\ \}_{i=0}^{\infty}$ *collapses iff*

$$Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H})) \ = \ Th_{\leq}(F_{\leq}^{(1)}(\mathcal{H}))$$

In fact, take a judgment from $Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H}))$, which does not belong to $Th_{\leq}(F_{\leq}^{(1)}(\mathcal{H}))$. Without loss of generality we may suppose that this judgment is of the form $\Gamma,\ \alpha \leq \tau,\ \Gamma' \ \vdash\ \alpha \ \leq\ \rho$. Now we may use these types τ and ρ instead of (4) and (5).

18 Substitution Property

Recall that we need subtyping theories with the following *Substitution property*:

$$\Gamma\ \vdash\ (\forall\gamma \leq \tau_1 \,.\, \tau_2)\ \leq\ (\forall\gamma \leq \rho_1 \,.\, \rho_2)\ \ \textit{and}\ \ \Gamma\ \vdash\ \sigma \leq \rho_1\ \ \textit{imply}\ \ \Gamma\ \vdash\ \tau_2[\sigma/\gamma] \leq \rho_2[\sigma/\gamma] \quad (7)$$

which is indispensable for the the typing proof normalization, see Section 2.

Proposition 18.1 *The substitution property (7) holds for* $F_{\leq}$. □

Proof . Simple Corollary 19.2 to our Substitution Lemma 19.1 below. □

19 Substitution Lemma

We are going to investigate the substitution property for hierarchies $\{\ Th_{\leq}(F_{\leq}^{(i)}(\mathcal{H}))\ \}_{i=0}^{\infty}$. Suppose, for some $k \in \mathbb{N}$:

$$\Gamma\ \vdash\ (\forall\gamma \leq \tau_1 \,.\, \tau_2)\ \leq\ (\forall\gamma \leq \rho_1 \,.\, \rho_2)\ \in\ Th_{\leq}(F_{\leq}^{(k)}(\mathcal{H}))$$

Then by definition (see also Proposition 5.4)

$$\Gamma,\ \gamma \leq \rho_1\ \vdash\ \tau_2\ \leq\ \rho_2\ \in\ Th_{\leq}(F_{\leq}^{(k)}(\mathcal{H})) \quad (8)$$

There might be two kinds of $\forall$-hypotheses used in the $Th_{\leq}(F_{\leq}^{(k)}(\mathcal{H}))$-proof of (8):

$$\Gamma,\ \gamma \leq \rho_1,\ \Gamma'\ \ \vdash\ \ \delta\ \leq\ (\forall\xi \leq \theta_1 \,.\, \theta_2)\quad (\delta \neq \gamma) \quad (9)$$

$$\Gamma,\ \gamma \leq \rho_1,\ \Gamma'\ \ \vdash\ \ \gamma\ \leq\ (\forall\xi \leq \theta_1 \,.\, \theta_2) \quad (10)$$

The $\forall$-hypotheses of the form (9) are good, whereas the $\forall$-hypotheses of the form (10) create an insurmountable obstacle for the substitution property, see Section 20. The problem is that substituting a type σ (satisfying $\Gamma \ \vdash \ \sigma \ \leq \ \rho_1$) instead of γ into an $\forall$-hypothesis of the form (10) produces an expression:

$$\Gamma,\ \Gamma'[\sigma/\gamma] \ \vdash \ \sigma \ \leq \ (\forall\xi \leq \theta_1[\sigma/\gamma] \,.\, \theta_2[\sigma/\gamma]) \ ,$$

which is not, in general, an $\forall$-hypothesis any more, and, moreover, this judgment is *not guaranteed to be provable*, see Proposition 20.1.

In a sense, adding the substitution property as a new inference rule may lead to contradictions.

This explains restrictions on the form of $\forall$-hypotheses in the Substitution Lemma below. In Section 20 we show that these restrictions are essential and cannot be removed, so the Substitution Lemma cannot be reinforced.

Lemma 19.1 (Substitution Lemma)

Let:

- $\Gamma,\ \alpha \leq \sigma,\ \Gamma' \ \vdash\ \phi\ \leq\ \psi\ \in\ Th_{\leq}(F_{\leq}^{(k)}(\mathcal{H}))$ *for some* $k \in \mathbb{N}$, (11)
- $\Gamma\ \vdash\ \rho\ \leq\ \sigma\ \in\ Th_{\leq}(F_{\leq}^{(l)}(\mathcal{H}))$ *for some* $l \geq k$, (12)
- *all* $Th_{\leq}(F_{\leq}^{(j)}(\mathcal{H}))$ *be transitive for* $0 \leq j \leq k$, (13)
- $Th_{\leq}(F_{\leq}^{(j+1)}(\mathcal{H}))\ \subseteq\ Th_{\leq}(F_{\leq}^{(j)}(\mathcal{H}))$ *for* $0 \leq j < l$, (14)
- *all the* $\forall$*-hypotheses used in the* $Th_{\leq}(F_{\leq}^{(k)}(\mathcal{H}))$*-proof of* (11) *are either of the form :*

  $$\Gamma,\ \alpha\ \leq\ \sigma,\ \Gamma''\ \vdash\ \alpha\ \leq\ \sigma\quad (\textit{where } \sigma \equiv (\forall\gamma \leq \sigma_1 \,.\, \sigma_2)), \tag{15}$$

  or of the form :

  $$\Gamma,\ \alpha \leq \sigma,\ \Gamma''\ \vdash\ \beta\ \leq\ (\forall\gamma \leq \theta_1 \,.\, \theta_2)\quad (\textit{with } \alpha \textit{ different from } \beta), \tag{16}$$
- *and each of these* $\forall$*-hypotheses remains a* $\forall$*-hypothesis after substituting* $[\rho/\alpha]$, *i.e.*,

  $$\Gamma,\ \Gamma''[\rho/\alpha]\ \vdash\ \beta\ \leq\ (\forall\gamma \leq \theta_1[\rho/\alpha] \,.\, \theta_2[\rho/\alpha])\ \in\ \mathcal{H}\quad (\alpha \neq \beta). \tag{17}$$

Then

$$\Gamma,\ \Gamma'[\rho/\alpha]\ \vdash\ \phi[\rho/\alpha]\ \leq\ \psi[\rho/\alpha]\ \in\ Th_{\leq}(F_{\leq}^{(k)}(\mathcal{H})) \tag{18}$$

Proof . See Appendix B.7. As becomes clear from the proof, the rôle of the assumption $\alpha \neq \beta$ in (16) is crucial. Proposition 20.1. □

As an immediate application we have:

Corollary 19.2 (Substitution Property for $F_{\leq}$) *The substitution property holds for* $F_{\leq}$.

Proof . Recall that:

$$Th_{\leq}(F_{\leq})\ =\ \bigcup_{i=0}^{\infty}\ Th_{\leq}(F_{\leq}{}^{(i)}(\emptyset))$$

Therefore, if $J \in Th_{\leq}(F_{\leq})$ then $J \in Th_{\leq}(F_{\leq}{}^{(m)}(\emptyset))$ for some $m \in \mathbb{N}$.

As $Th_{\leq}(F_{\leq}{}^{(m)}(\emptyset))$-inferences do not use any $\forall$-hypotheses at all, each of these (absent) $\forall$-hypotheses satisfy (16) and (17). So, Lemma 19.1 applies. □

20 Assumptions on ∀-Hypotheses in Substitution Lemma

One may ask whether the assumptions on ∀-hypotheses (16) in Lemma 19.1 could be relaxed. The analysis of the proof of Lemma 19.1 shows that this assumption is crucial for the ability to transform subtyping proofs after type substitutions.

Why cannot one admit arbitrary ∀-hypotheses of the form

$$\Gamma,\ \alpha\ \leq\ (\forall\gamma\leq\sigma_1\,.\,\sigma_2),\ \Gamma''\ \vdash\ \alpha\ \leq\ (\forall\gamma\leq\tau_1\,.\,\tau_2)$$

not necessarily with $(\forall\gamma\leq\sigma_1\,.\,\sigma_2)\equiv(\forall\gamma\leq\tau_1\,.\,\tau_2)$?

The immediate answer is that the proof of the Substitution Lemma does not work in this case since the general ∀-hypotheses after type substitutions may become non-hypotheses, and even non-provable judgments, and the proof transformation process fails.

To prove that the restrictions on the form of ∀-hypotheses (15) and (16) in Lemma 19.1 are *really essential* we have to construct for an arbitrary $n\geq 0$:

1. a judgment $\Gamma\ \vdash\ (\forall\gamma\leq\tau_1\,.\,\tau_2)\ \leq\ (\forall\gamma\leq\rho_1\,.\,\rho_2)\ \in\ Th_{\leq}(F_{\leq}^{(n)}(\mathcal{H}))$ and
2. a judgment $\Gamma\ \vdash\ \sigma\leq\rho_1\ \in\ Th_{\leq}(F_{\leq}^{(l)}(\mathcal{H}))$ for $l\geq n$,
3. such that $\Gamma\ \vdash\ \tau_2[\sigma/\gamma]\ \leq\ \rho_2[\sigma/\gamma]\ \notin\ Th_{\leq}(F_{\leq}^{(n)}(\mathcal{H}))$,
4. because the $Th_{\leq}(F_{\leq}^{(n)}(\mathcal{H}))$-proof of the first judgment contains a prohibited axiom of the form $\Gamma,\ \gamma\leq\rho_1,\ \Gamma''\ \vdash\ \gamma\ \leq\ \tau$, which becomes unprovable after substituting $[\rho_1/\gamma]$.

The "simplest" such judgments are constructed in Proposition 20.1 below.

Like this we demonstrate that the restrictions on the form of ∀-hypotheses in the Substitution Lemma are essential, could not be relaxed, and the Substitution Lemma could not be reinforced.

Proposition 20.1 *Let $k\in\mathbb{N}$. Define:*

$$\begin{array}{rcl}
\sigma & \equiv & \forall\gamma\leq\top\,.\,((\top\rightarrow\top)\rightarrow\top)\\
\tau & \equiv & \forall\gamma\leq\top\,.\,(\top\rightarrow\top)\\
\tau_1 & \equiv & \top\\
\tau_2 & \equiv & (\forall\delta_1\ldots\forall\delta_n\,.\,(\gamma\rightarrow\top))\rightarrow\top\\
\rho_1 & \equiv & \sigma\\
\rho_2 & \equiv & \beta_1\rightarrow\top\\
\Gamma & \equiv & \beta_n\leq(\forall\delta_n\,.\,(\tau\rightarrow\top)),\ \beta_{n-1}\leq(\forall\delta_{n-1}\,.\,\beta_n),\ldots,\beta_2\leq(\forall\delta_2\,.\,\beta_3),\beta_1\leq(\forall\delta_1\,.\,\beta_2)
\end{array}$$

Consider the $Th_{\leq}(F_{\leq}^{(n)}(\mathcal{H}))$-proof of the judgment:

$$\Gamma \vdash (\forall\gamma \leq \tau_1 . \tau_2) \leq (\forall\gamma \leq \rho_1 . \rho_2) \ , \tag{19}$$

which uses a $\forall$-hypothesis of the form $\Gamma,\ \gamma \leq \sigma,\ \Gamma'' \vdash \gamma \leq \tau$ disallowed by Substitution Lemma. Then:

1. *the judgment (19) belongs to $Th_{\leq}(F_{\leq}^{(n)}(\mathcal{H}))$,*
2. *the judgment $\Gamma \vdash \sigma \leq \sigma$ belongs to $Th_{\leq}(F_{\leq}^{(l)}(\mathcal{H}))$ for every $l \in \mathbb{N}$,*
3. *but the judgment $\Gamma \vdash \tau_2[\sigma/\gamma] \leq \rho_2[\sigma/\gamma]$ does not belong to $Th_{\leq}(F_{\leq}^{(n)}(\mathcal{H}))$.*

Therefore the restrictions (15) and (16) on the form of $\forall$-hypotheses in the Substitution Lemma are essential, could not be relaxed, and the Substitution Lemma could not be reinforced. □

Proof . See Appendix B.8. □

21 Conclusion

We investigated the structure and the properties of the hierarchies of decidable extensions of the $F_{\leq}$ subtyping relation. We demonstrated sufficient conditions guaranteeing that a hierarchy extends $F_{\leq}$, converges to $F_{\leq}$, every its member is transitive and satisfies the substitution property. We also suggested an infinite class of **SnS**-based non-logical hypotheses sets satisfying all these sufficient conditions.

We thus constructed the general theory of decidable extensions of the undecidable $F_{\leq}$ subtyping relation and demonstrated a particular class of such extensions. The next logical step will consist in applying these hierarchies to build extensions of the $F_{\leq}$ typing relation, which would satisfy the unique canonical typing proofs and the least type properties, and would be decidable too[3]. The progress in this direction will give us deeper understanding of the semantics and the structure of higher-order polymorphic type systems with subtyping.

[3] We implemented this program in [Vor94e]

A Appendix: SnS-interpretations of $F_{\leq}$

The *alphabet* of the n-successor monadic second-order logic **SnS** ($n = 0, 1, 2, \ldots, \omega$) consists of: 1) infinitely many object variables $x,\ y,\ z, \ldots$, 2) the equality predicate symbol $=$, 3) infinitely many unary (monadic) predicate variables $A,\ B,\ X,\ Y, \ldots$, 4) n successor function symbols $\{succ_i\}_{i<n}$, 5) all usual boolean connectives, parentheses, 6) first- and second-order universal and existential quantifiers $\forall^1$, $\exists^1$, $\forall^2$, $\exists^2$.

Terms are constructed as usual, starting from object variables by applying the successor function symbol(s).

Atomic formulas are either equalities of terms or expressions of the form $A(t)$, where A is a predicate variable and t is a term.

Formulas are constructed from atomic ones by the usual rules using boolean connectives, parentheses, first- and second-order quantifiers: $\forall^1 x\ \Phi$, $\exists^1 x\ \Phi$, $\forall^2 X\ \Phi$, $\exists^2 X\ \Phi$, (where x is an object and X is a predicate variable).

Interpretation. For an n-successor theory **SnS** consider the infinite n-ary tree T_n^∞. Interpret: 1) object variables as nodes of the tree, 2) $succ_i(t)$ as the i-th son of the node interpreting t, 3) equality, boolean connectives, and first-order quantifiers as usual, 4) predicate variables as arbitrary sets of nodes, 5) atomic formula $A(t)$ as the membership relation "the node t is in the set A"; 6) second-order quantifiers as quantifiers over sets of nodes.

Denote by $Th^2(\mathbf{SnS})$ or simply by **SnS** the set of all formulas valid in the above interpretation.

Replacing the interpretation 6) of the second-order quantifiers by the following clause:

6′) second-order quantifiers are interpreted as quantifiers over *finite* sets of nodes,

we get the *weak monadic second order arithmetic of n successors*, denoted by **WSnS**.

All theories **WSnS** and **SnS** are *decidable* [Rab69, Rab77].

The best known of all these are: Büchi's arithmetic **S1S**, Rabin's arithmetic **S2S**, and their weak counterparts **WS1S**, **WS2S**. The theory **S2S** is strictly more powerful than **WS2S**, **S1S**, and easily encodes all **SnS**.

Definition A.1 ($\mathbf{SnS}[F_{\leq}](\mathbf{f},\mathbf{g})$-interpretations of $F_{\leq}$) *Fix any $n = 0,1,2,\ldots,\omega$. Let* $\mathbf{f}$ *and* $\mathbf{g}$ *be two arbitrary strings composed of successor function symbols of* **SnS**. *Both may be equal to the empty string ε.*

For an arbitrary type ρ of $F_{\leq}$, the "Types-As-Propositions-Interpretation" *of ρ in* **SnS** *with parameters* $\mathbf{f}$ *and* $\mathbf{g}$ *(the* $\mathbf{SnS}[F_{\leq}](\mathbf{f},\mathbf{g})$*-interpretation for short) is defined as an* **SnS**-*formula $[\![\, \rho \,]\!]_{\mathbf{g}}^{\mathbf{f}}(x)$ with unique distinguished free object variable x by induction on the structure of the type ρ:*

1. $[\![\, \alpha \,]\!]_{\mathbf{g}}^{\mathbf{f}}(x) \quad \equiv_{df} \quad A(x)$

   (*a new predicate variable A for each type variable α*);

2. $[\![\, \top \,]\!]_{\mathbf{g}}^{\mathbf{f}}(x) \quad \equiv_{df} \quad x = x;$

3. $[\![\, \sigma \to \tau \,]\!]_{\mathbf{g}}^{\mathbf{f}}(x) \quad \equiv_{df} \quad [\![\, \sigma \,]\!]_{\mathbf{g}}^{\mathbf{f}}(x) \supset [\![\, \tau \,]\!]_{\mathbf{g}}^{\mathbf{f}}(\mathbf{f}(x));$

4. $[\![\, \forall\, \alpha \leq \sigma \,.\, \tau \,]\!]_{\mathbf{g}}^{\mathbf{f}}(x) \quad \equiv_{df} \quad \forall^2 A \left\{\forall^1 x \left(A(x) \supset [\![\, \sigma \,]\!]_{\mathbf{g}}^{\mathbf{f}}(x)\right) \supset [\![\, \tau \,]\!]_{\mathbf{g}}^{\mathbf{f}}(\mathbf{g}(x))\right\};$

The $\mathbf{SnS}[F_{\leq}](\mathbf{f},\mathbf{g})$*-interpretation is extended to all subtyping judgments by:*

5. $[\![\, \sigma \leq \tau \,]\!]_{\mathbf{g}}^{\mathbf{f}} \quad \equiv_{df} \quad \forall^1 x\, ([\![\, \sigma \,]\!]_{\mathbf{g}}^{\mathbf{f}}(x) \supset [\![\, \tau \,]\!]_{\mathbf{g}}^{\mathbf{f}}(x));$

6. $[\![\, \alpha_1 \leq \sigma_1, \ldots, \alpha_n \leq \sigma_n \;\vdash\; \sigma \leq \tau \,]\!]_{\mathbf{g}}^{\mathbf{f}} \quad \equiv_{df}$

$$[\![\, \alpha_1 \leq \sigma_1 \,]\!]_{\mathbf{g}}^{\mathbf{f}}, \ldots, [\![\, \alpha_n \leq \sigma_n \,]\!]_{\mathbf{g}}^{\mathbf{f}} \models_{\mathbf{SnS}} [\![\, \sigma \leq \tau \,]\!]_{\mathbf{g}}^{\mathbf{f}}. \;\square$$

Definition A.2 (Theory) *Define the* $\mathbf{SnS}[F_{\leq}](\mathbf{f},\mathbf{g})$*-theory as:*

$$\mathbf{SnS}[F_{\leq}](\mathbf{f},\mathbf{g}) \;\equiv_{df}\; \{\, \Gamma \vdash \sigma \leq \tau \mid [\![\, \Gamma \vdash \sigma \leq \tau \,]\!]_{\mathbf{g}}^{\mathbf{f}} \,\}$$

We say that a typing judgment is true or valid in (or with respect to) a $\mathbf{SnS}[F_{\leq}](\mathbf{f},\mathbf{g})$*-interpretation iff it belongs to the set* $\mathbf{SnS}[F_{\leq}](\mathbf{f},\mathbf{g})$. □

The **SnS**-interpretations enjoy the following important properties:

Lemma A.3 ([Vor94a]) *1) All axioms of $F_{\leq}$ are valid with respect to any $\mathbf{SnS}[F_{\leq}](\mathbf{f}, \mathbf{g})$.*

2) All inference rules of $F_{\leq}$ preserve validity with respect to any $\mathbf{SnS}[F_{\leq}](\mathbf{f}, \mathbf{g})$, i.e., if both premises of a rule are valid in $\mathbf{SnS}[F_{\leq}](\mathbf{f}, \mathbf{g})$, then so is the conclusion of the rule.

3) There exist infinitely many different theories $\mathbf{SnS}[F_{\leq}](\mathbf{f}, \mathbf{g})$ (for different choices of $\mathbf{f}$ and $\mathbf{g}$). □

Corollary A.4 (On Decidable Extensions of $F_{\leq}$) *Every theory $\mathbf{SnS}[F_{\leq}](\mathbf{f}, \mathbf{g})$ is a consistent decidable theory containing all $F_{\leq}$-provable subtyping judgments. Henceforth, the subtyping relation of $F_{\leq}$ is not essentially undecidable, possessing infinitely many consistent decidable extensions.* □

Corollary A.5 (SnS-interpretations and the hierarchy) *For an arbitrary theory $\mathbf{SnS}[F_{\leq}](\mathbf{f}, \mathbf{g})$ let $\mathcal{H}$ be $\mathbf{SnS}[F_{\leq}](\mathbf{f}, \mathbf{g})$ restricted to the judgments of the form $\Gamma \vdash \alpha \leq (\forall\gamma \leq \tau_1 \,.\, \tau_2)$. Then for all $n \in \mathbb{N}$:*

$$Th_{\leq}(F_{\leq}^{(n)}(\mathcal{H})) \;\subset\; \mathbf{SnS}[F_{\leq}](\mathbf{f}, \mathbf{g})$$

i.e., $\mathbf{SnS}[F_{\leq}](\mathbf{f}, \mathbf{g})$ is more powerful than all corresponding subtyping theories $Th_{\leq}(F_{\leq}^{(n)}(\mathcal{H}))$. □

The problem is that $\mathbf{SnS}[F_{\leq}](\mathbf{f}, \mathbf{g})$ are *too powerful*, and unstructural, proving too many undesirable subtyping judgments. They can, for example, subtype a universal and a functional types, see [Vor94a].

As an immediate consequence of Definition A.1 and Corollary A.5 we have:

Proposition A.6 (Closure conditions for SnS-interpretations) *Every theory $\mathbf{SnS}[F_{\leq}](\mathbf{f}, \mathbf{g})$ restricted to the subtyping judgments of the form $\Gamma \vdash \alpha \leq (\forall\gamma \leq \tau_1 \,.\, \tau_2)$ satisfies the natural closure conditions of Definition 9.1.* □

B Appendix: Proofs

B.1 Proof of Lemma 9.3 (Transitivity of $Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H}))$)

We have to consider the following cases for the types θ_1, θ_2, θ_3:

	θ_1	θ_2	θ_3
Case T	σ	τ	$\top$
Case V	α	β	γ
Case F1	$\sigma_1 \to \sigma_2$	$\tau_1 \to \tau_2$	$\rho_1 \to \rho_2$
Case F2	α	$\tau_1 \to \tau_2$	$\rho_1 \to \rho_2$
Case F3	α	β	$\rho_1 \to \rho_2$
Case U1	$\forall\gamma \leq \sigma_1 \,.\, \sigma_2$	$\forall\gamma \leq \tau_1 \,.\, \tau_2$	$\forall\gamma \leq \rho_1 \,.\, \rho_2$
Case U2	α	$\forall\gamma \leq \tau_1 \,.\, \tau_2$	$\forall\gamma \leq \rho_1 \,.\, \rho_2$
Case U3	α	β	$\forall\gamma \leq \rho_1 \,.\, \rho_2$

Note that we need not consider the following cases, because they cannot belong to $Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H}))$ by definition (see Propositions 4.3 and 5.2):

1. $\theta_i = \top$ and $\theta_{i+1} \neq \top$;
2. $\theta_i = \sigma_1 \to \sigma_2$ and $\theta_{i+1} = \beta$ (type variable);
3. $\theta_i = \forall\gamma \leq \sigma_1 \,.\, \sigma_2$ and $\theta_{i+1} = \beta$ (type variable);
4. $\theta_i = \forall\gamma \leq \sigma_1 \,.\, \sigma_2$ and $\theta_{i+1} = \tau_1 \to \tau_2$;
5. $\theta_i = \sigma_1 \to \sigma_2$ and $\theta_{i+1} = \forall\gamma \leq \tau_1 \,.\, \tau_2$

Case T is trivial, since $\Gamma \vdash \sigma \leq \top$ is always an axiom.

Cases U2 and U3 is covered by the second and the third natural closure conditions on the $\forall$-hypotheses assumed by the Lemma.

Let us consider the most interesting cases: **Case V**, **Case F2**, **Case U1** (**Case F1** is completely analogous), and **Case F3**. Like this everything will be covered.

All the proofs will be conducted on the complexity of the structure of the $F_{\leq}^{(0)}(\mathcal{H})$-proofs of subtyping judgments.

Case V

We should demonstrate that for any type variables α, β, and γ:

$$\Gamma \vdash \alpha \;\leq\; \beta \;\in\; Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H})), \tag{20}$$

$$\Gamma \vdash \beta \;\leq\; \gamma \;\in\; Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H})) \tag{21}$$

imply

$$\Gamma \vdash \alpha \leq \gamma \;\in\; Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H})) \tag{22}$$

The $F_{\leq}^{(0)}(\mathcal{H})$-proof of the judgment in (20) is just a finite sequence of $(AlgTrans\text{-}Var)$-applications finishing by an application of $(Refl)$ to $\Gamma \vdash \beta \leq \beta$. So, starting from the judgment in (22) by backward applications of the same sequence of $(AlgTrans\text{-}Var)$ we are guaranteed by to reach β on the left of $\leq$, i.e., to reach (21), which is $F_{\leq}^{(0)}(\mathcal{H})$-provable by assumption.

Case F2

We have to prove that:

$$\Gamma \vdash \alpha \;\leq\; \tau_1 \to \tau_2 \;\in\; Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H})) \tag{23}$$

and

$$\Gamma \vdash \tau_1 \to \tau_2 \;\leq\; \rho_1 \to \rho_2 \;\in\; Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H})) \tag{24}$$

imply

$$\Gamma \vdash \alpha \;\leq\; \rho_1 \to \rho_2 \;\in\; Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H})) \tag{25}$$

The $F_{\leq}^{(0)}(\mathcal{H})$-proof of (23) is uniquely determined:

$$\dfrac{\dfrac{\Gamma \vdash \overset{(*J1*)}{\tau_1 \;\leq\; \sigma_1} \qquad \Gamma \vdash \overset{(*J2*)}{\sigma_2 \;\leq\; \tau_2}}{\Gamma \vdash \Gamma^*(\alpha) \;\equiv\; \sigma_1 \to \sigma_2 \;\leq\; \tau_1 \to \tau_2}\,(Arrow)}{\Gamma \vdash \alpha \;\leq\; \tau_1 \to \tau_2}\,(AlgTrans\text{-}\to)$$

where by definition of the $F_{\leq}^{(0)}(\mathcal{H})$-proof, $(*J1*),\ (*J2*) \;\in\; Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H}))$.

The $F_{\leq}^{(0)}(\mathcal{H})$-proof of (24) is also uniquely determined:

$$\dfrac{\Gamma \vdash \overset{(*J3*)}{\rho_1 \;\leq\; \tau_1} \qquad \Gamma \vdash \overset{(*J4*)}{\tau_2 \;\leq\; \rho_2}}{\Gamma \vdash (\tau_1 \to \tau_2) \;\leq\; (\rho_1 \to \rho_2)}\,(Arrow)$$

where by definition of the $F_{\leq}^{(0)}(\mathcal{H})$-proof, $(*J3*)$, $(*J4*) \in Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H}))$.

Now by induction hypothesis from $(*J3*) \in Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H}))$ and $(*J1*) \in Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H}))$ we conclude:

$$\Gamma \vdash \rho_1 \leq \sigma_1 \in Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H})) \tag{26}$$

Also by induction hypothesis from $(*J2*) \in Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H}))$ and $(*J4*) \in Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H}))$ we derive:

$$\Gamma \vdash \sigma_2 \leq \rho_2 \in Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H})) \tag{27}$$

Now using (26) and (27) we construct the $F_{\leq}^{(0)}(\mathcal{H})$-proof for (25) as follows:

$$\cfrac{\cfrac{\overset{(*J5*)}{\Gamma \vdash \rho_1 \leq \sigma_1} \qquad \overset{(*J6*)}{\Gamma \vdash \sigma_2 \leq \rho_2}}{\Gamma \vdash \Gamma^*(\alpha) \equiv \sigma_1 \to \sigma_2 \leq \rho_1 \to \rho_2}\,(Arrow)}{\Gamma \vdash \alpha \leq \rho_1 \to \rho_2}\,(AlgTrans\text{-}\to)$$

As $(*J5*) \in Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H}))$ by (26) and $(*J6*) \in Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H}))$ by (27), we have (25).

Case U1

We have to prove that:

$$\Gamma \vdash (\forall\gamma \leq \sigma_1 . \sigma_2) \leq (\forall\gamma \leq \tau_1 . \tau_2) \in Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H})) \tag{28}$$

and

$$\Gamma \vdash (\forall\gamma \leq \tau_1 . \tau_2) \leq (\forall\gamma \leq \rho_1 . \rho_2) \in Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H})) \tag{29}$$

imply

$$\Gamma \vdash (\forall\gamma \leq \sigma_1 . \sigma_2) \leq (\forall\gamma \leq \rho_1 . \rho_2) \in Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H})) \tag{30}$$

The $F_{\leq}^{(0)}(\mathcal{H})$-proof of (28) is uniquely determined:

$$\cfrac{\overset{(*J1*)}{\Gamma \vdash \tau_1 \leq \sigma_1} \qquad \overset{(*J2*)}{\Gamma,\, \gamma \leq \tau_1 \vdash \sigma_2 \leq \tau_2}}{\Gamma \vdash (\forall\gamma \leq \sigma_1 . \sigma_2) \leq (\forall\gamma \leq \tau_1 . \tau_2)}\,(All) \tag{31}$$

where by definition of the $F_{\leq}^{(0)}(\mathcal{H})$-proof, $(*J1*)$, $(*J2*) \in Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H}))$.

The $F_{\leq}^{(0)}(\mathcal{H})$-proof of (29) is also uniquely determined:

$$\frac{\Gamma \vdash \overset{(*J3*)}{\rho_1 \leq \tau_1} \qquad \Gamma,\, \gamma \leq \rho_1 \overset{(*J4*)}{\vdash \tau_2 \leq \rho_2}}{\Gamma \vdash (\forall \gamma \leq \tau_1 \,.\, \tau_2) \leq (\forall \gamma \leq \rho_1 \,.\, \rho_2)}\,(All) \tag{32}$$

where by definition of the $F_{\leq}^{(0)}(\mathcal{H})$-proof, $(*J3*)$, $(*J4*) \in Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H}))$.

From $(*J2*)$, $(*J3*) \in Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H}))$ we conclude:

$$\Gamma,\, \gamma \leq \rho_1 \vdash \sigma_2 \leq \tau_2 \in Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H})) \tag{33}$$

In fact, the only difference between $(*J2*)$ and the judgment in (33) is the context bound on variable γ (τ_1 in $(*J2*)$ and ρ_1 in (33)). To transform an $F_{\leq}^{(0)}(\mathcal{H})$-proof of $(*J2*)$ into an $F_{\leq}^{(0)}(\mathcal{H})$-proof of (33), it suffices to replace $\gamma \leq \tau_1$ in all contexts of the $F_{\leq}^{(0)}(\mathcal{H})$-proof of $(*J2*)$ by $\gamma \leq \rho_1$, and every axiom $\Gamma, \gamma \leq \tau_1, \Gamma' \vdash \gamma \leq \tau_1$ by the $F_{\leq}^{(0)}(\mathcal{H})$-inference:

$$\frac{\Gamma,\, \gamma \leq \rho_1 \vdash \gamma \leq \rho_1 \qquad \Gamma,\, \gamma \leq \overset{(*J3'*)}{\rho_1 \vdash \rho_1 \leq \tau_1}}{\Gamma,\, \gamma \leq \rho_1 \vdash \gamma \leq \tau_1} \tag{34}$$

where $(*J3'*) \in Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H}))$ because $(*J3*) \in Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H}))$ and $\Gamma,\, \gamma \leq \rho_1 \vdash$ extends Γ.

Note that in this transformation every $\forall$-hypothesis $\Gamma,\, \gamma \leq \tau_1 \vdash \delta \leq \theta$ turns into an $\forall$-hypothesis $\Gamma,\, \gamma \leq \rho_1 \vdash \delta \leq \theta$, since the $\forall$-hypotheses satisfy the first natural closure condition of Definition 9.1 by assumption.

Note that (34) is the application of transitivity. This is lawful, since the premises of (34) are of smaller complexity than the conclusions of (31) and (32). So the inductive hypothesis works.

Now from (33) and $(*J4*) \in Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H}))$ by induction hypothesis we can conclude:

$$\Gamma,\, \gamma \leq \rho_1 \vdash \sigma_2 \leq \rho_2 \in Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H})) \tag{35}$$

Also by induction hypothesis from $(*J3*) \in Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H}))$ and $(*J1*) \in Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H}))$ we conclude:

$$\Gamma \vdash \rho_1 \leq \sigma_1 \in Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H})) \tag{36}$$

But (36) and (35) imply (30).

Case F3

Let θ_3 be $\rho_1 \to \rho_2$. We have to prove that

$$\begin{array}{rcl} \Gamma \vdash \alpha & \leq & \beta, \\ \Gamma \vdash \beta & \leq & \theta_3 \end{array} \qquad \begin{array}{r} (37) \\ (38) \end{array}$$

imply

$$\Gamma \vdash \alpha \leq \theta_3 \tag{39}$$

As (38) is $F^{(0)}_{\leq}(\mathcal{H})$-provable, so is $\Gamma \vdash \Gamma^*(\beta) \leq \theta_3$ (by application of ($AlgTrans$-$\to$)). But $\Gamma^*(\beta) = \Gamma^*(\alpha)$ since the $F^{(0)}_{\leq}(\mathcal{H})$-proof of (37) is a sequence of ($AlgTrans$-Var)-applications leading to $\Gamma \vdash \beta \leq \beta$. So, $\Gamma^*(\beta)$ and $\Gamma^*(\alpha)$ should coincide. Therefore, (38) is also $F^{(0)}_{\leq}(\mathcal{H})$-provable.

This finishes the proof of the Lemma 9.3. □

B.2 Proof of Lemma 10.1 (First Embedding Lemma)

By induction on the structure of $F^{(1)}_{\leq}(\mathcal{H})$-proofs. All axioms and $\forall$-hypotheses in the theories $Th_{\leq}(F^{(0)}_{\leq}(\mathcal{H}))$ and $Th_{\leq}(F^{(1)}_{\leq}(\mathcal{H}))$ are the same. The only rule, which can appear in an $F^{(1)}_{\leq}(\mathcal{H})$-proof and cannot appear in an $F^{(0)}_{\leq}(\mathcal{H})$-proof, is the rule ($AlgTrans$-$\forall$). But every application of

$$\frac{\Gamma \vdash \Gamma^*(\alpha) \leq (\forall\beta \leq \rho \,.\, \tau)}{\Gamma \vdash \alpha \leq (\forall\beta \leq \rho \,.\, \tau)} \qquad (AlgTrans\text{-}\forall)$$

in a $F^{(1)}_{\leq}(\mathcal{H})$-proof can be replaced by the following $F^{(0)}_{\leq}(\mathcal{H})$-proof:

$$\frac{\Gamma \vdash \alpha \overset{(*D1*)}{\leq} \Gamma^*(\alpha) \qquad \Gamma \vdash \Gamma^*(\alpha) \overset{(*D2*)}{\leq} (\forall\beta \leq \rho \,.\, \tau)}{\Gamma \vdash \alpha \leq (\forall\beta \leq \rho \,.\, \tau)} \qquad (Trans)$$

since $Th_{\leq}(F^{(0)}_{\leq}(\mathcal{H}))$ is closed with respect to transitivity by Lemma 9.3, $(*D1*)$ is in $\mathcal{H}$ by assumption, and the premise $(*D2*)$ is in $Th_{\leq}(F^{(0)}_{\leq}(\mathcal{H}))$ by inductive hypothesis. □

B.3 Proof of Lemma 11.1 (Second Embedding Lemma)

Consider a judgment $J \in Th_{\leq}(F_{\leq}^{(n+1)}(\mathcal{H}))$. If all the branches in its $F_{\leq}^{(n+1)}(\mathcal{H})$-proof tree has $\leq n$ applications of $(AlgTrans\text{-}\forall)$, then $J \in Th_{\leq}(F_{\leq}^{(n)}(\mathcal{H}))$. Otherwise, let

$$J_{n+1},\ J_n,\ \ldots,\ J_1,\ J_0$$

be the sequence of all conclusions of $(AlgTrans\text{-}\forall)$ (read bottom-up) applied on a branch of the $F_{\leq}^{(n+1)}(\mathcal{H})$-proof tree of J with $n+1$ applications of $(AlgTrans\text{-}\forall)$, and J_0 be either an instance of axioms $(Refl)$, (Top), or an $\forall$-hypothesis from $\mathcal{H}$.

By definition, $J_1 \in Th_{\leq}(F_{\leq}^{(1)}(\mathcal{H}))$. So, by Lemma 10.1, $J_1 \in Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H}))$, i.e., can be proved without any applications of $(AlgTrans\text{-}\forall)$. We thus can reduce the number of $(AlgTrans\text{-}\forall)$-applications by 1, and, therefore, J can be proved using at most n applications of this rule. Hence, $J \in Th_{\leq}(F_{\leq}^{(n)}(\mathcal{H}))$. $\square$

B.4 Proof of Lemma 12.1 (Transitivity Lemma)

By induction on $n \in \mathbb{N}$. The base case is established by Lemma 9.3. The inductive step is routine. We conduct it only for the most interesting case, when θ_1 is a variable, and θ_2 and θ_3 are universal types.

1. Suppose $\Gamma \vdash \alpha \leq (\forall\gamma \leq \tau_1 . \tau_2) \in Th_{\leq}(F_{\leq}^{(n+1)}(\mathcal{H}))$ and
2. $\Gamma \vdash (\forall\gamma \leq \tau_1 . \tau_2) \leq (\forall\gamma \leq \rho_1 . \rho_2) \in Th_{\leq}(F_{\leq}^{(n+1)}(\mathcal{H}))$.
3. We have to prove $\Gamma \vdash \alpha \leq (\forall\gamma \leq \rho_1 . \rho_2) \in Th_{\leq}(F_{\leq}^{(n+1)}(\mathcal{H}))$.
4. Let $\Gamma^*(\alpha) \equiv (\forall\gamma \leq \sigma_1 . \sigma_2)$ (if $\Gamma^*(\alpha)$ is not a universal type, 1 does not hold, see Propositions 4.3, 5.2).
5. From 1 by Proposition 5.4.1.a, $\Gamma \vdash \tau_1 \leq \sigma_1 \in Th_{\leq}(F_{\leq}^{(n)}(\mathcal{H}))$ and
6. $\Gamma,\ \gamma \leq \tau_1 \vdash \sigma_2 \leq \tau_2 \in Th_{\leq}(F_{\leq}^{(n)}(\mathcal{H}))$.
7. From 2 by Second Embedding Lemma 11.1, $\Gamma \vdash (\forall\gamma \leq \tau_1 . \tau_2) \leq (\forall\gamma \leq \rho_1 . \rho_2) \in Th_{\leq}(F_{\leq}^{(n)}(\mathcal{H}))$.
8. So, by Proposition 5.4.2, $\Gamma \vdash \rho_1 \leq \tau_1 \in Th_{\leq}(F_{\leq}^{(n)}(\mathcal{H}))$ and
9. $\Gamma,\ \gamma \leq \rho_1 \vdash \tau_2 \leq \rho_2 \in Th_{\leq}(F_{\leq}^{(n)}(\mathcal{H}))$.
10. From 8 and 5 by inductive assumption on transitivity of $Th_{\leq}(F_{\leq}^{(n)}(\mathcal{H}))$ we have:

    $\Gamma \vdash \rho_1 \leq \sigma_1 \in Th_{\leq}(F_{\leq}^{(n)}(\mathcal{H}))$.

11. From 6 and 10 we have: $\Gamma,\ \gamma \leq \rho_1\ \vdash\ \sigma_2\ \leq\ \tau_2\ \in\ Th_{\leq}(F_{\leq}^{(n)}(\mathcal{H}))$.

    In fact, the only difference between 11 and 6 is the context bound on γ (ρ_1 and τ_1 respectively). To transform the $F_{\leq}^{(n)}(\mathcal{H})$-proof of 6 into the $F_{\leq}^{(n)}(\mathcal{H})$-proof of 11 it suffices to replace $\gamma \leq \tau_1$ in all contexts of the $F_{\leq}^{(n)}(\mathcal{H})$-inference of 6 by $\gamma \leq \rho_1$, and any axiom $\Gamma,\ \gamma \leq \tau_1,\ \Gamma'\ \vdash\ \gamma\ \leq\ \tau_1$ by the $F_{\leq}^{(n)}(\mathcal{H})$-inference

$$\frac{\Gamma,\ \gamma \leq \rho_1\ \vdash\ \gamma\ \leq\ \rho_1 \qquad \Gamma,\ \gamma \leq \rho_1\ \vdash\ \rho_1\ \leq\ \tau_1}{\Gamma,\ \gamma \leq \rho_1\ \vdash\ \gamma\ \leq\ \tau_1}$$

    where $\Gamma, \gamma \leq \rho_1\ \vdash\ \rho_1\ \leq\ \tau_1 \in Th_{\leq}(F_{\leq}^{(n)}(\mathcal{H}))$ by 8, since $\Gamma,\ \gamma \leq \rho_1$ extends Γ.
    Note that in this transformation every $\forall$-hypothesis $\Gamma,\ \gamma \leq \tau_1\ \vdash\ \delta\ \leq\ \theta$ turns into an $\forall$-hypothesis $\Gamma,\ \gamma \leq \rho_1\ \vdash\ \delta\ \leq\ \theta$, since the $\forall$-hypotheses satisfy the first natural closure condition of Definition 9.1 by assumption.

    Note also, that to construct the latter inference we apply the inductive assumption on transitivity of $Th_{\leq}(F_{\leq}^{(n)}(\mathcal{H}))$.

12. From 11 and 9 we have $\Gamma\ \vdash\ (\forall\gamma \leq \sigma_1 \,.\, \sigma_2)\ \leq\ (\forall\gamma \leq \rho_1 \,.\, \rho_2)\ \in\ Th_{\leq}(F_{\leq}^{(n)}(\mathcal{H}))$ by the inductive assumption on transitivity of $Th_{\leq}(F_{\leq}^{(n)}(\mathcal{H}))$.

13. Henceforth, $\Gamma\ \vdash\ \alpha\ \leq\ (\forall\gamma \leq \rho_1 \,.\, \rho_2)\ \in\ Th_{\leq}(F_{\leq}^{(n+1)}(\mathcal{H}))$, which is needed. □

B.5 Proof of Proposition 15.2

The uniquely determined $Th_{\leq}(F_{\leq}^{(1)}(\mathcal{H}))$-derivation of J is:

$$\dfrac{\Gamma\ \vdash\ \top\ \leq\ \top \qquad \dfrac{\ldots \vdash \top\ \leq\ \top \to \top \qquad \ldots \vdash \top\ \leq\ \top}{\Gamma,\ \alpha_0\ \leq\ \tau,\ \gamma \leq \top\ \vdash\ (\top \to \top) \to \top\ \leq\ \top \to \top}\,(Arrow)}{\dfrac{\Gamma,\ \alpha_0\ \leq\ \tau\ \vdash\ \tau\ \leq\ \rho}{\Gamma,\ \alpha_0\ \leq\ \tau\ \vdash\ \alpha_0\ \leq\ \rho}\,(AlgTrans\text{-}\forall)}\,(All)$$

But the premise $\ldots \vdash \top\ \leq\ \top \to \top$ is unprovable. □

B.6 Proof of Proposition 16.1

1) Consider the proof tree for the judgment $\Gamma_1^n\ \vdash\ (\forall\alpha_0 \,.\, \alpha_0)\ \leq\ (\forall\alpha_0 \leq \tau \,.\, \rho)$:

$$\frac{\Gamma_1^n\ \vdash\ \tau\ \leq\ \top \qquad \Gamma_1^n,\ \alpha_0 \leq \tau\ \vdash\ \alpha_0\ \leq\ \rho}{\Gamma_1^n\ \vdash\ (\forall\alpha_0 \,.\, \alpha_0)\ \leq\ (\forall\alpha_0 \leq \tau \,.\, \rho)}\quad (All)$$

Now the conclusion follows from Proposition 15.2.

2) Consider the proof tree of the judgment $\Gamma^n_{n+1} \ \vdash \ (\forall \alpha_n \, . \, \alpha_n) \ \leq \ \theta_n$:

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cdots}{\Gamma^n_{n-1} \ \vdash \ (\forall \alpha_{n-2} \, . \, \alpha_{n-2}) \ \leq \ \theta_{n-2}}\;5}{\Gamma^n_{n-1} \ \vdash \ \alpha_{n-1} \ \leq \ \theta_{n-2}}\;4}{\Gamma^n_{n} \ \vdash \ (\forall \alpha_{n-1} \, . \, \alpha_{n-1}) \ \leq \ \theta_{n-1} \equiv (\forall \alpha_{n-1} \leq (\forall \alpha_{n-2} \, . \, \alpha_{n-2}) \, . \, \theta_{n-2})}\;3}{\Gamma^n_n \vdash \alpha_n \ \leq \ \theta_{n-1}}\;2}{\Gamma^n_{n+1} \ \vdash \ (\forall \alpha_n \, . \, \alpha_n) \ \leq \ \theta_n \equiv (\forall \alpha_n \leq (\forall \alpha_{n-1} \, . \, \alpha_{n-1}) \, . \, \theta_{n-1})}\;1
$$

where the odd numbers correspond to applications of the rule (*All*) and the even to the rule (*AlgTrans*-$\forall$).

This inference is the alternation of the rules (*All*) and (*AlgTrans*) At every odd step we have judgments of the form:

$$\Gamma^n_k \ \vdash \ (\forall \alpha_{k-1} \, . \, \alpha_{k-1}) \ \leq \ \theta_{k-1}$$

and after n times applying the pair of rules (*All*) and (*AlgTrans*-$\forall$) arrive at:

$$\Gamma^n_1 \ \vdash \ (\forall \alpha_0 \, . \, \alpha_0) \ \leq \ \theta_0$$

By 1) the latter belongs to $Th_{\leq}(F^{(0)}_{\leq}(\mathcal{H}))$ but does not belong to $Th_{\leq}(F^{(1)}_{\leq}(\mathcal{H}))$. Therefore, the initial judgment

$$\Gamma^n_{n+1} \ \vdash \ (\forall \alpha_n \, . \, \alpha_n) \ \leq \ \theta_n$$

belongs to $Th_{\leq}(F^{(n)}_{\leq}(\mathcal{H}))$ but does not belong to $Th_{\leq}(F^{(n+1)}_{\leq}(\mathcal{H}))$, since there is exactly one application of (*AlgTrans*-$\forall$) between $\Gamma^n_k \ \vdash \ (\forall \alpha_{k-1} \, . \, \alpha_{k-1}) \ \leq \ \theta_{k-1}$ and $\Gamma^n_{k+1} \ \vdash \ (\forall \alpha_k \, . \, \alpha_k) \ \leq \ \theta_k$. $\square$

B.7 Proof of Lemma 19.1 (Substitution Lemma)

Consider the judgment (11). If we formally replace all occurrences of α in its $F^{(k)}_{\leq}(\mathcal{H})$-proof tree $\mathcal{T}$ by ρ and discard $\rho \leq \sigma$ from all contexts, then we get a tree $\mathcal{T}'$ with the needed judgment (18) at the root (recall that Γ does not contain α by definition).

The only problem is that $\mathcal{T}'$ is not a correct $F^{(k)}_{\leq}(\mathcal{H})$-proof tree any more.

We demonstrate below how to transform the tree $\mathcal{T}'$ into an $F^{(k)}_{\leq}(\mathcal{H})$-proof tree of (18).

We have to prove that in the tree $\mathcal{T}'$, obtained after substituting $[\rho/\alpha]$ in $\mathcal{T}$ and discarding $\rho \leq \sigma$ from all contexts:

1. axioms remain axioms;
2. $\forall$-hypotheses remain $\forall$-hypotheses, or become $F_{\leq}^{(k)}(\mathcal{H})$-provable judgments;
3. rule applications could be transformed into correct inferences.

Case of axioms

It is evident that all axioms (γ may be equal to α or not):

$$\Gamma,\ \alpha \leq \sigma,\ \Gamma'' \ \vdash\ \gamma \leq \gamma \quad \text{and} \quad \Gamma,\ \alpha \leq \sigma,\ \Gamma'' \ \vdash\ \gamma \leq \top$$

remain axioms after substituting $[\rho/\alpha]$ and discarding $\rho \leq \sigma$ from the contexts.

Case of $\forall$-hypotheses

By assumption (17), the set of $\forall$-hypotheses of the form (16) used in the proof $\mathcal{T}$ is closed with respect to substitution $[\rho/\alpha]$. So each such $\forall$-hypothesis remains a $\forall$-hypothesis after substituting $[\rho/\alpha]$ and discarding $\rho \leq \sigma$ from the contexts.

By assumptions (12) and (14), all $\forall$-hypotheses of the form (15) used in the proof after substituting $[\rho/\alpha]$ and discarding $\rho \leq \sigma$ from the context ($\sigma \equiv (\forall\gamma \leq \sigma_1 . \sigma_2)$) become $F_{\leq}^{(j)}(\mathcal{H})$-provable judgments

$$\Gamma,\ \Gamma'' \ \vdash\ \rho\ \leq\ (\forall\gamma \leq \sigma_1 . \sigma_2)$$

for all $0 \leq j \leq l$, since $\Gamma,\ \Gamma''$ extends Γ.

Case of the rule (*Arrow*)

Every application of the rule (*Arrow*):

$$\frac{\Gamma, \alpha \leq \sigma, \Gamma'' \ \vdash\ \theta_1 \leq \tau_1 \qquad \Gamma, \alpha \leq \sigma, \Gamma'' \ \vdash\ \tau_2 \leq \theta_2}{\Gamma, \alpha \leq \sigma, \Gamma'' \ \vdash\ \tau_1 \to \tau_2\ \leq\ \theta_1 \to \theta_2} \quad (Arrow)$$

remains a correct application of (*Arrow*) after substituting $[\rho/\alpha]$ and discarding $\rho \leq \sigma$ from the contexts:

$$\frac{\Gamma, \Gamma''[\rho/\alpha] \ \vdash\ \theta_1[\rho/\alpha]\ \leq\ \tau_1[\rho/\alpha] \qquad \Gamma, \Gamma''[\rho/\alpha] \ \vdash\ \tau_2[\rho/\alpha]\ \leq\ \theta_2[\rho/\alpha]}{\Gamma, \Gamma''[\rho/\alpha] \ \vdash\ \tau_1[\rho/\alpha] \to \tau_2[\rho/\alpha]\ \leq\ \theta_1[\rho/\alpha] \to \theta_2[\rho/\alpha]} \quad (Arrow)$$

Case of the rule (All)

Every application of the rule (All):

$$\frac{\Gamma, \alpha \leq \sigma, \Gamma'' \vdash \theta_1 \leq \tau_1 \qquad \Gamma, \alpha \leq \sigma, \Gamma'', \gamma \leq \theta_1 \vdash \tau_2 \leq \theta_2}{\Gamma, \alpha \leq \sigma, \Gamma'' \vdash (\forall \gamma \leq \tau_1 \,.\, \tau_2) \ \leq \ (\forall \gamma \leq \theta_1 \,.\, \theta_2)} \quad (All)$$

remains a correct application of (All) after substituting $[\rho/\alpha]$ and discarding $\rho \leq \sigma$ from the contexts (γ is completely fresh variable):

$$\frac{\Gamma, \Gamma''[\rho/\alpha] \vdash \theta_1[\rho/\alpha] \leq \tau_1[\rho/\alpha] \qquad \Gamma, \Gamma''[\rho/\alpha], \gamma \leq \theta_1[\rho/\alpha] \vdash \tau_2[\rho/\alpha] \leq \theta_2[\rho/\alpha]}{\Gamma, \Gamma''[\rho/\alpha] \vdash (\forall \gamma \leq \tau_1[\rho/\alpha] \,.\, \tau_2[\rho/\alpha]) \ \leq \ (\forall \gamma \leq \theta_1[\rho/\alpha] \,.\, \theta_2[\rho/\alpha])} \quad (All)$$

Case of the rule $(AlgTrans)$

Let $\widehat{\Gamma}$ denote a context of the form $\Gamma, \alpha \leq \sigma, \Gamma''$. Let $\widehat{\Gamma}[\rho/\alpha]$ be obtained from $\widehat{\Gamma}$ by substituting ρ instead of free occurrences of α and discarding $\rho \leq \sigma$.

We have to prove that each application of $(AlgTrans)$ (where γ may be possibly equal to α):

$$\frac{\widehat{\Gamma} \vdash \widehat{\Gamma}(\gamma) \leq \theta}{\widehat{\Gamma} \vdash \gamma \leq \theta} \, (AlgTrans) \tag{40}$$

after substituting $[\rho/\alpha]$ and discarding $\rho \leq \sigma$ can be transformed into a correct $F_{\leq}^{(k)}(\mathcal{H})$-proof:

$$\begin{array}{c} \widehat{\Gamma}[\rho/\alpha] \vdash \widehat{\Gamma}(\gamma)[\rho/\alpha] \ \leq \ \theta[\rho/\alpha] \\ \hline \vdots \\ \hline \widehat{\Gamma}[\rho/\alpha] \vdash \gamma[\rho/\alpha] \ \leq \ \theta[\rho/\alpha] \end{array}$$

(with possible auxiliary subinferences).

Subcase $\gamma \neq \alpha$. If $\gamma \neq \alpha$ then (40) after substituting $[\rho/\alpha]$ and discarding $\rho \leq \sigma$ from the contexts:

$$\frac{\widehat{\Gamma}[\rho/\alpha] \vdash \widehat{\Gamma}(\gamma)[\rho/\alpha] \ \leq \ \theta[\rho/\alpha]}{\widehat{\Gamma}[\rho/\alpha] \vdash \gamma[\rho/\alpha] \ \leq \ \theta[\rho/\alpha]} \tag{41}$$

becomes a correct application of the rule $(AlgTrans)$.

Indeed, if $\gamma \neq \alpha$ then $\gamma[\rho/\alpha] \equiv \gamma$ and:

- either $\gamma \in Dom(\Gamma)$; in this case $\widehat{\Gamma}(\gamma) \equiv \Gamma(\gamma)$ does not contain α by definition of the context, so $\widehat{\Gamma}(\gamma)[\rho/\alpha] \equiv \Gamma(\gamma)$, and the application (41) of $(AlgTrans)$ is correct;
- or $\gamma \in Dom(\Gamma'')$; in this case $\widehat{\Gamma}(\gamma) \equiv \Gamma''(\gamma)$, so $\widehat{\Gamma}(\gamma)[\rho/\alpha] \equiv \Gamma''(\gamma)[\rho/\alpha]$, and the application (41) of $(AlgTrans)$ is also correct.

Case $\alpha = \gamma$. Suppose finally, that $\alpha = \gamma$. In this case the application (40) after substituting $[\rho/\alpha]$ and discarding $\rho \leq \sigma$ becomes **incorrect**:

$$\frac{\widehat{\Gamma}[\rho/\alpha] \vdash \sigma \leq \theta[\rho/\alpha]}{\widehat{\Gamma}[\rho/\alpha] \vdash \rho \leq \theta[\rho/\alpha]}\,(AlgTrans) \tag{42}$$

(recall that by definition of the context σ does not contain α).

To transform the incorrect inferences (42) into the correct ones we proceed by induction on the structure of the inference using the assumptions of the lemma. At each stage of the transformation we choose the topmost incorrect application (42) supposing that its premise is already correctly proved. We then transform this incorrect application into the correct one with the same conclusion:

$$\frac{\widehat{\Gamma}[\rho/\alpha] \vdash \rho \ \leq \ \sigma \qquad\qquad \widehat{\Gamma}[\rho/\alpha] \vdash \sigma \leq \theta[\rho/\alpha]}{\widehat{\Gamma}[\rho/\alpha] \vdash \rho \leq \theta[\rho/\alpha]} \tag{43}$$

In fact:

1. The left premise of (43) belongs to $Th_{\leq}(F_{\leq}^{(l)}(\mathcal{H}))$ by assumption (12), since $\widehat{\Gamma}$ extends Γ and $\alpha \notin FV(\Gamma)$.
2. The right premise of (43) belongs to $Th_{\leq}(F_{\leq}^{(j)}(\mathcal{H}))$ for some $0 \leq j \leq k$ by assumption (11) and by inductive assumption.
3. As $l \geq j$ by assumption (12), one has $Th_{\leq}(F_{\leq}^{(l)}(\mathcal{H})) \subseteq Th_{\leq}(F_{\leq}^{(j)}(\mathcal{H}))$ by assumption (14); henceforth, both premises are in $Th_{\leq}(F_{\leq}^{(j)}(\mathcal{H}))$.
4. Therefore, as $Th_{\leq}(F_{\leq}^{(j)}(\mathcal{H}))$ is transitive by assumption (13), the conclusion of (43) belongs to $Th_{\leq}(F_{\leq}^{(j)}(\mathcal{H}))$.

This finishes the proof of the Substitution Lemma. $\square$

B.8 Proof of Proposition 20.1

Consider the bottom-up inference of the judgment $\Gamma \vdash (\forall\gamma \leq \tau_1 . \tau_2) \leq (\forall\gamma \leq \rho_1 . \rho_2)$:

- First of all we get by the (*All*):

$$\Gamma,\ \gamma \leq \sigma \ \vdash\ (\forall\delta_1 \ldots \forall\delta_n . (\gamma \to \top)) \to \top \ \leq\ \beta_1 \to \top$$

- From this one by the (*Arrow*) we have:

$$\Gamma,\ \gamma \leq \sigma \ \vdash\ \beta_1 \ \leq\ (\forall\delta_1 \ldots \forall\delta_n . (\gamma \to \top))$$

- Applying for the first time (*AlgTrans*-$\forall$) we obtain:

$$\Gamma,\ \gamma \leq \sigma \ \vdash\ (\forall\delta_1 . \beta_2) \ \leq\ (\forall\delta_1 \ldots \forall\delta_n . (\gamma \to \top))$$

- Again applying (*All*) we get:

$$\Gamma,\ \gamma \leq \sigma, \delta_1 \leq \top \ \vdash\ \beta_2 \ \leq\ (\forall\delta_2 \ldots \forall\delta_n . (\gamma \to \top))$$

- Applying the rule (*AlgTrans*-$\forall$) for the second time we obtain:

$$\Gamma,\ \gamma \leq \sigma, \delta_1 \leq \top \ \vdash\ (\forall\delta_2 . \beta_3) \ \leq\ (\forall\delta_2 \ldots \forall\delta_n . (\gamma \to \top))\ ,$$

- which followed by (*All*) leads to:

$$\Gamma,\ \gamma \leq \sigma, \delta_1 \leq \top, \delta_2 \leq \top \ \vdash\ \beta_3 \ \leq\ (\forall\delta_3 \ldots \forall\delta_n . (\gamma \to \top))$$

- ... (the pattern of the repetition is now clear)
- Finally we arrive at:

$$\Gamma,\ \gamma \leq \sigma, \delta_1 \leq \top, \ldots, \delta_{n-1} \leq \top \ \vdash\ \beta_n \ \leq\ (\forall\delta_n . (\gamma \to \top))$$

- Applying the rule (*AlgTrans*-$\forall$) for the n-th time we obtain:

$$\Gamma,\ \gamma \leq \sigma, \delta_1 \leq \top, \ldots, \delta_{n-1} \leq \top \ \vdash\ (\forall\delta_n . (\tau \to \top)) \ \leq\ (\forall\delta_n . (\gamma \to \top))\ ,$$

- which followed by (*All*) produces:

$$\Gamma,\ \gamma \leq \sigma, \delta_1 \leq \top, \ldots, \delta_n \leq \top \ \vdash\ \tau \to \top \ \leq\ \gamma \to \top$$

- After the application of $(Arrow)$ we have:

$$\Gamma,\ \gamma \leq \sigma, \delta_1 \leq \top, \ldots, \delta_n \leq \top\ \vdash\ \gamma\ \leq\ \tau \tag{44}$$

Let $\mathcal{H}$ be any **SnS**-based set of $\forall$-hypotheses. By Proposition 15.2, the judgment (44) is in $Th_{\leq}(F_{\leq}^{(0)}(\mathcal{H}))$ but not in $Th_{\leq}(F_{\leq}^{(1)}(\mathcal{H}))$. So, the initial judgment $\Gamma\ \vdash\ (\forall\gamma \leq \tau_1 \,.\, \tau_2)\ \leq\ (\forall\gamma \leq \rho_1 \,.\, \rho_2)$ is in $Th_{\leq}(F_{\leq}^{(n)}(\mathcal{H}))$ but not in $Th_{\leq}(F_{\leq}^{(n+1)}(\mathcal{H}))$.

The above $F_{\leq}^{(n)}(\mathcal{H})$-proof uses the $\forall$-hypothesis (44) of the form disallowed by Substitution Lemma.

If we substitute everywhere in this $F_{\leq}^{(n)}(\mathcal{H})$-proof γ by σ, the resulting inference will show that $\Gamma\ \vdash\ \tau_2[\sigma/\gamma]\ \leq\ \rho_2[\sigma/\gamma]$ does not belong to $Th_{\leq}(F_{\leq}^{(n)}(\mathcal{H}))$ (see the proof of Proposition 15.2). $\square$

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