Approximation of the multidimensional Riemann problem in Compressible Fluid Mechanics by a Roe type method

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Abstract: We discuss the approximation of the solution of the multidimensional Riemann problem by a Roe type method. In a first part, we show that the exact solution of the problem obeys, for small times, a jump relation that generalizes exactly the one that helps to define the Roe averaged matrix in 1D. We show that the new linearization of Roe, Deconinck and Struijs does not follow this jump relation. Then, we show that there exists, in general, a solution to the problem. In a second part, we show the existence and unicity of the solution of the linearized hyperbolic Riemann problem, and, recalling the results of [1], we give its analytic solution. Then we provide some indications on how all this can be used in a numerical scheme.

Key-words: Godunov-type schemes, linear hyperbolic systems, Riemann problem, Roe linearisation

(Résumé : tsvp)

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Approximation du problème de Riemann multidimensionnel de la Mécanique des Fluides Compressible par une méthode de type Roe

Résumé : Nous discutons l'approximation du problème de Riemann multidimensionel par une méthode de type Roe. Dans un premier temps, nous montrons que la solution exacte du problème vérifie, pour des temps petits, une relation de saut qui généralise exactement celle permettant la définition la moyenne de Roe unidimensionelle. Nous montrons que la linéarisation proposée par Roe, Struijs et Deconinck ne satisfait pas cette relation de saut. Ensuite, nous montrons que notre nouvelle relation de saut possède toujours des solutions.

Dans un deuxième temps, nous montrons l'existence et l'unicité de la solution du problème de Riemann linéarisé, et rappelant les résultats de [1], nous construisons la solution analytique. Enfin, on esquisse comment utiliser nos résultats pour construire de nouveaux schémas d'intégration des équations de la Mécanique des Fluides

Mots-clé : Schémas de type Godunov, Systèmes hyperboliques linéaires, Problème de Riemann, Lin‘’earisation de Roe

Introduction

Many modern numerical schemes for compressible flows simulations are built on TVD and Riemann solvers techniques. These methods have been designed first for scalar nonlinear conservations equations and then, with the help of the finite volume formulation, extended to multidimensional flows. One must emphasis on the fact that though these schemes are devoted to multidimensional flows, the method is intrinsically one dimensional : at each interface of any control volume, the flux evaluation is done assuming a one dimensional structure of the acoustic and material waves. To cure to this default, a first idea is to design a mesh that follows the flow so that the interfaces of the control volumes are in some sense "orthogonal" to it and hence a 1D approximation is indeed valid. Another, and deeper, idea is to reexamine the problem of upwinding for multidimensional flows. A review of multidimensional upwinding may be found in [2]. A first try is to solve several Riemann problems in different directions at the interface, so that one may hope to better approximate the multidimensional character of the flow. Several attempts in that direction has been made, among them one may cite [3, 4, 5]. Another research direction is to derive truly multidimensional solvers. Among the most significant contribution in this topic, one has to consider the work by Deconinck, Roe and their coworkers [6, 7]. Their idea is to derive first a truly multidimensional solver for a linear convective equation. This has been carried out in [7] and their results are impressive. Then the idea is to use this convection scheme for compressible flows with the help of a wave decomposition. Several wave models has been yet considered by these researchers [8, 9].

To our opinion, the difficulty of this methodology is the wave decomposition because it carries arbitraryness in it. Up to some extend, this can be seen as the try to find a complete set of common eigenvectors of two matrices that does not commute. Moreover, this wave decomposition is often made with the help of gradient evaluations ; this may be to the detriment of the robustness of the scheme. Last the implementation of boundary conditions is not yet very clear.

In this paper, we try to restart from the original ideas of the finite volume methods. In any one-dimensional finite volume scheme, the flow is approximated by the averaged value of the conserved quantities (density, momentum, energy) in cells that exactly cover the computational domain. The temporal evolution of the flow is then seen as a succession of Riemann problems, that are independent if the time step is small enough. Of course the solution of the 1D Riemann problem is well known, but even in the case of a calorically perfect gas, where the ratio of specific heats is a constant, this is computationally demanding. For that reason, several approximations of its solution have been derived, among which the Roe's Riemann [10] solver is probably the most popular. This solver is obtained in three stages :

1. one approximates the exact solution of the Riemann problem by that of a linearized hyperbolic system where the linearized matrix is chosen in a consistent way with the exact solution, i.e., one imposes that the averaged value, on $\mathbb{R}$, of the approximated solution is equal to that of the exact one ;

2. one solves the approximated Riemann problem ;

3. the approximated solution is used in the flux evaluation instead of the exact one to get an approximated update at a later time.

In two dimensions, the situation is much more complicated, for at least two reasons :

1. The exact solution of the Riemann problem, where the initial condition is constant per angular sector, is not known. Only conjectures exists, at least up to our knowledge [11].

2. The Riemann problems, at the interface of two adjacent cells, are no longer independent : if $\mathcal{E}$ is a common edge, which endpoints are two common corners of two cells, according to [11] and all numerical simulations, there will be a set of (one dimensional) waves, parallel to $\mathcal{E}$; this is a one dimensional Riemann problem. But, at both corners the situation is no longer one dimensional, and the Riemann problem at two edges meeting at the same corner are not identical. In particular, the propagation speed of waves may be different.

Nevertheless, there may be advantages to take into account the corners. This is the case where an unstructured mesh is used. In this case, the mesh may really be distorted, hence the shape of the cells also, so that a one dimensional approach becomes really questionable.

As we said it before, we try to follow the original demarche of P.L. Roe. We assume that the initial condition, constant per angular sector, is in L^∞, and that the exact self-similar solution of the Riemann problem also belongs to L^∞. We also assume the maximum speed is finite. These assumptions are consistent with [11] and all known numerical simulations [12]. With these assumptions, we show that, for small times, the exact solution satisfies a jump relation that generalizes exactly a well known result for 1D systems. If we try to approximate the exact solution by that of a linear hyperbolic one with the same initial data, assuming moreover that both solutions are, in average, equal, we get a jump relation that generalizes exactly the one Roe has used in his scheme. Then we show that there is, in general, a solution to the problem. We also examines the validity of the new linearizations of Roe, Struijs and Deconinck [6] from our results.

In a second part, we solve the linearized hyperbolic system. We first start to show the existence and unicity of the solution in L^2_{loc}, despites the problem is ill-posed in any L^p, $p \neq 2$, [13]. Our results seem surprisingly new. Then we compute analytically the exact solution, and show that it belongs to L^∞. We end this paper by giving some indications on how to build a numerical scheme from our results.

Part I
The linearization problem

This paper only deals with the Riemann problem of the Euler equations in Fluid Mechanics :

$$\begin{aligned} &\frac{\partial W}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = 0 \\ &W(x, y, t = 0) = W_0(x, y) \end{aligned} \tag{1}$$

where the initial condition is assumed to be constant in angular sectors. In the following, we assume that the plane is divided into three angular sectors, but our results can easily be generalized . In the sequel, O stands for the intersection of the three half lines which are the frontiers of the angular sectors (see Figure 1), and is the origin of the plane.

$$W_0 = \begin{array}{lll} W_1 & \text{if} & (x,y) \in \mathcal{D}_1 \\ W_2 & \text{if} & (x,y) \in \mathcal{D}_2 \\ W_3 & \text{if} & (x,y) \in \mathcal{D}_3 \end{array} \tag{2}$$

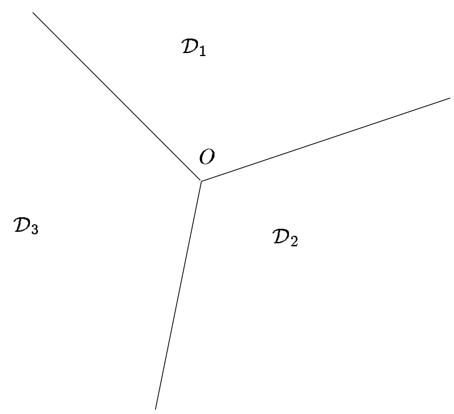

Figure 1: Definition of $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$.

Contrarily to what happens in $\mathbb{R}$, and as far as we know, the exact solution to this problem is still unknown. Nevertheless, several solutions have been described in the literature [11]. No result of existence and unicity is known, so we have to rely on conjectures concerning the behavior of the solution. We shall suppose that :

1. The solution exists in $L^\infty(\mathbb{R}^2 \times \mathbb{R})$ if $W_0 \in L^\infty(\mathbb{R}^2)$. Moreover it is unique and self-similar. The self-similar solution of the problem will be noted $\mathcal{R}(\xi, \nu, W_0)$.

2. There exists a constant C such that if $\sqrt{\xi^2+\nu^2} > C$, then the solution $\mathcal{R}(\xi, \nu, W_0)$ is constituted of planar waves.

The second condition means that above a certain distance, the corner O has no more influence. These conjectures are comforted by many numerical results (see e.g. [12]), and by those of [11] on the behavior of the solution. We want to approach W, solution of (1-2), with the solution of the following linear equation :

$$\begin{aligned} &\frac{\partial W}{\partial t} + \overline{A}\frac{\partial W}{\partial x} + \overline{B}\frac{\partial W}{\partial y} = 0 \\ &W(x,y,t=0) = W_0(x,y) \end{aligned} \tag{3}$$

It seems natural to impose that the problem (3) is hyperbolic. We want to determine which conditions to impose on $\overline{A}$ and $\overline{B}$ for the solution of (3) to be as close as possible to that of (1-2).

In $\mathbb{R}$, the answer to this question is clear : if the initial condition W_0 is W_L for $x < 0$ and W_R for $x > 0$, the self-similar solution of the Riemann problem, $\mathcal{R}(\xi, W_0)$ satisfies

$$\int_{-\infty}^{+\infty} (\mathcal{R}(\xi, W_0) - W_0)d\xi + F(W_L) - F(W_R) = 0. \tag{4}$$

If we want the average value on $\mathbb{R}$ of $\mathcal{R}(\xi, W_0)$ to be the same as the average value on $\mathbb{R}$ of the linearized problem, then we get the following condition for $\overline{A}$:

$$\overline{A}\,(W_L - W_R) = F(W_L) - F(W_R). \tag{5}$$

This is precisely Roe's jump condition. This equation has numerous solutions. But if we want that :

1. $\overline{A}$ is diagonalizable in $\mathbb{R}$,
2. The condition (5) is satisfied,
3. There exists a state $\overline{W}(W_L, W_R)$ such that

   $$\overline{A} = \frac{\partial F}{\partial W}\left(\overline{W}(W_L, W_R)\right)$$

   and $\overline{W}(W, W) = W$ for any W

then there is a *unique* solution : the Roe average.

This can be considered in a purely algebraic way, without using the "*parameter vector*" of Roe, which is anyway useless for more complex cases (see e.g. [14, 15]).

Actually, granted that $\overline{A} = \frac{\partial F}{\partial W}\left(\overline{W}(W_L, W_R)\right)$, the jump relation becomes :

$$\overline{u}^2\,(\kappa-2)\Delta\rho\,+\,2(2-\kappa)\,\overline{u}\,\Delta m + \kappa\Delta E = \Delta\{\rho u^2 + p\},$$

$$\overline{u}\,\left(-\overline{H}+\frac{\overline{u}^2}{2}\right)\Delta\rho + \left(\overline{H}-\kappa\overline{u}^2\right)\Delta m + \overline{u}(1+\kappa)\Delta E = \Delta m H,$$

where we set $\Delta a = a_L - a_R$ and ρ, m, u, H, p are the density, momentum,velocity, total enthalpy and pressure. The terms $\bar{u}$, $\bar{H}$ are the averaged velocities and enthalpy that we are looking for. The term κ is equal to $\gamma - 1$ where γ is the ratio of specific heats. For a perfect gas, the state law is $p = \kappa(E - 1/2\rho u^2)$. When eliminating the pressure in the system's first equation, we get a second degree equation :

$$(\kappa-2)\Delta\rho\,\overline{u}^2 + 2(2-\kappa)\Delta m\,\overline{u} + (\kappa-2)\Delta\rho u^2 = 0$$

The square root δ of the discriminant is always real for $0 \leq \kappa \leq 2$, and if :

$$\delta = (2-\kappa)\left|\sqrt{\frac{\rho_L}{\rho_R}}m_R - \sqrt{\frac{\rho_R}{\rho_L}}m_L\right|$$

the roots are :

$$\overline{u}_{1,2} = \frac{-(2-\kappa)\Delta m \pm \delta}{(2-\kappa)\Delta\rho}$$

After simplification, we get :

$$\overline{u}_1 = \frac{\sqrt{\rho_L}u_L + \sqrt{\rho_R}u_R}{\sqrt{\rho_L}+\sqrt{\rho_R}} \quad \text{or} \quad \overline{u}_2 = \frac{\sqrt{\rho_L}u_L - \sqrt{\rho_R}u_R}{\sqrt{\rho_L}-\sqrt{\rho_R}}$$

Now the equation is linear in $\overline{H}$ so there is only one solution,which is

$$\overline{H}_1 = \frac{\sqrt{\rho_L}H_L + \sqrt{\rho_R}H_R}{\sqrt{\rho_L}+\sqrt{\rho_R}} \quad \text{or} \quad \overline{H}_2 = \frac{\sqrt{\rho_L}H_L - \sqrt{\rho_R}H_R}{\sqrt{\rho_L}-\sqrt{\rho_R}}$$

The couple $\overline{u}_1, \overline{H}_1$ only, can be a solution to the problem because of the continuity condition. One can also verify that the matrix obtained this way is indeed diagonalizable in $\mathbb{R}$.

These results could have been found by using Roe's parameter vector $Z = (\sqrt{\rho}, \sqrt{\rho}u, \sqrt{\rho}H)$ [10]. One can easily prove that there exists matrices $D(Z)$ and $R(Z)$ such that

$$W = \frac{1}{2}D(Z).Z \quad , \quad F(W) = \frac{1}{2}R(Z).Z$$

Moreover, we have symmetry properties : For any Z et Z',

$$D(Z).Z' = D(Z').Z \quad , \quad R(Z).Z' = R(Z').Z$$

Thus, calculations can be made as though we had real quadratic forms and we get two solutions to our problem :

$$\overline{A}_1 = R(Z_R + Z_L)D^{-1}(Z_R + Z_L) \qquad \overline{A}_2 = R(Z_R - Z_L)D^{-1}(Z_R - Z_L)$$

but we can't insure they are the only ones.

Let us notice that, besides it does not guarantee continuity, the second solution doesn't either guarantee the linearized problem is hyperbolic, in general.

Roe's property guarantees a precise capture of discontinuities since if two states W_R and W_L lie on both sides of a discontinuity, one of the eigenvalues of $\overline{A}$ is precisely its propagation speed. Thus, it seems interesting to generalize this jump property, and if possible, calculate average matrices, $\overline{A}$ and $\overline{B}$. This is one the aims of this paper.

1 A Jump Condition

We shall first try to find a relation of type (4) which is verified by the exact solution.

Let T_1 be any triangle, T_λ will stand for the triangle obtained from T_1 by an similitude transformation of ratio λ and center O. Let us integrate the equation (1) in $\mathbf{T}_{t\lambda} \times [0,t]$, $t > 0$. Assuming $\vec{n}$ is the outward unit normal of T_1, we get :

$$\int_0^t \int_{T_{t\lambda}} \frac{\partial W(x,y,s)}{\partial s} ds + \int_0^t \int_{\partial T_{t\lambda}} F_{\vec{n}}\left(W(x,y,s)\right) dl = 0$$

and so

$$\int_{T_{t\lambda}} [W(x,y,t) - W_0(x,y)]\, dx\, dy + \int_0^t \int_{\partial T_{t\lambda}} F_{\vec{n}}\left(W(x,y,s)\right) dl = 0. \qquad (6)$$

Here, $W(x,y,s)$ is the solution of the problem (1). Since the solution is self-similar, we have :

$$W(x,y,t) = \mathcal{R}(\frac{x}{t}, \frac{y}{t}, W_0).$$

So we obtain :

$$\int_{T_{t\lambda}} [W(x,y,t) - W_0(x,y)]\, dx\, dy = t^2 \int_{T_\lambda} [\mathcal{R}(\xi,\nu,W_0) - W_0(\xi,\nu)]\, d\xi\, d\nu$$

and

$$\int_0^t \int_{\partial T_{t\lambda}} F_{\vec{n}}\left(W(x,y,s)\right) dl\, ds = \int_0^t s \left(\int_{\partial T_{t\lambda}} F_{\vec{n}} dl \right) ds = \frac{t^2}{2} \int_{\partial T_{t\lambda}} F_{\vec{n}} dl$$

so

$$\int_{T_\lambda} [\mathcal{R}(\xi,\nu,W_0) - W_0(\xi,\nu)]\, d\xi\, d\nu + \frac{1}{2} \int_{\partial T_{t\lambda}} F_{\vec{n}} dl = 0 \qquad (7)$$

Using the hypothesis made on the behavior of the exact solution, since for λ large enough, T_λ contains the area of the plane (ξ,ν) defined by $\xi^2 + \nu^2 \leq C$, we get that

$$\lim_{\lambda \to +\infty} \frac{1}{\lambda} \left(\int_{T_\lambda} [\mathcal{R}(\xi,\nu,W_0) - W_0(\xi,\nu)]\, d\xi\, d\nu \right)$$

is finite. Indeed, $\mathcal{R}(\xi,\nu,W_0) - W_0(\xi,\nu)$ is nil above the simple waves area (represented in dotted lines in Figure 2) and above the disc of center O and of radius C independent of λ : since $\mathcal{R}(\xi,\nu)$ and W_0 are bounded,

$$\lim_{\lambda \to +\infty} \left| \int_{\xi^2+\nu^2 \leq C} \mathcal{R}(\xi,\nu) - W_0 d\xi d\nu \right| = 0.$$

As well, we can evaluate the boundary integral. To do so, we just have to study what happens on one side of T_λ, for example the side $[A,B]$. We get :

$$\int_{AB} F_{\vec{n}}[\mathcal{R}(\xi,\nu,W_0)] dl = \overline{AA'} F_{\vec{n}}(W_A) + \overline{B'B} F_{\vec{n}}(W_B) + \int_{A'}^{B'} F_{\vec{n}}[\mathcal{R}(\xi,\nu,W_0)] dl$$

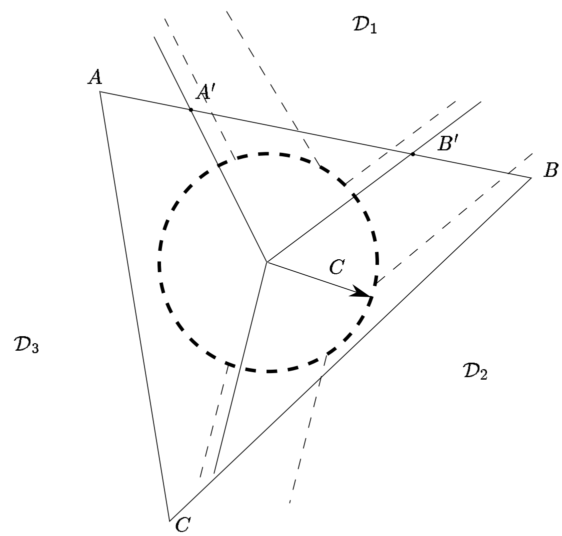

Figure 2: Behavior of the exact solution, in dotted lines : the planar waves

where W_A and W_B are the values of the solution in A and B. These values are constant on each segment $[A, A']$ and $[B', B]$ by assumption. Moreover, the length $\overline{A'B'}$ does not depend on λ : the waves are planar, parallel to the half-lines sharing the domains $\mathcal{D}_i$. It ensures, since we assumed the solution is bounded,

$$\lim_{\lambda \to +\infty} \frac{1}{\lambda} \int_{A'}^{B'} F_{\vec{n}}[\mathcal{R}(\xi, \nu, W_0)] dl = 0.$$

Moreover, the sum of the remaining terms divided by λ is equal, in the limit $\lambda \to +\infty$ to the integral, along T_1, of the normal flux of the initial condition. So we have the following result :

$$\lim_{\lambda \to +\infty} \frac{1}{\lambda} \left(\int_{T_\lambda} [\mathcal{R}(\xi, \nu, W_0) - W_0(\xi, \nu)] \, d\xi \, d\nu \right) + \frac{1}{2} \int_{T_1} F_{\vec{n}}(W_0) dl = 0 \qquad (8)$$

Applying the same method as before to the linearized equation, we find that the solution $\mathcal{R}_{app}(\xi, \nu, W_0)$ of (3) satisfies :

$$\lim_{\lambda\to+\infty}\frac{1}{\lambda}\left(\int_{T_\lambda}\left[\mathcal{R}_{app}(\xi,\nu,W_0)-W_0(\xi,\nu)\right]d\xi\,d\nu\right)+\frac{1}{2}\int_{T_1}A_{\vec{n}}(W_0)dl=0$$

where $A_{\vec{n}} = A\vec{n}_x + B\vec{n}_y$. In part II, we show that, provided (3) is hyperbolic, $\mathcal{R}_{app}$ satisfies the assumptions we have made on $\mathcal{R}$.

By analogy to what happens in $\mathbb{R}$, we can demand to the approximate solver to be in average equal to the exact solver which means :

$$\int_{T_1} A_{\vec{n}}(W_0)dl = \int_{T_1} F_{\vec{n}}(W_0)dl \tag{9}$$

This equation may be rewritten this way :

$$\sum_{i=1,3} A_{\vec{N}_i}(W_i) = \sum_{i=1,3} F_{N_i}(W_i) \tag{10}$$

where the outward unit normals $\vec{N}_i$ are defined, χ_i being the characteristic function of $\mathcal{D}_i$, by

$$\vec{N}_i = \int_{T_1}\chi_i\vec{n}dl.$$

These normals verify $\vec{N}_1 + \vec{N}_2 + \vec{N}_3 = 0$.

In the sequel, we shall take a set of outward normals, and look for the eventual solutions of the problem (10).

Remark :

1. If two states are the same, we find again the classical jump condition in a prescribed direction.

2. The computation, made for a triangle may be done as well for any polygon. As well we do not necessarily need to have three angular sectors.

2 Struijs, Roe and Deconinck linearization

Recently, Struijs, Roe and Deconinck have proposed a two dimensional linearization. It is based on the same ideas as the 1D linearization, using the parameter vector Z. Since the flux F and W are given by :

$$F(W) = \frac{1}{2}R(Z).Z \quad , \quad W = \frac{1}{2}D(Z).Z,$$

it ensures that

$$F_x = R(Z).Z_x \quad , \quad W_x = D(Z).Z_x$$

if we assume that Z is linear on $[0,1]$, with $Z(0)$ and $Z(1)$ that corresponds to W_R and W_L, we can get $\bar{A}(W_L - W_R) = F(W_L) - F(W_R)$.

In the case of $\mathbb{R}^2$, the same thing can be done. We have :

$$F(W) = \frac{1}{2}R(Z).Z \ , \ G(W) = \frac{1}{2}S(Z).Z \ , \ W = \frac{1}{2}D(Z).Z,$$

and so

$$F(W)_x = R(Z).Z_x \ , \ \ G(W)_y = S(Z).Z_y \quad , \quad \nabla W = D(Z).\nabla Z.$$

Assuming Z is linear on a triangle on which vertices W has the values W_1, W_2, W_3, we get, since ∇Z is constant in T :

$$\begin{aligned} \frac{1}{\text{area}(T)} \int_T \nabla.F(W) dx\, dy &= \frac{1}{\text{area}(T)} \int_T \left(R(Z).\nabla Z_x + S(Z).\nabla Z_y \right) dx\, dy \\ &= R(\overline{Z}).\nabla Z_x + S(\overline{Z}).\nabla Z_y \end{aligned}$$

where $\overline{Z} = \frac{Z_1+Z_2+Z_3}{3}$. As well, we have :

$$\frac{1}{\text{area}(T)} \int_T \frac{\partial W}{\partial x} dx\, dy = D(\overline{Z}) \frac{\partial Z}{\partial x} \quad , \quad \frac{1}{\text{area}(T)} \int_T \frac{\partial W}{\partial y} dx\, dy = D(\overline{Z}) \frac{\partial Z}{\partial x}$$

Mixing all this, we get

$$\begin{aligned} \frac{1}{\text{area}(T)} \int_T \nabla.F(W) dx\, dy &= R(\overline{Z}) D(\overline{Z})^{-1} \frac{1}{\text{area}(T)} \int_T \frac{\partial W}{\partial x} dx\, dy \\ &\quad + S(\overline{Z}) D(\overline{Z})^{-1} \frac{1}{\text{area}(T)} \int_T \frac{\partial W}{\partial x} dx\, dy \end{aligned} \tag{11}$$

It is clear that this solution is not consistent with Euler equations. Let us consider a stationary planar shock, separating two states W_1 for $x > 0$ and W_2 for $x < 0$. We share the half plane $x < 0$ into two parts $y > 0$ and $y < 0$. For the solution (11) to be consistent with Euler equations, the matrix $R(\overline{Z})D(\overline{Z})^{-1}$ with $\overline{Z} = (Z1 + 2\ Z2)/3$ should admit 0 as eigenvalue. A simple counter example is :

$$\rho_1 = \gamma, \quad p_1 = 1, \quad u_1 \in \mathbb{R}.$$

W_2 is defined by the Rankine-Hugoniot relations with a zero shock speed. Let us name $\lambda_\pm$ (resp. $\lambda'_\pm$) the eigenvalues $u \pm c$ of the classical Roe matrix (resp. the one defined by(11)). It is clear that $\lambda_- = 0$ for any u_1 and that $\lim_{u_1 \to +\infty} \lambda'_+ = +\infty$, $\lim_{u_1 \to +\infty} \lambda'_- = -\infty$. More precisely, we have ($\bar{c}$ and $\bar{u}$ are the averaged speed of sound and velocity) :

$$\begin{aligned}
\frac{\lambda_-}{\bar{c}} &= -\frac{-\sqrt{\gamma-1} - 2\sqrt{\gamma-1} + 4\sqrt{\sqrt{\gamma^2-1}+8\gamma}}{\sqrt{4\sqrt{\gamma^2-1}+8\gamma}} + O(\tfrac{1}{M^2}) < 0 \\
\frac{\bar{u}}{\bar{c}} &= \frac{\sqrt{\gamma+1} + 2\sqrt{\gamma-1}}{\sqrt{4\sqrt{\gamma^2-1}+8\gamma}} + O(\frac{1}{M^2}) > 0 \\
\frac{\lambda_+}{\bar{c}} &= 1 + \frac{\bar{u}}{\bar{c}} > 0.
\end{aligned}$$

On Figure 1, we show a plot of $\frac{\lambda_-}{\bar{c}}$ that should be identically zero.

On the other hand, we can see that the linearized system will always be hyperbolic. We shall have :

$$\frac{\overline{c}^2}{\kappa} = \overline{H} - \frac{1}{2}\left(\overline{u}^2 + \overline{v}^2\right) = \overline{h} + \delta$$

where $\overline{h}$ stands for the average specific enthalpy with the same weight coefficients. The rest δ is quadratic in speed and is equal to (u_i and v_i being the components of the velocity at each node) :

$$\delta\left(\sqrt{\rho_1} + \sqrt{\rho_2} + \sqrt{\rho_3}\right)^2 = (u_1, u_2, u_3)A(u_1, u_2, u_3)^T + (v_1, v_2, v_3)A(v_1, v_2, v_3)^T$$

if $z_1 = \sqrt{\rho_1}$, etc, we get :

$$A = \begin{pmatrix} z_1(z_2+z_3) & -z_1 z_2 & -z_1 z_3 \\ -z_1 z_2 & z_2(z_1+z_3) & -z_3 z_1 \\ -z_1 z_3 & -z_3 z_2 & z_3(z_1+z_2) \end{pmatrix}$$

This matrix is symmetric, and positive because its eigenvalues are :

$$\lambda_1 = 0, \quad \lambda_2 = \nu + \eta, \quad \lambda_3 = \nu - \eta$$

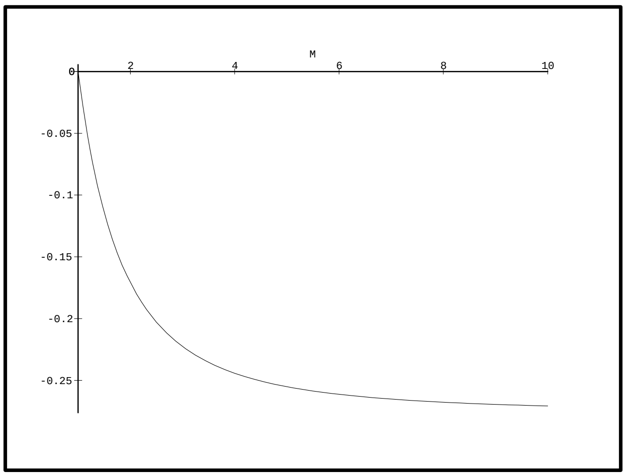

Figure 3: $\frac{\lambda_-}{\bar{c}}$ a $1 \leq M = \frac{u_1}{c_1} \leq 10$

with

$$\nu = z_1z_2 + z_1z_3 + z_3z_2$$
$$\eta^2 = z_1^2z_2^2 + z_1^2z_3^2 + z_2^2z_3^2 - z_1^2z_2z_3 - z_1z_2^2z_3 - z_3^2z_2z_1$$

We can see that $\eta^2 = (z_1z_2 + z_1z_3 + z_2z_3)^2 - 3z_1z_2z_3(z_1+z_2+z_3)$ is always positive. If we left z_2 and z_3 constant, η^2 becomes a second degree polynomial which discriminant $-3z_3^2z_2^2(z_2-z_3)^2$ is negative. The coefficient η^2 has thus a constant sign, positive. Moreover, $\eta^2 \leq z_1^2z_2^2 + z_1^2z_3^2 + z_2^2z_3^2 \leq (z_1z_2 + z_1z_3 + z_2z_3)^2$, so all the eigenvalues of A are positive. Thus, $\delta \geq 0$, and $\bar{c}^2 \geq 0$.

Proposition 2.1 *Roe-Struijs-Deconinck's linearization always leads to hyperbolic systems.*

3 Existence of the average

We shall look for a solution to the equation (10). In general, there exist many solutions (4 equations, 2×16 unknowns). We shall search $\overline{A}$ and $\overline{B}$ satisfying :

A1 the jump relation (10),

A2 there exists a continuous function of the variables $(W_1, W_2, W_3) \to \overline{W}(W_1, W_2, W_3)$ such that

$$\overline{A} = \frac{\partial F}{\partial W}\left(\overline{W}(W_1, W_2, W_3)\right) \qquad \overline{B} = \frac{\partial G}{\partial W}\left(\overline{W}(W_1, W_2, W_3)\right),$$

A3 the linearized system is hyperbolic.

Strictly speaking, we shall see that one may not fulfill conditions A1, A2, A3 for any choice of W_1, W_2, W_3. But we shall see that they are satisfied by most (W_1, W_2, W_3). Furthermore, for practical applications, we shall have to relax the continuity condition on $\bar{W}(W_1, W_2, W_3)$.

Let us give the following notations. The constant γ is the ratio of specific heats, and we set $\kappa = \gamma - 1$. If a a quantity taking 3 values $a_1,\ a_2,\ a_3$, we set

$$\nabla a = a_1\vec{N}_1 + a_2\vec{N}_2 + a_3\vec{N}_3.$$

If we have a frame, $\Delta_x a$ and $\Delta_y a$ will be the components on x and y of ∇a. Unfortunately, and contrarily to what happens in $\mathbb{R}$, *the operator* ∇ *is not a* *derivation.* Using property A1, let us write the relations defined by $\overline{u}$, $\overline{v}$ and $\overline{H}$. We obtain the following system, after having used the perfect gas law :

$$
\begin{array}{lll}
\overline{u}^2(\kappa-2)\Delta_x\rho - 2\overline{uv}\Delta_y\rho + \kappa\overline{v}^2\Delta_x\rho & +2\left\{(2-\kappa)\Delta_x m + \Delta_y n\right\}\overline{u} + 2\left\{\Delta_y m - \kappa\Delta_x n\right\}\overline{v} & \\
 & +(\kappa-2)\Delta_x\rho u^2 - 2\Delta_y\rho uv + \kappa\Delta_x\rho v^2 = 0 & (a) \\
\overline{u}^2\kappa\Delta_y\rho - 2\overline{uv}\Delta_x\rho + (\kappa-2)\overline{v}^2\Delta_y\rho & +2\left\{\Delta_x n - \kappa\Delta_y m\right\}\overline{u} + 2\left\{\Delta_x m + (2-\kappa)\Delta_y n\right\}\overline{v} & \\
 & +(\kappa-2)\Delta_y\rho v^2 - 2\Delta_x\rho uv + \kappa\Delta_y\rho u^2 = 0 & (b) \\
H\left\{-\overline{u}\Delta_x\rho - \overline{v}\Delta_y\rho + \Delta_x m + \Delta_y n\right\} & +\frac{\kappa}{2}\left(\overline{u}^2+\overline{v}^2\right)\left(\overline{u}\Delta_x\rho + \overline{v}\Delta_y\rho\right) - \kappa\left(\overline{u}^2\Delta_x m + \overline{v}^2\Delta_y n\right) & \\
 & -\kappa\overline{uv}(\Delta_y m + \Delta_x n) & \\
 & +(1+\kappa)\left\{\overline{u}\Delta_x E + \overline{v}\Delta_y E\right\} = \Delta_x m H + \Delta_y n H & (c)
\end{array}
\tag{12}
$$

where the notations are the same as the ones used previously and n stands for the quantity ρv.

We start by considering the equations (12-a) and (12-b). If we are able to prove the existence of a couple $(\overline{u}, \overline{v})$, it will be finished because the equation (12-c) is linear in $\overline{H}$, provided, of course that the denominator of the $\overline{H}$-term does not vanish.

3.1 Existence of a solution

We start with the following remark :

Lemma 3.1 *Let us suppose that the problem [A1-A2] has a solution $(\overline{A}, \overline{B})$. Let us consider the following matrix :*

$$
\mathcal{R}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

and, for any state W let us note, W^θ the state $\mathcal{R}(\theta)W$. The matrices $\mathcal{R}(\theta)\overline{A}\mathcal{R}(\theta)^{-1}$, $\mathcal{R}(\theta)\overline{B}\mathcal{R}(\theta)^{-1})$ are a solution of the problem, where the data are the $\mathcal{R}(\theta)W_i$ and the unit normals have rotated of θ. Moreover, the solution is invariant by homothety of the normals N_is. Finally, this system is invariant by any Galilean transformation

Proof : From the relation (10), we deduce that

$$\sum_{i=1,3} \mathcal{R}(\theta)\overline{A}_{\vec{N}_i} W_i = \sum_{i=1,3} \mathcal{R}(\theta) F_{\vec{N}_i}(W_i)$$

But since Euler equations are invariant by rotation, and because of A2, we may write

$$\mathcal{R}(\theta) F_{\vec{N}_i}(W_i) = F_{r(\theta)\vec{N}_i}(\mathcal{R}(\theta)W_i)$$

(where $r(\theta)$ is the rotation of angle θ). Last, for the same reason,

$$\mathcal{R}(\theta)\overline{A}_{\vec{N}_i}(\overline{W})W_i = \overline{A}_{r(\theta)\vec{N}_i}[\mathcal{R}(\theta)\overline{W}]\mathcal{R}(\theta)W_i$$

which gives the result. The Galilean invariance is obvious, as well as the homogeneity property. •

If we study the relations (12)-a and (12)-b, we get the following result :

Lemma 3.2 *If $0 \leq \kappa \leq 2$, the equations in $\overline{u}$ and $\overline{v}$ defined by (12-a) and (12-b) define hyperbolas (possibly degenerated) when $\Delta_x\rho^2 + \Delta_y\rho^2 \neq 0$*

Proof : The equation (12-a) may be rewritten, thanks to Gauss method :

$$[u\,\Delta_x\rho - v\,\Delta_y\rho + \alpha]^2 - \left(\frac{\kappa}{2-\kappa}(\Delta_x\rho)^2 + \left(\frac{\Delta_y\rho}{\kappa-2}\right)^2\right)(v-\beta)^2 + C = 0$$

where we set

$$\alpha = \frac{\Delta_y n + (2-\kappa)\Delta_x m}{\kappa - 2}$$

$$\beta = \frac{\Delta_y\rho\,(\Delta_y n + (2-\kappa)\Delta_x m) + \Delta_x\rho\,((\kappa-2)\Delta_y m + \kappa(\kappa-2)\Delta_x n)}{\kappa(2-\kappa)\{\Delta_x\rho\}^2 + \{\Delta_y\rho\}^2}$$

and C is a constant. In the same manner, the other conic may be rewritten this way (C and C' being two other constants) :

$$[v\,\Delta_y\rho - u\,\Delta_x\rho + \alpha']^2 - \left(\frac{\kappa}{2-\kappa}(\Delta_y\rho)^2 + \left(\frac{\Delta_x\rho}{\kappa-2}\right)^2\right)(u-\beta')^2 + C' = 0$$

and

$$\alpha' = \frac{\Delta_x m + (2-\kappa)\Delta_y n}{\kappa - 2}$$

$$\beta' = \frac{\Delta_x \rho\left(\Delta_x m + (2-\kappa)\Delta_y n\right) + \Delta_y \rho\left((\kappa-2)\Delta_x n + \kappa(\kappa-2)\Delta_y m\right)}{\kappa(2-\kappa)\{\Delta_y \rho\}^2 + \{\Delta_x \rho\}^2}$$

which gives the result. •

From this canonic writing, we deduce the following result :

Lemma 3.3 *If* $0 < \kappa < 2$ *and* $\{\Delta_x \rho\}^2 + \{\Delta_y \rho\}^2 \neq 0$ *then the problem has at least two distincts solutions.*

Proof : Using Lemma 3.1, one uses a frame where $\nabla \rho = (\Delta_x \rho, 0)$. We can also suppose that $\Delta_x \rho > 0$. Using lemma 3.2, we can see that the conics may be written this way :

$$\begin{cases} (2-\kappa)\left(u - \dfrac{\Delta_x m}{\Delta_x \rho} + \dfrac{\Delta_y n}{(\kappa-2)\Delta_x \rho}\right)^2 - \kappa\left(v - \dfrac{\Delta_x n}{\Delta_x \rho} + \dfrac{\Delta_y m}{\kappa \Delta_x \rho}\right)^2 - C = 0 \\ \left(u - \dfrac{\Delta_x m}{\Delta_x \rho} + (2-\kappa)\dfrac{\Delta_y n}{\Delta_x \rho}\right)\left(v - \dfrac{\Delta_x n}{\Delta_x \rho} + \kappa\dfrac{\Delta_y m}{\Delta_x \rho}\right) - C' = 0 \end{cases}$$

Then, we do the change of variable $U = u - \dfrac{\Delta_x m}{\Delta_x \rho} + (2-\kappa)\dfrac{\Delta_y n}{\Delta_x \rho}$, $V = v - \dfrac{\Delta_x n}{\Delta_x \rho} + \kappa\dfrac{\Delta_y m}{\Delta_x \rho}$. The two hyperbolas then have equations of type :

$$\begin{cases} -(2-\kappa)\left(U-\alpha\right)^2 + \kappa\left(V-\beta\right)^2 = -C \\ U\,V = C' \end{cases} \tag{13}$$

The first conic admits $\sqrt{2-\kappa}U \pm \sqrt{\kappa}V = 0$ as asymptotes, the second one, the lines $U = 0$ and $V = 0$. Let us suppose $C < 0$ to simplify.

1st case : $C' \neq 0$ **and** $C \neq 0$ We consider, for any $U > 0$ the function ϕ defined by :

$$\phi(U) = \frac{C'}{U} - \frac{1}{\sqrt{\kappa}}\sqrt{-C + (2-\kappa)\left(U-\alpha\right)^2} + \beta.$$

This function tends to $\pm\infty$ if U tends to 0 and to $\mp\infty$ for U tending to infinity. Rolle's theorem insures the existence of a solution. If $C > 0$, we consider the analog equation where $V > 0$ is the unknown and we still have a solution. In these two cases, the intersection of the two conics is in fact the study of a fourth degree equation with real coefficients. The product of its roots is negative for $\kappa \in]0, 2[$. Since we already have a real root $U_0 > 0$, there is a second real root U_1. If there are only two real roots, they have opposite signs. If there are four of them, possibly identical, the same argument shows that three of them have the same sign, the fourth one having the opposite sign.

2nd case : $C' = 0$ and $C \neq 0$ In this case, U or V are nil. If $C > 0$, then, there are at least two distinct solutions to $V = 0$, if $C < 0$, we have the same result for $U = 0$. If $C = 0$, since $0 < \kappa < 2$, the two degenerated hyperbolas have at least two distinct points in common.

3rd case : $C = 0$ This case can be studied from the previous ones by a linear change of variable. If $C' = 0$, there are still solutions because the conics degenerate into the asymptotes that are not parallels because $\kappa - 2 \neq 0$ and $\kappa \neq 0$ if $\kappa \in]0, 2[$.

•

Lemma 3.4 *If $\{\Delta_x \rho\}^2 + \{\Delta_y \rho\}^2 = 0$ the problem still has a solution under the condition $(\Delta_x m, \Delta_x n, \Delta_y m, \Delta_y n)$ does not belong to $(\mathcal{E})$ which equation is :*

$$(\mathcal{E}) \; : \; \begin{aligned} &(2-\kappa)\left[(\Delta_x m + \frac{(2-\kappa)^2+1}{2(2-\kappa)}\Delta_y n)^2 + \left\{1 - (\frac{(2-\kappa)^2+1}{2(2-\kappa)})^2\right\}\Delta_y n^2\right] + \\ &\kappa\left[(\Delta_x n - \frac{\kappa^2+1}{2\kappa}\Delta_y m)^2 + \left\{1 - (\frac{\kappa^2+1}{2\kappa})^2\right\}\Delta_y m^2\right]^2 = 0 \end{aligned} \tag{14}$$

Proof : The problem becomes :

$$\begin{aligned}2\{(2-\kappa)\Delta_x m+\Delta_y n\}\,\overline{u} &+ 2\{\Delta_y m-\kappa\Delta_x n\}\,\overline{v}\\ &+ (\kappa-2)\Delta_x\rho u^2-2\Delta_y uv+\kappa\Delta_x\rho u^2=0 \quad (a)\end{aligned}$$

$$\begin{aligned}2\{\kappa\Delta_x n-\kappa\Delta_y m\}\,\overline{u} &+ 2\{\Delta_x m(2-\kappa)\Delta_y n\}\,\overline{v}\\ &+ \kappa\Delta_y\rho v^2-2\Delta_x uv+(\kappa-2)\Delta_y\rho u^2=0 \quad (b).\end{aligned}$$

The equation (14) of $(\mathcal{E})$ is precisely the determinant of the system. • **Remark** : The interpretation of equation (14) is not really clear. We may assume that $\Delta_y m=0$ and that $\Delta_x m>0$ with an appropriate rotation. The equation (14) shows that $\Delta_y n<0$. The case $\Delta_x m=0$ is impossible. Physically, it seems to be like a point source divergent.

Gathering all these results, we get the

Theorem 3.5 *If $0<\kappa<2$ and $\{\Delta_x\rho\}^2+\{\Delta_y\rho\}^2\neq 0$ then the problem [(12)-a,(12)-b] has at least two solutions. If $\{\Delta_x\rho\}^2+\{\Delta_y\rho\}^2=0$ this problem has a single solution if and only if and $(\Delta_x m,\Delta_x n,\Delta_y m,\Delta_y n)$ does not belong to $(\mathcal{E})$.*

Now, we shall assume $\{\Delta_x\rho\}^2+\{\Delta_y\rho\}^2\neq 0$. The topology of the set of solutions is rather tough, especially in the neighborhood of multiple points of conics intersection. Except for these pathological points , the theorem of implicit functions is valid and for each set of states (W_1,W_2,W_3), there exists a neighborhood and two or four regular parametrizations of the solution. The set of pathological points is included in the set of states anulating the discriminants of resultants in u and v of the two polynomials defining the conics. We shall first give a result on the behavior of the solutions when $\{\Delta_x\rho\}^2+\{\Delta_y\rho\}^2$ tends to 0, and when we do not have a pathological point.

Proposition 3.6 *One of the solutions is bounded in the neighborhood of $\nabla\rho=0$, whereas the others are not (and behave like $C/||\nabla\rho||$) if $(\Delta_x m,\Delta_y m,\Delta_x n,\Delta_y n)$ is not on $(\mathcal{E})$.*

Proof : Let us suppose that $(\Delta_x m,\Delta_y m,\Delta_x n,\Delta_y n)$ is not on $(\mathcal{E})$. Let us examine u^2+v^2. We take a frame for which $\Delta_y\rho$ is zero. We may multiply

(12-a) and (12-b) by $\Delta_x\rho$, and we get a similar system, where $\Delta_x\rho = 1$ and (u, v) has been replaced by (U, V), $U = \Delta_x\rho\ u$ and $V = \Delta_x\rho\ v$:

$$
\begin{array}{lll}
(\kappa-2)U^2+\kappa V^2 & +2\left\{(2-\kappa)\Delta_x m+\Delta_y n\right\}U+2\left\{\Delta_y m-\kappa\Delta_x n\right\}V & \\
 & +\Delta_x\rho\left\{(\kappa-2)\Delta_x\rho u^2-2\Delta_y\rho uv+\kappa\Delta_x\rho v^2\right\}=0 & (a)\\
 & & \\
-2UV & +2\left\{\Delta_x n-\kappa\Delta_y m\right\}U+2\left\{\Delta_x m+(2-\kappa)\Delta_y n\right\}V & \\
 & +\Delta_x\rho\left\{(\kappa-2)\Delta_y\rho v^2-2\Delta_x\rho uv+\kappa\Delta_y\rho u^2\right\}=0 & (b)
\end{array}
\tag{15}
$$

When $\Delta_x\rho \to 0$, the problem reduces to know how many solutions of (15) are equal to $(0, 0)$. It is also equivalent to say that $(\Delta_x m, \Delta_y m, \Delta_x n, \Delta_y n) \notin (\mathcal{E})$ or to say that $(0, 0)$ is a simple intersection point. Thus, all the other intersection points are different of $(0, 0)$. So, we may apply the implicit functions theorem to the origin, so one of the solutions will in norm, behave like $||J^{-1}.\nabla\rho||$ and the other ones like $||(U_0, V_0) + J^{-1}\nabla\rho||$ (J is the Jacobian). Going back to the standard coordinates, we can see that one of the roots is bounded while the other one are not and behaves as we said it would. •

Remark : The asymptotical behavior of the solutions that may lie on the pathological points set is much more complicated. In general, for a double point we get a behavior in $||\nabla\rho||^{-1/2}$ and for a triple point in $||\nabla\rho||^{-2/3}$ [16]. From Goursat [16], one can easily see that if (u_0, v_0) is a solution for a given $\Delta\rho\ |_0$, whatever the nature of this point (simple or not), there is a continuous function $\Delta\rho \to (u, v) = \phi(\Delta\rho)$ of solutions to our problem such that $(u_0, v_0) = \phi(\Delta\rho\ |_0)$. This comes from the fact that both u and v obey a polynomial equation of degree 4, the resultant in u or v of (12-a) and (12-b).

3.2 Study of the enthalpy's equation

The equation (12-c) shows that if $Q_1(\overline{u}, \overline{v}) = -\Delta_x m - \Delta_y n + \overline{u}\,\Delta_x\rho + \overline{v}\,\Delta_y\rho \neq 0$, then $\overline{H}$ exists. The problem is to know what happens when Q_1 is equal to zero. This case may occur, for example when the velocities of the three states are the same, or when there is a contact discontinuity which is parallel to the density gradient. As usual, let us take a frame linked to the density gradient (so we assume that it is not zero). Let P_1 be the polynomial given by (12-a), P_2 the polynomial given by (12-b) and Q_2 the polynomial :

$$\begin{aligned} Q_2(\overline{u},\overline{v}) = \tfrac{\kappa}{2}\left(\overline{u}^2+\overline{v}^2\right)\overline{u}\Delta_x\rho \quad & -\kappa\left(\overline{u}^2\Delta_x m+\overline{v}^2\Delta_y n\right)-\kappa\overline{u}\overline{v}(\Delta_y m+\Delta_x n) \\ & +(1+\kappa)\left\{\overline{u}\Delta_x E+\overline{v}\Delta_y E\right\}-(\Delta_x mH+\Delta_y nH) \end{aligned} \tag{16}$$

With these notations, the equation (12-c) may be written

$$Q_2(\overline{u},\overline{v})-\overline{H}Q_1(\overline{u},\overline{v})=0$$

Two cases may occur, if $\Delta_x\rho^2+\Delta_y\rho^2\neq 0$:

1. if none of the solutions of (12-a), (12-b) annihilates Q_1, then we may associate to each one a value of $\overline{H}$.

2. if one of them annihilates Q_1, then there is at least another one which does not annihilate Q_1. Indeed, the problem has at least two solutions. Not all the solutions of the problem are on the line $Q_1=0$ because the equation (12-b), for $\Delta_y\rho=0$ is linear in each variable. So, to each value of u, we associate a single value of v, and so a unique point on the plane (u,v). Let u^* be the root of $Q_1=0$ and v^* the other component of the velocity. We make the Euclidian division of $Q_2(u,v^*)$ by Q_1 :

   $$Q_2(u,v^*)=M(u)Q_1(u,v^*)+Q_2(u^*,v^*)$$

   Two cases may occur

   - $Q_2(u^*,v^*)=0$. Then $\bar{H}=M(u^*)$ is the expected solution.
   - $Q_2(u^*,v^*)\neq 0$. Then there is no *bounded* solution corresponding to u^*.

Remark : In the 1–D case, if the solution annihilates Q_1, then we *always* have $Q_2(u^*,v^*)=0$. This is due to the properties of the operator Δ which then is a derivation. But in our case, this property is lost. If now $\Delta_x\rho^2+\Delta_y\rho^2=0$, similar statements may be written, but here, since there is either one solution, no solution or an infinite number of solutions, the existence of $\bar{H}$ is not necessarily insured if $\Delta_x m+\Delta_y n=0$.

We have, if $\Delta_x\rho^2+\Delta_y\rho^2\neq 0$, the following result :

Proposition 3.7 *In the neighborhood of $\nabla\rho = 0$, the specific enthalpy, $h = H - 1/2(u^2+v^2)$ is negative for all the unbounded solutions of the problem if the hypothesis of the Proposition 3.6 are satisfied.*

Proof : We have the following relation :

$$Q_2 = -\frac{\kappa}{2}\left(u^2+v^2\right)Q_1 + H_2$$

where

$$H_2 = -\frac{u^2-v^2}{2}\left(\Delta_x m - \Delta_y n\right) - \kappa uv\left(\Delta_x m + \Delta_y n\right) + Au + B'v + C''$$

and A, B, B',C, C', C'' are constants. Because of that, we get :

$$\bar{H} - \frac{1}{2}(\overline{u}^2+\overline{v}^2) = \frac{1-\kappa}{2}(\overline{u}^2+\overline{v}^2) + \frac{H_2}{Q_1}$$

It ensures that $h = H - \frac{1}{2}(u^2+v^2)$ is negative, in the neighborhood of $\nabla\rho = 0$, for all the unbounded solutions, from the proposition 3.6. •

3.3 Choice of the solution and main result

In principle, it would be interesting to study the intersection (15) : the "true" solution belongs to the curve passing through $(0,0)$ when $\Delta\rho \to 0$. It is clear that this criterion is not numerically efficient, even if analytical expression of the solution are here existing (because the resultants are only of degree 4). An other reason not to push further in this way is that, even if the selection of the true solution of (15) can be made easy by some tricky argument, it is likely that one cannot prove, in general, that the exists a bounded solution to (12-c) corresponding to this "true" solution.

A more heuristic criterion that reduces to the previous rigorous one in 1D is the following

Criterion 3.8 *find among all the possible solutions $(\overline{u},\overline{v},\bar{H})$, the closer one to the convex hull of W_1, W_2 and W_3.*

This criterion give the true solution if $||\nabla\rho||$ is small enough (because of Proposition 3.7) or when two states are equal (because it reduces to a 1D problem, where the solution lies in this convex hull) and hence in the vicinity of such states.

Let us notice that the continuity criterion of the function may not always be satisfied, as well as the hyperbolicity of the linearized system even though we have not been able to find a numerical counter example.

We get the result :

Theorem 3.9 *If $\nabla\rho \neq 0$, then, the problem (12)-(a,b,c) has always a solution that satisfies Criterion 3.8. When $\nabla\rho = 0$, the problem has still a solution if $(\Delta_x m, \Delta_y m, \Delta_x n, \Delta_y n) \notin \mathcal{E}$, and if $\Delta_x m + \Delta_y n \neq 0$.*

Part II

Solution of the linearized problem

In this part, we want to solve the Riemann problem for system (3) when it is hyperbolic. Since there is no ambiguity, A will stand for $\bar{A}$ and B for $\bar{B}$, hence, that is for any vector $\vec{n} = (n_x, n_y)$ the matrix $n_x A + n_y B$ is diagonalizable with real eigenvalues. The matrices may be constructed either by the method of Roe, Struijs and Deconinck [6], or by ours, or by any other method that ensure the hyperbolicity. Furthermore, we assume here that these matrices are obtained by evaluating the Jacobian matrices of the Euler equation (1) at an averaged state. It is uniquely defined by an averaged velocity $(\overline{u}, \overline{v})$ and an averaged enthalpy $\overline{H}$. This defines an averaged speed of sound $\bar{c}$. As before, the initial condition U_0 is assumed to be :

$$W_0 = \begin{array}{lll} W_1 & \text{if} & (x,y) \in \mathcal{D}_1 \\ W_2 & \text{if} & (x,y) \in \mathcal{D}_2 \\ W_3 & \text{if} & (x,y) \in \mathcal{D}_3 \end{array} \tag{17}$$

where the W_i's are constant states and the sets $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ are angular sectors as defined on Figure 1. The point O is the origin. We first study the existence and unicity problem of weak solutions of (3), when $W_0 \in L^\infty$. Brenner [13] has shown that this problem is ill-posed in L^∞ (and in fact in any L^p, $p \neq 2$ for initial condition in L^p). We shall relax the L^∞ condition to a L^2_{loc} one, which is far enough for our purpose. With this result in hand, any solution of (3-17) we find will be the right one. Thus, we will proceed formally, the justification of everything is done at the end.

4 Existence and unicity of the solution of the Cauchy problem

In all what follows, if $f \in L^2$, $\widehat{f}$ is its Fourier transform :

$$\widehat{f}(\vec{k}) = \int_{\mathbb{R}^2} f(x,y) \exp\left(i\vec{k}.(x,y)\right) dx\, dy$$

Since it is an isometry of L^2, since the set of compactly supported C^∞ function is included in L^2, a weak solution of (3) with initial condition in L^2 also satisfies ($\vec{k} = (k_1, k_2)$) :

$$\frac{d\widehat{W}}{dt}(\vec{p}) = i\left(k_1 A + k_2 B\right) \widehat{W}(\vec{p}).$$

Now, let us define ξ_R^M the characteristic function of $D_R(M)$, the disc of center M and radius R. If $W_0 \in L^\infty$, $W_{0,R} = \xi_R W_0 \in L^2$, whatever $R > 0$. Then, we can consider W_R, the solution to (3) with initial condition $W_{0,R}$. We shall show that $\lim_{R\to+\infty} W_R$ converges in L^2_{loc} to a weak solution of (3), and that this solution is unique.

We start with the following lemma :

Lemma 4.1 *Let $\epsilon_1, \cdots, \epsilon_4$ be the real numbers 0 or 1 and V_R defined on $\mathbb{R}^2$ by*

$$V_R(x,y) = \begin{matrix} (\epsilon_1 \cdots \epsilon_4)^T & \text{if} & (x,y) \in D_R(O) \\ 0 & \text{else} & \end{matrix}$$

Then, for all $t > 0$, the solution $V \in L^2$ of the adjoint equation of (3) :

$$V_t - A^* V_x - B^* V_y = 0 \tag{18}$$

is supported in $D_{R+ct}(\bar{u}t, \bar{v}t)$. Moreover, if $t < R/c$, then $V = (\epsilon_1 \cdots \epsilon_4)^T$ in $D_{R-ct}(ut, vt)$.

Proof : This is a well known result : one first look for a solution in polar coordinates r, θ that does not depend on r. The problem reduces to a classical 1-D problem Riemann problem. • We can now state the following lemma :

Lemma 4.2 *If $W_0 \in L^\infty$, and ξ_R^O is the characteristic function of $D_R(O)$, and if, for $R' > R$, $v_{R'}$ is defined as in Lemma 4.1, we have, for $T < R/c$,*

$$\int_{|(x-uT,y-vT)|\leq R'+cT} u_R(x,y,T).v_{R'+cT}(x,y,T)dxdy$$
$$=$$
$$\int_{|(x,y)|\leq R} u_R(x,y,0).v_{R'+cT}(x,y,0)\ dxdy.$$

Proof : The proof is inspired of Courant-Hilbert [17]. One has

$$\int_{\mathbb{R}^2} u_R(x,y,T)\ .\ v_{R'+cT}(x,y,T)\ dxdy = \int_{\mathbb{R}^2} u_R(\widehat{x,y},T)\ .\ v_{R'+cT}\widehat{(x},y,T)\ dk_1\ dk_1$$

because because u_R, $v_{R'+cT} \in L^2$ and because the Fourier transform is an isometry of L^2. Moreover, one sees that

$$\begin{array}{rcl}\widehat{u_R}_t\ .\ \widehat{v_{R'+cT}} + \widehat{u_R}\ .\ \widehat{v_{R'+cT}}_t & = & i\,(Ak_1 + Bk_2)\,\widehat{u_R}\ .\ \widehat{v_{R'+cT}} \\ & - & i\,(A^*k_1 + B^*k_2)\,\widehat{v_{R'+cT}}\ .\ \widehat{u_R} \\ & = & 0\end{array}$$

Then $\widehat{u}_R\ .\ \widehat{v}_{R'+cT}$ remains constant. Using back the Fourier transform, one gets

$$\int_{\mathbb{R}^2} u_R(x,y,T)\ .\ v_{R'+cT}(x,y,T)\ dxdy = \int_{\mathbb{R}^2} u_R(x,y,0)\ .\ v_{R'+cT}(x,y,0)\ dxdy.$$

Last, by Lemma 4.1, applied to $v_{R'+cT}$ with A^* and B^*, one gets

$$\int_{\mathbb{R}^2} u_R(x,y,T)\ .\ v_{R'+cT}(x,y,T)\ dxdy = \int_{|(x-\bar{u}t,y-\bar{v}t)|} u_R(x,y,T)\ .\ v_{R'+cT}(x,y,T)\ dxdy.$$

The assumption made on u_R enables to conclude the proof. •

An easy consequence is that if f, $g \in L^\infty$ are equal in $D(0,R)$, then the solutions of the Cauchy problem (3) with initial conditions f and g are equal in $D((u,v), R + cT)$.

This gives the following result :

Theorem 4.3 *Let $W_0 \in L^\infty$, and define W_R as above, and consider the solution W_R of the Cauchy problem with initial data $W_{0,R} = \xi_R^0\, W_0$. Then*

1. *$W = \lim_{R\to\infty} W_R$ exists in L^2_{loc},*
2. *W is a solution of (3) with initial condition W_0,*
3. *W is unique.*

Remark : Nevertheless, as pointed out by Brenner [13], the problem is ill-posed in L^∞ with the sup-norm. If one consider the norm

$$|||f||| = \sup_{R>1} \sqrt{\frac{\int_{D_R(O)} f^2 dx}{\pi R^2}}$$

that is defined for any $f \in L^\infty$, L^∞ is still Banach, if a sequence converge for the sup-norm it also converges for $|||\ |||$ (the reverse is of course wrong) and the Cauchy problem is well posed now.

5 Solution of the Riemann problem

If one find a solution of the Riemann problem, the theorem 4.3 enable to state that it is the solution. To get it, we start with the following remarks that enable to consider only one particular case : **Remark** :

Proposition 5.1 *Since the Euler equations are invariant under Galilean transformation, it follows easily that the vector*

$$\widetilde{W} = \begin{pmatrix} A \\ B - \overline{u}A \\ C - \overline{v}A \\ D - \frac{1}{2}(\overline{u}^2 + \overline{v}^2)A - \overline{u}B - \overline{v}C \end{pmatrix} \quad \text{for} \quad W = \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix}$$

satisfies

$$\widetilde{W}_\tau + \widetilde{A}\widetilde{W}_{x'} + \widetilde{B}\widetilde{W}_{y'} = 0 \tag{19}$$

if W satisfies (1). In equation (19), the matrices $\tilde{A}$ and $\tilde{B}$ are given by

$$\widetilde{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma - 1 \\ 0 & 0 & 0 & 0 \\ 0 & \bar{h} & 0 & 0 \end{pmatrix} \quad \text{and} \quad \widetilde{B} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma - 1 \\ 0 & 0 & \bar{h} & 0 \end{pmatrix}$$

with $\bar{h} = \bar{H} - \frac{1}{2}(\bar{u}^2 + \bar{v}^2)$. *We also have use the new variables* τ, $x' = x - \overline{u}t$ *and* $y' = y - \overline{v}t$.

Proof : One has : $W = \widetilde{W} + \mathcal{M}\widetilde{W}$ where $\mathcal{M}$ is the *constant* matrix

$$\mathcal{M} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \overline{u} & 0 & 0 & 0 \\ \overline{v} & 0 & 0 & 0 \\ \frac{1}{2}(\overline{u}^2 + \overline{v}^2) & \overline{u} & \overline{v} & 0 \end{pmatrix},$$

so that $\widetilde{W}$ satisfies :

$$(Id + \mathcal{M})\widetilde{W}_t + A(Id + \mathcal{M})\widetilde{W}_x + B(Id + \mathcal{M})\widetilde{W}_y = 0$$

or, which is equivalent,

$$\widetilde{W}_t + (Id + \mathcal{M})^{-1}A(Id + \mathcal{M})\widetilde{W}_x + (Id + \mathcal{M})^{-1}B(Id + \mathcal{M})\widetilde{W}_y = 0$$

It is straightforward to see that $(Id + \mathcal{M})^{-1}A(Id + \mathcal{M}) = \widetilde{A} + \overline{u}Id$ and $(Id + \mathcal{M})^{-1}B(Id + \mathcal{M}) = \widetilde{B} + \overline{v}Id$. The use of the new variables τ, x, y enable to get the result. • **Remark** : It is also possible to simplify the problem as follows. Instead of considering problem (1-17), we consider equation (1) with the following three initial conditions :

- Condition (a)

$$W_{0-a} = \begin{cases} W_1 & \text{if} \quad (x, y) \in \mathcal{D}_1 \\ 0 & \text{elsewhere} \end{cases} \tag{20}$$

- Condition (b)

$$W_{0-b} = \begin{cases} W_2 & \text{if} \quad (x, y) \in \mathcal{D}_2 \\ 0 & \text{elsewhere} \end{cases} \tag{21}$$

- Condition (c)

$$W_{0-c} = \begin{cases} W_3 & \text{if} \quad (x,y) \in \mathcal{D}_3 \\ 0 & \text{elsewhere} \end{cases} \tag{22}$$

Equation (1) is linear. Since $U_0 = U_{0-a} + U_{0-b} + U_{0-c}$, the solution of problem (1-17) is the sum of the solutions of the problems (1-20), (1-21), (1-22). We will specialize now to the solution of problem (1-20).

6 A simplified Riemann Problem

6.1 Geometrical description

Because of the first remark, we may assume that $\overline{u} = \overline{v} = 0$. Following Courant and Hilbert [17], The structure of the solution is given in Figure 4 : each points of the lines D and D' are acoustic sources so that their influence domain is the envelop of the circles which center lie on D or D' and which radius is c, the speed of sound which square is : $\bar{c}^2 = (\gamma - 1)\bar{h}$.

One can distinguish five zones in the $(x/t, y/t)$- plane :

- zones I and II where the solution is not modified,
- zone III where only the influence of D is visible,
- zone IV where only the influence of D' is visible,
- zone V where D and D' plays a role,
- zone VI where planar waves interact. It is limited by the circle $\mathcal{C}$: $(x/t)^2 + (y/t)^2 = \bar{c}^2$.

7 Analytical description of the solution

We are looking for a self-similar solution, $W(x/t, y/t)$ of the problem

$$U_t + AU_x + BU_y = 0$$

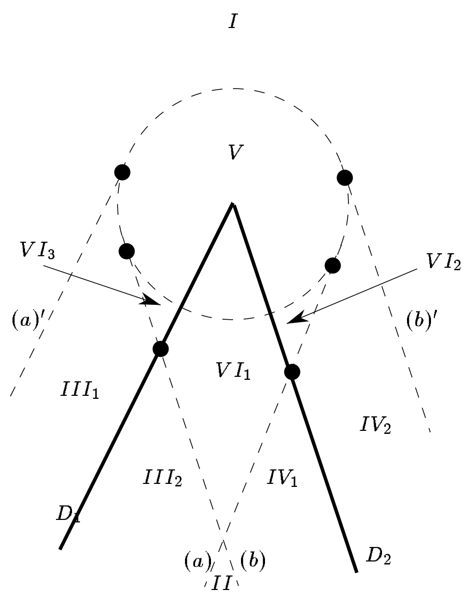

Figure 4: The solution of the Riemann problem, stationary case

with

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \gamma-1 \\ 0 & 0 & 0 & 0 \\ 0 & \bar{h} & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma-1 \\ 0 & 0 & \bar{h} & 0 \end{pmatrix}$$

If one set $\xi = x/t$ and $\nu = y/t$, the solution W must also fulfill :

$$-\xi W_\xi - \nu W_\nu + AW_\xi + BW_\nu = 0 \tag{23}$$

In the following, the parameters r and θ will stand for :

$$r = \sqrt{\xi^2+\nu^2}, \xi = r\cos\theta, \nu = r\sin\theta \tag{24}$$

with $\theta \in [0, 2\pi[$.

7.1 Preliminary results

Let us begin with some notations. For any angle θ, we consider the eigenvectors of $A\cos\theta + B\sin\theta$:

$$R_\epsilon = \begin{pmatrix} 1 \\ \epsilon\,\bar{c}\cos\theta \\ \epsilon\,\bar{c}\sin\theta \\ H \end{pmatrix} \quad R_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad R_t = \begin{pmatrix} 0 \\ -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix} \tag{25}$$

The vectors R_ϵ $(\epsilon = \pm 1)$ are associated to the eigenvalue $\lambda = \epsilon\bar{c}$ while R_0 and R_t are associated to the eigenvalue $\lambda = 0$.

For any function f, regular enough, we have the identities :

$$\begin{aligned} f_\xi &= \cos\theta f_r - \frac{\sin\theta}{r} f_\theta \\ f_\nu &= \sin\theta f_r + \frac{\cos\theta}{r} f_\theta \end{aligned} \tag{26}$$

so that, in particular,

$$r f_r = \xi f_\xi + \nu f_\nu. \tag{27}$$

From this relation, since the eigenvectors only depend on θ, we get

$$\xi R_\xi + \nu R_\nu = 0 \tag{28}$$

for $R = R_\epsilon, R_0, R_t$.

Moreover, because of (26), we have :

$$(Af_\xi + Bf_\nu)\,R_\epsilon = \epsilon\,\bar{c}f_r R_\epsilon + \frac{f_\theta}{r}\,(-A\sin\theta + B\cos\theta)\,R_\epsilon.$$

A short calculation gives :

$$(-A\sin\theta + B\cos\theta)\,R_\epsilon = \bar{c}^2 R_t$$

so that

$$(Af_\xi + Bf_\nu)\,R_\epsilon = \epsilon c f_r R_\epsilon + R_\epsilon + \frac{c^2 f_\theta}{r} R_t. \tag{29}$$

The same method gives :

$$(Af_\xi + Bf_\nu)\,R_t = \frac{f_\theta}{2r}\,(R_1 + R_{-1}) \tag{30}$$

because $(A\cos\theta + B\sin\theta)\,R_t = 0$ and last

$$(Af_\xi + Bf_\nu)\,R_0 = 0 \tag{31}$$

since $(A\cos\theta + B\sin\theta)\,R_t = (-A\sin\theta + B\cos\theta)\,R_0 = 0$. There is another interesting equation on R_ϵ :

$$\frac{dR_\epsilon}{d\theta} = \epsilon\,\bar{c}R_t\ ;$$

together with (30), it implies :

$$AR_{\epsilon\ \xi} + BR_{\epsilon\ \nu} = \epsilon\,\bar{c}\frac{R_1 + R_{-1}}{2}. \tag{32}$$

7.2 Rewriting of system (9)

Any function W may be expressed as a linear combinations of the eigenvectors :

$$W(\xi,\nu) = f(\xi,\nu)R_1 + g(\xi,\nu)R_{-1} + h(\xi,\nu)R_t + k(\xi,\nu)R_0$$

where the functions f, g, h, k are assumed regular enough if W is. Putting this in equation (23) gives :

$$\begin{array}{llll}
-\left(\xi f_\xi+\nu f_\nu\right) R_1 & -f\left(\xi R_{1\ \xi}+\nu R_{1\ \nu}\right) & +\left(A f_\xi+B f_\nu\right) R_1 & +f\left(A R_{1\ \xi}+B R_{1\ \nu}\right) \\
-\left(\xi g_\xi+\nu g_\nu\right) R_{-1} & -g\left(\xi R_{-1\ \xi}+\nu R_{-1\ \nu}\right) & +\left(A g_\xi+B g_\nu\right) R_{-1} & +g\left(A R_{-1\ \xi}+B R_{-1\ \nu}\right) \\
-\left(\xi h_\xi+\nu h_\nu\right) R_t & -h\left(\xi R_{t\ \xi}+\nu R_{t\ \nu}\right) & +\left(A h_\xi+B h_\nu\right) R_t & +h\left(A R_{t\ \xi}+B R_{t\ \nu}\right) \\
-\left(\xi k_\xi+\nu k_\nu\right) R_0 & -k\left(\xi R_{0\ \xi}+\nu R_{0\ \nu}\right) & +\left(A k_\xi+B k_\nu\right) R_0 & +k\left(A R_{0\ \xi}+B R_{0\ \nu}\right) \\
=0 & & &
\end{array}$$

Then we use the preliminary results of equations (27), (28), (29), (30), (31), (32) and obtain :

$$\begin{array}{lllll}
-r f_r R_1 & +\bar{c} f_r R_1 & +\bar{c}^2 \frac{f_\theta}{r} R_t & +\frac{\bar{c}\ f}{2r}\left(R_1+R_{-1}\right) & (a) \\
-r g_r R_{-1} & -\bar{c} g_r R_{-1} & +\bar{c}^2 \frac{g_\theta}{r} R_t & -\frac{\bar{c}\ g}{2r}\left(R_1+R_{-1}\right) & (b) \\
-r h_r R_t & & +\frac{h_\theta}{2r}\left(R_1+R_{-1}\right) & & (c) \\
-r k_r R_0 & & & \qquad\qquad =0 & (d)
\end{array}$$

Since $\{R_1, R_{-1}, R_t, R_0\}$ is a basis, we get

$$\left\{\begin{array}{l}
2r(\bar{c}-r) f_r+\bar{c}(f-g)+h_\theta=0 \\
-2r(r+\bar{c}) g_r+\bar{c}(f-g)+h_\theta=0 \\
f_\theta+g_\theta-\frac{r^2}{\bar{c}^2} h_r=0 \\
r k_r=0
\end{array}\right. \tag{33}$$

It is convenient to introduce $u=f-g$, $v=f+g$, $w=h/\bar{c}$, and the variable $R=r/c$, so that (33) becomes :

$$\left\{\begin{array}{l}
\left(1-R^2\right) v_R+u+w_\theta=0 \\
v_R-R u_R=0 \\
v_\theta-R^2 w_R=0 \\
R k_R=0
\end{array}\right. \tag{34}$$

Until the end, we drop k, since the equation on k is totally decoupled from the others. It gives $k \equiv k(\theta)$, left invariant in each angular sector.

Some algebraic manipulations of system (34) enable to write an equation on v only :

$$v_{\theta\theta} + R^2(1 - R^2)v_{RR} + R(1 - 2R^2)v_R = 0 \tag{35}$$

This problem is elliptic for $R < 1$ and hyperbolic for $R > 1$. We first compute the solution in the hyperbolic domain and then we concentrate on the elliptic problem. We will give the boundary conditions on the unit circle in an other paragraph. The only condition we impose on u, v and W is to remain in L^2 in the unit circle.

7.3 Solution of the problem outside of the unit circle

Outside of the unit circle, the problem is hyperbolic. Referring to Figure 4, in areas I and II, the solution is the initial one. In areas III and IV, the solution is the classical 1D solution. The only remaining problem is to evaluate the solution in VI. The solid and dotted lines that separate the subarea are discontinuity lines. This enable to get the following Lemma

Lemma 7.1 *Let us give U, V, W and X four constant states that are separated by admissible straight discontinuities D_1 and D_2 as in Figure 5, then one has*

$$X + V = U + W.$$

Proof : We denote by $(n_{1,x}, n_{1,y})$ and $(n_{2,x}, n_{2,y})$ the components of the normal to D_1 and D_2. The discontinuity D_1, between U and V is associated to an eigenvalue λ_1 of $An_{1,x} + Bn_{1,x}$:

$$(An_{1,x} + Bn_{1,x})(U - V) = \lambda_1(U - V).$$

Similarly, one has

$$(An_{2,x} + Bn_{2,x})(V - W) = \lambda_2(V - W).$$

Since $X - U = W - V$, one sees that the discontinuity D_2 between X and U is admissible with eigenvalue λ_2. Since $X - W = U - V$, D_1 is admissible with eigenvalue λ_1. This concludes the proof. •

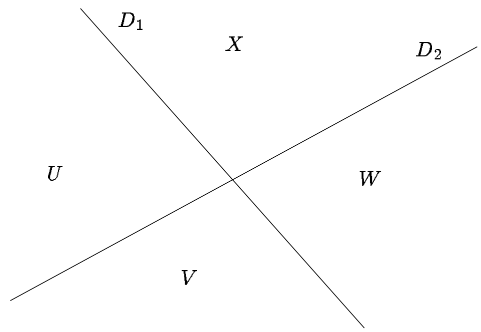

Figure 5:

7.4 Solution of the elliptic problem in the unit circle

7.4.1 Boundary conditions

We have to set the following boundary condition on the unit circle. They are obtained from the Rankine Hugoniot relations applied on the unit circle : the unit circle is a characteristic line of the system associated with the eigenvalue $\bar{c}$, hence the jump of W is collinear to R_1. This means that the jumps of g and w are equal to 0.

7.4.2 Solution

We set $\rho = \log\left(\frac{1}{R} + \sqrt{\frac{1}{R^2} - 1}\right)$, that is the inverse function of the hyperbolic cosine applied to $\frac{1}{R}$. This function satisfies

$$\rho_R^2 R^2(1 - R^2) = 1. \tag{36}$$

We make now the change of variable $(R, \theta) \to (\rho, \theta)$ that is valid. It transforms the unit disk into $\mathbb{R}^2 - D(0,1)$. The equation (34) writes now :

$$v_{\rho\rho} + v_{\theta\theta} = 0 \quad , \rho \geq 1, \quad \theta \in [0, 2\pi[. \tag{37}$$

We get

$$v(R, \theta) = \mathcal{R}e\left(\sum_{n \in \mathbf{Z}} a_n e^{in\theta} \left\{\frac{R}{1 + \sqrt{1 - R^2}}\right\}^{|n|}\right) \tag{38}$$

with, of course $\overline{a_n} = a_{-n} \in \mathbb{C}$.
Remark : This sum is indeed defined if the Fourier serie $\sum_{n\in\mathbf{Z}} a_n e^{in\theta}$ converges because $\frac{R}{1+\sqrt{1-R^2}} < 1$ if $R < 1$.

7.4.3 Evaluation of u and w

It is more simple to make this evaluation in the coordinates ρ, θ. In these variables, the system takes the form :

$$\begin{cases} -\sinh(\rho)\, v_\rho + u + w_\theta = 0 \\ \cosh(\rho)\;\; v_\rho - u_\rho = 0 \\ w_\rho + \sinh(\rho)\, v_\theta = 0 \end{cases}$$

We want u,v and w in L^2 so that we finally get :

$$v = \textstyle\sum_{|n|\neq 1} a_n e^{in\theta} \left\{\dfrac{R}{1+\sqrt{1-R^2}}\right\}^{|n|}$$

$$u = -\frac{1}{2}\textstyle\sum_{|n|>1} |n| a_n e^{in\theta}\left(\dfrac{1}{1-|n|}\dfrac{1+\sqrt{1-R^2}}{R} - \dfrac{1}{1+|n|}\dfrac{R}{1+\sqrt{1-R^2}}\right)\left\{\dfrac{R}{1+\sqrt{1-R^2}}\right\}^{|n|}$$
$$+H(\theta)$$

$$w = -\frac{1}{2}\textstyle\sum_{|n|>1} in e^{in\theta}\left(\dfrac{1}{1-|n|}\dfrac{1+\sqrt{1-R^2}}{R} + \dfrac{1}{1+|n|}\dfrac{R}{1+\sqrt{1-R^2}}\right)\left\{\dfrac{R}{1+\sqrt{1-R^2}}\right\}^{|n|}$$
$$+G(\theta) \tag{39}$$

In equation (39), the complex a_n satisfy $\overline{a_n} = a_{-n}$, G and H are two arbitrary functions that fulfill :

$$H(\theta) + G'(\theta) = 0. \tag{40}$$

Last, we have removed the terms with $n = 1$ in order to fulfill the L^2 condition.
Remark : The equation (40) implies :

$$\int_0^{2\pi} H(\theta)d\theta = 0$$

so that the constant term of its Fourier serie is nil.

Now, we want to show that a solution exits, indicate how to construct it and show that it is in L^∞. We set $\Phi = \widehat{D_1 D_2}$, and, according to Figure 6, we consider the points $A_1, \cdots, A_6$ that are the intersection of the circle and the lines D_1, D_2 and their parallels tangent to the unbit circle. We choose a clockwise orientation of the circle, and identify, for any point P and Q of the circle, the segment $[P, Q]$ and the arc PQ. One has to consider two cases,

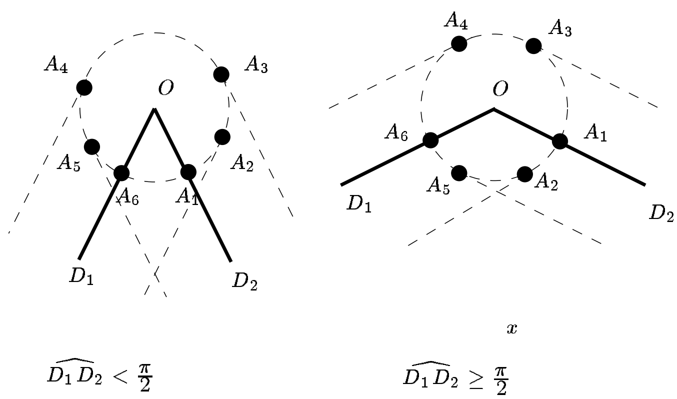

Figure 6:

depending on whether Φ is smaller or larger than $\frac{\pi}{2}$

If $\Phi \geq \frac{\pi}{2}$, then the solution W on the sonic circle $\mathcal{C}$ is

1. in $[M_6, M_1]$, $W_{[M_6,M_1]} = W_{VI_1} = W_{III_2} + W_{IV_1} - W_0$,
2. in $[M_1, M_2]$, $W_{[M_1,M_2]} = W_{VI_2} = W_{VI_1} + W_{IV_2} - W_{IV_1} = W_{III_2} + W_{IV_2} - W_0$,
3. in $[M_2, M_3]$, $W_{[M_2,M_3]} = W_{IV_2}$,
4. in $[M_3, M_4]$, $W_{[M_3,M_4]} = 0$,
5. in $[M_4, M_5]$, $W_{[M_4,M_5]} = W_{III_1}$,
6. in $[M_5, M_6]$, $W_{[M_5,M_6]} = W_{VI_3} = W_{III_1} + W_{VI_1} - W_{III_2} = W_{III_1} + W_{IV_1} - W_0$.

If $\Phi \leq \frac{\pi}{2}$, then the solution W on $\mathcal{C}$ is

1. in $[M_5, M_2]$, $W_{[M_5,M_2]} = W_{VI_1} = W_{III_2} + W_{IV_1} - W_0$,

2. in $[M_1, M_2]$, $W_{[M_1,M_2]} = W_{VI_1}$,
3. in $[M_1, M_3]$, $W_{[M_1,M_3]} = W_{IV_2}$,
4. in $[M_3, M_4]$, $W_{[M_3,M_4]} = 0$,
5. in $[M_4, M_6]$, $W_{[M_4,M_6]} = W_{III_1}$,
6. in $[M_6, M_5]$, $W_{[M_5,M_6]} = W_{III_2}$.

If $\xi_{[A,B]}$ is the characteristic function of the arc $[A, B]$, the solution can then be written as a sum

$$W_{\text{circle}^+} = \sum_{i=1}^{6} \xi_{[M_{i^+},M_{i^-}]} W_{[M_{i^+},M_{i^-}]}.$$

The $W_{[M_{i^+},M_{i^-}]}$s depend linearly on the components A_k of W_0 :

$$W_{[M_{i^+},M_{i^-}]} = \sum_{k=1,4} A_k X^k_{[M_{i^+},M_{i^-}]}$$

so that,

$$W_{\text{circle}^+} = \sum_{k=1}^{4} A_k \sum_{i=1}^{6} \xi_{[M_{i^+},M_{i^-}]} X^k_{[M_{i^+},M_{i^-}]}. \tag{41}$$

To describe u, w, w on $\mathcal{C}$, we use complex notations and set

$$\begin{aligned}
v(R = \bar{c}, \theta) &\equiv v(\theta) = \left(\textstyle\sum_{|n|\neq 1} a_n e^{i\ n\theta}\right) \\
u(R = \bar{c}, \theta) &\equiv u(\theta) = \left(\textstyle\sum_{|n|>1} \left\{\frac{n^2}{n^2-1} a_n + \alpha_n\right\} e^{i\ n\theta} + \alpha_1 e^{i\theta}\right) \\
w(R = \bar{c}, \theta) &\equiv w(\theta) = \left(\textstyle\sum_{|n|>1} i \left\{\frac{n^2}{n^2-1} a_n - \frac{\alpha_n}{n}\right\} - i\alpha_1 e^{i\theta}\right) + \beta_0
\end{aligned}$$

In these formula, the α_n's stand for the Fourier development of H, β_0 is the additional constant for G. We then get the Fourier development of f, g and h :

$$\begin{aligned}
f(R = \bar{c}, \theta) &\equiv f(\theta) = \tfrac{1}{2}\left(\textstyle\sum_{|n|>1} \left\{\frac{2n^2-1}{n^2-1} a_n + \alpha_n\right\} e^{i\ n\theta} + \alpha_1 e^{i\theta}\right) + \frac{a_0}{2} \\
g(R = \bar{c}, \theta) &\equiv g(\theta) = \tfrac{1}{2}\left(\textstyle\sum_{|n|>1} \left\{\frac{1}{n^2-1} a_n + \alpha_n\right\} e^{i\ n\theta} + \alpha_1 e^{i\theta}\right) - \frac{a_0}{2} \\
h(R = \bar{c}, \theta) &\equiv h(\theta) = \bar{c}\left(\textstyle\sum_{|n|>1} i \left\{\frac{n^2}{n^2-1} a_n - \frac{\alpha_n}{n}\right\} e^{in\theta} - i\alpha_1 e^{i\theta}\right) + \bar{c}\beta_0
\end{aligned} \tag{42}$$

We know that g and h are *continuous* on $\mathcal{C}$. Since, for any θ,and for any state $W=(A_1,A_2,A_3,A_4)^T$, one has :

$$\begin{array}{c} h(\theta)=\frac{1}{2}\left(\frac{A_4}{\bar{h}}-\frac{A_2\,\cos\theta++A_3\sin\theta}{\bar{c}}\right),\\ g(\theta)=-A_2\sin\theta+A_3\cos\theta, \end{array} \tag{43}$$

the equation (41) enable easily to get h and g on $\mathcal{C}$. They depend linearly on the components of Q_0 through four universal vectors. One can then write the Fourier series of g and h :

$$\begin{array}{l} g=Re\left(\sum_{n\geq 0}\hat{g}_n e^{i\;n\theta}\right)\\ \\ h=Re\left(\sum_{n\geq 0}\hat{h}_n e^{i\;n\theta}\right) \end{array} \tag{44}$$

and then we get :

$$\begin{array}{lll} a_n=\dfrac{2(n+1)}{n^2+n+1}\left(\hat{g}_n-in\dfrac{\hat{h}_n}{c}\right),\;|n|>2 & & (a)\\ \\ \alpha_n=-\dfrac{2(n+1)}{n^2+n+1}\left(\hat{g}_n+in\dfrac{\hat{h}_n}{c}\right),\;|n|>2 & & (b)\\ \\ \alpha_1=2\hat{g}_1 & & (c)\\ \\ \alpha_0=2\hat{g}_0 & & (d)\\ \\ \beta_0=\dfrac{\hat{h}_0}{c} & & (e) \end{array} \tag{45}$$

We note that the form of h and g in equation (43) as well as equation (41) guaranties the necessary relation $i\bar{c}\,\hat{g}_1=\hat{g}_1$. Since the universal vectors we have introduced above are piecewise continuous functions of θ, we see that the solution in $\mathcal{C}$ is bounded.

Theorem 7.2 *The solution of the Riemann problem (1) is given by the results of §7.3 and in particular Lemma 7.1, and equations (39), (40), (41), (44) and (45). It is in* L^∞.

Remark : In the case of a general system, with a non zero velocity $(\overline{u}, \overline{v})$, the solution is

$$W(x,y,t) = (Id + \mathcal{M})U(\frac{x}{t} - \overline{u}, \frac{y}{t} - \overline{v})$$

where U is the solution of the modified system (19).

8 Conclusion

In this paper, we have studied the approximation of the genuinely multidimensional Riemann problem by a Roe type method. We have focused on two subproblems : the linearization of the system, and the solution of the approximated Riemann problem. Our results generalize well known results in 1D. The next step is to build a numerical scheme from them. It is clear that the exact solution of the linearized Riemann problem is of no use, because of its assumed heavy computational cost, and because all the details of the solution utilized in a first order scheme are probably unnecessary. Hence a simplified version must be derived. An idea is to substitute the exact solution in the sonic circle by its average. This has the advantage of using only the first Fourier coefficients of the functions f and g. This study will be made in a forthcoming paper. This scheme should be evaluated in situations where the classical Riemann solvers are known to be uneffective, for example with highly stretched meshes. It is not clear how the well known entropy fix problem will occur and be solved with this new scheme. Its extension to higher order of accuracy must be studied, and its coupling with ENO type methods [18] must be done. Last, a 3D version of this solver can probably be obtained with the same method.

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