

Self-dual codes which are principal ideals of the group algebra $\mathbb{F}_2[\mathbb{F}_2^m, +]$

Pascale Charpin

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INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

*Self-Dual Codes which
are Principal Ideals
of the Group Algebra
 $\mathbb{F}_2[\{\mathbb{F}_{2^m}, +\}]$*

Pascale CHARPIN

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Codes principaux autoduaux de l'algèbre de groupe

$$\mathbb{F}_2[\{\mathbb{F}_{2^m}, +\}]$$

Self-dual codes which are principal ideals of the
group algebra $\mathbb{F}_2[\{\mathbb{F}_{2^m}, +\}]$ *

Pascale CHARPIN**

Résumé

Cet article est un prolongement du travail de Paul CAMION portant sur les codes autoduaux, idéaux principaux de l'algèbre $\mathbb{F}_2[\{\mathbb{F}_{2^m}, +\}]$. Dans [5], ces codes sont introduits et appelés H -codes. Notre principal résultat ici est que les H -codes sont asymptotiquement mauvais, au sens de la borne de Gilbert-Varshamov. Pour obtenir ce résultat, nous exhibons d'abord une borne supérieure pour la distance minimale de tout H -code donné. Nous caractérisons les H -codes extrémaux et montrons que leurs générateurs sont liés à certains ensembles à différence.

Abstract

This paper follows up CAMION's contribution on self-dual codes which are principal ideals of the algebra $\mathbb{F}_2[\{\mathbb{F}_{2^m}, +\}]$, the so-called H -codes. Our main result is that this class of codes does not meet the Gilbert-Varshamov bound. We obtain this result by giving an upper bound on the minimal distance of any H -code. We characterize extremal H -codes and link up their generators with certain difference sets.

Keywords: self-dual codes, extremal codes, group algebra, difference sets.

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** INRIA, projet CODES, Domaine de Voluceau, Rocquencourt, BP 105, 78153 Le Chesnay Cedex, FRANCE

Self-dual codes which are principal ideals of the group algebra $\mathbb{F}_2[\{\mathbb{F}_{2^m}, +\}]$

1 Introduction

Let C be a binary linear code of length n . The code C is said to be *self-dual* if and only if it is equal to its dual; in this case its dimension is $n/2$. A code is said to be t -divisible when the distance between codewords is always a multiple of t . By convention a t -divisible code is not t' -divisible for $t' > t$.

Our main reference on self-dual codes is [17, chapter 19], where an extensive study of the work of GLEASON is given. A binary t -divisible self-dual code exists for only two values of t : $t = 2$ or 4 . When $t = 2$ we say that the code is *even*; when $t = 4$ we say that the code is *doubly-even*. An important result on these self-dual codes was obtained by GLEASON when he established strong constraints on their weight enumerators [14]. Then an upper bound on the minimal distance of these codes appeared; a code which meets this bound is said to be *extremal*. Many questions on the existence of extremal self-dual codes remain open.

Binary H-codes were introduced by CAMION in [5]. They form a class of binary self-dual codes which are principal ideals of a modular algebra of type $\mathbb{F}_2[G]$, $G = \{\mathbb{F}_{2^m}, +\}$. Basic properties of such codes are given in [5]. Above all it is shown that it is very easy to construct an H -code; it is sufficient to choose a hyperplane H of G and a codeword x of odd weight whose support is contained in H . A generator matrix is directly obtained. The code is even if and only if the weight of x is congruent to 1 modulo 4; otherwise the code is doubly-even. It is also shown in [5] that there are many extremal doubly-even H -codes of length 32. In fact many will be equivalent because there are only five extremal codes of length 32 [13, 15]. Our purpose in this paper is to study asymptotic properties of the minimal distance of H -codes.

We will denote by \mathcal{A} the algebra $\mathbb{F}_2[G]$. In Section 2, we present the class of H -codes, giving a brief review of properties of the algebra \mathcal{A} ; more details on codes of \mathcal{A} can be found in [2, 8, 11]. Recall that the powers of the radical of \mathcal{A} are the Reed-Muller codes [4].

Extremal H -codes are studied in Section 3. We first give an upper bound on the minimal distance of any H -code (Theorem 3). We can then deduce that H -codes are asymptotically bad; moreover we prove that there are no extremal H -codes of length greater than 32.

The last section treats the extremal H -codes of length 32. Our main result is presented in Theorem 5, linking extremal H -codes to difference sets of the abelian group $\{\mathbb{F}_{16}, +\}$.

The usual terminology of coding theory can be found in [17]. In this paper we only treat binary linear codes of length 2^m . The distance is always the Hamming distance.

2 The class of H-codes

2.1 The algebra \mathcal{A}

Let G be the additive group of the finite field \mathbb{F}_{2^m} . We denote by \mathcal{A} the algebra of the group G over the field \mathbb{F}_2 . An element of \mathcal{A} is a formal sum

$$x = \sum_{g \in G} x_g X^g, \quad x_g \in \mathbb{F}_2.$$

For any $a \in \mathbb{F}_2$, $b \in \mathbb{F}_2$, $x \in \mathcal{A}$ and $y \in \mathcal{A}$, we have:

$$ax + by = a \sum_{g \in G} x_g X^g + b \sum_{g \in G} y_g X^g = \sum_{g \in G} (ax_g + by_g) X^g,$$

$$xy = \sum_{g \in G} x_g X^g \sum_{g \in G} y_g X^g = \sum_{h \in G} \left(\sum_{g \in G} x_g y_{h+g} \right) X^h,$$

The additive and multiplicative unit elements are $0 = \sum_{g \in G} 0 X^g$ and $1 = X^0$. In this paper, the algebra \mathcal{A} is the ambient space of the codes. A code of \mathcal{A} is an \mathbb{F}_2 -subspace. A codeword is an element of \mathcal{A} . Note that an ideal of \mathcal{A} is a code invariant under the multiplication by X^g , for any $g \in G$. The scalar product is $\langle x, y \rangle = \sum_{g \in G} x_g y_g$. The dual of a code C will be denoted by C^\perp . If C is an ideal, the annihilator of C is:

$$\text{Ann } C = \{ x \in \mathcal{A} \mid xy = 0 \text{ for all } y \in C \}.$$

Since $xy = \sum_{h \in G} \langle x, X^h y \rangle X^h$, the following proposition is obvious:

Proposition 1 *Let C be an ideal of \mathcal{A} . Then $\text{Ann } C = C^\perp$.*

Since G is an elementary abelian 2-group and $a^2 = a$ for $a \in \mathbb{F}_2$, we have for any element x of \mathcal{A} :

$$x^2 = \left(\sum_{g \in G} x_g X^g \right)^2 = \sum_{g \in G} x_g X^{2g} = \left(\sum_{g \in G} x_g \right) X^0.$$

That means that x is invertible in \mathcal{A} if and only if its weight is odd. The set of non invertible codewords is the only maximal ideal of \mathcal{A} , its so-called *radical*. We will denote by \mathcal{P} the radical of \mathcal{A} ; that is

$$\mathcal{P} = \{ x \in \mathcal{A} \mid \sum_{g \in G} x_g = 0 \} = \{ x \in \mathcal{A} \mid x^2 = 0 \}.$$

For any $j \in [1, m]$ we define the ideal which is the j -power of \mathcal{P} . The ideal \mathcal{P}^j is generated linearly by elements of the form $\prod_{i=1}^j x_i$ where $x_i \in \mathcal{P}$. For any basis $\{e_1, \dots, e_m\}$ of G , the 2^m elements

$$\left\{ \prod_{i=1}^m (X^{e_i} + 1)^{\lambda_i} \mid \lambda_i \in \{0, 1\} \right\}$$

form a linear basis for \mathcal{A} . Moreover, for all $j \in [1, m]$, the elements of

$$\mathcal{B}^j = \left\{ \prod_{i=1}^m (X^{e_i} + 1)^{\lambda_i} \mid j \leq \sum_{i=1}^m \lambda_i, \lambda_i \in \{0, 1\} \right\} \quad (1)$$

form a linear basis of \mathcal{P}^j (by convention, $\mathcal{P}^0 = \mathcal{A}$) [8]. We now define a useful parameter of an element or of a subset of \mathcal{A} :

Definition 1 Let $x \in \mathcal{A}$; let U be a subset of \mathcal{A} . Let $j \in [0, m]$.

- We will say that the depth of x equals j if and only if x is in \mathcal{P}^j and not in \mathcal{P}^{j+1} .
- We will say that the depth of U equals j if and only if U is included in \mathcal{P}^j and not included in \mathcal{P}^{j+1} .

In the following we will use several representations of codewords of depth 1. Two of them are given in Lemma 1. It is clear, by means of the basis \mathcal{B}^0 , that a codeword z of depth 0 is of the form:

$$z = X^0 + z' = 1 + z', \quad z' \in \mathcal{P}. \quad (2)$$

Lemma 1 Let $x \in \mathcal{A}$ be a codeword of depth 1. By using the basis \mathcal{B}^1 (cf. (1)), we have:

$$x = \sum_{i=1}^m a_i (X^{e_i} + 1) + y, \quad y \in \mathcal{P}^2, \quad a_i \in \mathbb{F}_2, \quad (3)$$

where at least one a_i is not zero. Set $h = \sum_{i=1}^m a_i e_i$. Then

$$x = (X^h + 1) + x', \quad x' \in \mathcal{P}^2. \quad (4)$$

Proof: We can assume that $a_1 \neq 0$ in (3). Since $e_1 = h + \sum_{i=2}^m a_i e_i$, we have

$$\begin{aligned} X^{e_1} + 1 &= X^h \prod_{i=2}^m X^{a_i e_i} + 1 = ((X^h + 1) + 1) \left(\prod_{i=2}^m ((X^{e_i} + 1)^{a_i} + 1) \right) + 1 \\ &= (X^h + 1) + \sum_{i=2}^m a_i (X^{e_i} + 1) + y, \quad y \in \mathcal{P}^2. \end{aligned}$$

Then, according to (3), $x = (X^h + 1) + y$.

□

BERMAN proved in [4] that the powers of the radical of \mathcal{A} are the Reed-Muller codes (RM-codes). More precisely:

$$\mathcal{P}^j = R(m - j, m), \quad \text{for all } j \in [1, m], \quad (5)$$

where $R(m - j, m)$ is the RM-code of length 2^m and order $m - j$ (see an easy proof in [1]).

Remark 1: From well-known properties of RM-codes or by using the bases \mathcal{B}^j , it follows easily that :

1. $\mathcal{P}^{m+1} = \{0\}$ and $\mathcal{P}^m = \{ a \sum_{g \in G} X^g \mid a \in \mathbb{F}_2 \}$;
2. The minimal distance of \mathcal{P}^j equals 2^j .
3. $(\mathcal{P}^j)^\perp = \text{Ann } \mathcal{P}^j = \mathcal{P}^{m-j+1}$;
4. Since \mathcal{P}^{m-1} is the RM-code of order 1, the set $\mathcal{P}^{m-1} \setminus \mathcal{P}^m$ contains the elements of \mathcal{A} whose supports are the affine hyperplanes of G ;
5. Any ideal of \mathcal{A} of depth j contains \mathcal{P}^m and is such that its intersection with \mathcal{P}^k , $k \geq j$ is not empty.

The *support* of a codeword $x = \sum_{g \in G} x_g X^g$ is:

$$\text{supp}(x) = \{ g \in G \mid x_g = 1 \} .$$

Recall that the cardinal of $\text{supp}(x)$ is the *weight* of x , which we will denote by $\omega(x)$. We will use also the notation $\mathbb{F}_2 G$ instead of \mathcal{A} . More generally the notation $\mathbb{F}_2 V$, where V is an \mathbb{F}_2 -subspace of G , will appear. It is because we will often need to study some codeword whose support is contained in some V . For instance the *restriction* of x to V is $x' = \sum_{g \in V} x_g X^g$ and we can identify it with a codeword of the algebra $\mathbb{F}_2 V$. The following lemma will be very useful for the proof of Theorem 3.

Lemma 2 *Let V be a subspace of G of dimension k . Let W be a subset of G of 2^{m-k} elements, which contains one representative of each coset of V : W is such that G is equal to $\cup_{h \in W} (h + V)$. Let x in \mathcal{A} and $y = \sum_{g \in V} X^g$. Let us denote by $x(h)$, $h \in W$, the restriction of $X^h x$ to V . So x can be written as follows :*

$$x = \sum_{h \in W} x(h) X^h \quad , \quad x(h) \in \mathbb{F}_2 V .$$

Then either $x(h)$ is invertible and the restriction of yx to $h + V$ equals yX^h or $x(h) \in \mathcal{P}$ and the restriction of yx to $h + V$ equals 0. The weight of yx equals $\lambda 2^k$, where λ is the number of $h \in W$ such that $x(h)$ is invertible.

Proof. The codeword y can be considered as an element of $\mathbb{F}_2 V$; it is the all-one vector of $\mathbb{F}_2 V$. Note that, for any basis (e_1, \dots, e_k) of V , y equals $\prod_{i=1}^k (X^{e_i} + 1)$. So $y \in \mathcal{P}^k$ and $\mathcal{P}^{k+1} \cap \mathbb{F}_2 V = \{0\}$. We have

$$yx = \sum_{h \in W} yx(h) X^h \quad , \quad yx(h) \in \mathcal{P}^k \cap \mathbb{F}_2 V .$$

Whenever $x(h) \in \mathcal{P}$ the product $yx(h)$ is zero, since its depth is $k + 1$. If $x(h)$ is invertible, then $x(h) = 1 + z$, $z \in \mathcal{P}$, which yields $yx(h) = y$.

□

2.2 Principal ideals of depth 1

The definition of the H -codes and the properties given in this section are mainly due to CAMION [5]. We denote by (x) the principal ideal generated by an element x of \mathcal{A} . That is

$$(x) = \{ yx \mid y \in \mathcal{A} \}.$$

A generator of (x) is an element yx such that y has depth 0 (i.e. is invertible); all generators have the same depth. Since the ideal (x) equals (yx) for all such y , we always will consider x as a formal generator that we can fix when it is necessary. In accordance with Proposition 1, an element z is in the dual of the ideal (x) if and only if it is in its annihilator. So

$$(x)^\perp = \text{Ann}(x) = \{ z \in \mathcal{A} \mid zx = 0 \}.$$

The algebra \mathcal{A} itself is the only principal ideal of depth 0. When $x \in \mathcal{P}$, then $x^2 = 0$ which yields that the ideal (x) is contained in its dual; the dimension of (x) is less than or equal to 2^{m-1} .

From now on we assume that x has depth 1 - i.e. $x \in \mathcal{P} \setminus \mathcal{P}^2$. We assume that x is written as in (3); we can suppose, without loss of generality, that $a_1 = 1$. Hence the restriction of x to the hyperplane H generated by (e_2, \dots, e_m) is invertible. Indeed set $z = \sum_{g \in H} X^g$; using (3), with $a_1 = 1$, we obtain:

$$zx = \prod_{i=2}^m (X^{e_i} + 1) x = \prod_{i=1}^m (X^{e_i} + 1) = \sum_{g \in G} X^g,$$

which implies, with the notation of Lemma 2, that

$$x = x(0)X^0 + x(h)X^h, \quad h \notin H,$$

where $x(0)$ and $x(h)$ are invertible elements of \mathbb{F}_2H . Note that if $x \in \mathcal{P}^2$ the product zx is zero for any hyperplane H . By multiplying x by $x(0)$, we obtain a generator of (x) of the form

$$X^0 + x'X^h, \quad h \notin H, \quad x' \text{ invertible}, \quad x' \in \mathbb{F}_2H. \quad (6)$$

Now it is clear that the set $\{ X^g + (x'X^g)X^h \mid g \in H \}$ is a linearly independent subset of (x) containing 2^{m-1} elements. Since $\dim(x) \leq 2^{m-1}$, it is a basis of (x) . More generally the set $\{ X^g x \mid g \in H \}$ it is a basis of (x) . Then we have proved the following theorem.

Theorem 1 *Let x in \mathcal{A} . Recall that $\mathcal{A} = \mathbb{F}_2G$ where G is the additive group of \mathbb{F}_2^m .*

(i) *The element x is of depth 1 if and only if there exists a hyperplane H of G such that the restriction of x to H (respectively to $h + H$, $h \notin H$) is invertible.*

(ii) *Assume that x is of depth 1. Then the ideal (x) is self-dual. Its dimension equals 2^{m-1} . The code (x) is said to be an H -code, where H is a hyperplane of G such that*

$$x = x' + x''X^h, \quad h \notin H, \quad \text{supp}(x') \subset H, \quad \text{supp}(x'') \subset H,$$

where x' and x'' are invertible. The set $\{ X^g x \mid g \in H \}$ is a basis of (x) . The generators of (x) are the

$$\sum_{g \in H} a_g X^g x, \quad a_g \in \mathbb{F}_2, \quad \text{where} \quad \sum_{g \in H} a_g X^g \text{ is invertible}.$$

There is a generator of (x) of the form (6).

The H -codes are the principal ideals of depth 1.

Remark 2: Principal ideals of depth 1 of the algebra $K[G]$, $K = \mathbb{F}_{2^r}$ and $G = \mathbb{F}_{2^m}$, are self-dual. WOLFMANN proved that the binary image of such codes, relative to a convenient basis, are binary self-dual codes, which can be doubly-even [19]. For instance an extremal [64, 32, 12] code was constructed in this way [18].

An H -code (x) is contained in \mathcal{P} ; thus it is an even self-dual code. Moreover (x) has a generator matrix all of whose rows have the same weight as x . So it is easy to determine if the code is doubly-even or not, since a binary self-dual code is doubly-even if and only if every row of its generator matrix has a weight divisible by 4.

Corollary 1 *Let x be an element of \mathcal{A} of depth 1. Then the principal ideal (x) , generated by x , is an even self-dual code. Moreover (x) is doubly-even if and only if the weight of x is divisible by 4.*

Constructing a doubly-even self-dual code of length 2^m . We can summarize as follows the construction of the generator matrix of an H -code of length 2^m :

- Choose a hyperplane H of the \mathbb{F}_2 -vector space \mathbb{F}_{2^m} .
- Get a codeword x' of odd weight whose support is contained in H . That means that x' is an invertible element of $\mathbb{F}_2 H$.
- Compute the $2^{m-1} \times 2^{m-1}$ matrix M whose rows are the codewords $X^h x'$, $h \in H$.

Let I be the $2^{m-1} \times 2^{m-1}$ identity matrix. Then the code with generator matrix $[I \mid M]$ is an even self-dual $[2^m, 2^{m-1}]$ code. This code is doubly-even if and only if the weight of x' satisfies : $\omega(x') \equiv 3 \pmod{4}$.

3 Extremal H-codes

One corollary of GLEASON's result [14] is an upper bound on the minimal distance of the even (respectively doubly-even) self-dual codes. The proof of the following theorem can be found in [17, ch. 19].

Theorem 2 *The minimum distance of a binary self-dual code with all weights divisible by t (and not by $t' > t$) is at most d^* where:*

- if $t = 2$ then $d^* = 2\lfloor n/8 \rfloor + 2$;

- if $t = 4$ then $d^* = 4\lfloor n/24 \rfloor + 4$.

A self-dual code which has minimal distance d^* is said to be extremal. All extremal even (respectively doubly-even) codes of the same length have the same weight enumerator.

In this section we want to prove that there are no extremal H -codes of length greater than 32. Moreover we point out that the class of H -codes is asymptotically bad. In order to do that we first determine an upper bound for the minimal distance of every H -code. This bound is given in Table 1 for $m < 14$.

Theorem 3 Recall that $\mathcal{A} = \mathbb{F}_2G$, with $G = \mathbb{F}_2^m$, $m > 0$. Let (x) be an H -code in \mathcal{A} . Then, for each i , $i \geq 0$, such that $i + 2^{i-1} \leq m$, there exists an element y of (x) whose support is a subspace of G of dimension $m - i$.

Let d be the minimal distance of (x) and set

$$i^* = \max \{ i \in \mathbb{N} \mid i + 2^{i-1} \leq m \} .$$

Then $d \leq 2^{m-i^*}$.

Proof: Throughout the proof we will assume that x is an element of depth 1 of \mathcal{A} and we will study the principal ideal (x) . We will prove the first part of the theorem by induction on i .

The property is satisfied when $i = 0$, for any m , since each ideal of \mathcal{A} contains the all-one vector $\sum_{g \in G} X^g$. If $i = 1$ the property is satisfied for any $m > 1$ since (x) contains some codewords whose support is a hyperplane of G (see Remark 1).

We suppose now that $i > 1$ and that the property is satisfied up to $i - 1$. Hence we assume that

$$(i - 1) + 2^{i-2} \leq m \implies \text{there exists } y \in (x) \text{ such that } \text{supp}(y) \text{ is a subspace of } G \text{ of dimension } m - (i - 1) . \quad (7)$$

Now we want to prove that the property is satisfied for i .

1) We first suppose that $m = i + 2^{i-1}$. In accordance with (7) there exists an $y \in (x)$ such that $\text{supp}(y)$ is a subspace V of G of dimension $k = m - i + 1$. Let W be the set of representatives of cosets of V . With notation of Lemma 2 we have

$$x = \sum_{h \in W} x(h)X^h \quad \text{where } x(h) \in \mathcal{P} \cap \mathbb{F}_2V .$$

Indeed $y \in (x)$ implies $yx = 0$ which means that each $x(h)$ is in \mathcal{P} (from Lemma 2). Note that W contains $2^{m-k} = 2^{i-1}$ elements. Whenever $x(h)$ is of depth 1, $x(h)$ can be written as follows (according to (4)):

$$x(h) = (X^{h'} + 1) + x' \quad , \quad h' \in V \setminus \{0\} \quad , \quad x' \in \mathcal{P}^2 . \quad (8)$$

Let Q be the subspace of G generated by these elements h' . Since there are 2^{i-1} elements $x(h)$, the dimension of Q is less than or equal to 2^{i-1} . By hypothesis, the dimension of V

m	i^*	$d \leq$	m	i^*	$d \leq$	m	i^*	$d \leq$
2	1	2	3	1	4	4	2	4
5	2	8	6	2	16	7	3	16
8	3	32	9	3	64	10	3	128
11	3	256	12	4	256	13	4	512

Table 1: The upper bound of the minimum distance of any H -codes of length 2^m , $m \leq 13$. Notation is that of Theorem 3.

equals $2^{i-1} + 1$. Hence there exists a hyperplane V' of V containing Q . Set $z = \sum_{g \in V'} X^g$ and note that $z \in \mathcal{P}^{k-1} \cap \mathbb{F}_2 V$; then we have

$$zx = \sum_{h \in W} zx(h)X^h \quad \text{where} \quad zx(h) \in \mathcal{P}^{k-1+r} \cap \mathbb{F}_2 V,$$

r being the depth of $x(h)$. If $x(h)$ is in \mathcal{P}^2 then $zx(h) = 0$. Whenever $x(h)$ is as in (8) we have $zx' = 0$ and

$$(X^{h'} + 1)z = \sum_{g \in V'} X^{g+h'} + z = 2z = 0,$$

since by construction $h' \in V'$. So $zx = 0$ which is equivalent to $z \in (x)$. Since the support of z is the subspace V' whose dimension equals $m - i$, the property is proved for all i , when $m = i + 2^{i-1}$.

2) We suppose now that $m > i + 2^{i-1}$ and we assume that the property is satisfied up to $m - 1$. In accordance with (6), there are a hyperplane H of G , an invertible element x' of $\mathbb{F}_2 H$ and $h \notin H$ such that $x = X^h + x'$. As $x' = 1 + x''$ with $x'' \in \mathcal{P}$ (see (2)), $x = (1 + X^h) + x''$. We can choose h such that x'' has depth 1. Indeed (x) contains the half of elements of \mathcal{P}^{m-1} , because $x\mathcal{P}^{m-1} = \{0, \sum_{g \in G} X^g\}$. So there exists a hyperplane H' , different from H , which is the support of a codeword $z \notin (x)$. Getting h in H' (and not in H) we obtain $zx = zx'' \neq 0$ which yields $x'' \in \mathcal{P} \setminus \mathcal{P}^2$. So we have:

$$x = (X^h + 1) + x'' \quad \text{where} \quad x'' \in \mathcal{P} \setminus \mathcal{P}^2 \quad \text{and} \quad \text{supp}(x'') \subset H.$$

Since $x'' \in \mathbb{F}_2 H$, where H is identified with the additive group of \mathbb{F}_2^{m-1} , there exists (by induction on m) an element y'' in (x'') whose support is a subspace of H of dimension $(m - 1) - i$. Then we have $y''x'' = 0$ which yields that the codeword $y''x$ of (x) equals $y''(X^h + 1)$. Since $h \notin H$, the support of $y''x$ is a subspace of G of dimension $m - i$. That completes the proof of the first part of the theorem.

Actually we have stated an upper bound for the minimum distance d of (x) . Indeed we have proved that (x) contains a codeword of weight 2^{m-i} , for any i such that $i + 2^{i-1} \leq m$. Hence $d \leq 2^{m-i^*}$, where i^* is the biggest i .

□

Corollary 2 Set $N = 2^m$, $m > 1$. Let d_m be the minimal distance of any H -code of length N . Then the quotient d_m/N approaches zero as m approaches infinity. The class of H -codes does not meet the Gilbert-Varshamov bound : H -codes are asymptotically bad.

Proof: In accordance with Theorem 3, $d_m \leq 2^{m-i^*}$, where i^* is the biggest i such that $i + 2^{i-1} \leq m$. Then $d_m/N \leq 2^{-i^*}$. By definition i^* approaches infinity as m approaches infinity. Then we have clearly:

$$\lim \frac{d_m}{N} = 0, \quad \text{as } m \rightarrow \infty.$$

So we have proved that the class of H -codes does not meet the Gilbert-Varshamov bound (cf. [17, p.557]). A H -code is an $[N, N/2, d_m]$ -code; then the rate k/N , where k is the dimension of the code, equals $1/2$. Let $\{C_m \mid m > 1\}$ be any infinite sequence of H -codes. Actually we have proved that such a sequence cannot satisfy

$$1 - K\left(\frac{d_m}{N}\right) \leq \frac{1}{2} \quad \text{where } K(s) = -s \log_2(s) - (1-s) \log_2(s-1), \quad (9)$$

for all m . Indeed $K(2^{-3}) = 0.54$ and $K(2^{-4}) = 0.33$. According with Table 1, that means that (9) is not satisfied for $m \geq 12$.

□

The last corollary of Theorem 3 will give the characterization of extremal H -codes. For its proof, we must treat separately length 64.

Lemma 3 *Any H -code of length 64 contains a codeword whose support is a 3-dimensional subspace. Hence its minimal distance is less than or equal to 8.*

Proof: Let (x) be an H -code of length 64 - i.e. G is the additive group of \mathbb{F}_{64} . Let $z \in (x)$ such that the support of z is an hyperplane V of G ; there exists such z from Theorem 3. Since $zx = 0$, we can write x as follows (see Lemma 2):

$$x = x' + x''X^g, \quad g \notin V, \quad x' \text{ and } x'' \text{ in } \mathbb{F}_2V \cap \mathcal{P}.$$

According to (4) we have:

$$x' = (X^u + 1) + y' \quad \text{and} \quad x'' = (X^v + 1) + y'',$$

with $u \in V, v \in V, y' \in \mathcal{P}^2, y'' \in \mathcal{P}^2$. If x' (resp. x'') is in \mathcal{P}^2 , then u (resp. v) is zero. Since x has depth 1, u and v cannot be zero together. Set

$$a' = (X^u + 1)(X^v + 1) x' \quad \text{and} \quad a'' = (X^u + 1)(X^v + 1) x''.$$

If $u = 0$ or $u = v$ we take another u in V and replace it in the expression of a' and a'' . So it is clear that a' and a'' are in $\mathbb{F}_2V \cap \mathcal{P}^4$. Since V is here the additive group of \mathbb{F}_{32} , a' and a'' are element of the Reed-Muller code of order 1 and length 32; the support of such element (when it is a non zero element) is an affine subspace of V of dimension 4 or 5. In all cases there exists $w \neq 0$ such that the codeword

$$a = (X^u + 1)(X^v + 1)(X^w + 1)$$

has weight 8 and satisfies $aa' = 0$ and $aa'' = 0$. That yields $ax = 0$, which means $a \in (x)$, completing the proof.

□

Corollary 3 *Let (x) be an extremal H -code.*

(i) *If (x) is an even self-dual code then it is a $[4, 2, 2]$ code.*

(ii) *If (x) is a doubly-even self-dual code, then it is either an $[8, 4, 4]$ code, a $[16, 8, 4]$ code or a $[32, 16, 8]$ code.*

Proof: (i) Assume that (x) is an extremal even self-dual code. In accordance with Theorem 2, its minimal distance d equals $2^{m-2} + 2$. But, from Theorem 3, we know that $d \leq 2^{m-2}$ as soon as $4 \leq m$. When $m = 3$, (x) is a $[8, 4, 4]$ code, which is clearly doubly-even. Indeed its generator matrix is of the form

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{array} \right]$$

(see in Section 2.2). So there is only one possibility : $m = 2$ and (x) is a $[4, 2, 2]$ code.

(ii) We now assume that (x) is an extremal doubly-even self-dual code. Then, from Theorem 2, its minimal distance d equals $4\left\lceil \frac{2^{m-3}}{3} \right\rceil + 4$. From Theorem 3, $d \leq 2^{m-3}$ as soon as $7 \leq m$. So (x) is a code of length 2^m , with $m \leq 6$. We have seen that it is very easy to construct a doubly-even self-dual $[8, 4, 4]$ code. It is also easy to construct a doubly-even self-dual $[16, 8, 4]$ code. Indeed whenever the weight of x is 4, we are sure that (x) is doubly-even; therefore (x) has minimal distance 4. If $m = 6$ then $d = 12$; but, in this case, we know from Lemma 3 that d is less than or equal to 8. There is no extremal $[64, 32, 12]$ H -code. In the following section we will show how to construct a $[32, 16, 8]$ H -code.

□

4 On generators of extremal H -codes of length 32

We first recall the definition of *difference sets* in elementary abelian groups and the theorem of H.B. MANN. An overall review on difference sets can be found in [6] and [16].

Definition 2 *Let $\{G, +\}$ be a finite abelian group and let D be a subset of G . Then D is a difference set of G with parameter λ if and only if it satisfies:*

$$\text{card} (D \cap (h + D)) = \lambda \quad \text{for all } h \text{ in } G \setminus \{0\} .$$

Theorem 4 [16] *Let $\{G, +\}$ be an abelian group of order 2^k and let D be a difference set of G with parameter λ . For any subgroup V of G , we denote by \bar{V} the set $G \setminus V$. If D is equal to $\{0\} \cup \bar{V}$, for any V , we will say that D is a trivial difference set. If D is not trivial, then the parameters of D are:*

$$k = 2t, \quad \text{card } D = 2^{t-1}(2^t + \epsilon) \quad \text{and} \quad \lambda = 2^{t-1}(2^{t-1} + \epsilon),$$

where $\epsilon = \pm 1$.

From now on G is the additive group of the finite field \mathbb{F}_{32} .

Lemma 4 *Let (x) be an H -code of length 32, where $x = 1 + X^h x'$, $h \notin H$, x' invertible in $\mathbb{F}_2 H$. Then (x) is a $[32, 16, 8]$ code if and only if:*

(i) *The weight of x' is 7 or 11.*

(ii) *For all $z = 1 + X^g$, $g \in H \setminus \{0\}$, the weight of the product zx' is greater than 2.*

Proof: Let $[I|M]$ be the generator matrix of (x) , where I is the identity matrix and $M = \{ X^u x' \mid u \in H \}$ (see Section 2). Let y in $\mathbb{F}_2 H$; then the product of the vector y by M is in fact the codeword yx' . Any codeword of (x) is of the form (y, yx') and its weight equals $\omega(y) + \omega(yx')$. By construction $yx' \neq 0$ whenever $y \neq 0$; moreover the depth of yx' is equal to the depth of y . So yx' has depth 1 for any y of weight 2.

If (x) has minimal distance 8 it is clear that (i) and (ii) are satisfied. Note that (ii) is not satisfied when $\omega(x')$ is 15.

Suppose now that (i) and (ii) are satisfied. Condition (i) implies that (x) is doubly-even. Then we must only prove that (x) does not contain a codeword of weight 4. First it is impossible to have $\omega(y) = 3$ and $\omega(yx') = 1$, since we cannot have $\omega(y) = 1$ and $\omega(yx') = 3$. Condition (ii) means that $\omega(y) = 2$ and $\omega(yx') = 2$ cannot appear, completing the proof.

□

Theorem 5 *We denote by H any hyperplane of G . Then an H -code of length 32 is an extremal doubly-even self-dual code (i.e. a $[32, 16, 8]$ code) if and only if it has a generator of the form*

$$x = 1 + X^h(1 + y), \quad h \notin H, \quad y \in \mathbb{F}_2 H, \quad (10)$$

where the weight of $1 + y$ is 7 or 11 and the support of y is a non trivial difference set of H , where H is identified with the additive group of \mathbb{F}_{16} .

Proof: 1) Let (x) be an H -code such that x is of the form (10). Set $D = \text{supp}(y)$ and $z = 1 + X^g$, $g \in H \setminus \{0\}$. In accordance with Definition 2 and Theorem 4, we have

$$\text{card } D = 6 \text{ (resp. 10)} \quad \text{and} \quad \text{card } (D \cap (D + h)) = 2 \text{ (resp. 6)}.$$

Hence

$$\omega(zy) = \omega(y + X^g y) = 2\text{card } D - 2\text{card } (D \cap (D + h)) = 8.$$

So, for all h , the weight of $z + zy$ is greater than 2. From Lemma 4, (x) is extremal.

2) Let C be an extremal H -code. According to (4), there is a generator of C of the form

$$a = (1 + X^h) + a', \quad a' \in \mathcal{P}^2.$$

where clearly $h \notin H$; then $a = (1 + X^h)a_1 + a_2$, where a_1 is invertible in $\mathbb{F}_2 H$ and $a_2 \in \mathcal{P}^2 \cap \mathbb{F}_2 H$. Now we obtain another generator of C :

$$x = X^h a a_1 = X^h((1 + X^h) + a_1 a_2) = 1 + X^h(1 + y), \quad y \in \mathcal{P}^2 \cap \mathbb{F}_2 H.$$

Since C is extremal, the weight of $1 + y$ is 7 or 11. Now we want to prove that the support D of y is a difference set.

It is sufficient to prove that for any $z = 1 + X^g$, $g \in H \setminus \{0\}$, $\omega(zy) = 8$. As $zy \in \mathcal{P}^3 \cap \mathbb{F}_2 H$, $\omega(zy) \in \{0, 8, 16\}$ (the three weights of the Reed-Muller code of order 1 and length 16). We know that $\omega(zy) > 2$. Suppose that $\omega(zy) = 16$. The element $y \in \mathcal{P}^2$ can be identified to a boolean function f on \mathbb{F}_2^4 ($y = \sum_{g \in G} f(g)X^g$) which is quadratic. Remark that zy corresponds to the function $u \rightarrow f(u) + f(u + h)$. If f is a bent function then the weight of zy is always 8 and the weight of y is 6 or 10 (see [17, p. 426-30]). If f is not a bent function, the kernel of its symplectic form is at least of dimension 2. That means that there exists $g' \in H$, $g' \neq g$, such that $\omega((1 + X^{g'})y)$ is either 0 or 16. It is impossible to have 0; moreover

$$(1 + X^g + 1 + X^{g'})y = (X^g + X^{g'})y = 2 \sum_{g \in H} X^g = 0$$

proves that the value 16 is also not available. Thus we have proved that only weight 8 is possible; f is a bent function and D is a difference set.

□

Remark 3: The quadratic bent functions are completely characterized. For instance the general form of these functions is known; from [17, p. 429] we obtain the general form of the generators of the extremal [32, 16, 8] H -codes. Let $\{e_1, \dots, e_5\}$ be a basis of G and let H be the hyperplane generated by the e_i , $i \in [1, 4]$. Then a generator of an extremal H -code can be written as follows :

$$x = 1 + X^{e_5} (1 + (X^{e_1} + 1)(X^{e_2} + 1) + (X^{e_3} + 1)(X^{e_4} + 1))$$

where $\omega(x) = 8$. An extremal H -code with generator of weight 12 is easily obtained. For instance the following x is such generator:

$$x = 1 + X^{e_5} \left(1 + (X^{e_1} + 1)(X^{e_2} + 1) + (X^{e_3} + 1)(X^{e_4} + 1) + X^{e_4} \left(\prod_{i=1}^3 (X^{e_i} + 1) \right) \right).$$

5 Conclusion

In this paper we studied a special class of ideals of the algebra $\mathbb{F}_2[\{\mathbb{F}_2^m, +\}]$. We gave an upper bound on the minimal distance of these codes which allowed us to prove that they are asymptotically bad. We think that our bound can be improved; maybe it would be possible to find an upper bound for any ideal of the algebra which is self-dual. We conjecture that the minimum-weight codewords of the RM-code $[2^m, 2^{m-1}, 2^{(m+1)/2}]$ are important for this problem. For example, we proved in [12] that all affine invariant self-dual codes of length 128 have minimal distance 16; the value 16 is the upper bound on the minimal distance of H -codes and the minimal distance of the RM-code (for the length 128). In all cases there is a codeword of weight 16 whose support is a vector-space of dimension 4.

The complete classification of all binary, self-dual, doubly-even [32, 16] codes is due to CONWAY and PLESS [13]; they proved that there are five non-equivalent extremal codes (another proof was given later in [15]). Two of the five codes are well-known extended cyclic

codes; that is the Reed-Muller code and the quadratic residue code. There are three other extremal codes of length 32, namely the *code F*, the *code G* and the *code U*. We proved that the generator of any extremal $[32, 16]$ *H*-code is completely defined by a quadratic bent function on \mathbb{F}_{16} . These functions correspond to the codewords of weight 6 or 10 of the RM-code $R(2, 4)$. All codewords of weight 6 (resp. 10) are equivalent. So there are at most two non-equivalent *H*-codes; from recent explorations it seems that all *H*-codes are equivalent to the code *G*. A full explanation of these facts in a more general context will be presented in a forthcoming paper.

The complete coset weight distributions of the five extremal $[32, 16]$ codes are given in [7] (see also [3] for the quadratic residue code).

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Unité de recherche INRIA Rocquencourt
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Unité de recherche INRIA Lorraine - Technopôle de Nancy-Brabois - Campus scientifique
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