Asymptotic Forms and Algebraic Differential equations

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Abstract

We analyse the complexity of a simple algorithm for computing asymptotic solutions of an algebraic differential equation. This analysis is based on a computation of the number of possible asymptotic monomials of a certain order, and on the study of the growth of this number as the order of the equation grows.

Comportements asymptotiques et équations différentielles algébriques

Résumé

Nous analysons la complexité d'un algorithme simple de calcul des solutions asymptotiques d'équations différentielles algébriques. L'analyse est basée sur la détermination du nombre de monômes asymptotiques d'ordre donné, et sur l'étude de la croissance de ce nombre lorsque l'ordre de l'équation croît.

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Asymptotic Forms and Algebraic Differential Equations

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We analyse the complexity of a simple algorithm for computing asymptotic solutions of an algebraic differential equation. This analysis is based on a computation of the number of possible asymptotic monomials of a certain order, and on the study of the growth of this number as the order of the equation grows.

Introduction

An algebraic differential equation over a ring of functions, $\mathcal{F}$, is an equation of the form

$$F(y, y', \ldots, y^{(r)}) = 0, \tag{1}$$

where F is a polynomial over $\mathcal{F}$ in $r+1$ variables, with $r \geq 0$.

In this paper, we only consider real solutions of algebraic differential equations that are continuous for sufficiently large values of their argument. In this case, for equations over the field of rational functions, $\mathbb{R}(x)$, Borel proved that if $r = 1$ then the increase of a solution could not be larger than $e_2(x)$ and that if $r = 2$ the increase could not be larger than $e_3(x)$. Here we have used the classical notation $e_k(x)$ to denote the kth iterate of the exponential and similarly $l_k(x)$ for the logarithm; i.e. $e_0(x) = l_0(x) = x$ and for $k \geq 1$, $e_k(x) = \exp(e_{k-1}(x))$ and $l_k(x) = \log(l_{k-1}(x))$.

Some of Borel's results were made more precise in Hardy (1911) where it was shown that when $r = 1$ and F is of degree 1 in y', then the possible modes of increase of a solution are restricted to the following list

$$Ax^d, \quad A\log^d x, \quad Ax^d \log^{d_1} x, \quad e^{Ax^{d_2}(1+o(1))};$$

here A, d, d_1 and d_2 are real constants with $d_2 > 0$.

The asymptotic behaviour of arbitrary solutions of an algebraic differential equation may be almost entirely unrestricted. This is a consequence of Rubel (1981) where a fourth-order differential equation with integer coefficients is exhibited such that, given any continuous function ϕ defined over $\mathbb{R}$, there is a solution of the equation lying arbitrarily close to ϕ in the L^∞ metric on $\mathbb{R}$. In view of this, it is natural to consider only solutions of differential equations that satisfy some additional property.

A key step in Hardy's proof is to show that any solution y of the equation satisfies the following condition $(\mathcal{P})$:

> For any positive integer k, and any rational function H in $k+2$ variables, $H(x, y, y', \ldots, y^{(k)})$ is ultimately monotonic, unless its denominator vanishes identically.

Not all continuous real solutions of algebraic differential equations possess this property, but it is known to hold for functions built over $\mathbb{R}(x)$ by exp, log and field operations, provided sub-expressions are real. This is Hardy's class of L-functions, see Hardy (1910). M. Singer proved that this property also holds for solutions of equations $y' = F(y)$, where F is a rational function in y with coefficients depending on x that satisfy $(\mathcal{P})$. In modern language, functions having this property are said to belong to a *Hardy field*, see Bourbaki (1961), and Rosenlicht (1983). Although most of the articles we refer to use the theory of Hardy fields, we make an effort in this paper to avoid any explicit mention of Hardy fields. However we reduce consideration to functions belonging to them by specifying that from now on, we shall only consider solutions having the property $(\mathcal{P})$, leaving out any oscillating solutions.

When applied to differential equations over $\mathbb{R}(x)$, the main theorem of Shackell (1993) can be seen as a generalisation of the aforementioned result of Hardy to algebraic differential equations of any order. In fact we concern ourselves here with equations over $\mathbb{R}$. This allows some slight simplification of the treatment, although the method can equally well be applied to equations over $\mathbb{R}(x)$ and over a large range of other Hardy fields. Before describing the results of Shackell (1993), it is necessary to define more precisely what kinds of asymptotic behaviour are admissible. A *nested form* for a positive function ϕ which tends to 0 or infinity, is a functional form for ϕ given by a finite sequence, $\{\phi_i\}_{i=0}^{k}$, of nested functions. These functions are defined by

$$\phi_0 = \phi, \qquad \phi_{i-1}(x) = e_{s_i}^{\epsilon_i}(l_{m_i}^{d_i}(x)\phi_i(x)), \quad i = 1, \ldots, k, \tag{2}$$

subject to the following conditions:

(a) For each $i = 1, \ldots, k$, s_i and m_i are non-negative integers, d_i is a positive real number and $\epsilon_i = \pm 1$.

(b) For each i, and any integer N, $\phi_i^N(x) = o(l_{m_i}(x))$ as $x \to \infty$.

(c) ϕ_k tends to a finite non-zero real constant.

(d) $d_k \neq 1$ unless $s_k = 0$ or $m_k = 0$.

Associated with a nested form is the sequence of tuples $\{(\epsilon_i, s_i, m_i, d_i), i = 1, \ldots, k\}$. Since there is a 1:1 correspondence between a nested form and its associated sequence of tuples, it is convenient to refer to either as the nested form.

The corresponding nested form for negative functions tending to 0 or $-\infty$ is $-\{(\epsilon_i, s_i, m_i, d_i), i = 1, \ldots, k\}$, while for functions tending to a non-zero finite limit, A, it is $A \pm \{(\epsilon_i, s_i, m_i, d_i), i = 1, \ldots, k\}$. We define k to be the *the length* of the nested form (2). To write down the nested form of a function ϕ, we rewrite ϕ by expanding the functions ϕ_i. Thus, the function $\exp[x/(\log x - 1)]$, has nested form $\exp[x \cdot \log^{-1} x \cdot (1 + 1/(\log x - 1))]$, corresponding to $\{(1,1,0,1),(-1,0,1,1)\}$ with $\phi_2 = 1 + 1/(\log x - 1)$.

In general, every L-function has a nested form. The nested forms corresponding to the functions in Hardy's theorem are thus $x^d \cdot A$, $\log^d x \cdot A$, $x^d \cdot \log^{d_1} x \cdot A$, $e^{x^{d_2} \cdot \phi_1(x)}$, with $\phi_1(x) = A + o(1)$ as $x \to \infty$.

Building on the works of Rosenlicht, Shackell (1993) proved that, for algebraic differential equations over $\mathbb{R}$ of order r, the asymptotic solutions which tend to 0 or ∞ (and satisfy $\mathcal{P}$ of course), have nested forms $\{(\epsilon_i, s_i, m_i, d_i), i = 1, \ldots, k\}$, that satisfy the conditions

$$\begin{aligned} &m_{i+1} \geqslant m_i + \max(1, s_{i+1}), \quad 1 \leqslant i < k, \\ &\sum_{i=1}^{k} s_i \leqslant r - m_k - \delta, \end{aligned} \tag{3}$$

where $\delta = 1$ unless $d_k = 1$ and $m_k = 0$ (in which case $\delta = 0$). The first condition is valid for all nested forms and the second one "stratifies" the nested forms according to r, which in this case is equal to the order of the equation (1). One can obtain similar conclusions concerning equations over other function fields. For example, for a solution of an equation of order r_0 over $\mathbb{R}(x)$, one should replace r by $r_0 + 1$ in (3).

If the function ϕ_k defined by the above nested form is not equal to a constant, then $\phi_k - \lim \phi_k$ will itself have a nested form, $\{(\delta_i, t_i, n_i, c_i), i = 1, \ldots, j\}$, defining associated functions ψ_i. Similarly, if ψ_j is not constant, $\psi_j - \lim \psi_j$ will have a nested form, and so on (see Appendix). Such a sequence of nested forms is called a *nested expansion* and generalises the classical notion of asymptotic expansion. It is not hard to see that any function having an asymptotic expansion in terms of L-functions has a nested expansion. However the converse is not true, as is exemplified by the function $\exp[x/(\log x - 1)]$.

The method of Shackell (1993) to find the asymptotic solutions of (1) obeying $(\mathcal{P})$ involves the testing of all nested forms satisfying (3), with the d_i's remaining as variables. It is therefore important to obtain some estimate of how many cases need to be considered, and of the growth of this number as higher- and higher-order estimates are computed. This will give us an estimate of the complexity of this method.

The main result of the paper is that the number of possible nested forms satisfying (3) grows exponentially with the r. However, the appropriate value of r might at worst double each time a new term in a nested expansion is required (see the Appendix). It follows that if all the possibilities are to be tested, the cost of computing the successive parts of a nested expansion will grow in a doubly exponential fashion. From the point of view of practical computation this is rather depressing, but it should be borne in mind that the class of differential equations concerned is very general and that even a second-order estimate may be useful in practice. Techniques to reduce the amount of computation and to extend the current results to algebraic differential equations over Hardy fields are the object of ongoing work, see Shackell (1993b).

There is genuine ambiguity as to precisely how the different cases should be counted. For example should $l_1^{d_1}(A + o(1))$ be considered as a distinct form from $x^{d_0} l_1^{d_1}(A + o(1))$ or as a special case obtained by putting $d_0 = 0$? A rather limited experience with second-order equations (see Shackell (1993) pp. 592–595) suggests that it might be appropriate to regard them as separate, and this is the viewpoint we have taken here. However, the method we use applies equally well with different hypotheses, and the growth in the number of cases is still—under reasonable assumptions—exponential.

Results related to Hardy field theory, that prove the existence of nested expansions for elements of a Rosenlicht field are given in an appendix.

The first author would like to take this opportunity to thank Max Rosenlicht both for the original inspiration given by his papers Rosenlicht (1983a)-Rosenlicht (1987) and for his encouragement.

The number of nested forms

Let $N(r)$ be the number of nested forms obeying (3), the d_i's remaining as variables. The first values are $N(1) = 3$, $N(2) = 7$, $N(3) = 19$, corresponding to the nested forms

$$\begin{array}{ll}
r = 1 & x^{d_1}(A + o(1)), \quad e^{\pm x^{d_1}(A+o(1))}, \\
r = 2 & x^{d_1}(A + o(1)), \quad \log^{d_1} x(A + o(1)), \quad x^{d_1} \log^{d_2} x(A + o(1)), \\
 & e^{\pm x^{d_1}(A+o(1))}, \quad e^{\pm e^{x^{d_1}(A+o(1))}}, \\
r = 3 & x^{d_1}(A + o(1)), \quad \log^{d_1} x(A + o(1)), \quad l_2^{d_1}(x)(A + o(1)), \\
 & x^{d_1} \log^{d_2} x(A + o(1)), \quad x^{d_1} l_2^{d_2}(x)(A + o(1)), \quad \log^{d_1} x l_2^{d_2}(x)(A + o(1)),
\end{array}$$

$$x^{d_1}\log^{d_2} x l_2^{d_3}(x)(A+o(1)), \quad e^{\pm x^{d_1}(A+o(1))}, \quad e^{\pm x^{d_1}\log^{d_2} x(A+o(1))},$$
$$e^{\pm \log^{d_1} x(A+o(1))}, \quad e^{\pm e^{x^{d_1}(A+o(1))}}, \quad e_3^{\pm 1}(x^{d_1}(A+o(1))), \quad x^{d_1}e^{\pm l_1^{d_2}(A+o(1))}.$$

Note that in some of the cases, the definition of δ implies that d_k is further constrained to be 1. This happens for instance when $r=0$, and then $\epsilon_1 d_1 \in \{-1,1\}$. Nevertheless, we need to include these among the different nested forms which have to be substituted into the equation in order to check whether or not they can correspond to the asymptotic behaviour of a solution. Note also that condition (b) in the definition of a nested form may also constrain some d_is. For example in the last form given for $r=3$, one must have $0<d_2<1$.

We prove the following theorem.

Theorem 1 *The number of possible positive nested forms of solutions of algebraic differential equations of order r over $\mathbb{R}$ which tend to zero or infinity is*

$$N(r)=\frac{99+109\lambda-34\lambda^2}{158}\lambda^{-r}+\frac{99+109\mu-34\mu^2}{158}\mu^{-r}+\frac{99+109\nu-34\nu^2}{158}\nu^{-r},$$

where λ, μ and ν are the roots of $2x^3-3x^2-2x+1$, and $\lambda\approx 0.355$, $\mu\approx -0.745$, $\nu\approx 1.889$.

Of course it is understood that the number of nested forms is counted in the way indicated above. A numerical version of this result, which gives an idea of the complexity of the algorithm in Shackell (1993), is

$$N(r)=0.845\,e^{1.034r}+O(e^{0.295\,r}), \quad r\to\infty.$$

Theorem 1 is a consequence of the following result.

Theorem 2 *The number of nested forms of length k obeying (3) is the coefficient of $z^k u^r$ in the Taylor expansion at the origin of*

$$C(z,u)=\frac{z(1+u)(1-zu-2zu^2+u^3+zu^3)}{(1-u-u^2+u^3-zu-2zu^2+zu^3)(1-u)}. \tag{4}$$

PROOF. Define $f_i=m_i-m_{i-1}-s_i$ for $i\geqslant 2$. Note that once s_i (and of course m_{i-1}) is fixed, there are as many choices for f_i as for m_i. The conditions (3) now translate into

$$s_i=0 \Longrightarrow f_i\geqslant 1, \quad i\geqslant 2, \tag{5}$$
$$s_i\neq 0 \Longrightarrow f_i\geqslant 0, \quad i\geqslant 2, \tag{6}$$
$$m_1+s_1+\sum_{i=2}^{k}(f_i+2s_i)+\delta\leqslant r. \tag{7}$$

Let $f_{k,r}$ be the number of nested forms of length k obeying conditions (5–7). By definition, the *generating function* of $f_{k,r}$ is

$$\sum_{k,r\geqslant 0} f_{k,r}z^k u^r, \tag{8}$$

and our aim is to show that this is $C(z,u)$.

The proof uses an encoding of the constraints (5–7) in the language of *regular expressions*, a classical tool in computer science and combinatorics (see, e.g. Eilenberg (1974) p. 167).

We consider the finite alphabet $\{z, P, M, f, s, s_1, m_1, \delta\}$. In the encoding of a nested form such as

$$\exp[-e^{x^{\alpha}e^{\log\log^{\beta}x/\log\log\log^{\gamma}x}}], \tag{9}$$

or equivalently $\{(-1,2,0,\alpha),(1,1,2,\beta),(-1,0,3,\gamma)\}$, we shall use the letter z each time a new quadruple begins. Then if $s_i \neq 0$, the letters P for 'plus' and M for 'minus' will be used to encode the value of $\epsilon_i = \pm 1$. If $s_i = k$, then we concatenate k times the letter s. Similarly, $f_i = k$ will be coded by the k-fold concatenation of f. Because of the special rôle played by $i = 1$ in condition (7), we use separate letters s_1 and m_1 to encode increments of s_1 and m_1. Finally we shall use the letter δ every time we have to account for $\delta = 1$ instead of 0 in (7). With these conventions to each nested form (where the d_i are unspecified) we associate a unique word. For instance, the nested form (9) is associated to the word

$$zMs_1s_1zPsfzMf\delta. \tag{10}$$

We now proceed in two steps: First we encode (5-6) as constraints on the words of our language using the language of *regular expressions*. Then we compute the generating function of this language, whose coefficients are $f_{k,r}$, the number of words with k 'z' satisfying these constraints.

The case when $k = 1$ needs special treatment. If $m_1 = 0$ then s_1 is arbitrary, $\delta = 1$ unless $d_1 = 1$ and the possible words are described by $z[\epsilon|(P|M)s_1^+](\epsilon|\delta)$. We have made use of the classical notations for regular expressions, where ϵ denotes the empty word, $|$ denotes an alternative, and $^+$ denotes repetition an arbitrary, but positive, number of times. To simplify the expressions, we shall set $s_1^{(\star)} := \epsilon|(P|M)s_1^+$. Thus for $k = 1$ and $m_1 = 0$ we have the description

$$z[\epsilon|(P|M)s_1^+](\epsilon|\delta) = zs_1^{(\star)}(\epsilon|\delta)$$

The case when $s_1 = 0$ is distinguished from the others because here we can take $\epsilon_1 = 1$ and regard any negative sign as being subsumed into the value of d_1. If $k = 1$ and $m_1 \neq 0$ then $\delta = 1$, and so this case is described by $zs_1^{(\star)}m_1^+\delta$. Thus for $k = 1$ we have the description

$$[zs_1^{(\star)}(\epsilon|\delta)] \mid [zs_1^{(\star)}m_1^+\delta] = zs_1^{(\star)}[\epsilon|(m_1^\star\delta)].$$

We now move to the case when $k > 1$. Then the possible first steps are encoded by $zs_1^{(\star)}m_1^\star$, the notation * corresponding to unrestricted repetition. The following steps are encoded by $z\{f^+|(P|M)s^+f^\star\}$, which translates conditions (5-6). We now have a complete description as a regular expression of the possible choices of s_i and m_i,

$$zs_1^{(\star)}\left(\epsilon|m_1^\star\left(z\{f^+|(P|M)s^+f^\star\}\right)^\star\delta\right). \tag{11}$$

It is not difficult to see that there is a one-to-one correspondence between the words belonging to this language and nested forms.

We now compute the generating function $\sum f_{k,r}z^ku^r$ where $f_{k,r}$ is the number of words of the language (11) such that (7) holds. This is done by first computing the generating function of $g_{k,r}$ the number of words such that (7) holds with an equality. For instance, the word (10) will be enumerated by z^3u^7 because the length of the nested form is 3 and, by (7) the least order of an equation having a solution of asymptotic growth (9) is 7. The generating function we are seeking is the sum of the contributions of all the words of the language (11). This is obtained by the general method (see e.g., Eilenberg (1974) pp. 196–198), which consists in using the fact that when two sub-languages $\mathcal{A}$ and $\mathcal{B}$ have generating functions $A(z,u)$ and $B(z,u)$, then the alternative $\mathcal{A}|\mathcal{B}$ has for generating function $A(z,u)+B(z,u)$ and the concatenation $\mathcal{AB}$ has for generating function $A(z,u)B(z,u)$. The work is therefore reduced to computing the generating functions of the letters of the language. Because they all increase r by 1 in (7), s_1, f, m_1 and δ have u for generating function, while u^2 is that of s. Obviously the generating function of z is z and those of P and M are 1. Combining the rules on alternative and concatenation yields rules for repetition:

$$\begin{aligned}
\mathcal{A}^\star &\mapsto \frac{1}{1-A(z,u)}\\
\mathcal{A}^+ &\mapsto \frac{1}{1-A(z,u)}-1=\frac{A(z,u)}{1-A(z,u)}\\
\mathcal{A}^{(\star)} &\mapsto 1+\frac{2A(z,u)}{1-A(z,u)}
\end{aligned}$$

Putting this together we get the following translation of (11):

$$z\left(1+2\frac{u}{1-u}\right)\left(1+\frac{u}{(1-u)\left(1-z\left(\frac{u}{1-u}+2\frac{u^2}{(1-u^2)(1-u)}\right)\right)}\right). \tag{12}$$

To account for the $\leqslant$ sign in (7) we have to sum all these contributions except that the cases when $m_k=0$ and $d_k=1$ should only be counted when $s_1=r$. This case is first set aside by suppressing the ϵ in (11). The corresponding generating function is reduced to

$$z\left(1+2\frac{u}{1-u}\right)\frac{u}{(1-u)\left(1-z\left(\frac{u}{1-u}+2\frac{u^2}{(1-u^2)(1-u)}\right)\right)}.$$

Then summation of the contributions over r is obtained by dividing the new generating function by $1-u$ and finally the extra case is recovered by adding back the generating function of $zs_1^{(\star)}$, namely $z[1+2u/(1-u)]$. After simplification we get the function $C(z,u)$ of the theorem. ∎

The variable z was not strictly necessary for our proofs, but it allows for a simple check by computing some of the coefficients of $C(z,u)$. Thus[1]

$$\begin{aligned}
[z^1]C(z,u) &= 1+3u+6u^2+11u^3+18u^4+O(u^5),\\
[z^2]C(z,u) &= u^2+7u^3+24u^4+O(u^5),\\
[z^3]C(z,u) &= u^3+10u^4+O(u^5).
\end{aligned}$$

Summing these first coefficients we get the first values of $N(r)$ above, with more precision since we can distinguish the nested forms by their length (the number of free variables d_i).

PROOF. [of Theorem 1] Setting $z=1$ in $C(z,u)$ corresponds to summing over all possible values of k and thus we get the generating function of $N(r)$:

$$C(u)=\sum_{r\geqslant 0}N(r)u^r=\frac{(1+u)(1-2u^2)}{1-2u-3u^2+2u^3}. \tag{13}$$

[1] We use the classical notation $[z^k]f(z)$ to denote the kth Taylor coefficient of $f(z)$ at the origin.

Since the denominator is irreducible, the partial fraction decomposition of C is obtained by reducing $(1+u)(1-2u^2)/(6u^2-6u-2)$ modulo $1-2u-3u^2+2u^3$, and this gives

$$C(u) = \sum_{1-2\alpha-3\alpha^2+2\alpha^3=0} \frac{99+109\alpha-34\alpha^2}{158(1-\frac{u}{\alpha})}. \tag{14}$$

Let λ, μ and ν be the three roots of $2x^3-3x^2-2x+1$, then extracting the coefficients from (14) we get Theorem 1. ∎

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Appendix

In Shackell (1993) a *Rosenlicht field* was defined to be a Hardy field of finite rank which contains the real constants and also arbitrary real powers of positive elements. It follows from Rosenlicht (1984) that any Hardy field of finite transcendence degree over its field of constants is contained in a Rosenlicht field of rank no greater than that transcendence degree. An important property of Rosenlicht fields is that one can obtain fairly precise information about their comparability classes. Thus in Shackell (1993) it was shown that any function belonging to a Rosenlicht field of rank r has a nested form obeying (3).

We complete this article by showing that, at least in principle, the techniques of Shackell (1993) can be used to compute the successive nested forms in the nested expansion of a Hardy-field solution of a differential equation. First we require the following lemma.

Lemma 1 *Let f be a non-zero element of a Rosenlicht field $\mathcal{F}$ of rank r. Then* $\exp(f)$ *and* $\log|f|$ *each belong to a Rosenlicht field of rank at most $r+1$.*

Proof. It follows from Theorem 2 of Rosenlicht (1983a) that $\mathcal{F}(\exp(f))$ and $\mathcal{F}(\log|f|)$ are each Hardy fields. By Corollary 2 of Rosenlicht (1984) and Proposition 5 of Rosenlicht (1983b), these may be enlarged to Rosenlicht fields whose rank is no greater than r plus the transcendence degree of $\mathcal{F}(\exp(f))$ (or of $\mathcal{F}(\log|f|)$ respectively) over $\mathcal{F}$. The lemma now follows. ∎

If $\mathcal{F}$ and $\mathcal{G}$ are two Hardy fields with $\mathcal{F} \subset \mathcal{G}$ and $\phi \in \mathcal{G}$, we write $\mathcal{F}\langle\phi\rangle$ for the differential field generated by $\mathcal{F}$ and ϕ.

Proposition 1 *Suppose f belongs to a Rosenlicht field of rank r and has a nested form defined by $\{(\epsilon_i, s_i, m_i, d_i),\ i = 1, \ldots, k\}$. Then $\phi = \phi_k - \lim \phi_k$ belongs to a Rosenlicht field of rank at most*

$$\sum_{i=1}^{k} s_i \ + \ rank(\mathcal{F}\langle l_{m_k}\rangle)$$

and this is no greater than $m_k + 1 + \sum s_i + r$. In the case when $m_k = 0$ and $d_k = 1$, ϕ belongs to a Rosenlicht field of rank at most $m_k + \sum s_i + r$.

Proof. The proof follows from reverting the sequence of ϕ_i's of the nested form, so that ϕ is expressed as an L-function of f. Careful consideration of what happens to the logarithms and exponentials during this process yields the estimate on the rank. ∎

Corollary 1 *If f belongs to a Rosenlicht field then f has a nested expansion.*

This is now a consequence of the general theorem of Shackell (1993) showing that every element of a Rosenlicht field has a nested form, applied to ϕ and subsequent functions.

Corollary 2 *The number of cases to be considered when computing the J-th term of the nested expansion of f is at most $N(2^{J-1}r)$.*

Proof. From Proposition 1, the rank of $\mathbb{R}\langle\phi\rangle$ is at most $2r$. Corollary 2 then follows by induction. ∎