An Additive Standpoint in Parallel Two-Level Multigrid Algorithms

Bruno Koobus, Marie-Hélène Lallemand

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Abstract: The purpose of this work is to study a parallel two-grid algorithm with residual splitting for the solution of partial differential equations.
In standard Multigrid (MG) schemes, if the correction on the two levels is computed in parallel, the low frequency components of the error are corrected twice, which may generate inconsistency and thus loss in efficiency. In the parallel MG method proposed by Chan-Tuminaro [1, 2], a residual splitting is then used to avoid this problem. This is the initial point of our study.
In this report, we present an abstract analysis of a parallel MG solver based on residual splitting. In particular, we give a convergence proof based on a smoothing property and an approximation property like assumptions. In this approach, a filtering of the correction is done on the fine level.
We also show examples of application of such an algorithm. It turns out that a method without correction filtering seems to be more efficient.

Key-words: Additive Multigrid Methods, Residual Splitting, Correction Filtering, Diffusion Equation, Advection Equation, Numerical Analysis

(Résumé : tsvp)

*INRIA Sophia Antipolis,B.P. 93, 06902 Sophia-Antipolis Cedex,France
**INRIA Rocquencourt, B.P. 105, 78153 Le Chesnay Cedex, France

Unité de recherche INRIA Sophia-Antipolis
2004 route des Lucioles, BP 93, 06902 SOPHIA-ANTIPOLIS Cedex (France)
Téléphone : (33) 93 65 77 77 – Télécopie : (33) 93 65 77 65

Un Point de Vue Additif sur des Algorithmes Bigrille Parallèles

Résumé : L'objet de ce travail est l'étude d'un algorithme bigrille avec décomposition du résidu pour la solution d'équations aux dérivées partielles. Pour les schémas Multigrille (MG) classiques, si la correction sur les deux niveaux est calculée en parallèle, les composantes basses fréquences de l'erreur sont corrigées deux fois. Ceci peut générer une inconsistance et une perte d'efficacité. Dans la méthode MG parallèle proposée par Chan-Tuminaro [1, 2], une décomposition du résidu est utilisée pour éviter ce problème. C'est le point initial de notre étude.

Dans ce rapport, nous présentons une analyse abstraite d'un solveur MG parallèle fondée sur une décomposition du résidu. En particulier, on donne une preuve de convergence fondée sur une propriété de lissage et une propriété d'approximation comme hypothèses. Dans cette approche, un filtrage de la correction est effectué sur le niveau fin.

On présentera des exemples d'application d'un tel algorithme. De par ces expériences numériques, il s'est avéré qu'une méthode sans filtrage de la correction semble être plus efficace.

Mots-clé : Méthodes Multigrille Additives, Décomposition du Résidu, Filtrage de la Correction, Equation de Diffusion, Equation d'Advection, Analyse Numérique

Contents

1 Introduction

Multi-Grid (MG) solvers are known to be very efficient for the solution of partial differential equations (even in the non elliptic case) and, in particular, allow a solution in $O(N)$ operations ("Full Multigrid", [3]), where N is the number of vertices of the mesh.
The interest of MG algorithms is to treat the smooth and the oscillatory part of the error in an uncoupled way. For a 2-level MG, the high frequency components of the error are treated on the fine grid, while the low frequency part of the error is approximated on the coarse grid [4]. In standard MG, these two problems are solved sequentially.
A standard way of parallelizing MG algorithms [5, 6, 7] consists in the parallelization of its classic version within each level. However, most processors are idle [7] when dealing with the coarse grid problems since they contain fewer points. In [8], Frederickson and McBryan propose a "parallel superconvergent MG algorithm" (PSMG) where all points are retained on all grid levels, keeping therefore all processors busy, but increasing the computational complexity. This is somehow contradictory in the multi-level philosophy and it is not much more efficient than a parallel implementation of standard MG methods, as it is claimed by Decker in [9].
In this paper, we choose an approach close to the algorithm of Chan and Tuminaro [1, 2] : the residual equations are solved in parallel by approximating on the different levels the smooth and the oscillatory parts of the error. The resulting algorithm belongs to the class of so called Parallel Subspace Correction (PSC) methods, as defined by Xu in [14], while standard MG algorithms belong to the class of Successive Subspace Correction (SSC) methods, therefore, we do not use the *multiplicative* theory [3] in our convergence result. As a matter of fact, the standard two-grid iteration error during one cycle (with no post-smoothing) is sequencially reduced by the pre-smoothing operator and the coarse grid correction operator, therefore, the resulting two-grid iterator can be expressed as the multiplication of these two operators. In the PSC approach, the 2-grid iterator is splitted as the sum of a smoothing operator (related to the fine grid residual equation), and a coarse grid correction projection operator (related to the coarse grid residual equation), therefore, an *additive*approach has to be used here. This can be roughly expressed by the

following :
Let E be a Hilbert space and let E_1 and E_2 be two orthogonal subspaces allowing the direct sum decomposition $E = E_1 \oplus E_2$. Then the original problem

$$A\,x = b \text{ in } E \ , \tag{1}$$

is decomposed into two subproblems

$$\begin{gathered} A\,x_1 = b_1 \ , \\ A\,x_2 = b_2 \ , \\ \text{with } b = b_1 + b_2 \ , \end{gathered}$$

which are obtained by projecting the initial problem 1 on E_1 and E_2. So, finding x in E such than $A\,x = b$ is equivalent to finding x_1 and x_2 such that $A\,x_1 = b_1$ and $A\,x_2 = b_2$, then x is given by $x = x_1 + x_2$. In our two-level algorithm, $A\,x_1 = b_1$ is solved on the fine grid and $A\,x_2 = b_2$ on the coarse one. Then the two-grid iteration error is parallely reduced on both fine and coarse levels.

This is the main idea of Chan and Tuminaro [1], who define a parallel MG algorithm by solving, in parallel, complementary residual equations on the different grids. This avoids the idle processor problem which occurs on the coarse grids when parallelization is done within each level. Therefore, for a successful application of such a solver, it is essential to define a filtering of the fine grid residual which separates its low frequency components from its high frequency components. In that way, we can compute in parallel the "high-frequency residual equation" on the fine grid (that is the fine grid correction equation, where the right hand side (RHS) is the oscillatory part of the residual) and the "low-frequency residual equation" on the coarse grid (that is the coarse grid correction equation, where the RHS is the restriction of the smooth part of the fine grid residual).
In this paper, we first precise the framework of our parallel MG theory. In Section 3, we present an algorithm of a parallel two-grid method with residual splitting and correction filtering. We give, in Section 4, the main assumptions of our convergence theory. Section 5 is devoted to a mesh-independent convergence proof of the parallel MG algorithm relying on a residual splitting and a correction filtering. The convergence theory is based on a smoothing property

and an approximation property. Following Guillard's idea [15], in this analysis a splitting of the original space into a direct sum of orthogonal subspaces is used. At last, Section 6 illustrates our study by giving some numerical results for 1D diffusion and advection equations. We find that a scheme without correction filtering is more efficient.

2 Problem framework

Let Ω be an open bounded domain in $\mathbb{R}^2$ with regular boundary Γ. Let $\mathcal{V}$ be the vector space of functions u defined on Ω and with values in $\mathbb{R}$. We assume $\mathcal{V}$ to be a Hilbert space, and denote its inner scalar product by $< \cdot , \cdot >$, and the associated norm by $\| \cdot \|$. Let A be a linear operator defined on $\mathcal{V}$, with values in $\mathcal{V}$, that we assume to be continuous, self-adjoint and positive definite in the following meaning:

$$< A\,u\,,\,u >\geq 0\ ,\ \forall u \in \mathcal{V}\ ,$$
$$< A\,u\,,\,u >= 0 \Rightarrow u = 0_{\mathcal{V}}\ .$$

Let $< \cdot , \cdot >_A$ be the new scalar product defined by

$$\forall u \in \mathcal{V}\ ,\ < u\,,\,u >_A =< A\,u\,,\,u >\ .$$

Let f be a given function in $\mathcal{V}$, then the problem:

$$\left\{ \begin{array}{l} \text{Find } u \text{ in } \mathcal{V} \text{ such that} \\ \qquad A\,u = f\ , \end{array} \right.$$

has a unique solution $u^* = A^{-1} f$.

Let Ω_h be a discretization of Ω, and let $\mathcal{V}_h$ be a vector subspace dense in $\mathcal{V}$. We denote by A_h the discrete operator defined on $\mathcal{V}_h$ as the restriction of A to $\mathcal{V}_h$ ($A_h\,u_h = A\,u_h\ ,\ \forall u_h \in \mathcal{V}_h$). The initial problem, restricted to $\mathcal{V}_h$, can then be written by

$$\left\{ \begin{array}{l} \text{Find } u_h \text{ in } \mathcal{V}_h\ ,\ \text{such that :} \\ \qquad A_h\,u_h = f_h\ . \end{array} \right. \tag{2}$$

Let $\mathcal{V}_H$ be a non-null vector space of $\mathcal{V}_h$ and let I_h^H (resp. I_H^h) be the restriction (resp. prolongation) transfer operator mapping from $\mathcal{V}_h$ (resp. $\mathcal{V}_H$) to $\mathcal{V}_H$ (resp. $\mathcal{V}_h$). We define the operator A_H on $\mathcal{V}_H$ by

$$A_H \stackrel{\text{def}}{=} I_h^H \, A_h \, I_H^h \ ,$$

then, a sufficient condition under which A_H keeps the properties of A_h is to define :

$$I_h^H = c \, (I_H^h)^* \ ,$$

where c a suitable scaling factor [10] and to impose that I_h^H to have full rank (i.e. $R(I_H^h) = \mathcal{V}_H$). But this property (setting the restriction operator to be the adjoint of the prolongation operator) is not necessary to insure the MG convergence [10].
Let U_h be another non-null vector subspace of $\mathcal{V}_h$, and let us consider the orthogonal projection operator T_h, defined on $\mathcal{V}_h$, with values in U_h, then we have the following direct sum decomposition :

$$\mathcal{V}_h = U_h \oplus U_h^{\perp} \ ,$$

with

$$U_h = R(T_h) \ , \ U_h^{\perp} = R(S_h) \ ,$$

where $S_h \stackrel{\text{def}}{=} I_h - T_h$ is the complementary orthogonal projection operator mapping from $\mathcal{V}_h$ to the kernel of T_h, and where I_h is the identity application in $\mathcal{V}_h$.

Suppose now that we have an iterative process with iteration matrix G_h, which allows the approximation of the exact solution $u_h^* = A_h^{-1} f_h$. Then the iterative process associated to G_h can be written by :

$$\left[\begin{array}{l} u^{(0)} \text{ given in } \ \mathcal{V}_h \ , \\ u^{(l)} = G_h \, u^{(l-1)} + b_h \ , \text{ for } \ l = 1, 2, \dots \ , \end{array} \right] \tag{3}$$

where, because of consistency, b_h is necessarily defined by :

$$b_h = (I_h - G_h) \, A_h^{-1} \, f_h \ .$$

Let $\overline{T}_h$ (resp. $\overline{S}_h$) be the projection operator defined by :

$$\overline{T}_h \stackrel{\text{def}}{=} A_h^{-1} T_h A_h \ ,$$

(resp.

$$\overline{S}_h \stackrel{\text{def}}{=} A_h^{-1} S_h A_h = I_h - \overline{T}_h \ ) .$$

3 Parallel two-grid algorithm

Let $u_h^0 \in \mathcal{V}_h$ be the initial guess and $\varepsilon > 0$ be the tolerance for convergence stop, then we propose to solve problem (2) (i.e. to find an approximated solution $\overline{u}_h$ of u_h^*), by the following iterative algorithm :

1. Initialization $n \leftarrow 0$;
2. Residual computation and convergence test

   $$r_h^n \leftarrow f_h - A_h u_h^n \ ,$$

   if $\|r_h^n\| \leq \varepsilon$, stop : $\overline{u}_h \leftarrow u_h^n$;
3. Parallel solution of the two following sub-steps

   (a) Approximate solution in $\mathcal{V}_h$

   $$A_h e_h = T_h r_h^n \ , \tag{4}$$

   in one iteration of algorithm (3), with $e^{(0)} = 0$ in $\mathcal{V}_h$ and $b_h = (I_h - G_h) A_h^{-1} T_h r_h^n$, and set :

   $$e_h^{n+1,1} \leftarrow \overline{T}_h e^{(1)} \ ;$$

   (b) Exact solution in $\mathcal{V}_H$

   $$A_H e_H = I_h^H S_h r_h^n \ , \tag{5}$$

   and denote its exact solution by $e_H^* = A_H^{-1} I_h^H S_h r_h^n$, then set :

   $$e_h^{n+1,2} \leftarrow \overline{S}_h I_H^h e_H^* \ ;$$

4. Solution updating, next iteration
   First set
   $$e_h^{n+1} \leftarrow e_h^{n+1,1} + e_h^{n+1,2} \ ,$$
   then define the new approximated solution of (2) by :
   $$u_h^{n+1} \leftarrow u_h^n + e_h^{n+1} \ ,$$
   and go to the next iteration : $n \leftarrow n+1$, go to 2.

4 Main assumptions

We may notice that the algorithm defined above is close to the classical two-grid Correction Scheme algorithm, except that, on $\mathcal{V}_h$, we do not compute an approximate solution of the initial problem, but only an approximation of its component in $R(T_h)$; likewise, on $\mathcal{V}_H$, e_H^* stands for the exact error in $\mathcal{V}_H$, of the component of $e_h^{*,n} (\stackrel{\text{def}}{=} A_h^{-1} r_h^n)$ in $R(S_h)$, restricted to $\mathcal{V}_H$. Here, the orthogonal projection operators (T_h et S_h) have to be seen as operators allowing the decomposition of the full residual in a high frequency component (in $R(T_h)$), and a low frequency component (in $R(S_h)$). To insure the efficiency of this parallel two-grid process, the iterative algorithm (3) must effectively reduce the component of the full error in $R(T_h)$ and the residual equation to be solved in $\mathcal{V}_H$ must be a good approximation of the residual equation associated with the component of the full error in $R(S_h)$. In other words, we may assume that the following smoothing property, related to the correction equation in $\mathcal{V}_h$, is satisfied :

$$\exists \alpha_1 \in]0,1[\ , \ \forall e_h \in \mathcal{V}_h, \quad \|\overline{T}_h \, G_h \, \overline{T}_h \, e_h\| \leq \alpha_1 \, \|\overline{T}_h \, e_h\| \ , \tag{6}$$

where α_1 does not depend on h. The projection in $R(\overline{S}_h)$ of the error defined as the difference between the original exact error in $\mathcal{V}_h$ and the exact solution in $\mathcal{V}_H$ of problem (5) is assumed to satisfy the following approximation property :

$$\exists \alpha_2 \in]0,1[\ , \ \forall e_h \in \mathcal{V}_h \ , \quad \|\overline{S}_h \left(I_h - I_H^h \, A_H^{-1} \, I_h^H \, A_h \right) \, \overline{S}_h \, e_h\| \leq \alpha_2 \, \|\overline{S}_h \, e_h\| \ , \tag{7}$$

where α_2 is independent of h. In the sequel, we shall show that these two properties are sufficient to make sure that the parallel 2-grid algorithm is convergent with a convergence rate bounded above by a constant $\alpha < 1$, independent of h.

5 Abstract convergence result

Let $e_h^{*,n}$ be the exact error in $\mathcal{V}_h$ defined by :

$$e_h^{*,n} \stackrel{\text{def}}{=} u_h^* - u_h^n = A_h^{-1} r_h^n \ .$$

Then we can state the following proposition :

Proposition 5.1 *Under assumptions (6–7), the parallel 2-grid algorithm described in Section 3 is convergent and we have :*

$$\forall n \geq 0 \ , \ \|e_h^{*,n+1}\| \leq \alpha \, \|e_h^{*,n}\| \ , \tag{8}$$

with $\alpha = \max(\alpha_1, \alpha_2)$ *less than unity and independent of* h,
and where α_1 *and* α_2 *are defined in the Inequalities (6–7).*

Proof :

Proving that the parallel 2-grid process is convergent and the limit is the exact solution of the problem in $\mathcal{V}_h$, is equivalent to proving that :

$$\lim_{h \to \infty} e_h^{*,n} = 0 \text{ in } \mathcal{V}_h \ .$$

If we notice that :

$$\forall n \geq 0 \ , \ e_h^{*,n+1} = e_h^{*,n} - e_h^{n+1} \ ,$$

then we have :

$$\forall n \geq 0 \ , \ e_h^{*,n+1} = e_h^{*,0} - \sum_{k=1,n+1} e_h^k \ ,$$

and therefore, proving that $\lim_{n \to \infty} e_h^{*,n} = 0$ in $\mathcal{V}_h$, is equivalent to proving that the series of general term e_h^n converges to $e_h^{*,0}$, i.e. :

$$\lim_{N \to \infty} \sum_{n=1,N} e_h^n = e_h^{*,0} \ . \tag{9}$$

First let us write down the two following lemmas :

Lemma 5.2 *The iteration matrix M_h corresponding to the parallel 2-grid operator and defining the iterate e_h^{n+1} from e_h^n for $n \geq 1$ can be written by:*

$$M_h = [I_h - (D_h + E_h)\ A_h] \ , \tag{10}$$

where:

$$\left\{ \begin{array}{ll} D_h & = \overline{T}_h\ (I_h - G_h)\ A_h^{-1}\, T_h \ , \\ E_h & = \overline{S}_h\, I_H^h\, A_H^{-1}\, I_h^H\, S_h \ ; \end{array} \right. \tag{11}$$

Proof :
Let $n \geq 1$ be fixed. By definition of e_h^{n+1}, we have:

$$\begin{aligned} e_h^{n+1,1} &= \overline{T}_h\, e_h^{n+1} = D_h\, r_h^n \ , \\ e_h^{n+1,2} &= \overline{S}_h\, e_h^{n+1} = E_h\, r_h^n \ , \end{aligned}$$

on the other hand:

$$r_h^n = f_h - A_h\, u_h^n = r_h^{n-1} - A_h\, e_h^n \ ,$$

whence:

$$\begin{aligned} e_h^{n+1,1} &= D_h\, r_h^{n-1} - D_h\, A_h\, e_h^n \\ &= \left(\overline{T}_h - D_h\, A_h\right)\ e_h^n \ , \\ e_h^{n+1,2} &= E_h\, r_h^{n-1} - E_h\, A_h\, e_h^n \\ &= \left(\overline{S}_h - E_h\, A_h\right)\ e_h^n \ . \end{aligned}$$

Therefore:

$$\begin{aligned} e_h^{n+1} &= e_h^{n+1,1} + e_h^{n+1,2} \\ &= (I_h - (D_h + E_h)\, A_h)\ e_h^n \ . \end{aligned}$$

□

Notice that, for $n = 0$, the first iterate can be written by:

$$e_h^1 = (D_h + E_h)\ A_h\, e_h^{*,0} \ ,$$

and therefore:

$$e_h^{n+1} = [I_h - (D_h + E_h)\ A_h]^n\ (D_h + E_h)\ A_h\, e_h^{*,0} \ .$$

Lemma 5.3 *The decomposition of M_h on $R(\overline{T}_h) \oplus R(\overline{S}_h)$ is given by:*

$$\begin{aligned} \overline{T}_h\, M_h &= \overline{T}_h\, G_h\, \overline{T}_h \ , \\ \overline{S}_h\, M_h &= \overline{S}_h\, \left(I_h - I_H^h\, A_H^{-1}\, I_h^H\, A_h\right)\, \overline{S}_h \ . \end{aligned}$$

Proof :

$$\begin{aligned} \overline{T}_h\, M_h &= \overline{T}_h\, \left[I_h - (D_h + E_h)\, A_h\right] \\ &= \overline{T}_h\, \left[I_h - D_h\, A_h\right] \\ &= \overline{T}_h\, \left[I_h - \overline{T}_h\, (I_h - G_h)\, A_h^{-1}\, T_h\, A_h\right] \\ &= \overline{T}_h\, G_h\, \overline{T}_h \end{aligned}$$

and

$$\begin{aligned} \overline{S}_h\, M_h &= \overline{S}_h\, \left[I_h - (D_h + E_h)\, A_h\right] \\ &= \overline{S}_h\, \left[I_h - E_h\, A_h\right] \\ &= \overline{S}_h\, \left[I_h - \overline{S}_h\, I_H^h\, A_H^{-1}\, I_h^H\, S_h\, A_h\right] \\ &= \overline{S}_h\, \left(I_h - I_H^h\, A_H^{-1}\, I_h^H\, A_h\right)\, \overline{S}_h \end{aligned}$$

□

The first part of the demonstration of (9) is a direct consequence of assumptions (6-7) and lemma (5.2-5.3).
Indeed, we can write that :

$$\begin{aligned} \forall n \geq 1,\ \|e_h^{n+1}\|^2 &= \|M_h\, e_h^n\|^2 \\ &= \|\overline{T}_h\, M_h\, e_h^n\|^2 + \|\overline{S}_h\, M_h\, e_h^n\|^2 \\ &= \|\overline{T}_h\, G_h\, \overline{T}_h\, e_h^n\|^2 + \|\overline{S}_h\, \left(I_h - I_H^h\, A_H^{-1}\, I_h^H\, A_h\right)\, \overline{S}_h\, e_h^n\|^2 \\ &\leq \alpha_1^2\, \|\overline{T}_h\, e_h^n\|^2 + \alpha_2^2\, \|\overline{S}_h\, e_h^n\|^2 \\ &\leq \max(\alpha_1^2, \alpha_2^2)\, \left(\|\overline{T}_h\, e_h^n\|^2 + \|\overline{S}_h\, e_h^n\|^2\right) \\ &= \max(\alpha_1^2, \alpha_2^2)\, \|e_h^n\|^2 \end{aligned}$$

We can deduce that :

$$\|M_h\| \leq \alpha < 1 \ , \tag{12}$$

with $\alpha = \max(\alpha_1, \alpha_2)$ independent of h .

A direct consequence of (12), is that the operator $(D_h + E_h)\, A_h$ is invertible and its inverse can be written by :

$$[(D_h + E_h)\ A_h]^{-1} = \lim_{n\to\infty} \sum_{k=0,n} \left(I_h - (D_h + E_h)\ A_h\right)^k \ ;$$

but, since :

$$\sum_{k=0,n} e_h^{k+1} = \left(\sum_{k=0,n} M_h^k\right)\ [D_h + E_h]\ A_h\, e_h^{*,0} \ ,$$

we can deduce that :

$$\begin{aligned} \lim_{n\to\infty} \sum_{k=0,n} e_h^{k+1} &= [(D_h + E_h)\ A_h]^{-1}\ (D_h + E_h)\ A_h\, e_h^{*,0} \\ &= e_h^{*,0} \ . \end{aligned}$$

Now we shall prove that the convergence rate of the exact error $e_h^{*,n}$ is bounded above by $\alpha = \max(\alpha_1, \alpha_2)$:
for $n \geq 0$, we have :

$$\begin{aligned} e_h^{*,n+1} &= e_h^{*,0} - \sum_{k=1,n+1} e_h^k \\ &= \sum_{k=n+2,\infty} e_h^k \\ &= \sum_{k=n+2,\infty} M_h\, e_h^{k-1} \\ &= \sum_{k=n+1,\infty} M_h\, e_h^k \\ &= M_h\, e_h^{*,n} \end{aligned}$$

For getting the last equality, the sum and the operator M_h have been permuted since this operator is continuous linear and the series of general term e_h^n is normally convergent.

Therefore :

$$\begin{aligned} \forall n \geq 0 \quad \|e_h^{*,n+1}\| &\leq \|M_h\|\, \|e_h^{*,n}\| \\ &\leq \alpha\, \|e_h^{*,n}\| \end{aligned}$$

which ends the demonstration of the proposition. □

6 Numerical experiments

For each experiment, the problem is considered to be solved if the tolerance $\epsilon = 10^{-10}$ is reached by the normalized iterative residual denoted by $Res(\alpha)$, which is defined at the iteration α by :

$$Res(\alpha) = \frac{\|f - A\,u^{\alpha+1}\|_{l^2}}{\|f - A\,u^{0}\|_{l^2}} \ ,$$

where u^{α} denotes the α^{th} multigrid solution iterate which approximates the exact solution of $A\,u = f$.
For both advective and diffusive problems, the multigrid framework is the volume agglomeration multigrid approach defined as in [11], i.e. the coarse grid equations are discretized on coarser meshes, where the coarse grid volumes are obtained by grouping the finite volumes of the fine grid.
For the restriction operator, the value of the restricted residual on a coarse cell is defined as the sum of the values of this residual corresponding to the fine cells contained in this coarse cell. The prolongation operator is defined via a weighting coefficient β as the composition of the natural injection operator and an averaging operator. Consequently, this prolongation operator smoothens the resulting interpolated error on the fine grid. The value of the weighting coefficient β is set to obtain optimal results.

6.1 1-D diffusion problem

We consider the following periodic problem :

$$-\mu\,u_{xx} = f \ , \ \text{on }]0, 2\pi[\ ,$$

with the following finite difference/finite volume discretization :

$$-\mu\,\frac{u^n_{i+1} - 2\,u^n_i + u^n_{i-1}}{h^2} = f_i \ , \ \text{for } i = 1, \ldots, n-1 \ , \qquad (13)$$

where n denotes the number of nodes that we suppose to be odd, $n = 2N + 1$, h is the uniform mesh size given by $h = \frac{\pi}{N}$, $x_i = (i-1)\,h$ is the discretization node in the interval $]0, 2\pi[$ and u_i stands for $u(x_i)$, $i = 1, \ldots, n-1$.
Equation (13) leads to a linear system with a circulant $(n-1) \times (n-1)$ tridiagonal matrix the row of which is given by:

$$[-\frac{\mu}{h^2} \quad \frac{2\mu}{h^2} \quad -\frac{\mu}{h^2}] \,.$$

We assume that the coarse grid is made of cells $\{C_p^c\}_{p=1,..,N}$ obtained by agglomerating two fine cells C_{2p-1}^f and C_{2p}^f, where N is the number of coarse cells given by $N = \frac{n-1}{2}$.
To solve this linear system, we consider the following MG algorithms:

Algorithm (a) A standard ideal two-grid scheme in which convergence is required on the coarse grid and where one pre-smoothing is performed on the fine grid. The prolongation operator used here is the natural injection operator.

Algorithm (b) A parallel ideal two-level algorithm with residual splitting and correction filtering as defined in section 3.
The low-frequency splitting operator S is defined as follows:

$$(S\,r)_{2p-1} = (1-\bar{\beta})\,r_{2p-1} + \bar{\beta}\,(\tfrac{r_{2p-1}+r_{2p}}{2}) \,,$$
$$(S\,r)_{2p} = (1-\bar{\beta})\,r_{2p} + \bar{\beta}\,(\tfrac{r_{2p-1}+r_{2p}}{2}) \,,\; p = 1, \ldots, N,$$

where $\bar{\beta}$ is a positive weighting parameter, the optimal value of which is found to be 0.6.
T is given by $T = I - S$.
We set $\bar{S} = S$ and $\bar{T} = T$.
The weighting coefficient β of the prolongation operator is set to 1.
One iteration of Jacobi is done for the high-frequency residual equation on the fine grid and convergence is required for the low-frequency residual equation on the coarse grid.

Algorithm (c) The same algorithm as (b) except that no correction filtering on the fine grid is done i.e. $\bar{S}$ and $\bar{T}$ are not used.
The weighting coefficients $\bar{\beta}$ and β are then set to 0.8.

Algorithm (d) A parallel ideal two-grid scheme where no splitting of the residual is done, that is the correction equation is computed in parallel on the fine and coarse level. No correction filtering on the fine grid is done either. The prolongation operator is the natural injection operator.

The variation of the reduction factor versus the Jacobi relaxation parameter is studied for each of the previous MG algorithms.
For each experiment, the initial solution is the Heavyside function and the RHS f is set to zero.

Algorithm (d) does not converge for any value of the Jacobi relaxation parameter. This is not surprising, since, in this parallel scheme, low frequencies of the error are addressed by both coarse and fine grids, leading therefore to some inconsistency.
In Figure 1, note that Algorithm (b) is less efficient than the two other algorithms for values of the Jacobi relaxation parameter ω less than 0.75. For larger values, the standard ideal two-grid scheme (Algorithm(a)) is the less efficient. Note also that Algorithm(c) is much more efficient than the others for ω greater than 0.7.
The optimal value (with ω) of the mean reduction factor is about 0.25 for the parallel ideal two-level scheme with no correction filtering (Algorithm(c)), and about 0.42 for the standard ideal two-grid scheme (Algorithm(a)).
This improvement is the most sensitive since the computation on the different grids can be done in parallel for Algorithm(c), whereas, for Algorithm(a) the computations on the different grids must be performed sequentially.

6.2 1-D advection problem

We consider the following periodic problem :

$$c\,u_x = f \text{ on }]0, 2\pi[,\ c > 0\ ,$$

with the following finite difference/finite volume discretization :

$$c\,\frac{u_i^n - u_{i-1}^n}{h} = f_i\ ,\ \text{for } i = 1, \ldots, n-1 \tag{14}$$

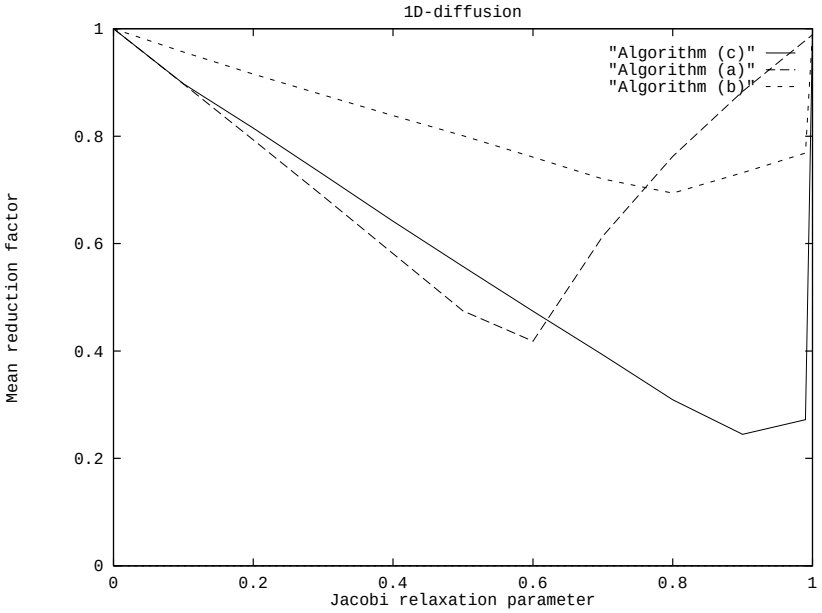

Figure 1: Mean reduction factor versus Jacobi relaxation parameter for 1-D-diffusion; Algorithm (a): Standard ideal 2-grid scheme; Algorithm (b): Parallel 2-grid scheme with residual splitting and correction filtering; Algorithm (c): Parallel 2-grid scheme with residual splitting and no correction filtering

where n, h, x_i and u_i are defined as in the previous subsection. Equation (14) leads to a linear system with a circulant $(n-1) \times (n-1)$ bidiagonal matrix the row of which is given by:

$$[-\frac{c}{h} \quad \frac{c}{h}] \, .$$

The variation of the reduction factor versus the Jacobi relaxation parameter is studied for the four 2-grid algorithms defined in the 1-D-diffusion subsection. For Algorithm(b) and (c), $\beta = 0$ and $\bar{\beta} = 0.5$ give the best results.
For each experiment, the initial solution is the Heavyside function and the RHS f is set to zero.
In Figure 2, we can note that Algorithm(d) diverges for big values of the Jacobi relaxation parameter ω; this is again due to the inconsistent processing of the low frequencies of the error specified previously.
Note that for ω greater than 0.8, Algorithm(b) is the optimal scheme. For middle values of ω, Algorithm(a) is the most efficient.
For $\omega = 0.5$ the standard ideal two-grid scheme (Algorithm(a)), due to aliasing, becomes even a direct solver in this particular case (see for example [12]).
Figure 3 points out that if we perform more smoothing sweeps (10 Jacobi relaxations) on the fine level, the parallel scheme without residual splitting becomes less efficient. Another remark may be done on the area of divergence associated to the Jacobi relaxation parameter which is much larger. Indeed, in that case, low frequencies of the error are better addressed on the fine grid and so, the inconsistency due to the processing of the smooth part of the error on both fine and coarse levels, is stronger.

7 Conclusion

A convergence theory for an abstract parallel two-level multigrid algorithm with residual splitting and correction filtering has been proposed. For this algorithm, we do not require pre-smoothing on the full fine grid residual equation in order to have a full parallel solution of the different level problems.
The convergence proof is based on a smoothing property and an approximation property. A "diagonalization" of the iterative operator of the parallel 2-grid algorithm on two orthogonal subspaces corresponding to the "low" and "high

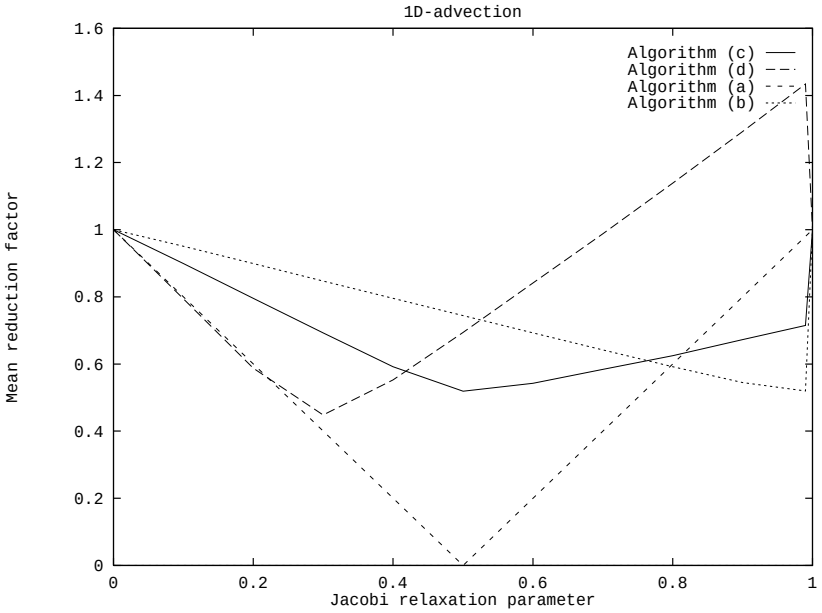

Figure 2: Mean reduction factor versus Jacobi relaxation parameter for 1-D-diffusion; Algorithm (a): Standard ideal 2-grid scheme; Algorithm (b): Parallel 2-grid scheme with residual splitting and correction filtering; Algorithm (c): Parallel 2-grid scheme with residual splitting and no correction filtering; Algorithm (d): Parallel 2-grid scheme without residual splitting and without correction filtering

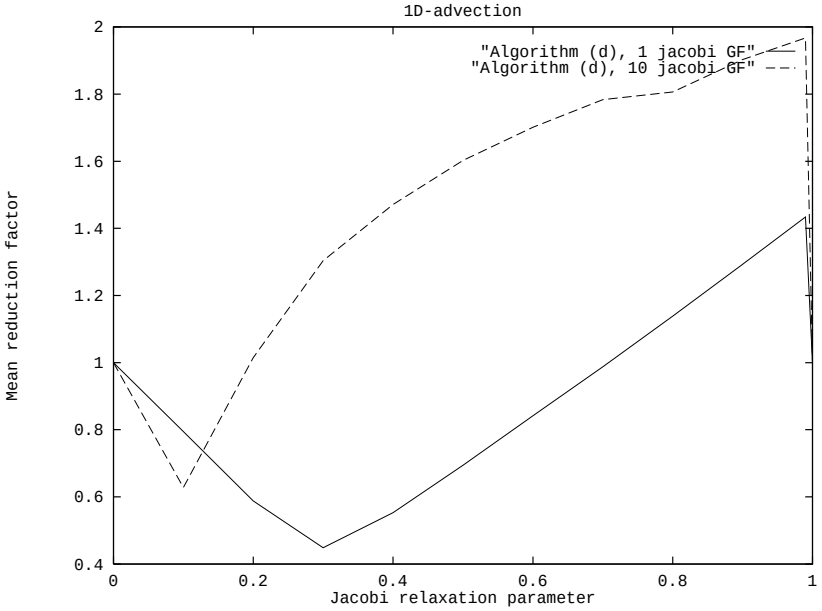

Figure 3: Parallel 2-grid algorithm without residual splitting and without correction filtering; 1 and 10 Jacobi smoothing sweeps are performed on the fine grid.

frequency spaces" is proposed.
Numerical experiments (1-D) have also been achieved for diffusion and advection problems.
For the diffusion case, the convergence rate of the multigrid algorithm is improved by the splitting of the residual. It turns out that the scheme without correction filtering gives the best results.
For the advection case, compared to the standard MG algorithm, the parallel two-level algorithm with residual splitting and correction filtering does not improve the convergence rate except for large values of the Jacobi relaxation parameter.
With such a simple solver as Jacobi, the results obtained here are globally as good as those given for sequential algorithms. We can expect that additional work will yield a more efficient parallel treatment of the different levels.
A similar standpoint is adopted in the context of multi-level optimization [13].
Numerical applications to CFD flow problems will be performed in order to complete this study.

References

[1] CHAN T. and TUMINARO R., "Design and Implementation of Parallel Multigrid Algorithms", *In Proc. Third Copper Mountain Conference on Multigrid Methods*, S.F. McCORMICK (ed), Marcel Dekker, New York and Basel, 1987.

[2] TUMINARO R., "A highly parallel Multi-like method for the solution of the Euler equations", *SIAM J. SCI. COMPUT.*, Vol. 13, N^o 1, pp. 88-100, January 1992.

[3] HACKBUSCH W., "Mutigrid Methods and Applications", *Springer Series in Computational Mathematics, Springer Verlag*, Vol.4, 1985.

[4] BRIGGS W.L., "A multigrid tutorial", *SIAM*, Philadelphia, 1987.

[5] McBRIAN O., FREDERICKSON P., LINDEN J., SCHULLER A., SOLCHENBACH K., STUBEN K., THOLE C.-A. and TROTTENBERG U.,

"Multigrid methods on parallel computers. A survey of recent developments", *Impact of Computing in Science and Engineering 3, 1-75* (1991).

[6] LINDEN J., STECKEL B. and STUBEN K., "Parallel multigrid solution of the Navier-Stokes equations on general 2D domains", *Parallel Computing 7* (1988) 461-475, North-Holland.

[7] CHAN J. and TUMINARO R., "A survey of parallel multigrid algorithms", *RIACS Technical Report 87.22*, August 1987.

[8] FREDERICKSON P. and McBRYAN O.,"Parallel Superconvergent Mutigrid", *In Proc. Third Copper Mountain Conference on Multigrid Methods*, S.F. McCORMICK (ed), Marcel Dekker, New York and Basel, 1987.

[9] DECKER N.,"On the Parallel Efficiency of the Frederickson-McBrian Multigrid", *ICASE report No. 90-17*, February 1990.

[10] WESSELING P.,"An introduction to multigrid methods",*Pure and applied mathematics*, John Wiley and Sons, 1992.

[11] KOOBUS B., LALLEMAND M.-H. and DERVIEUX A., "Unstructured Volume-agglomeration MG: solution of the Poisson equation", *International Journal for Numerical Methods in Fluids*, Vol. 18, 27-42, 1994.

[12] MORANO E., "Résolution des équations d'Euler par une méthode multigrille stationnaire", Thesis, University of Nice, 1992.

[13] MARCO N., KOOBUS B. and DERVIEUX A., "An additive multilevel preconditioning", *Rapport de Recherche INRIA*, in preparation.

[14] XU J., "Iterative methods by space decomposition and subspace correction", *SIAM Review*, Vol. 34, **4**, 581-613, 1992.

[15] GUILLARD H., "Convergence analysis of a multi-level relaxation method", *Rapport de recherche INRIA*, N^o 1884, 1993.

Unité de recherche INRIA Lorraine, Technopôle de Nancy-Brabois, Campus scientifique,
615 rue du Jardin Botanique, BP 101, 54600 VILLERS LÈS NANCY
Unité de recherche INRIA Rennes, Irisa, Campus universitaire de Beaulieu, 35042 RENNES Cedex
Unité de recherche INRIA Rhône-Alpes, 46 avenue Félix Viallet, 38031 GRENOBLE Cedex 1
Unité de recherche INRIA Rocquencourt, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex
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