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Abstract: We study theoretically and numerically the convergence of non-overlapping domain decomposition methods with coupled transmission conditions applied to the optimal control of systems governed by elliptic pde's.

keywords : Domain Decomposition, Optimal Control, Elliptic PDE, Coupled Systems.

AMS(MOS) subject classification : 49M99, 65J10, 65N30, 65Y05.

(Résumé : tsvp)

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Méthodes de Décomposition de Domaine avec Conditions de Transmissions couplées pour le Contrôle Optimal de Systèmes Gouvernés par des Equations aux dérivées partielles Elliptiques

Résumé : Nous étudions la convergence théorique et numérique de méthodes de décomposition de domaine sans recouvrement avec conditions de transmissions couplées appliquées au contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles elliptiques.

Mots-Clés : Décomposition de Domaine, Contrôle Optimal , EDP Elliptique, Système Couplé.

1 Introduction

Domain decomposition methods for elliptic problems have been widely studied during the last decade. Let us rapidly mention P.L. Lions [7], [8], [9], Dryja and Widlund [10] which provide a good survey of the field and detailed references.
The resolution of optimal control problems of systems governed by second order linear elliptic partial differential equations problem leads to coupled systems of elliptic PDE's and variational inequalities (the basic reference is J.L. Lions [6]). To the best of our knowledge, domain decomposition methods for such problems have only been proposed by Bensoussan et Al [1] and the author [2]. This paper proposes a more comprehensive and comparative analysis of these methods and a numerical study.
When there is no constraint on the set of controls, the optimality condition can be used to eliminate the control variable from the state equation and we simply deal with a coupled system of elliptic equations. We can apply the non-overlapping version of the Schwarz method with Robin transmission condition between subdomains introduced in [9] to this coupled system. This algorithm has also been interpreted by Le Tallec and Glowinski [14] as a classical saddle-point algorithm or as a time integration scheme of Peaceman-Rachford type for the minimization of an augmented Lagrangian arising from the decomposition of the variational formulation of an elliptic problem. A powerful proof of the convergence of this algorithm for the Helmholtz equation has recently been given by Despres in [3]. We adapt this proof to our problem and the demonstration precisely establish convergence for the original Robin transmission conditions and also for a family of transmission condition coupling the direct and adjoint states which we believe to be new.
Trying to extend the convergence result to the case with constraints on the set of controls, we are led to introduce subproblems which formulation is different from (but consistent with) the original problem. We establish that the simple 'coupled' transmission conditions introduced by the author in [2] is the natural choice. This domain decomposition strategy defines a sequence of local problems which themselves can be reformulated as optimal control problems. We solve numerically a test problem (proposed in [1]) using these methods. A massively parallel approach is selected where the domain decomposition coin-

cide with the finite elements of the approximation and the subproblems are solved explicitly (implemented on the CM200). The convergence of the algorithms are discussed from a numerical point of view.

The paper is organized as follows : Section 2 presents the simple academic problem we will work on. Section 3 details the basic domain decomposition method with Robin transmission conditions in the case without constraint. We give in section 4 a proof of convergence for this method and obtain the convergence for our method with 'coupled' transmission conditions. We extend this result to the general (with constraints) case in section 5. In section 6, the subproblems are reformulated as local optimal control problems. We briefly describe in section 7 the method proposed in [1] and its main features. Section 8 presents the test problem we solve numerically and the selected strategy. We give the numerical results and discuss them in section 9.

2 The academic problem

We consider the simple academic problem

$$\min_{u \in U} \int_\Omega |y(u) - z|^2 + \ \nu\, |u|^2 \, dx$$

with Ω a bounded smooth open subset of $\mathbb{R}^d$, z given in $L^2(\Omega)$, U a closed convex subset of $L^2(\Omega)$, ν a strictly positive parameter and $y(u) \in H_0^1(\Omega)$ given by :

$$-\Delta y(u) = f + u \ \ in\ \Omega,$$

with $f \in L^2(\Omega, \mathbb{R})$.
This problem is known to have a unique solution (see [6] chap. 2) characterized as the solution of the set of equations :

$$-\ \Delta y(u) = f + u \ \ in\ \Omega, \ \ y(u) \in H_0^1(\Omega, \mathbb{R}), \tag{1}$$

$$-\ \Delta p(u) = y(u) - z \ \ in\ \Omega, \ \ p(u) \in H_0^1(\Omega, \mathbb{R}), \tag{2}$$

$$\int_\Omega (p(u) + \nu\, u)(v - u)\, dx \geq 0, \ \ \forall v \in U, \tag{3}$$

y is the direct state, p the adjoint state and inequation (3) the optimality condition.

When $U = L^2(\Omega)$ (i.e. there is no constraints on the set of controls) (3) gives $u = -\frac{p}{\nu}$ and equations (1) (2) reduce to the coupled system (we drop the dependence in u) :

$$-\Delta y = f - \frac{p}{\nu} \ \ in \ \Omega, \ \ y \in H_0^1(\Omega, \mathbb{R}), \tag{4}$$

$$-\Delta p = y - z \ \ in \ \Omega, \ \ p \in H_0^1(\Omega, \mathbb{R}). \tag{5}$$

This system has a standard variational formulation with a continuous coercive bilinear form and several domain decomposition methods can be applied to such problems.

Remark 1. The following domain decomposition methods can be extended to the more general case of distributed and boundary observation and control of systems governed by linear second order elliptic equations (as described in chap.2 [6]).

3 The basic domain decomposition method

In this section we focus on the non-overlapping version of the Schwarz algorithm with Robin transmission condition between subdomains introduced in [9].

Ω is decomposed into m non-overlapping subdomains ω_i and we introduce the following notations (see fig.1) :

$\gamma_i = \partial\omega_i \cap \partial\Omega$,
$\gamma_{ij} = \gamma_{ji} = \gamma_i \cap \gamma_j$ for $i \neq j$,
$\Gamma_i = \partial\omega_i \cap \partial\Omega$,
n_i is the external normal to γ_i and γ_{ij}.

When one considers the restriction of the solution of a Poisson equation on

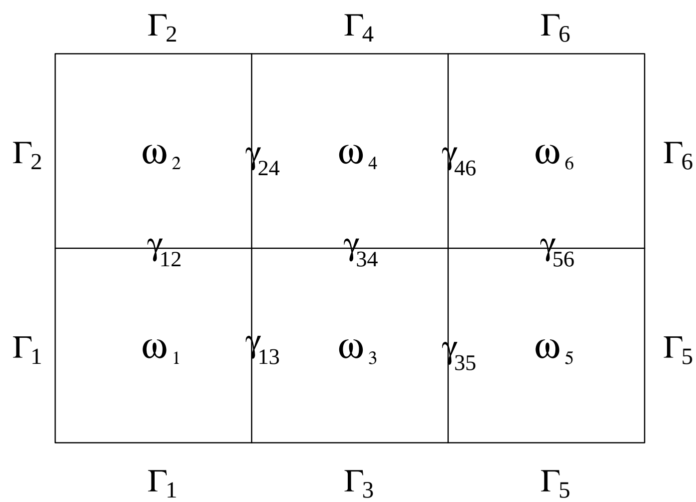

Figure 1: Decomposition of Ω

non overlapping subdomains, the Neumann and Dirichlet boundary conditions match at the interfaces of adjacent subdomains. The algorithm proposed and analyzed in [9] adjust iteratively these quantities by using as boundary conditions on the interface Robin (or mixed) transmission conditions. More precisely we solve iteratively on each subdomain ω_i subproblems of the form :

$$-\Delta y_i^{n+1} = f - \frac{p_i^{n+1}}{\nu}, \quad y_i^{n+1} \in H^1(\omega_i, \mathbb{R}), \tag{6}$$

$$-\Delta p_i^{n+1} = y_i^{n+1} - z, \quad p_i^{n+1} \in H^1(\omega_i, \mathbb{R}), \tag{7}$$

with the boundary conditions :

$$\begin{array}{l} y_i^{n+1} = 0 \quad on\ \Gamma_i, \\ p_i^{n+1} = 0 \quad on\ \Gamma_i, \\ \dfrac{\partial}{\partial n_i} y_i^{n+1} + \lambda\, y_i^{n+1} = -\dfrac{\partial}{\partial n_j} y_j^n + \lambda\, y_j^n \quad on\ \gamma_{ij}, \\ \dfrac{\partial}{\partial n_i} p_i^{n+1} + \lambda\, p_i^{n+1} = -\dfrac{\partial}{\partial n_j} p_j^n + \lambda\, p_j^n \quad on\ \gamma_{ij}. \end{array} \tag{8}$$

The boundary conditions on γ_{ij} are called the transmission conditions. λ may depend on n and i, j. The choice of this parameter is closely linked with the proof of convergence given in the next section.

4 A convergence proof and new 'coupled' transmission conditions

We give this proof for a more general setting of the problem. We want to solve :

$$-\Delta z + k\, z = g, \quad z \in H_0^1(\Omega, \mathbb{C}), \tag{9}$$

k is a complex parameter, using the following domain decomposition algorithm (Ω is decomposed as in section 3). Solve iteratively on each subdomain ω_i subproblems of the form :

$$-\Delta z_i^{n+1} + k\, z_i^{n+1} = g, \quad z_i^{n+1} \in H^1(\omega_i, \mathbb{C}) \tag{10}$$

with the boundary conditions :

$$\begin{aligned} &\frac{\partial}{\partial n_i} z_i^{n+1} + l\, z_i^{n+1} = -\frac{\partial}{\partial n_j} z_j^n + l\, z_j^n \quad on\ \gamma_{ij}, \\ &z_i^{n+1} = 0 \quad on\ \Gamma_i. \end{aligned} \tag{11}$$

Remark 2. Setting $z = y + \frac{\mathrm{i}}{\sqrt{\nu}}\, p$, $k = -\frac{\mathrm{i}}{\sqrt{\nu}}$ and $g = f - \frac{\mathrm{i}}{\sqrt{\nu}}\, z$, we recover our optimal control problem.

Remark 3. We remark that it is enough to take the initial traces $\frac{\partial}{\partial n_i} z_i^0$ and z_i^0 in $L^2(\gamma_{ij}, \mathbb{C})$ to ensure recursively this regularity for all n. The following convergence result (theorem 2) is actually independent of the initialization (for example $z_i^0 = 0$ is a suitable initialization).

Theorem 1 establish existence and uniqueness for the following problem :

$$-\Delta z + k\, z = g, \quad z \in H^1(\omega, \mathbb{C}), \tag{12}$$

with boundary conditions $\partial\omega = \Gamma^d \cup \Gamma^m$:

$$\begin{aligned} &\frac{\partial}{\partial n} z + l\, z = f \quad on\ \Gamma^m, \\ &z = 0 \quad on\ \Gamma^d. \end{aligned} \tag{13}$$

Theorem 1 *We assume that* $g \in L^2(\Omega, \mathbb{C})$*,* $f \in L^2(\Gamma^m, \mathbb{C})$ *and* $\frac{\partial}{\partial n_i} z_i^0$ *and* z_i^0 *in* $L^2(\gamma_{ij}, \mathbb{C})$*.*

If k *and* l *satisfy :*

$$\Re e(l) \geq 0, \quad \Im m(k) \Im m(l) \geq 0, \quad \Im m(k) + \Im m(l) \neq 0, \tag{14}$$

then, the problem (12) (13) has a unique solution.

Proof

The proof rely on a standard transformation of the considered operator into a coercive plus compact operator, the use of the Freedholm alternative and a unique continuation principle. Let us consider the problem :

$$-\Delta z + z = \tilde{g}, \;\; z \in H^1(\omega, \mathbb{C}),$$

$$\begin{array}{l} \frac{\partial}{\partial n} z + l\, z = f \;\; \text{on } \Gamma^m, \\ z = 0 \;\; \text{on } \Gamma^d, \end{array}$$

where $\tilde{g} \in L^2(\omega, \mathbb{C})$.

As $\Re e(l) \geq 0$ by assumption, the problem is coercive in H^1. Considering the regularity of the second hand terms and using the Lax-Milgramm theorem we can define the continuous linear mapping :

$$\begin{array}{rrcl} R: & L^2(\omega, \mathbb{C}) \times L^2(\Gamma^m, \mathbb{C}) & \rightarrow & H^1(\omega, \mathbb{C}) \\ & \tilde{g}, f & \rightarrow & z. \end{array}$$

We set $\phi = R(g, f)$ and define :

$$\begin{array}{rrcl} T: & H^1(\omega, \mathbb{C}) & \rightarrow & H^1(\omega, \mathbb{C}) \\ & z & \rightarrow & R((1-k)\, z, f). \end{array}$$

Our original problem may now be written :

$$(I - T)\, z = \phi,$$

where T is a continuous linear and compact operator from $H^1(\omega, \mathbb{C})$ into $H^1(\omega, \mathbb{C})$ (we recall that ω is bounded). We apply the Freedholm alternative and we have to check the uniqueness of the problem to ensure existence of a solution.

We consider our original problem with $f = g = 0$ multiply the equation by $\bar{z}$, integrate by part and equate to 0 the resulting imaginary part. We obtain ($|z|$ denotes the usual modulus of $z \in \mathbb{C}$) :

$$\Im m(k) \int_{\omega} |z|^2 \, dx + \Im m(l) \int_{\Gamma^m} |z|^2 \, d\sigma = 0.$$

By assumption (14) these two term have the same sign. They are equal to 0. When $\Im m(k) \neq 0$ then $z = 0$ in L^2 and we conclude that $z = 0$ in H^1 using the equation.
When $\Im m(k) = 0$, $\Im m(l) \neq 0$ and therefore $z = \dfrac{\partial}{\partial n} z = 0$. We conclude that $u = 0$ in H^1 using a unique continuation principle (see [5]).

■

Both the original problem (9) and the subproblems (10) (11) are particular settings of (12) (13). Under the assumptions of theorem 1, the domain decomposition method is well defined. We now turn to the problem of convergence. We write the sequence of problems satisfied by the errors $e_i^n = z - z_i^n$:

$$-\Delta e_i^{n+1} + k\, e_i^{n+1} = 0, \quad e_i^{n+1} \in H^1(\omega_i, \mathbb{C}), \tag{15}$$

with the boundary conditions :

$$\begin{aligned} & e_i^{n+1} = 0 \quad on\ \Gamma_i, \\ & \frac{\partial}{\partial n_i} e_i^{n+1} + l\, e_i^{n+1} = -\frac{\partial}{\partial n_j} e_j^n + l\, e_j^n \quad on\ \gamma_{ij}. \end{aligned} \tag{16}$$

We have the following convergence result :

Theorem 2 *If k and l satisfy :*

$$\Re e(\bar{l}) \geq 0, \quad \Re e(k\bar{l}) \geq 0, \quad \Re e(\bar{l}) + \Re e(k\bar{l}) \neq 0, \tag{17}$$

then, $\forall\, i$,

$$e_i^n \xrightarrow{n} 0 \ in\ H^1(\omega_i).$$

Proof

Considering the regularity assumed for $\frac{\partial}{\partial n_i} e_i^{n+1}$ and e_i^{n+1} (remark 3) we can define :

$$E^{n+1} = \sum_i \sum_{j, \gamma_{ij} \neq \emptyset} \int_{\gamma_{ij}} |\frac{\partial}{\partial n_i} e_i^{n+1}|^2 + |l\, e_i^{n+1}|^2 \, d\sigma.$$

Using (16) we rewrite E^{n+1} as ($(p,q) = 2\Re e(p\bar{q})$ is a bilinear form on $\mathbb{C} \times \mathbb{C}$) :

$$\begin{aligned} E^{n+1} = \sum_i \sum_{j, \gamma_{ij} \neq \emptyset} & (\int_{\gamma_{ij}} |\frac{\partial}{\partial n_j} e_j^n|^2 + |l\, e_j^n|^2 \, d\sigma - \\ & - \int_{\gamma_{ij}} (\frac{\partial}{\partial n_j} e_j^n, l\, e_j^n) + (\frac{\partial}{\partial n_i} e_i^{n+1}, l\, e_i^{n+1}) \, d\sigma). \end{aligned} \tag{18}$$

Multiplying (15) by $\overline{l\, e_i^{n+1}}$ and integrating by part we obtain ($\|.\|_i$ denotes $\|.\|_{L^2(\omega_i, \mathbb{C})}$ or $\|.\|_{(L^2(\omega_i, \mathbb{C}))^2}$ following the context) :

$$\sum_{j, \gamma_{ij} \neq \emptyset} \int_{\gamma_{ij}} (\frac{\partial}{\partial n_i} e_i^{n+1}, l\, e_i^{n+1}) \, d\sigma = 2\Re e(\bar{l}) \, \|\nabla e_i^{n+1}\|_i^2 + 2\Re e(k\bar{l}) \, \|e_i^{n+1}\|_i^2. \tag{19}$$

The same operation can be done on ω_j at step n. Plugging (19) into (18) and noticing that $\sum_i \sum_{j, \gamma_{ij} \neq \emptyset} = \sum_j \sum_{i, \gamma_{ji} \neq \emptyset}$, we get the estimate :

$$\begin{aligned} E^{n+1} + \sum_i (2\Re e(\bar{l}) \, \|\nabla e_i^{n+1}\|_i^2 + 2\Re e(k\bar{l}) \, \|e_i^{n+1}\|_i^2) + \\ + \sum_j (2\Re e(\bar{l}) \, \|\nabla e_j^n\|_i^2 + 2\Re e(k\bar{l}) \, \|e_j^n\|_i^2) = E^n. \end{aligned} \tag{20}$$

Summing over n this equation gives :

$$\begin{aligned} E^{n+1} + \sum_{s=0}^{n} (\sum_i (2\Re e(\bar{l}) \, \|\nabla e_i^{s+1}\|_i^2 + 2\Re e(k\bar{l}) \, \|e_i^{s+1}\|_i^2) + \\ + \sum_j (2\Re e(\bar{l}) \, \|\nabla e_j^s\|_i^2 + 2\Re e(k\bar{l}) \, \|e_j^s\|_i^2)) = E^0. \end{aligned} \tag{21}$$

Thanks to the conditions imposed on k and l, the coefficients in front of the series are positive. All the terms on the left hand side of the above equation are positive and therefore bounded independently of n. The series are convergent and the generic term go to 0.

We now have to consider three cases :

i) $\Re e(\bar{l}) > 0$ and $\Re e(k\bar{l}) = 0$.
(21) only gives the convergence for the gradient of the error : for all i,

$$\nabla e_i^n \xrightarrow{n} 0 \; in \; L^2(\omega_i).$$

We then use a generalization of the Poincaré Lemma [11], first in the subdomains neighboring Γ :

$$\|e_i^n\|_i \leq C\,(\|\nabla e_i^n\|_i + \|e_i^n\|_{L^2(\Gamma_i)}).$$

As $e_i^{n+1} = 0$ on Γ_i, we obtain the wanted convergence. Then, we penetrate in the interior subdomains using a similar inequality :

$$\|e_i^n\|_i \leq C\,(\|\nabla e_i^n\|_i + \|e_i^n\|_{L^2(\gamma_{ij})}).$$

ii) $\Re e(\bar{l}) = 0$ and $\Re e(k\bar{l}) > 0$.
(21) gives :

$$e_i^n \xrightarrow{n} 0 \; in \; L^2(\omega_i).$$

Multiplying (15) by $\overline{e_i^{n+1}}$ and integrating by part we obtain :

$$\sum_{j,\gamma_{ij}\neq\emptyset} \|\nabla e_i^{n+1}\|_i^2 = -2\Re e(k)\,\|e_i^{n+1}\|_i^2 - \int_{\gamma_{ij}} (\frac{\partial}{\partial n_i} e_i^{n+1}, e_i^{n+1})\; d\sigma.$$

As E^{n+1} is uniformly bounded, we can extract a weakly convergent subsequence in $L^2(\gamma_{ij})$ for $(\frac{\partial}{\partial n_i} e_i^{n+1})$. (e_i^{n+1}) is actually bounded in $H^1(\omega_i)$. (e_i^{n+1}) therefore goes strongly to 0 in $L^2(\gamma_{ij})$ and $(\frac{\partial}{\partial n_i} e_i^{n+1})$ in $L^2(\gamma_{ij})$. Consequently all the sequence $(\frac{\partial}{\partial n_i} e_i^{n+1})$ converge in $L^2(\gamma_{ij})$. Passing to the limit in the above equation gives the convergence to 0 in $H^1(\omega_i, \mathbb{C})$ for (e_i^{n+1}).

iii) $\Re e(\bar{l}) > 0$ and $\Re e(k\bar{l}) > 0$.
(21) gives straightforwardly the convergence in $H^1(\omega_i, \mathbb{C})$ for (e_i^{n+1}).

■

Remark 4. k and l may depend on n and i,j. As long as they satisfy uniformly one of the three former conditions, this does not affect the proof of convergence.

Remark 5. With mixed boundary conditions $\frac{\partial}{\partial n_i} e_i^{n+1} + l\, e_i^{n+1}$ on Γ, we can extend the proof of convergence to the case $\Re e(\bar{l}) = \Re e(k\bar{l}) = 0$, l is pure imaginary and k is real positive. We are in the case of the Helmholtz equation. The 'pseudo-energy' E^n was originally introduced in [3] to prove the convergence of the same domain decomposition method in this context. k was $({k_w}^2)$ where k_w is the wave number, l was chosen to be $(\mathrm{i}\, k_w)$ which corresponds to a first order transparent conditions. The mixed boundary conditions on the external boundary Γ are interpreted as transparent boundary conditions. $\Re e(\bar{l})$ and $\Re e(k\bar{l})$ do not satisfy (17) but a similar proof shows that the errors satisfy homogeneous Helmholtz equations and their trace and normal derivative on Γ goes to 0. This is enough to prove the convergence using a unique continuation principle from the external boundary.

We come back to our optimal control problem (see remark 2). Setting $l = \lambda + \mathrm{i}\,\mu$ the convergence conditions (17) are (we recall that $k = -\frac{\mathrm{i}}{\sqrt{\nu}}$) :

$$\lambda \geq 0, \quad -\mu \geq 0, \quad \lambda - \mu \neq 0.$$

Note that the conditions (14) of theorem 1 are satisfied. Let us describe the different transmission conditions on the interfaces associated with different values of λ an μ. We distinguish the three cases of the above proof :

i) $\lambda > 0$, $\mu = 0$.
(11) reduces to :

$$\begin{aligned} \frac{\partial}{\partial n_i} y_i^{n+1} + \lambda\, y_i^{n+1} &= -\frac{\partial}{\partial n_j} y_j^n + \lambda\, y_j^n \quad on\ \gamma_{ij}, \\ \frac{\partial}{\partial n_i} p_i^{n+1} + \lambda\, p_i^{n+1} &= -\frac{\partial}{\partial n_j} p_j^n + \lambda\, p_j^n \quad on\ \gamma_{ij}. \end{aligned}$$

This is the original method of [9] described in section 3.

ii) $\lambda = 0$, $-\mu > 0$.
(11) reduces to :

$$\begin{aligned}
&\frac{\partial}{\partial n_i} y_i^{n+1} - \frac{\mu}{\sqrt{\nu}}\, p_i^{n+1} = -\frac{\partial}{\partial n_j} y_j^n - \frac{\mu}{\sqrt{\nu}}\, p_j^n \;\; on\ \gamma_{ij}, \\
&\frac{\partial}{\partial n_i} p_i^{n+1} + \mu\sqrt{\nu}\, y_i^{n+1} = -\frac{\partial}{\partial n_j} p_j^n + \mu\sqrt{\nu}\, y_j^n \;\; on\ \gamma_{ij}.
\end{aligned}$$

These are the 'coupled' Robin-like transmission conditions proposed in [2].

iii) $\lambda > 0$, $-\mu > 0$.
(11) reduces to :

$$\begin{aligned}
&\frac{\partial}{\partial n_i} y_i^{n+1} + \lambda\, y_i^{n+1} - \frac{\mu}{\sqrt{\nu}}\, p_i^{n+1} = -\frac{\partial}{\partial n_j} y_j^n + \lambda\, y_j^n - \frac{\mu}{\sqrt{\nu}}\, p_j^n \;\; on\ \gamma_{ij}, \\
&\frac{\partial}{\partial n_i} p_i^{n+1} + \mu\sqrt{\nu}\, y_i^{n+1} + \lambda\, p_i^{n+1} = -\frac{\partial}{\partial n_j} p_j^n + \mu\sqrt{\nu}\, y_j^n + \lambda\, p_j^n \;\; on\ \gamma_{ij}.
\end{aligned}$$

We obtain more general 'coupled' transmission conditions.

Remark 6. We point out that in case *iii)* the choice $|\lambda| = |\mu|$ leads to $l = |\mu| \exp^{-\mathrm{i}\frac{\pi}{4}}$. $\exp^{-\mathrm{i}\frac{\pi}{4}}$ is the square root of $-\mathrm{i}$ $(= k)$ which satisfies the convergence conditions (17), we recall that for the Helmholtz equation the choice in [3] was precisely $l = \sqrt{k}$ corresponding to a first order transparent conditions (see remark 5). This choice is also linked to the trace operator arising from the factorization of the operator considered in the problem (see [4]).

5 Extension to the constrained case

We now try to extend the results of section 4 to the constrained (U is a closed convex subset of $L^2(\Omega, \mathbb{R})$) optimal control problem (1)-(3). The domain is decomposed as in section 3 and a sequence of subproblems with transmission conditions at the interfaces must be defined to approximate the global solution.

We assume that we can define m sets U_i which are closed convex subsets of $L^2(\omega_i)$ and satisfy the following compatibility conditions with U :

$$\begin{aligned}
&\forall u \in U, \;\; u|_{\omega_i} \in U_i \;\; and \\
&\forall u_i \in U_i, \;\; u \; such\ that\ u|_{\omega_i} = u_i \; \forall\, i \; belongs\ to\ U.
\end{aligned} \tag{22}$$

We then consider the sequence of subproblems on each subdomain ω_i :
find $(y_i^{n+1}, p_i^{n+1}, u_i^{n+1}) \in H^1(\omega_i, \mathbb{R}) \times H^1(\omega_i, \mathbb{R}) \times U_i$ such that :

$$- \Delta y_i^{n+1} = f + u_i^{n+1}, \tag{23}$$

$$- \Delta p_i^{n+1} = y_i^{n+1} - z, \tag{24}$$

$$\int_{\omega_i} (p_i^{n+1} + \nu\, u_i^{n+1})(v_i - u_i^{n+1})\, dx \geq 0, \quad \forall v_i \in U_i. \tag{25}$$

We have to add boundary conditions on the interfaces γ_{ij}. If we consider transmission conditions of type *i)* section 4, for any suitable u_i^{n+1} equations (23) and (24) can be solved one after the other.
Conversely, 'coupled' transmissions such as *ii)* and *iii)* section 4 lead to a coupled system of elliptic equation (23) (24) plus the considered boundary conditions. Existence and uniqueness can be established as in theorem 1.

The unconstrained case of section 4 suggests to replace (23) by :

$$- \Delta y_i^{n+1} + \frac{p_i^{n+1}}{\nu} = f + u_i^{n+1} + \frac{p_i^n}{\nu}. \tag{26}$$

Consistency with the original equation (1) is preserved and the coupled system (26) (24) (25) with suitable boundary conditions can be solved uniquely (this question is treated in proposition 1, section 6).

The choice of (26) instead of (23) is justified in remark 7 and leads to better theoretical convergence properties.

We examine the problem of convergence. As in section 4, we write the equations satisfied by the errors. We set $ey_i^n = y - y_i^n$, $ep_i^n = p - p_i^n$, $eu_i^n = u - u_i^n$ and $e_i^n = ey_i^n + \frac{\mathrm{i}}{\sqrt{\nu}}\, ep_i^n$:

$$- \Delta e_i^{n+1} - \frac{\mathrm{i}}{\sqrt{\nu}}\, e_i^{n+1} = eu_i^{n+1} + \frac{ep_i^n}{\nu}, \tag{27}$$

with the boundary conditions :

$$\begin{aligned} &e_i^{n+1} = 0 \quad on\ \Gamma_i, \\ &\frac{\partial}{\partial n_i} e_i^{n+1} + l\, e_i^{n+1} = -\frac{\partial}{\partial n_j} e_j^n + l\, e_j^n \quad on\ \gamma_{ij}. \end{aligned} \tag{28}$$

We copy the proof of theorem 2 and obtain (instead of (20), $k = -\frac{\mathrm{i}}{\sqrt{\nu}}$) :

$$\begin{gathered} E^{n+1} + \sum_i (2\Re e(\bar{l})\,\|\nabla e_i^{n+1}\|_i^2 + 2\Re e(k\bar{l})\,\|e_i^{n+1}\|_i^2 - \\ -2\int_{\omega_i}(eu_i^{n+1} + \frac{ep_i^n}{\nu})\,\Re e(\overline{l\,e_i^{n+1}})\,dx) + \sum_j (2\Re e(\bar{l})\,\|\nabla e_j^n\|_i^2 + \\ +2\Re e(k\bar{l})\,\|e_j^n\|_i^2 - 2\int_{\omega_i}(eu_j^n + \frac{ep_j^{n-1}}{\nu})\,\Re e(\overline{l\,e_j^n})\,dx) = E^n \end{gathered} \tag{29}$$

We have to deal with new terms of the form :

$$\int_{\omega_i}(eu_i^{n+1} + \frac{ep_i^n}{\nu})\Re e(\overline{l\,e_i^{n+1}})\,dx. \tag{30}$$

Let us explicit them. We set again $l = \lambda + \mathrm{i}\mu$, then :

$$\Re e(\overline{l\,e_i^{n+1}}) = \lambda\, ey_i^{n+1} - \frac{\mu}{\sqrt{\nu}}\, ep_i^{n+1},$$

and (30) gives :

$$\begin{gathered} \textstyle\int_{\omega_i} eu_i^{n+1}(-\frac{\mu}{\sqrt{\nu}}\, ep_i^{n+1}) + \frac{ep_i^n}{\nu}\,(-\frac{\mu}{\sqrt{\nu}}\, ep_i^{n+1})\,dx + \\ +\textstyle\int_{\omega_i} eu_i^{n+1}(\lambda\, ey_i^{n+1}) + \frac{ep_i^n}{\nu}\,(\lambda\, ey_i^{n+1})\,dx. \end{gathered} \tag{31}$$

When $\lambda = 0$ and $-\mu > 0$ (i.e. case *ii*) p. 13), we are only concerned with the first line of (31). The compatibility conditions (22) allow us to take $v = u_i^n$ on ω_i and 0 elsewhere in (3) and $v_i = u$ in (25). Adding these inequalities, we get :

$$\nu\|eu_i^{n+1}\|^2_{L^2(\omega_i,\mathbb{R})} \le -\int_{\omega_i} ep_i^{n+1} eu_i^{n+1}\,dx. \tag{32}$$

We make use of (32) to bound the first term of (31) :

$$-\mu\sqrt{\nu}\|eu_i^{n+1}\|^2_{L^2(\omega_i,\mathbb{R})} \le -\int_{\omega_i} eu_i^{n+1}\,(-\frac{\mu}{\sqrt{\nu}}\, ep_i^{n+1})\,dx. \tag{33}$$

We combine the second term with the norm of ep_i^{n+1} extracted from the norm of e_i^{n+1} in (29) to get (use the identity $2a(a-b) = (a-b)^2 + (a-b)(a+b)$) :

$$\begin{gathered} 2\int_{\omega_i} -\frac{\mu}{\nu^{\frac{3}{2}}}\,(ep_i^{n+1})^2 - \frac{ep_i^n}{\nu}\,(-\frac{\mu}{\sqrt{\nu}}\, ep_i^{n+1})\,dx = \\ -\frac{\mu}{\nu^{\frac{3}{2}}}\int_{\omega_i}(ep_i^{n+1} - ep_i^n)^2 + (ep_i^{n+1})^2 - (ep_i^n)^2\,dx. \end{gathered} \tag{34}$$

We finally obtain from (29) (the same calculations are done on e_j^n) :

$$
\begin{gathered}
E^{n+1}+\\
+\sum_i -\frac{\mu}{\nu^{\frac{3}{2}}}\,(2\|ey_i^{n+1}\|_i^2+\|ep_i^{n+1}\|_i^2+\|ep_i^{n+1}-ep_i^n\|_i^2-\|ep_i^n\|_i^2)-2\mu\sqrt{\nu}\,\|eu_i^{n+1}\|_i^2)+\\
+\sum_j -\frac{\mu}{\nu^{\frac{3}{2}}}\,(2\|ey_j^{n}\|_j^2+\|ep_j^{n}\|_j^2+\|ep_j^{n}-ep_j^{n-1}\|_j^2-\|ep_j^{n-1}\|_j^2)-2\mu\sqrt{\nu}\,\|eu_j^{n}\|_j^2)\\
\leq E^n.
\end{gathered}
$$

We sum over n this inequality and deduce :

i) The error ep_i^n satisfies $ep_i^{n+1}-ep_i^n \xrightarrow{n} 0 \; in \; L^2(\omega_i)$. It is a Cauchy sequence and therefore has a limit in $L^2(\omega_i)$.

ii) ey_i^n goes to 0 in $L^2(\omega_i)$.

This is enough to prove the following convergence results using the same techniques as in the proof of theorem 2 :

Assume that $l=\lambda+\mathrm{i}\,\mu$ with $\lambda=0$ and $-\mu>0$, then

Theorem 3 *$\forall\, i$, we have :*

$$
\begin{aligned}
ey_i^n &\xrightarrow{n} 0 \; in \; H^1(\omega_i),\\
ep_i^n &\xrightarrow{n} 0 \; in \; H^1(\omega_i),\\
eu_i^n &\xrightarrow{n} 0 \; in \; L^2(\omega_i).
\end{aligned}
$$

Remark 7. If we had kept equation (23) instead of replacing it by (26) the same demonstration would have led to

$$
\begin{gathered}
E^{n+1}+\\
+\sum_i -\frac{\mu}{\nu^{\frac{3}{2}}}\,2\|ey_i^{n+1}\|_i^2-2\mu\sqrt{\nu}\,\|eu_i^{n+1}\|_i^2)+\\
+\sum_j -\frac{\mu}{\nu^{\frac{3}{2}}}\,2\|ey_j^{n}\|_j^2-2\mu\sqrt{\nu}\,\|eu_j^{n}\|_j^2)\\
\leq E^n.
\end{gathered}
$$

We have no indication of the convergence of the adjoint variable in this context.

Remark 8. When $\lambda > 0$ (i.e. case *i*) and *iii*) of section 4). We have no clue on how to deal with the second term of (31). Case *ii*) (i.e. $\lambda = 0$ and $-\mu > 0$) is well suited to the constrained optimal control problem as it directly exploits the optimality conditions through (32). It seems to be the most natural choice and it is the only one for which we showed convergence of the algorithm.

6 Reformulation of the subproblems

The algorithm therefore consists in the iterative resolution of the subproblems :

Find $(y_i^{n+1}, p_i^{n+1}, u_i^{n+1}) \in H^1(\omega_i, \mathbb{R}) \times H^1(\omega_i, \mathbb{R}) \times U_i$ such that,

$$-\Delta y_i^{n+1} + \frac{p_i^{n+1}}{\nu} = f + u_i^{n+1} + \frac{p_i^n}{\nu}, \tag{35}$$

$$-\Delta p_i^{n+1} = y_i^{n+1} - z, \tag{36}$$

$$\int_{\omega_i} (p_i^{n+1} + \nu\, u_i^{n+1})(v_i - u_i^{n+1})\, dx \geq 0, \ \ \forall v_i \in U_i, \tag{37}$$

with boundary conditions (we recall that $-\mu > 0$) :

$$\begin{array}{l} y_i^{n+1} = 0 \ \ on\ \Gamma_i, \\ p_i^{n+1} = 0 \ \ on\ \Gamma_i, \\ \dfrac{\partial}{\partial n_i} y_i^{n+1} - \dfrac{\mu}{\sqrt{\nu}}\, p_i^{n+1} = -\dfrac{\partial}{\partial n_j} y_j^n - \dfrac{\mu}{\sqrt{\nu}}\, p_j^n \ \ on\ \gamma_{ij}, \\ \dfrac{\partial}{\partial n_i} p_i^{n+1} + \mu\sqrt{\nu}\, y_i^{n+1} = -\dfrac{\partial}{\partial n_j} p_j^n + \mu\sqrt{\nu}\, y_j^n \ \ on\ \gamma_{ij}. \end{array} \tag{38}$$

We reformulate this subproblems as optimal control problems. It ensures, in particular, that the algorithm is well defined.

Proposition 1 *Each subproblem (35)-(38) has a unique solution which solves the optimal control problem :*

$$\begin{aligned} \min_{u_i^{n+1} \in U_i} \quad & \int_{\omega_i} \frac{1}{2} \left(|\frac{p_i^{n+1}}{\sqrt{\nu}}|^2 + |(y_i^{n+1} - z)|^2 + \nu\, |u_i^{n+1}|^2 \right) dx\, + \\ & + \sum_{j, \gamma_{ij} \neq \emptyset} \int_{\gamma_{ij}} \frac{1}{-2\mu\sqrt{\nu}}\, |\frac{\partial}{\partial n_i} p_i^{n+1}|^2 + \frac{-\mu}{2\sqrt{\nu}}\, |p_i^{n+1}|^2\, d\sigma, \end{aligned}$$

where y_i^{n+1} and p_i^{n+1} are the solution of (35) (36) (38).

Before giving the proof of this proposition, we want to point out that our domain decomposition algorithm has changed the nature of the adjoint variable p. As we coupled the equation in y and p, this two variables are now the direct state of the optimal control formulated in proposition 1. The new cost functional on each subdomain involves terms in p together with the terms in y and u of the original global problem.

Remark 9. We did not discuss the choice of the parameter μ only saying that it can depend on the subdomains and the rank of iteration. Proposition 1 suggests a method of steepest descent type where we would perform the minimization of the functional with respect to u_i^{n+1} and μ. In the unconstrained case, i.e. when $u_i^{n+1} = -\frac{p_i^{n+1}}{\nu}$, it is simply a one dimensional minimization over μ.

Proof of proposition 1. For all $u_i^{n+1} \in U_i$ The coupled system (35) (36) (38) has a unique solution. This is a straightforward application of theorem 1. Setting $z = y_i^{n+1} + \frac{\mathrm{i}}{\sqrt{\nu}}\, p_i^{n+1}$, our problem enters this framework with $k = -\frac{\mathrm{i}}{\sqrt{\nu}}$ and $l = -\frac{\mathrm{i}}{\sqrt{\nu}}$. We easily check that conditions (14) holds.

We skip the standard proof (application of theorem 2.1, chap. 2, [6]) of existence and uniqueness of $u_i^{n+1} \in U_i$ minimizing the functional :

$$\begin{aligned} J(u_i^{n+1}) = \; & \int_{\omega_i} \frac{1}{2}\,(|\frac{p_i^{n+1}}{\sqrt{\nu}}|^2 + |(y_i^{n+1} - z)|^2 + \nu\,|u_i^{n+1}|^2)\; dx\; + \\ & + \sum_{j,\gamma_{ij}\neq\emptyset} \int_{\gamma_{ij}} \frac{1}{-2\mu\sqrt{\nu}}\,|\frac{\partial}{\partial n_i} p_i^{n+1}|^2 + \frac{-\mu}{2\sqrt{\nu}}\,|p_i^{n+1}|^2\; d\sigma, \end{aligned}$$

over U_i where (y_i^{n+1}, p_i^{n+1}) is the solution of (35) (36) (38).

This unique solution is characterized by the optimality condition ([6] chap. 1 theorem 1.2) :

$$J'(u^{n+1}).(v_i - u_i^{n+1}) \geq 0, \;\; \forall v_i \in U_i. \tag{39}$$

We show that (39) and (37) are equivalent. We set $\delta u = v_i - u_i^{n+1}$, $\delta y = y_i - y_i^{n+1}$ and $\delta p = p_i - p_i^{n+1}$ where (y_i, p_i) is the solution of (35) (36) (38) associated to the control v_i. Using (35) we rewrite (37)

$$\int_{\omega_i} (p_i^{n+1} + \nu\, u_i^{n+1})\, \delta u\; dx = \int_{\omega_i} \nu\, u_i^{n+1}\, \delta u\; dx + \int_{\omega_i} p_i^{n+1}\, (\Delta \delta y + \frac{\delta p}{\nu})\; dx$$

Integrating twice by part and using (36) we get :

$$\begin{aligned} \int_{\omega_i} (p_i^{n+1} + \nu\, u_i^{n+1})\, \delta u\; dx = & \int_{\omega_i} \nu\, u_i^{n+1}\, \delta u + p_i^{n+1} \frac{\delta p}{\nu} + \delta y\, (y_i^{n+1} - z)\; dx + \\ & + \sum_{j, \gamma_{ij} \neq \emptyset} \int_{\gamma_{ij}} \frac{\partial}{\partial n_i} p_i^{n+1}\, \delta y - \frac{\partial}{\partial n_i} \delta y\, p_i^{n+1}\; d\sigma. \end{aligned}$$

We use (38) to replace in the integrals on the boundaries and obtain :

$$\begin{aligned} \int_{\omega_i} (p_i^{n+1} + \nu\, u_i^{n+1})\, \delta u\; dx = & \int_{\omega_i} \nu\, u_i^{n+1}\, \delta u + p_i^{n+1} \frac{\delta p}{\nu} + \delta y\, (y_i^{n+1} - z)\; dx + \\ & + \sum_{j, \gamma_{ij} \neq \emptyset} \int_{\gamma_{ij}} \frac{-\mu}{\sqrt{\nu}}\, p_i^{n+1}\, \delta p + \frac{1}{-\mu \sqrt{\nu}}\, \frac{\partial}{\partial n_i} p_i^{n+1}\, \frac{\partial}{\partial n_i} \delta p\; d\sigma. \end{aligned}$$

The above quantity can be written as $J'(u^{n+1}).(v_i - u_i^{n+1})$ with :

$$\begin{aligned} J(u_i^{n+1}) = & \int_{\omega_i} \frac{1}{2} \, (|\frac{p_i^{n+1}}{\sqrt{\nu}}|^2 + |(y_i^{n+1} - z)|^2 + \nu\, |u_i^{n+1}|^2)\; dx + \\ & + \sum_{j, \gamma_{ij} \neq \emptyset} \int_{\gamma_{ij}} \frac{1}{-2\mu \sqrt{\nu}}\, |\frac{\partial}{\partial n_i} p_i^{n+1}|^2 + \frac{-\mu}{2\sqrt{\nu}}\, |p_i^{n+1}|^2\; d\sigma. \end{aligned}$$

It ends our proof.

■

7 Another method

For the sake of completeness we review in the section the method proposed in [1] for the problem described in section 2. We consider the same domain decomposition as in section 5. The method defines a sequence of subproblems on each subdomain which have matching Neumann boundary conditions and iteratively adjust the Dirichlet boundary conditions. More precisely, solve the sequence of subproblems on each subdomain ω_i :

Find $(y_i^{n+1}, p_i^{n+1}, u_i^{n+1}) \in H^1(\omega_i, \mathbb{R}) \times H^1(\omega_i, \mathbb{R}) \times U_i$ such that,

$$-\Delta y_i^{n+1} = f + u_i^{n+1},$$

$$-\Delta p_i^{n+1} = y_i^{n+1} - z,$$

$$\int_{\omega_i} (p_i^{n+1} + \nu\, u_i^{n+1})(v_i - u_i^{n+1})\, dx \geq 0, \;\; \forall v_i \in U_i,$$

with the boundary conditions

$$\begin{array}{l} y_i^{n+1} = 0 \;\; on\ \Gamma_i, \\ p_i^{n+1} = 0 \;\; on\ \Gamma_i, \\ \dfrac{\partial}{\partial n_i} y_i^{n+1} = -\dfrac{\partial}{\partial n_j} y_j^n + \rho(y_i^n - y_j^n) \;\; on\ \gamma_{ij}, \\ \dfrac{\partial}{\partial n_i} p_i^{n+1} = -\dfrac{\partial}{\partial n_j} p_j^n + \rho(p_i^n - p_j^n) \;\; on\ \gamma_{ij}. \end{array} \tag{40}$$

ρ is a positive parameter and may depend as λ on the subdomains and the rank of iteration. This algorithm is well defined, i.e. each subproblem is the unique solution of an optimal control corresponding to the minimization over U_i of the functional :

$$\int_{\omega_i} \frac{1}{2} \left((y_i^{n+1} - z)|^2 + \nu\, |u_i^{n+1}|^2 \right) dx \;+ \sum_{j, \gamma_{ij} \neq \emptyset} \int_{\gamma_{ij}} (-\frac{\partial}{\partial n_j} p_j^n + \rho(p_i^n - p_j^n))\, y_i^{n+1}\, d\sigma.$$

This algorithm is shown to be convergent if $4\nu C^2 > 1$ where C is a constant such that the Poincaré inequality :

$$\int_{\omega_i} |\nabla y|^2\, dx \geq C(\int_{\omega_i} |y|^2\, dx + \int_{\gamma_i} |y|^2\, d\sigma),$$

is satisfied.

Remarks on this method :

The transmission conditions of case i) section 4 are actually an implicit version of (40).

The Neumann boundary conditions match. i.e. at each step :

$$\frac{\partial}{\partial n_i} y_i^{n+1} = -\frac{\partial}{\partial n_j} y_j^{n+1} \quad on\ \gamma_{ij},$$
$$\frac{\partial}{\partial n_i} p_i^{n+1} = -\frac{\partial}{\partial n_j} p_j^{n+1} \quad on\ \gamma_{ij}.$$

There is no coupling between the direct and adjoint states in the subproblems which can be interpreted as local optimal control problems where (y_i^{n+1}, p_i^{n+1}) are respectively the direct and adjoint states. In our 'coupled' method (section 6) (y_i^{n+1}, p_i^{n+1}) becomes the direct state of a new optimal control problem.

The theoretical convergence depends on the size of the parameter ν whereas our method unconditionally converges. The proof of [1] also relies on estimates of 'pseudo-energies' supported by the interfaces. More precisely they show that the sum of the norms of the fluxes of the error on the interfaces at each iteration constitute the general term of a convergent serie.

Remark 10. A domain decomposition in vertical strips has been used in [1] so that the subproblems which retain Dirichlet boundary conditions are well posed. A decomposition with interior subdomains will result into Neumann problems for which we have no guarantees that the usual compatibility conditions are satisfied.

8 The test problem and the numerical strategy

This section and the following are devoted to the numerical resolution of a test problem using the domain decomposition algorithms we studied. The results provide information on the convergence properties of these methods.

The 2-D test problem is a boundary control problem proposed in [1]. Ω is the rectangle $]0,4[\times]0,1[$ with boundaries Γ_N, $\Gamma_E, \Gamma_W, \Gamma_S$ (NEWS for North, East, West, South). (x_1, x_2) is the coordinate system. We solve the optimal control problem :

$$\min_{u \in U} \int_\Omega |y(u) - z|^2 + \int_{\Gamma_E \cup \Gamma_W} \nu \, |u|^2 \, d\sigma,$$

where $U = L^2(\Gamma_E \cup \Gamma_W)$ and $y(u) \in H_0^1(\Omega)$ are given by :

$$-\Delta y(u) = f \;\; in \; \Omega,$$

$$\frac{\partial}{\partial n} y(u) = u \;\; on \; \Gamma_E \cup \Gamma_W,$$
$$y(u) = 0 \;\; on \; \Gamma_N \cup \Gamma_S.$$

When

$$f(x_1, x_2) = 2(-x_1^2 - x_2^2 + 4x_1 + x_2)$$
$$z(x_1, x_2) = (x_2^2 - x_2)(x_1^2 - 4x_1) - 8\nu$$

the analytic solution is given by :

$$y(x_1, x_2) = (x_2^2 - x_2)(x_1^2 - 4x_1),$$
$$p(x_1, x_2) = -4\nu(x_2^2 - x_2),$$
$$u(x_2) = 4(x_2^2 - x_2).$$

We are in the unconstrained case and the control is given by $u = -\frac{p}{\nu}$ where p is the solution of the adjoint equation :

$$-\Delta p(u) = y(u) - z \;\; in \; \Omega,$$

$$\frac{\partial}{\partial n} p(u) = 0 \;\; on \; \Gamma_E \cup \Gamma_W,$$
$$p(u) = 0 \;\; on \; \Gamma_N \cup \Gamma_S.$$

The domain is decomposed into a $4N \times N$ regular squares ω_i, $h = \frac{1}{N}$ is the size of the subdomains. γ_{ij} is the interface between ω_i and ω_j and Γ_{Ni} (respectively Γ_{Ei}, Γ_{Wi}, Γ_{Si}) the northern (respectively eastern, western, southern) external boundary of ω_i (see figure 2).
We apply the non overlapping domain decomposition algorithm. The subproblems will a priori be of the form :

$$-\Delta y_i^{n+1} = f, \;\; y_i^{n+1} \in H^1(\omega_i, \mathbb{R}),$$

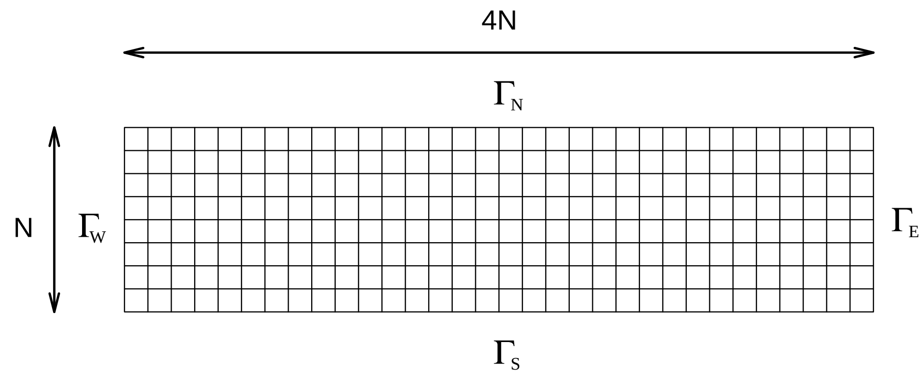

Figure 2: Decomposition of Ω

$$
\begin{aligned}
-\Delta p_i^{n+1} = y_i^{n+1} - z, \;\; p_i^{n+1} \in H^1(\omega_i, \mathbb{R}),\\
\frac{\partial}{\partial n_i} y_i^{n+1} = -\frac{p_i^{n+1}}{\nu} \;\; on\ \Gamma_{Ei} \cup \Gamma_{Wi},\\
\frac{\partial}{\partial n_i} p_i^{n+1} = 0 \;\; on\ \Gamma_{Ei} \cup \Gamma_{Wi},\\
y_i^{n+1} = 0 \;\; on\ \Gamma_{Ni} \cup \Gamma_{Si},\\
p_i^{n+1} = 0 \;\; on\ \Gamma_{Ni} \cup \Gamma_{Si},
\end{aligned}
$$

plus transmission conditions on γ_{ij}.

We proceed as in section 5 where we substituted (26) to (23) and consider, instead of the above equations, subproblems of the form :

$$
-\Delta y_i^{n+1} + \frac{1}{\sqrt{\nu}} \frac{p_i^{n+1}}{\sqrt{\nu}} = f + \frac{p_i^n}{\nu}, \;\; y_i^{n+1} \in H^1(\omega_i, \mathbb{R}),
$$

$$
-\Delta \frac{p_i^{n+1}}{\sqrt{\nu}} - \frac{1}{\sqrt{\nu}} y_i^{n+1} = -\frac{z}{\sqrt{\nu}}, \;\; p_i^{n+1} \in H^1(\omega_i, \mathbb{R}),
$$

$$
\begin{aligned}
y_i^{n+1} = 0 \;\; on\ \Gamma_{Ni} \cup \Gamma_{Si},\\
p_i^{n+1} = 0 \;\; on\ \Gamma_{Ni} \cup \Gamma_{Si}.
\end{aligned}
$$

In order to get symmetric boundary conditions on $\Gamma_E \cup \Gamma_W$ we also transform these boundary conditions :

$$\begin{aligned}
\frac{\partial}{\partial n_i} y_i^{n+1} + \frac{1}{\sqrt{\nu}} \frac{p_i^{n+1}}{\sqrt{\nu}} &= 0 \;\; \textit{on } \Gamma_{Ei} \cup \Gamma_{Wi}, \\
\frac{\partial}{\partial n_i} \frac{p_i^{n+1}}{\sqrt{\nu}} - \frac{1}{\sqrt{\nu}} y_i^{n+1} &= -\frac{1}{\sqrt{\nu}} y_i^n \;\; \textit{on } \Gamma_{Ei} \cup \Gamma_{Wi}.
\end{aligned}$$

We will use three different transmission conditions on γ_{ij} corresponding to case $i)$ section 4 the non-coupled method and case $ii)$ and $iii)$ section 4 for coupled methods. These references also gives the sign conditions to be satisfied by the parameters λ and μ. For each case, the analysis of section 4 for can be done, i.e. : existence and uniqueness of the global problem and of the subproblems, convergence of the algorithm.

The three different transmission conditions are :

Alg 1.

$$\begin{aligned}
\frac{\partial}{\partial n_i} y_i^{n+1} + \lambda\, y_i^{n+1} &= -\frac{\partial}{\partial n_j} y_j^n + \lambda\, y_j^n \;\; \textit{on } \gamma_{ij}, \\
\frac{\partial}{\partial n_i} p_i^{n+1} + \lambda\, p_i^{n+1} &= -\frac{\partial}{\partial n_j} p_j^n + \lambda\, p_j^n \;\; \textit{on } \gamma_{ij}.
\end{aligned}$$

Alg 2.

$$\begin{aligned}
\frac{\partial}{\partial n_i} y_i^{n+1} - \frac{\mu}{\sqrt{\nu}}\, p_i^{n+1} &= -\frac{\partial}{\partial n_j} y_j^n - \frac{\mu}{\sqrt{\nu}}\, p_j^n \;\; \textit{on } \gamma_{ij}, \\
\frac{\partial}{\partial n_i} p_i^{n+1} + \mu\sqrt{\nu}\, y_i^{n+1} &= -\frac{\partial}{\partial n_j} p_j^n + \mu\sqrt{\nu}\, y_j^n \;\; \textit{on } \gamma_{ij}.
\end{aligned}$$

Alg 3.

$$\begin{aligned}
\frac{\partial}{\partial n_i} y_i^{n+1} + \lambda\, y_i^{n+1} - \frac{\mu}{\sqrt{\nu}}\, p_i^{n+1} &= -\frac{\partial}{\partial n_j} y_j^n + \lambda\, y_j^n - \frac{\mu}{\sqrt{\nu}}\, p_j^n \;\; \textit{on } \gamma_{ij}, \\
\frac{\partial}{\partial n_i} p_i^{n+1} + \mu\sqrt{\nu}\, y_i^{n+1} + \lambda\, p_i^{n+1} &= -\frac{\partial}{\partial n_j} p_j^n + \mu\sqrt{\nu}\, y_j^n + \lambda\, p_j^n \;\; \textit{on } \gamma_{ij}.
\end{aligned}$$

The adimensionalization of our subproblems indicates that :

$i)$ ν has to be of order h^4. We will take $\nu = h^4$.

ii) $\frac{p_i^{n+1}}{\sqrt{\nu}}$ and y_i^{n+1} are of the same order. We will formulate and solve numerically our subproblem with respect to these adimensional variables.

iii) Concerning the parameters of the transmissions conditions, $\lambda = \frac{1}{h}$ has the correct dimension for Alg 1. Considering that $\sqrt{\nu} = h^2$, we chose $-\mu = h$ in Alg 2 and $\lambda = -\mu = \frac{h}{2}$ in Alg 3.

Remark 11. In Alg 3 we chose $|\lambda| = |\mu|$ (see remark 6 on this choice).

Remark 12. For all the algorithms we transformed the boundary conditions on $\Gamma_{Ni} \cup \Gamma_{Si}$ in order to get boundary conditions of the same type as the transmission conditions. We add terms from step $n+1$ on the right side of the equality and terms from step n on the left. We get :

$$\begin{aligned}
\frac{\partial}{\partial n_i} y_i^{n+1} + \lambda\, y_i^{n+1} - \frac{\mu}{\sqrt{\nu}}\, p_i^{n+1} &= \frac{\partial}{\partial n_i} y_i^n \quad \textit{on } \Gamma_{Ni} \cup \Gamma_{Si}, \\
\frac{\partial}{\partial n_i} p_i^{n+1} + \mu\sqrt{\nu}\, y_i^{n+1} + \lambda\, p_i^{n+1} &= \frac{\partial}{\partial n_i} p_i^n \quad \textit{on } \Gamma_{Ni} \cup \Gamma_{Si},
\end{aligned}$$

instead of :

$$\begin{aligned}
y_i^{n+1} &= 0 \quad \textit{on } \Gamma_{Ni} \cup \Gamma_{Si}, \\
p_i^{n+1} &= 0 \quad \textit{on } \Gamma_{Ni} \cup \Gamma_{Si}.
\end{aligned}$$

This transformation does not affect convergence and allows a more symmetric numerical implementation.

The chosen strategy is to discretize each step of the algorithm such that each finite element is a subdomain. The subproblems can be solved explicitly and no optimization algorithm is required. We use a first order finite element method with mass lumping on a mixt hybrid formulation of the problem. This approach is well suited to our problem for it uses in particular, as degrees of freedom, the fluxes of the normal derivative and the average values of the trace of the direct and adjoint states on the interfaces which are the natural unknowns of our transmission conditions. The proof of convergence in the continuous case is easily extended to this mixt hybrid discrete formulation as it allows an exact discrete integration by part. For more details on this discretization strategy we refer to [3] and its bibliography (see also [12] [13] for the use of mixt finite elements in domain decomposition methods).

The algorithm now reduce to calculate explicitly the solution on each subdomain at each iteration and transmit the necessary data to the adjacent subdomains for the next iteration. This method is particularly well suited to the Connection Machine as it needs only communication to adjacent virtual processors (one processor per subdomain) and performs the same instructions on every virtual processors.

Remark 13. We also implemented Bensoussan et al algorithm with a four subdomain decomposition (see also remark 12) as a reference algorithm. The subproblem are solved by a conjugate gradient method and the equation are discretized by finite difference. The resulting linear system is solved using a LU factorization which can be made once for all during the initialization. Results are given in the next section (Remark 14).

9 Numerical results

We now present the numerical results ran on 8k processors of the CM200 of INRIA Sophia-Antipolis for the three algorithms.

We ran the algorithms for a large number of iterations and different values of N in order to check the convergence to the exact solution and to get an estimate of the accuracy of our method. Figures 3, 4, 5 show the relative $L^2(\Omega)$-error with the exact solution versus the number of iteration for the different values of N.

The three algorithms decrease rapidly the error and then increase lightly before stabilizing. Alg 2 shows small oscillations which amplitude decrease. Alg 1 is a little less accurate than the two other methods. We have no explanation for this unconsistency but it disappears as we refine the subdomains.

Alg 2 and Alg 3 are slightly faster than Alg 1. A temptative explanation may be found in the proof of convergence (theorem 2). The quantity which

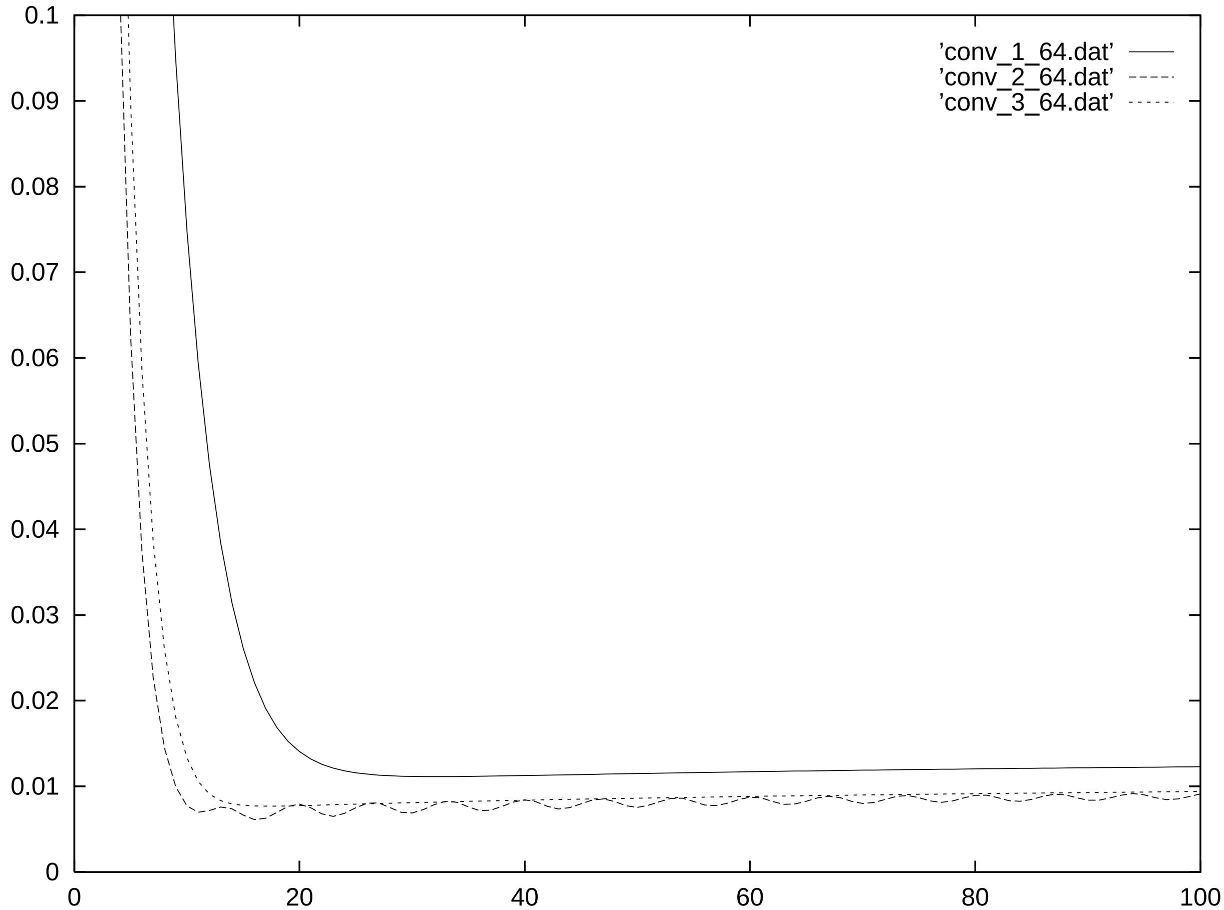

Figure 3: Convergence of Alg 1, Alg 2 and Alg 3, $N = 64$.

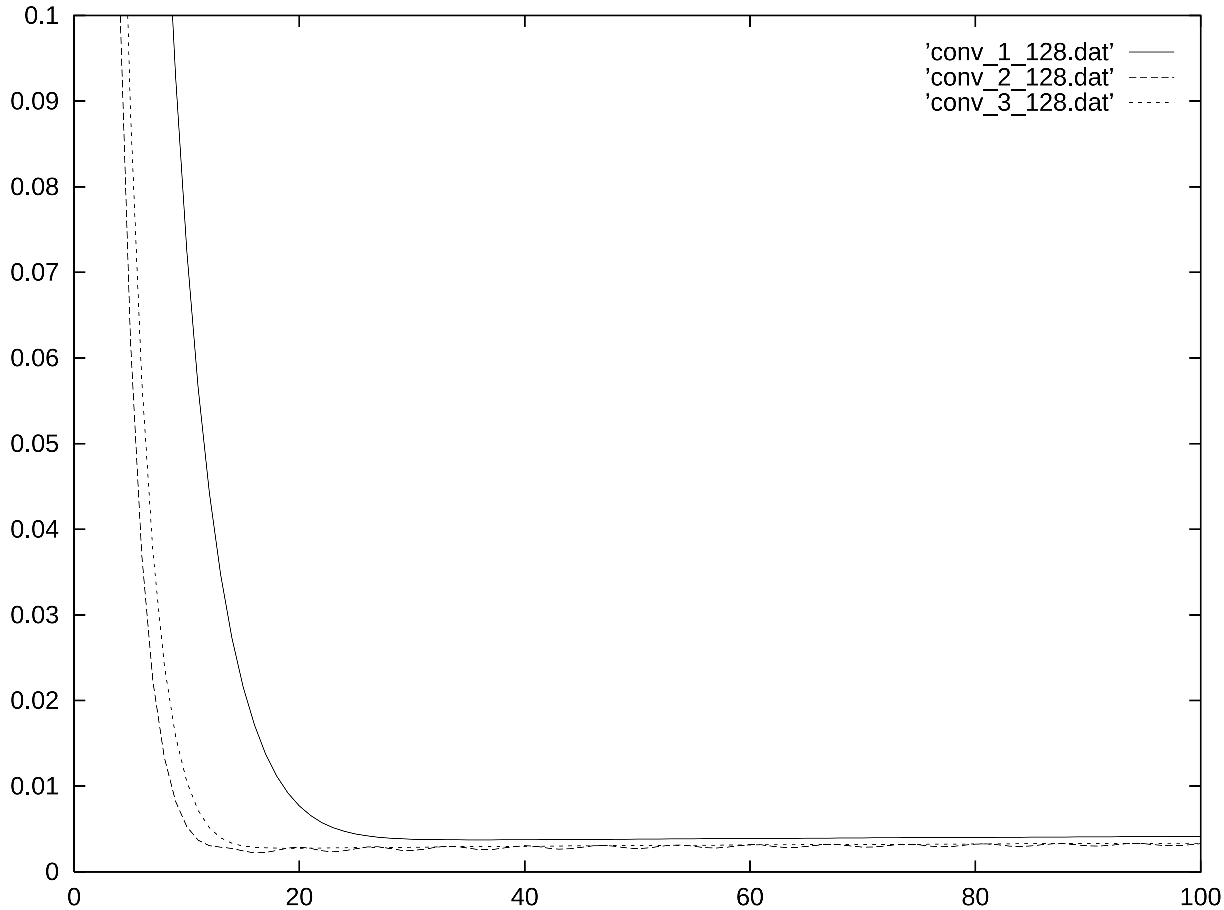

Figure 4: Convergence of Alg 1, Alg 2 and Alg 3, $N = 128$.

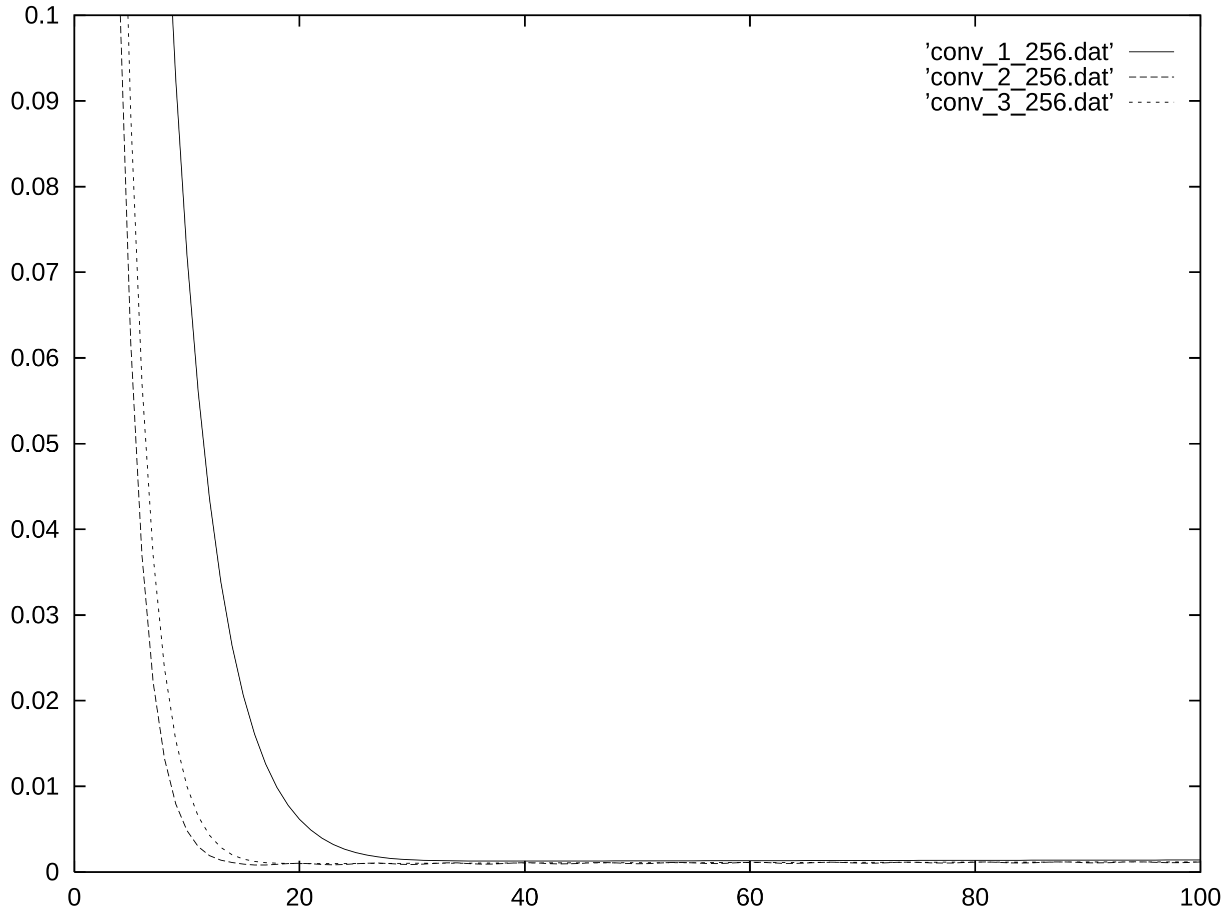

Figure 5: Convergence of Alg 1, Alg 2 and Alg 3, $N = 256$.

decreases our pseudo energy E^n from step to step in (20) is

$$\begin{aligned}\sum_i (2\Re e(\bar{l})\,\|\nabla e_i^{n+1}\|_i^2 + 2\Re e(k\bar{l})\,\|e_i^{n+1}\|_i^2)+ \\ +\sum_j (2\Re e(\bar{l})\,\|\nabla e_j^{n}\|_i^2 + 2\Re e(k\bar{l})\,\|e_j^{n}\|_i^2)\end{aligned} \tag{41}$$

The conditions of convergence (14) ensure that these terms are strictly positive. The three algorithms correspond to the following cases :

Alg 1. $\Re e(\bar{l}) > 0$ and $\Re e(k\bar{l}) = 0$. (41) only involves the L^2 norm of the gradient of the error. The convergence of the algorithm is obtained by the combination of these semi-norms with the homogeneous Dirichlet conditions on the external boundary (see proof i) section 4).

Alg 2. $\Re e(\bar{l}) = 0$ and $\Re e(k\bar{l}) > 0$. (41) only involves the L^2 norm of the error.

Alg 3. $\Re e(\bar{l}) > 0$ and $\Re e(k\bar{l}) > 0$. (41) involves the full H^1 norm of the error.

It seems natural that Alg 2 and Alg 3 behaves better than Alg 1 as a real norm of the error decreases the 'pseudo-energy' E^{n+1} in each subdomain. Note that unlike for Alg 1 and Alg 3, the gradient of the error does not appear in the proof of convergence of Alg 2. This could be an explanation of the above mentionned oscillations.

When $h = \frac{1}{N}$ is divided by 2, the error is divided by 3. Our methods are more than $O(h)$-accurate which is usually expected when one uses first order finite element.

As we expect a better error than $O(h)$ with the exact solution, we now decide to stop the iterative process when the precision quantity $pre = \frac{\|y^{n+1}-y^n\|_{L^2(\Omega)}}{\|y^{n+1}\|_{L^2(\Omega)}}$ is of the same order. We precisely take $pre = 6\,h^2$. The algorithm is now independent of the exact solution. Results are displayed in tables 1, 2 and 3. For different values of N we give err : the relative $L^2(\Omega)$ error with the exact

Table 1: Alg 1.

N	*err*	*nbr of iter.*	*time* (s.)
64	1.3191726E-002	22	0.4241651
128	4.1968077E-003	27	1.048676
256	1.3389768E-003	33	2.859681

Table 2: Alg 2.

N	*err*	*nbr of iter.*	*time* (s.)
64	7.3801433E-003	15	0.3515814
128	2.2592354E-003	18	0.6567114
256	9.7718020E-004	22	2.066284

solution, the number of required iteration to reach the fixed precision, and the time of the full process (including the initialization of the algorithm).

The peak flops rate for the iteration kernel of Alg 3 is 1.321 Gflops for 8k processors (which is theoretically multiplied by 4 on a 64k machine), which prove that our algorithm is well suited (as expected) to the CM.

Remark 14. As an element of comparison we give here results obtained on one processor of the CRAY C98 with the algorithm of section 7. See also

Table 3: Alg 3.

N	*err*	*nbr of iter.*	*time* (s.)
64	8.3214156E-003	14	0.3289179
128	2.7926140E-003	18	0.6046120
256	9.9027425E-004	21	2.007038

remark 13. For a $N = 64$ point discretization in the x_2 direction and $4N$ in the x_1 direction, the algorithm converged in 10 iterations, reached an error of $4.043E - 003$ with the exact solution and the process took $3.435s$.

10 Conclusion

This paper provide a study of the non-overlapping domain decomposition of [9] applied to the optimal control of system governed by second order linear elliptic partial differential equations. We give a convergence proof in the manner of [3] for the original Robin transmission conditions and establish convergence for new coupled transmission conditions in the unconstrained case. We confirmed these result by a numerical study which highlighted the efficiency of the coupled transmission condition.

We illustrated on the constrained case how the subproblems can be constructed in a non exact but consistent way in order to ensure better convergence properties for the algorithm. Convergence was proved for only for the 'simple' coupled transmission conditions (Alg 2). It would be interesting to test numerically these methods on a constrained case.

These domain decomposition methods offer an alternative to the more classical gradient type methods where one has to iterate the resolution of large linear systems. We iterate the resolution of small subproblems solving at the same time the equations and the optimal control problem. We therefore avoid the problem of storing and inverting large stiffness matrices associated to the global equations.

The use of a massively parallel computer makes possible the resolution of large problem. The resolution of 3-D inverse scattering problems under the Born approximation using this domain decomposition method is investigated.

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