Differential algebraic equations : a new look at the index

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Abstract: Systems of Differential-Algebraic Equations (DAE's) have been widely investigated, during the last twenty years, from a numerical point of view. The notion of *index* has shown to be efficient for the classification of such systems with respect to the difficulties when trying to solve them with the standard discretization schemes. In this paper, our interest is in the study of formal ("structural") properties of DAE systems : we make the link between the index of a DAE system and general notions coming from the formal theory of PDE's. This interpretation gives new insights for the understanding of DAE's and allows us to give rigorous justification of manipulations that are of usual practice in this area.

Key-words: Index, DAE systems, formal theory, jet spaces

(Résumé : tsvp)

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Equations différentielles algébriques
Un point de vue nouveau sur la notion d'indice

Résumé : Les deux dernières décennies ont vu un grand nombre de travaux porter sur l'analyse numérique des systèmes d'équations différentielles algébriques (DAE). La notion d'*indice* s'est progressivement révélée être un concept important pour caractériser le degré de difficulté dans la résolution numérique de ces systèmes. Nous nous intéressons dans ce travail aux propriétés formelles ("structurelles") des DAE, au moyen d'outils plus généraux venant de la théorie formelle des EDP. Ce cadre permet une définition rigoureuse de l'indice qui soit applicable aussi bien aux systèmes sur-déterminés et ne nécessite pas la réduction préalable à un système du premier ordre. Plusieurs autres notions, apparaissant dispersées dans la littérature, sont replacées dans ce cadre et reçoivent de ce fait un statut précis.

Mots-clé : Indice, DAE, intégrabilité formelle, espaces de jets

1 Introduction

It is now a well-known fact that systems of Differential Algebraic Equations (DAE's) are, in the general case, far more difficult to solve numerically than systems of ODE's. A lot of work ([15], [2], [10] e.g.) has been done during the last twenty years, studying the efficiency and the difficulties of the classical numerical discretization schemes (BDF, Runge-Kutta, General Linear, extrapolation), when applied to DAE's [6] : they originated in the early 70's with the now famous and often cited [4]. One essential result in this area is the concept of *index*, an integer used to characterize the "solvability" of a DAE. On another hand, it has been known for long that dynamical systems "with constraints" (semi-explicit systems, a particular case of DAE in the usual sense) may give rise to discontinuous solutions : [21] e.g. studies such systems in the context of bifurcation theory of dynamical systems, that is from a qualitative point of view. In this last context, [8] studies numerical methods for bifurcation problems near singular points, mainly path following methods. [20] studies continuation methods too for DAE's and consider these as differential equations on manifolds, for "existence and uniqueness of solutions" purposes, that is from a functional analysis point of view. With the notion of index at hand, most of the works ([15], [11], ...) study the problems of convergence for numerical schemes and have attempted to classify them with respect to the index : only systems of index less than two may be solved easily with standard numerical solvers so that index reduction techniques must be used in general. Other works study the problem of consistency of initial data for DAE systems ([14], [9]) which is in fact a closely related one, as we shall see below. Unfortunately, the computation of the index itself seems presently unreachable or at least very difficult in the general case. Some attempts have been made to compute it in very specific although frequent cases : semi-explicit, or linear in the derivative. An inportant contribution in that direction may be found in [5] and [3] gives an algorithmic approach in the particular case of linear systems. The main conclusion is that, although the knowledge of the index is crucial for the numerical solution of systems of DAE, there is no practical procedure for its computation in the general case. An interesting paper is the recent [12] which gives a procedure to reduce an index n system to an index 1 which can then be solved numerically. In that paper, the notion of "dummy" derivative is in fact what is known in modern differential geometry as a jet coordinate and this is related to the context of our work.

It is a fact that the notion of index is of a pure formal (or structural) nature because it only depends on the given equations and not on the solutions themselves, although in the usual definition some regularity conditions on the solutions are always required ([2, 9]). Our goal is here to give a good framework for dealing with this notion. On another hand, in the context of simulation, many DAE systems come from the space discretization of partial differential equations, so it would be interesting to relate, in a way, formal properties of ordinary systems (in fact the index) to more general ones. This is the goal of our contribution : we give in this work a new definition of the index, in a differential geometric framework, with the use of *jet theory* which is, in our opinion, the right one to study differential equations - in particular, the vague notion of constrained equation is best formulated as a submanifold of a jet space. With this definition, we show that there is no need to transform a n^{th} order system of p

equations in a first order system of $n \times p$ equations for such a computation, as is always done in the literature (except in [12] but with no justification), which is satisfactory because, as we said, the index is a "structural" property of the system under study. Our contribution could be viewed as a geometric, algebraic one, in that only the equations are of interest and not the solutions, from the index viewpoint : we think that there is a need for a formal theory of DAE systems as a complementary tool of the numerical approach and the qualitative theory, for a better understanding of dynamical systems, in view of their numerical resolution for simulation purposes.

The paper is organized as follows : in section 2 we recall the "classical" definition of the index (in fact several definitions may be found in the literature but the one given below will suffice for our exposition). Section 3 gives the main results adapted from the formal theory of PDE's : we show that the notion of index is related to the "formal integrability" of the system and we give a rigorous definition for it, exhibiting at once some lacks in the usual definition as well as its limitations. In section 4 we illustrate the power of our point of view on several examples limited neither to order one nor semi-explicit systems, although fully implicit systems are not so frequent in engineering. The last example is in fact a PDE system briefly presented in order to introduce to make the link with the general theory.

2 The classical point of view

We briefly recall in this section the classical definition of the *index*, mainly for notations purposes. Let us mention once again that the usual definition, as given for example in [5] or [2], suppose that the given DAE system has first been reduced to a first order one, a requirement which can be easily relaxed, as we shall show in section 3. Let be given a system of first order non linear DAE's :

$$F(t,y,y') = 0 \tag{1}$$

F being differentiable enough (see [2] for the details). We then consider the following system :

$$\begin{array}{rcl} F(t,y,y') & = & 0 \\ \frac{d}{dt}F(t,y,y') & = & 0 \\ & \vdots & \\ \frac{d^{j-1}}{dt^{j-1}}F(t,y,y') & = & 0 \end{array} \tag{2}$$

Following ([2]), this system writes :

$$\begin{array}{rcl} F_{[0]}(t,y,y') & = & 0 \\ F_{[1]}(t,y,y',y'') & = & 0 \\ & \vdots & \\ F_{[j-1]}(t,y,y',\ldots,y^{(j)}) & = & 0 \end{array} \tag{3}$$

with for example :$F_{[1]} = F_{y'}y'' + F_y y' + F_t$. Then we write it :

$$\mathbf{F}_j(t, y, \mathbf{y}_j) = 0 \tag{4}$$

where by definition :

$$\mathbf{y}_j = [y', \dots, y^{(j)}]^T$$

The *index* is defined as follows :

Definition 2.1 The *index* ν of the system (1) is the least integer ν such that $\mathbf{F}_j(t, y, \mathbf{y}_{\nu+1})$ uniquely determines y' as a continuous function of y and t, the other derivatives being considered as independent variables.

This means that with ν differentiations and eliminations, we get from (1) a system to which the implicit function theorem may be applied and solved for y'. Nothing is said of the higher order derivatives except that they are considered as independent variables, giving rise to the question : why keep those derivatives if they are of no use in the definition of the index ? When dealing with explicit or semi-explicit systems, it is easy to see that higher order derivatives can be eliminated at each differentiation step but in the case of fully implicit, nonlinear systems, this task cannot be achieved (the most tricky examples found in the literature are in fact "linearly in the first derivative" implicit systems). We shall give in the sequel an answer to that question in the framework of *jet theory*. Nevertheless, the index is a useful concept when solving numerically a DAE system, essentially because of the following ([2], p. 144) :

Theorem 2.1 The condition number of the iteration matrix (in the Newton step) for a system with index ν is $O(h^{-\nu})$ (where h is the numerical step)

So it is important to have a good understanding of the notion and we show in the sequel that this can be achieved without particular assumptions on the solutions and that a lot of information is contained in the equations themselves. Other similar definitions have been given in [2] (more or less algorithmic) or [9] (with the introduction of a "pseudoderivative", which we show, in the next section, to be in fact the formal derivative that we present below). The notion of "dummy derivative", introduced in the recent [12], is on another hand nothing else than what is known as a *jet coordinate*. We refer the reader to [2] and the numerous references therein for a detailed exposition of practical situations. In the following section we give precise definitions for all these notions in the framework of jet spaces.

3 The formal theory of ordinary differential systems

We recall in this section some essential facts from the formal theory of differential equations(see [17] e.g.). As we are only interested in ordinary differential systems (and DAE's more precisely), we shall just have to specialize the general results to our particuliar situation, refering the interested reader to [16] or [17] for the general theory; [13] may be

consulted too on certain aspects (prolongation, invariants ...) of ODE's. All the results that we need have been established in this far more general framework. Let us just point out the main ingredients : *jet theory*, *formal theory of PDES's* and *differential algebraic geometry*. The formal theory of PDE's studies the PDE's and their spaces of solutions without explicit integration, by "simple inspection" of the equations, viewed as defining some submanifold of a convenient manifold (the bundle of q-jets as we shall see). It does not study problems like convergence of solutions which remain then to be studied afterwards. On another side, the index of a DAE is a structural ("formal") property that comes before any numerical computation and must be, at least theoretically, reachable by simple inspection of the equations too. Another crucial problem is the one of giving a "consistent" set of initial conditions for a DAE system : it can be given an answer with the same approach. The present section is adapted from [17] that we particularize to the case of ODE's/DAE's.
For the description of non linear systems of ODE's or DAE's, it is convenient to work with jet coordinates instead of derivatives. Moreover, it is the right framework to deal with differential systems, whatever they are ODE's, DAE's or PDE's, from a formal point of view, because it shows common features of all those systems that could not appear otherwise (ODE's and DAE's are obviously particular cases of PDE's). We first begin with some definitions essential to our purpose, postponing to the appendix notations and basic facts from differential geometry.

3.1 Nonlinear systems of DAE's

When doing local computations on differential systems, two operations are very frequent : differentiation and substitution of derivatives of least order into subsequent equations obtained through differentiation. In the context of jet theory, these operations translate into *prolongation* and *projection* on a submanifold, respectively. We give here their definitions after having defined what we mean by a *nonlinear system of DAE's* and its solutions.
Let X be a manifold of dimension $n = 1$ (there is just one independent variable in ODE's or DAE's), with local coordinate x and $\mathcal{E}$ be a fibered manifold over X with fiber dimension m and local coordinates (x, y^k). We consider $J_q(\mathcal{E})$, the q-jet bundle of $\mathcal{E}$. We then have :

Definition 3.1 A nonlinear system of DAE's (including of course ODE's), of order q, is a fibered submanifold $\mathcal{R}_q$ of $J_q(\mathcal{E})$, which is defined by a system of local equations :

$$\Phi^\tau(x, y_q^k) = 0 \tag{5}$$

Where τ is the codimension of the system and y_q^k stands for all the derivatives up to order q of the dependent variables. The general system defined by equations (1) is a particular case of DAE with $q = 1$.

Definition 3.2 A solution of $\mathcal{R}_q$ is a local section f of $\mathcal{E}$ such that $j_q(f)$ is a local section of $\mathcal{R}_q$.

Definition 3.3 The *r-prolongation* of $\mathcal{R}_q$ is the subset $\rho_r(\mathcal{R}_q) = J_r(\mathcal{R}_q) \cap J_{q+r}(\mathcal{R}_q) \subset J_r(J_q(\mathcal{E}))$. It will be simply noted $\mathcal{R}_{q+r}$, as we deal with only one system of order q at a time.

In other words, the r-prolongation of $\mathcal{R}_q$ is obtained by substituting derivatives for jet coordinates, differentiating the original system r-times and then substituting back the derivatives with corresponding *jet coordinates*. This is exactly what is done in [12] (see p.678, last paragraph) : their notion of "dummy derivatives" replacing a true derivative is nothing else than a jet coordinate. We then introduce the second operation, the projection, by mean of the following subsets :

Definition 3.4 $\forall r, s \in \mathbb{N}$, we define :

$$\mathcal{R}_{q+r}^{(s)} \quad \stackrel{\text{def}}{=} \quad \pi_{q+r}^{q+r+s}(\mathcal{R}_{q+r+s}) \subset \mathcal{R}_{q+r} \tag{6}$$

This set of systems has the following meaning : whenever we have a system $\mathcal{R}_q$, we prolongate (differentiate) it $r+s$ times and then project it onto $\mathcal{R}_{q+r}$; thus the process of differentiation-elimination may be viewed as a purely algebraic procedure of prolongation-projection.

Another useful notion, related to the prolongation, is the *formal derivative* of a DAE system :

Definition 3.5 Let Φ a function defined on $J_q(\mathcal{E})$. The *formal derivative* of Φ with respect to x is a function $d\Phi$ defined on $J_{q+1}(\mathcal{E})$ as follows :

$$d\Phi \circ j_{q+1}(f) \quad = \quad \partial_x(\Phi \circ j_q(f)) \tag{7}$$

which gives in local coordinates :

$$d \quad = \quad \frac{\partial}{\partial x} + \sum_{\mu=0}^{q} \sum_{k=1}^{m} y_{\mu+1}^k \frac{\partial}{\partial y_\mu^k} \tag{8}$$

We must notice that the summation (Einstein) convention of indices is always used in the context of PDE's. We do not use it here as it does not simplifies the presentation. Now, with this definition at hand, we are able (see [17]) to give local equations for $\mathcal{R}_{q+r}$ in the following very simple and intrinsic form (recall that $\mathcal{R}_q$ is defined by equation (5)):

$$d^\nu \Phi^\tau = 0 \; with \; 0 \leq \nu \leq r \tag{9}$$

where d^ν stands for the composition of d with itself ν times. At this point, it is worth noting that this definition corresponds exactly to the one given for a *pseudoderivative* in [9] (see def. 2.3, p. 207) but now, we have it in a precise setting, in the context of jet spaces, moreover independent of any coordinate system (by equation (7)). Morever, we shall see in the sequel that we do not have to consider $\mathcal{R}_{q+r}$, for increasing r, thanks to the projections $\mathcal{R}_{q+r}^{(s)}$ that play a very important role and this could not be seen in the context as settled in [9].

3.2 Formal integrability and the index

In this section, we make the link between the index of a DAE system and the *formal integrability* as given by the formal theory. This will allow us to give intrinsic properties, independent of the local coordinates (as we did above for the formal derivative). The most interesting result is a "dictionnary" in which we shall translate results scattered in the classical literature on DAE's into our framework. We first introduce, in a loose manner, the *symbol* of a system (see [17] for a more precise presentation, or [1]) :

Definition 3.6 The *symbol* M_q of $\mathcal{R}_q$ is a family of vector spaces over $\mathcal{R}_q$, defined in local coordinates by the following linear equations :

$$\frac{\partial \Phi}{\partial y_q^k}(x, y_q) v_q^k = 0 \; (x, y_q) \in \mathcal{R}_q \tag{10}$$

Although in an unusual form, we recognize the jacobian matrix of the system with respect to the highest order derivatives. In the particular case $q = 1$, we see the jacobian matrix $\partial\Phi/\partial y^{'}$ appear (compare with definition 2.1) and we start feeling some relationship with the index. The interest of the symbol comes from a well-known fact in differential algebra : the derivative of a differential polynomial of order n in an indeterminate z is linear in z_{n+1}. This has an important consequence : the formal study of a differential system (whatever it is an ODE, DAE or PDE system), through prolongations, will reduce to pure linear algebra. One consequence is that the symbol M_{q+r} of the r-prolongation $\mathcal{R}_{q+r}$ of $\mathcal{R}_q$ is defined by the following equations :

$$\frac{\partial \Phi}{\partial y_q^k}(x, y_q) v_{q+r}^k = 0 \; (x, y_q) \in \mathcal{R}_{q+r} \tag{11}$$

Let us illustrate how to compute the symbol on two simple examples. The case of first order, semi-explicit systems is trivial because the symbol is zero. In the case of a constrained Lagrangian system (which is semi-explicit) :

$$\begin{aligned} M(q)\ddot{q} &= f(q, \dot{q}) - G^T(q)\lambda \\ g(q) &= 0 \end{aligned}$$

the symbol is still zero. More generally, every system of order q, having equations of order less than q has a null symbol (this is also true for PDE systems). Now, we have the following definition :

Definition 3.7 A system $\mathcal{R}_q \subset J_q(\mathcal{E})$ is *formally integrable* if $\mathcal{R}_{q+r}$ is a fibered submanifold of $J_{q+r}(\mathcal{E})$ and if $\pi_{q+r}^{q+r+1} : \mathcal{R}_{q+r+1} \to \mathcal{R}_{q+r}$ is an epimorphism $\forall r \geq 0$.

The second condition means in our context that, for some integer r, subsequent prolongations of the system do not give supplementary equations of order $q+r$: for a system to be formally integrable, prolongation at any order must not give new equations of order less. This is exactly the situation encountered when dealing with semi-explicit systems (see below) : the zero-order equations are differentiated until a derivative appears and cannot be eliminated with the help of preceding equations. Nevertheless, the preceding definition is of no practical use because it involves an infinite number of steps, the same situation as with the classical definition of the index : when to stop differentiating the equations ? Of course, we shall stop at least after a number of steps equal to the dimension of the jet space minus the codimension of the system (this is an obvious bound for the index). The solution is given by the following theorem (see [17]), giving a practical criterion to test for the formal integrability :

Theorem 3.1 Let be given a non linear system $\mathcal{R}_q$ of DAE's. Then if $\mathcal{R}_q^{(1)} = \mathcal{R}_q$, the system is formally integrable.

It means that if prolongating and projecting once a system does not give new equations of order q then it is formally integrable. Now we understand what was missing in the use of the pseudoderivative in [9] : the derivatives of order greater than q (in fact [9] studies the case $q = 1$) are of no more relevance as such but only through the spaces $\mathcal{R}_{q+r}^{(s)}$. In other words, "we know now what to do with the higher order derivatives" : prolongating only is not enough and we need the couple (prolongation,projection) to achieve a "good system".

Remark 3.1 We have not introduced here the notions of *n-acyclicity* and of *involutiveness* of the symbols, because they are trivially satisfied in the case of a one dimensional base manifold, so we do not need them in our context. Nevertheless it is worth noting here that these two notions are essential in the context of PDE's, as it appears in [16]. For ODE's and DAE's, a formally integrable system is indeed involutive but the two properties have to be verified separatly for systems of PDE's. In particular, for the preceding theorem to be valid, it must be supposed that the symbol of the system under consideration is involutive.

Let $\mathcal{R}_q$ be a non linear system, $\mathcal{R}_{q+r}$ its r-prolongation. We have the following important result ([17]) :

Theorem 3.2 If $\mathcal{R}_q \subset J_q(\mathcal{E})$ is sufficiently regular (which means that $\mathcal{R}_{q+r}^{(s)}$ is a fibered manifold $\forall r, s \geq 0$), there exists a finite algorithm providing two integers $r, s \geq 0$ such that the system $\mathcal{R}_{q+r}^{(s)}$ is formally integrable with the same solutions as the original system $\mathcal{R}_q$.

Remark 3.2 In the particular case of a semi-explicit system, we see that r must be zero, because of a null symbol, so that only equations of order less than q must be prolongated in order to bring it to a formally integrable one.

Now it is clear that the notion of formal integrability allows to completely explicit the whole set of equations defining a DAE system. In a given DAE system, some of them may not appear : they are sometimes called "hidden constraints". In fact, the formal integrable system obtained from an original one will bring an ODE system (explicit or not) together with a set of lower order equations defining the submanifold on which evolves the corresponding dynamical system. Of course, the set of equations so obtained will certainly be overdetermined. We are now in position to give our new definition for the index :

Definition 3.8 Let $\mathcal{R}_q \subset J_q(\mathcal{E})$ be a non linear, sufficiently regular system of DAE's of order q. The *index* of $\mathcal{R}_q$ is $r+s$, the number of prolongation needed to bring $\mathcal{R}_q$ into a formal integrable system. In the particular case of semi-explicit systems, it reduces to the integer s as given by theorem 3.2.

It is through the hypothesis "sufficiently regular" that regularity conditions are introduced but we see that nothing is said about the solutions themselves. Of course, this means that problems may occur if we met a singularity, in which case the rank of the tangent map is reduced but in such a situation the equations do not define a manifold so the preceding results do not remain valid. This is interesting in that it makes a real link between the three points of view : *formal, qualitative (bifurcations)* and *numerical*. Anyway, as we shall see in the first example below, this definition gives a somewhat different result than with the usual definition and will coincide only for the case of first order systems.

3.3 A small dictionnary

Before treating some examples in the next section, we are now in position to summarize the preceding results in a small "dictionnary" giving the translation of results concerning DAE's from the usual viewpoint to the formal one. It is useful to make links between several notions that appear scattered in the literature on one hand and the corresponding definitions of the formal theory that we gave above on the other hand. It is given by the following table :

Classical viewpoint	**Formal viewpoint**
$F(t,y,y')$	$\Phi(x,y_q^k)$
Pseudoderivative	Formal derivative
Dummy derivative	Jet coordinate
Differentiation	Prolongation
Elimination	Projection
Jacobian wrt first order derivatives	Symbol
Index	$r+s$ in $\mathcal{R}_q^{(r+s)}$

In fact, this table is more than a simple changing in the words, as we saw, because most of the notions were used without a clear framework (jet manifolds) and were restricted to first order systems. So this table should be seen as making a correspondence between notions particular to first order systems, scattered in various papers, and their generalization formulated in an intrinsic way.

4 Examples

In this section, we illustrate our results on several examples coming from mechanics, control theory. We present another interesting example given in [16] : it is an interesting one because it is overdetermined, a case not treated in the numerical analysis literature, to our knowledge. The last example extends beyond the primary goal of this paper and concerns a PDE system, mainly for illustration of the way the computations are carried out.

4.1 The pendulum

We naturally begin with the case of the simple pendulum, a prototype of lagrangian mechanical systems, the equations of which, in "descriptor form", we recall now :

$$\begin{aligned} x^{''} + Tx &= 0 \\ y^{''} + Ty + g &= 0 \\ x^2 + y^2 - 1 &= 0 \end{aligned} \tag{12}$$

where x, y are the cartesian coordinates, T is the tension and g the earth acceleration. The length is supposed to be equal to one. This system is very interesting because of its simplicity but also in that it reveals some subtleties that does not appear in the usual treatment.
We first rewrite this system to fit the notations introduced previously for the ease of computations with jet coordinates, so we consider the system $\mathcal{R}_2 \subset J_2(\mathcal{E})$:

$$\mathcal{R}_2 \left\{ \begin{aligned} y^1_{xx} + y^3 y^1 &= 0 \\ y^2_{xx} + y^3 y^2 + g &= 0 \\ (y^1)^2 + (y^2)^2 - 1 &= 0 \end{aligned} \right. \tag{13}$$

where the independent variable has been noted x, the superscripts and subscripts being in accordance with the general notation y^k_μ. The system has clearly a null symbol. Moreover it is not formally integrable as we check it at the first step (note that these computations are always done in the literature by first reducing the system to a first order one but in our framework it is not necessary and possibly worse as we shall see !). We give first a bound on the index s : the dimension of $J_2(\mathcal{E})$ is 9 (number of derivatives up to order 2). The system having codimension 3, we have : $1 \leq s \leq 6$, by counting the parametric jet components. So we begin the process, eliminating constants in factor at each step :

$$\mathcal{R}_2^{(1)} \left\{ \begin{aligned} y^1_{xx} + y^3 y^1 &= 0 \\ y^2_{xx} + y^3 y^2 + g &= 0 \\ (y^1)^2 + (y^2)^2 - 1 &= 0 \\ y^1 y^1_x + y^2 y^2_x &= 0 \end{aligned} \right. \tag{14}$$

where we recall that $\mathcal{R}_2^{(1)}$ stands for $\pi_2^3(\mathcal{R}_2)$. We prolongate more :

$$\mathcal{R}_2^{(2)} \left\{ \begin{aligned} y^1_{xx} + y^3 y^1 &= 0 \\ y^2_{xx} + y^3 y^2 + g &= 0 \\ (y^1)^2 + (y^2)^2 - 1 &= 0 \\ y^1 y^1_x + y^2 y^2_x &= 0 \\ (y^1_x)^2 + (y^2_x)^2 + y^1 y^1_{xx} + y^2 y^2_{xx} &= 0 \end{aligned} \right. \tag{15}$$

Projecting now gives :

$$\mathcal{R}_2^{(2)} \left\{ \begin{array}{rcl} y^1_{xx} + y^3 y^1 & = & 0 \\ y^2_{xx} + y^3 y^2 + g & = & 0 \\ (y^1)^2 + (y^2)^2 - 1 & = & 0 \\ y^1 y^1_x + y^2 y^2_x & = & 0 \\ (y^1_x)^2 + (y^2_x)^2 - y^3 - g y^2 & = & 0 \end{array} \right. \tag{16}$$

We continue prolongating :

$$\mathcal{R}_2^{(3)} \left\{ \begin{array}{rcl} y^1_{xx} + y^3 y^1 & = & 0 \\ y^2_{xx} + y^3 y^2 + g & = & 0 \\ (y^1)^2 + (y^2)^2 - 1 & = & 0 \\ y^1 y^1_x + y^2 y^2_x & = & 0 \\ (y^1_x)^2 + (y^2_x)^2 - y^3 - g y^2 & = & 0 \\ y^1_x y^1_{xx} + y^2_x y^2_{xx} - y^3_x - g y^2_x & = & 0 \end{array} \right. \tag{17}$$

that is, after projection :

$$\mathcal{R}_2^{(3)} \left\{ \begin{array}{rcl} y^1_{xx} + y^3 y^1 & = & 0 \\ y^2_{xx} + y^3 y^2 + g & = & 0 \\ (y^1)^2 + (y^2)^2 - 1 & = & 0 \\ y^1 y^1_x + y^2 y^2_x & = & 0 \\ (y^1_x)^2 + (y^2_x)^2 - y^3 - g y^2 & = & 0 \\ y^3_x + 3g y^2_x & = & 0 \end{array} \right. \tag{18}$$

Obviously we have to prolongate still, obtaining after projection and reordering of the equations :

$$\mathcal{R}_2^{(4)} \left\{ \begin{array}{rcl} y^1_{xx} + y^3 y^1 & = & 0 \\ y^2_{xx} + y^3 y^2 + g & = & 0 \\ y^3_{xx} + 3g y^2_{xx} & = & 0 \\ (y^1)^2 + (y^2)^2 - 1 & = & 0 \\ y^1 y^1_x + y^2 y^2_x & = & 0 \\ (y^1_x)^2 + (y^2_x)^2 - y^3 - g y^2 & = & 0 \\ y^3_x + 3g y^2_x & = & 0 \end{array} \right. \tag{19}$$

and now the system is formally integrable (in fact involutive, that is formally integrable with an involutive symbol) because it is obviously not possible to get more equations of order less by prolongating further. In other words, we have $\mathcal{R}_2^{(5)} = \mathcal{R}_2^{(4)}$. So we had to prolongate **four** times in order to get a formally integrable system : the system is *index four* and not three ! The difference comes from the fact that usually the computations are done by first reducing the system to a first order one, in order to suit the usual numerical solvers. In our example, the computations give the following equations (in usual notations) :

$$\left\{\begin{array}{rcl} \dot{x}-u & = & 0 \\ \dot{y}-v & = & 0 \\ \dot{u}+Tx & = & 0 \\ \dot{v}+Ty+g & = & 0 \\ x^2+y^2-1 & = & 0 \\ xu+yv & = & 0 \\ u^2+v^2+T+gy & = & 0 \\ \dot{T}+3gv & = & 0 \end{array}\right. \tag{20}$$

and the usual definition of the index, as well as ours, leads to an index three. But this is not sufficient ! The preceding system defines a submanifold of $J_1(X\times\mathcal{E})$, this one having $(t,x,y,T,\dot{x},\dot{y},\dot{T})$ as local coordinates or in usual notations (t,x,y,T,u,v,w) with $(\dot{T}=w)$, that is the dimension of the target is 6. But now it is necessary to add the equation $\dot{T}-w=0$, because we have $J_2(X\times\mathcal{E})\subset J_1(J_1(X\times\mathcal{E}))$ (we are dealing with an original **second** order system), defining $\dot{T}$ as a *new* indeterminate (even if it does not appear explicitly in the equations !). Thus, the system must be written as follows :

$$\left\{\begin{array}{rcl} \dot{x}-u & = & 0 \\ \dot{y}-v & = & 0 \\ \dot{T}-w & = & 0 \\ \dot{u}+Tx & = & 0 \\ \dot{v}+Ty+g & = & 0 \\ \dot{w}+3g(Ty+g) & = & 0 \\ x^2+y^2-1 & = & 0 \\ xu+yv & = & 0 \\ u^2+v^2+T+gy & = & 0 \\ w+3gv & = & 0 \end{array}\right. \tag{21}$$

That is we have now a simple ordinary dynamical system evolving on the submanifold defined by the last four equations in a six dimensional space, leading so to an overdetermined system as it will always be the case with the formal approach.

4.2 Example 2

This example is taken from [12]. It is defined by the following linear system :

$$\mathcal{R}_2\left\{\begin{array}{rcl} y^1+y^2 & = & u^1 \\ y^1+y^2+y^3 & = & u^2 \\ y^1+y^3_x+y^4 & = & u^3 \\ 2y^1_{xx}+y^2_{xx}+y^3_{xx}+y^4_x & = & u^4 \end{array}\right. \tag{22}$$

where we have simply changed the sign of $u^i, i=1,..,4$ in order to separate the input from the output. The u^i are input functions so this is a simple linear control system. It is clear that we must prolongate twice the first two equations and the third once :

$$\mathcal{R}_2^{(2)} \left\{ \begin{array}{rcl} y^1 + y^2 & = & u^1 \\ y^1 + y^2 + y^3 & = & u^2 \\ y^1 + y_x^3 + y^4 & = & u^3 \\ y_x^1 + y_x^2 & = & u_x^1 \\ y_x^1 + y_x^2 + y_x^3 & = & u_x^2 \\ y_{xx}^1 + y_{xx}^2 & = & u_{xx}^1 \\ y_{xx}^1 + y_{xx}^2 + y_{xx}^3 & = & u_{xx}^2 \\ y_x^1 + y_{xx}^3 + y_x^4 & = & u_x^3 \\ 2y_{xx}^1 + y_{xx}^2 + y_{xx}^3 + y_x^4 & = & u^4 \end{array} \right. \tag{23}$$

and this system is clearly formally integrable ($\mathcal{R}_2^{(3)} = \mathcal{R}_2^{(2)}$) so that $\mathcal{R}_2$ is index two. In [12], it is said (p. 682) that the prolongated system is index 0 because the jacobian wrt highest derivatives is nonsingular. It must be said that the conclusion is not true in general and the next example will show it and there seems to be a confusion between the involutiveness of the symbol and the formal integrability (although not formulated as such). In fact, we chose this example to illustrate another feature of the formal theory related to control theory. One interesting question concerns the inputs : how many of them may be given arbitrarily ? On one hand, it is clear that if all the outputs are given, the inputs are uniquely determined. On the other hand, we may ask if for any inputs we are able to compute the outputs in a unique fashion or in other words if a resolvent system exists for the inputs of this system. The answer is not evident a priori, although it is of usual practice in control community to suppose that the u^i may be arbitrary. In the present case, it can be shown that ([19]) the system is controllable, having no such resolvent system for the inputs nor the outputs. The next example will illustrate the opposite case. The problem of compatibility conditions for differential operators has been addressed theoretically for linear systems ($\mathcal{D}y = \mathcal{D}'u$ for linear operators) in [7]. The case of inhomogeneous linear systems that are not formally integrable (as $\mathcal{R}_2$ here) has been studied in [16] and used in [17, 19] in the framework of control theory. In the same manner, the choice of a set of initial values for the dependent variables is a closely related problem that may be set in this framework and for which we refer to the same. What can be simply said is that a set of consistent initial value must lay on $\mathcal{R}_q$ and then must satisfy all the definition equations of the submanifold. For example, in the case of the pendulum, we obtained a formally integrable system with four zero order equations in a space of dimension six so we can fix arbitrarily any two dependent variables and deduce the others to obtain the desired set of initial values. Several other questions about control systems such as causality, invertibility, controllability can be answered in the unique framework of formal theory (see [17, 18, 19]) reinforcing in our opinion its interest.

4.3 The Vessiot example

This example is taken from [16] (J.F. Pommaret suggested it to us) and was first given by E. Vessiot, a french mathematician, at the beginning of the century in another context. It is an interesting system because of its overdetermination, a case never treated in the numerical analysis literature to our knowledge. In such a situation, the classical definition is of no use because all the first derivatives may be explicit in the original system although other equations of order less may be present. The following example is overdetermined, with three linear equations of order three and one nonlinear of order zero :

$$\mathcal{R}_3 \left\{ \begin{array}{rcl} y^1_{xxx} - p(x)y^1_{xx} + q(x)y^1_x - r(x)y^3 & = & 0 \\ y^2_{xxx} - p(x)y^2_{xx} + q(x)y^2_x - r(x)y^2 & = & 0 \\ y^3_{xxx} - p(x)y^3_{xx} + q(x)y^3_x - r(x)y^1 & = & 0 \\ (y^3)^2 - y^1y^2 & = & 0 \end{array} \right. \tag{24}$$

Its symbol is obviously zero. We must point out that with the classical approach of the index and after reduction to a first order one, with 9 ODE and one nonlinear algebraic one, there is no way to even define what the index is (all the first derivatives are already explicit), although this system is perfectly meaningful. On the other hand, the formal theory point of view leads to a similar treatment than for "square" systems. To show it, we present the calculations made in [16] (p. 378-380). First, certain intermediary quantities are to be defined :

$$\left\{ \begin{array}{rcl} a & \equiv & (y^3)^2 - y^1y^2 \\ b & \equiv & (y^3_x)^2 - y^1_xy^2_x \\ c & \equiv & (y^3_{xx})^2 - y^1_{xx}y^2_{xx} \\ A & \equiv & y^3_xy^3_{xx} - \frac{1}{2}(y^1_xy^2_{xx} + y^2_xy^1_{xx}) \\ B & \equiv & y^3y^3_{xx} - \frac{1}{2}(y^1y^2_{xx} + y^2y^1_{xx}) \\ C & \equiv & y^3y^3_x - \frac{1}{2}(y^1y^2_x + y^2y^1_x) \end{array} \right.$$

The symbol of $\mathcal{R}_3$ is, as we said before, obviously zero, because $a = 0$. In order to get a formally integrable system, we just have to prolongate the last equation ($a = 0$) until a new prolongation does not bring new equations of order less : when this is achieved, we just have to count the number of successive prolongations, which is given by the exponent s in $\mathcal{R}^{(s)}_3$:

$$\left\{ \begin{array}{rcl} \mathcal{R}_3 & \Rightarrow & a = 0 \\ \mathcal{R}^{(1)}_3 & \Rightarrow & C = 0 \\ \mathcal{R}^{(2)}_3 & \Rightarrow & b + B = 0 \\ \mathcal{R}^{(3)}_3 & \Rightarrow & 3A + p(x)B = 0 \\ \mathcal{R}^{(4)}_3 & \Rightarrow & -3q(x)b + 3c + 4p(x)A + (\partial_x p(x) + p^2(x))B = 0 \\ \mathcal{R}^{(5)}_3 & \Rightarrow & -(4p(x)q(x) + 3\partial_x q(x))b + 10p(x)c \\ & & +(5\partial_x p(x) + 5p^2(x) - 12q(x))A \\ & & +(\partial_{xx}p(x) + 3p(x)\partial_x p(x) + p^3(x) + 6r(x))B = 0 \end{array} \right. \tag{25}$$

It is worth noting that until the fifth step, in the general case, we only have equations of order strictly less than 3 and consequently the systems then obtained are not formally integrable, leading to a new prolongation. Now, this set of 6 equations is a linear system in the 6 unknowns a, b, c, A, B, C, the determinant of which is :

$$D \equiv 9\partial_{xx}p(x) - 18p\partial_x p(x) + 27\partial_x q(x) + 4p^3(x) - 18p(x)q(x) + 54r(x)$$

and this gives the (non obvious) compatibility condition on p, q, r in order for the system to be formally integrable : depending on wether this condition is satisfied or not, the system will not have the same index. Hence we consider two cases : $D = 0$ or $D \neq 0$.

1. $D = 0$
   Then we can verify that $\mathcal{R}_3^{(4)} = \mathcal{R}_3^{(5)}$ and $\mathcal{R}_3^{(4)}$ is formally integrable : the system $\mathcal{R}_3$ is index 4 in our sense.

2. $D \neq 0$
   In that case, the only solution to the linear system is the zero solution : $a = b = c = A = B = C = 0$: $\mathcal{R}_3^{(2)}$ to $\mathcal{R}_3^{(5)}$ are of no more relevance and the system is index one. Then, using $a = b = 0$ and substituting in $C^2 = 0$, we obtain the following relations :

$$(y^1 y_x^2 - y_x^1 y^2)^2 = 0 \Rightarrow \frac{y_x^1}{y^1} = \frac{y_x^2}{y^2} = \frac{y_x^3}{y^3}$$

We see that such a system could not be correctly investigated without the formal approach although the “structural” questions are in that case as relevant as for “square” systems.

4.4 A PDE example

The last example we present is in fact a PDE system. It is presented for example in [2] and concerns the modelisation of combustion in a cylindric vessel. It is beyond the scope of the present paper to work out such an example in full detail, because we have restricted the definitions and theorems to the case with just one independent variable. Nevertheless, we think that it is likely that a PDE system such that the one below should be first prolonged to a formally integrable form and only then discretized in order to solve it numerically; otherwise missing equations in the PDE system will be missed too in a DAE system obtained for example through the method of lines (see [2]), leading probably to untractable problems during the numerical resolution. Let us just give the first step of the procedure, in order to show that it works the same albeit more tricky way. The original system is the following:

$$\left\{ \begin{array}{rcl} \frac{\partial T}{\partial t} - \frac{1}{\rho c_p}\frac{\partial p}{\partial t} & = & \frac{1}{c_p}\frac{\partial}{\partial \psi}(\rho r^2 \lambda \frac{\partial T}{\partial \psi}) \\ \frac{\partial r}{\partial \psi} - \frac{1}{\rho r} & = & 0 \\ \frac{\partial p}{\partial \psi} & = & 0 \\ p - \rho \frac{RT}{W} & = & 0 \end{array} \right. \tag{26}$$

This system has 2 independent variables (t, ψ), 3 dependent variables (T, ρ, r, p), the other terms being constants. Modifying the notations in order to fit jet notations and eliminating the dependent variable p with the help of the last equation, we obtain the following system:

$$\left\{ \begin{array}{rcl} y_1^1 - \frac{a}{y^2}(y_1^2 y^1 + y_1^1 y^2) & = & b(y^2 (y^3)^2 y_2^1)_2 \\ y_2^3 - \frac{1}{y^2 y^3} & = & 0 \\ y_2^2 y^1 + y_2^1 y^2 & = & 0 \end{array} \right. \tag{27}$$

with: $x^1 = t$, $x^2 = \psi$, $y^1 = T$, $y^2 = \rho$, $y^3 = r$.

This system has a null symbol so that it is trivially involutive. We then have to prolong it in order to find a formally integrable system with the same solutions. This gives:

$$\left\{ \begin{array}{rcl} by^2(y^3)^2 y^1_{22} + b(y^2(y^3)^2)_2 y^1_2 - y^1_1 + \frac{a}{y^2}(y^2_1 y^1 + y^1_1 y^2) & = & 0 \\ y^3_{12} + \frac{y^2_1 y^3 + y^2 y^3_1}{(y^2 y^3)^2} & = & 0 \\ y^3_{22} + \frac{y^2_2 y^3 + y^2 y^3_2}{(y^2 y^3)^2} & = & 0 \\ y^2_{12} y^1 + y^2_2 y^1_1 + y^1_{12} y^2 + y^1_2 y^2_1 & = & 0 \\ y^2_{22} y^1 + y^1_{22} y^2 + 2 y^1_2 y^2_2 & = & 0 \\ y^3_2 - \frac{1}{y^2 y^3} & = & 0 \\ y^2_2 y^1 + y^1_2 y^2 & = & 0 \end{array} \right. \tag{28}$$

We see that every prolongation is made with respect to all the independent variables and that is one reason for the greater complexity of the formal study of PDE systems.

5 Conclusion

The notion of index has proved to be crucial for characterizing DAE systems with respect to their numerical solution. In our opinion, it suffers until now of a lack of rigorous definition because of no well established foundations. Our first purpose in this paper was to gather several notions related to the index but scattered in the literature and with no general framework to relate them, except local coordinates computations. We think that the right framework for dealing with formal ("structural") properties of differential systems is the modern differential geometry, including jet theory : we have used all this material to present a unified vision of some computations in relation with the index. This led us to an intrinsic definition through a test of algebraic nature (see theorem 3.1) that could be applied the same way to overdetermined systems. We have shown that the test procedure for formal integrability of DAE systems stops in a finite number of steps, through simple dimension computations in jet spaces. Finally, our interpretation of the index was made in a "top-down" fashion, using concepts coming from PDE's : this approach makes links between the theory of ODE's (DAE's) and the theory of PDE's that could not have appeared with the classical definition of the index (in a "bottom-up" fashion). Finally, we think that this approach could be very useful, for example when studying DAE's coming from the discretization of PDE's (method of lines e.g.) and we have presented on a simple PDE example one possible direction in which the formal theory could be used in a PDE altogether with DAE context.

6 Appendix : Basic facts from differential geometry

X is a connected paracompact manifold with dimension 1.

Definition 6.1 $\mathcal{E}$ is said to be a *fibered manifold* over X with *projection* π if, for any point of $\mathcal{E}$ there exists a coordinate neighbourhood $\mathcal{U}$ of this point in $\mathcal{E}$ and a local chart $(\mathcal{U},\varphi)$

of X with $\mathcal{U}=\pi(\mathcal{U})$ such that the following diagram commutes.

$$\begin{array}{ccccccc} & \mathcal{E} & \supset & \mathcal{U} & \xrightarrow{\Phi} & \mathbb{R}^n \times \mathbb{R} \\ \pi & \downarrow & & \downarrow & \varphi & \downarrow \\ & X & \supset & U & \longrightarrow & \mathbb{R}^n \end{array}$$

Definition 6.2 X is called the *base manifold*, $\mathcal{E}$ the *total manifold*. $\forall x \in X$, the closed submanifold $\mathcal{E}_x = \pi^{-1}(x) \subset \mathcal{E}$is called the *fiber* over X. When the fiber over x is an affine (resp. vector) space, $\mathcal{E}$ is called an affine (resp. vector) bundle.

Example 6.1 If X and Y are two manifolds, their product, together with the projection onto the first factor $\pi : X \times Y \to X$, is a fibered manifold.

Definition 6.3 A *local section* of $\mathcal{E}$ over an open set $U \subset X$ is a map $f : U \to \mathcal{E}$ such that $\pi \circ f(x) = x, \ \forall x \in U$.

Definition 6.4 Let f and g be two sections of a fibered manifold $\pi : \mathcal{E} \to X$ and $x \in dom(f) \cap dom(g)$. f and g are said to be equivalent up to order q (or q-equivalent) if $\partial_\mu f^k(x) = \partial_\mu g^k(x), \ \forall 0 \leq |\mu| \leq q$. This clearly defines an equivalence relation between sections of $\mathcal{E}$.

Definition 6.5 Let X be a manifold of dimension $n = 1$ with local coordinate x and $\mathcal{E}$ be a fibered manifold over X with fiber dimension m and local coordinates (x, y^k). The *q-jet bundle of* $\mathcal{E}$ is the bundle of classes of sections of $\mathcal{E}$ that are q-equivalent in x. It is a fibered manifold, $J_q(\mathcal{E})$ with local coordinates (x, y^k_μ) with : $\mu \leq q$. For $\mu = 0$, we define $y^k_0 = y^k$.

This notion of jet is useful to work in a geometric framework : it allows the translation of the vague notion of "condition on the derivatives" into the more precise one of submanifold (of a convenient manifold : the jet bundle). For example, the 1-jet of a real function $f(x)$ is defined by giving x, the value of $f(x)$ and the value of $f^{'}(x)$. Equation (1) is then nothing else than a submanifold of the bundle of 1-jets $J_1(\mathbb{R})$.
A section of $J_q(\mathcal{E})$will be denoted by f_q and to any section f of $\mathcal{E}$ corresponds a section $j_q(f)$ of $J_q(\mathcal{E})$, defined as follows :

$$j_q(f) : x \longrightarrow (x, \partial_q f^k(x))$$

We define the canonical projections π^{q+r}_q , $\forall q, r \geq 0$:

$$\begin{array}{rcl} \pi^{q+r}_q : J_{q+r}(\mathcal{E}) & \longrightarrow & J_q(\mathcal{E}) \\ (x, y_{q+r}) & \longrightarrow & (x, y_q) \end{array}$$

Intuitively, this corresponds to a truncation at order q of Taylor series at order $q + r$.
For more details on these aspects we refer the reader to [17] in which he will find the necessary material and to [1] for a more theoretical exposition.

7 Notations

Superscripts denote components of vectors and subscripts denote order of differentiation or of a jet.

x	local coordinate for the ground manifold (independent variable)
f	local section of a manifold
j_q	jet of order q (q-jets)
r, s	order of prolongations
$\partial_q y^k$ or $y^k_{xx..x}$	q-th derivative of y^k with respect to x or q-jet component
y^k_q	q-jet component of y^k
y_x or $\dot{y}$	first derivative of y with respect to x (1-jet)
y_{xx} or $\ddot{y}$	second derivative of y with respect to x (2-jet)
π	projection of a fibered manifold
$J_q(\mathcal{E})$	q-jets manifold
$\mathcal{R}_q$	DAE system as submanifold of $J_q(\mathcal{E})$
$\rho_r(\mathcal{R}_{q+r})$	r-prolongation of $\mathcal{R}_q$
$\mathcal{R}^{(s)}_{q+r}$	projection $\pi^{q+r+s}_{q+r}(\mathcal{R}_{q+r+s})$

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