Asymptotic expansions for functionals processes

Alexander A. Borovkov

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

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Abstract: The article provides the direct approach to obtaining formulas for derivatives of functionals of point processes in rare perturbation analysis ([1], [2]). Results are obtained for arbitrary (not necessarily stationary) point processes in R and R^d, $d \geq 2$, under transparent conditions, close to minimal. Formulas for higher order derivatives allow to construct asymptotical expansions.

The results can be useful in sensitivity analysis, in light traffic theory for queues and for computation by simulation of derivatives at positive intensity while the computation of the derivatives via statistical estimation of the functional itself and its increments usually gives poor results.

Key-words: Sensitivity analysis, rare perturbation analysis, light traffic, derivatives of functionals of point processes, asymptotic expansions for functionals of point processes. AMS 1991 Subject Classification: 60G55, 68M20

(Résumé : tsvp)

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*Institute of Mathematics,Novosibirsk

Unité de recherche INRIA Sophia-Antipolis
2004 route des Lucioles, BP 93, 06902 SOPHIA-ANTIPOLIS Cedex (France)
Téléphone : (33) 93 65 77 77 – Télécopie : (33) 93 65 77 65

Développements Asymptotiques de Fonctionnelles de Processus Ponctuels Dilatés

Résumé : Cet article propose une approche directe pour obtenir les dérivées de fonctionnelles de processus ponctuels dans le cadre de l'analyse des perturbations rares ([1], [2]). Les formules donnant les dérivées sont obtenues pour des processus ponctuels non nécessairement stationnaires, dans R ou R^d, sous des conditions proches des conditions minimales. Les formules obtenues pour les dérivées de divers ordres permettent aussi de construire des développements asymptotiques.

Ces résultats peuvent être utiles en analyse de sensibilité, dans le cadre de l'analyse de files d'attente avec trafic faible, et dans le cadre du calcul de gradients par simulation, notamment dans le cas où le calcul des incréments des estimateurs donne de mauvais résultats.

Mots-clé : Analyse de sensibilité, analyse des perturbations rares, trafic faible, dérivées de fonctionnelles de processus ponctuels, développements asymptotiques.

0 Introduction

Consider a marked point process Φ on R. We will represent it as a sum

$$\Phi = \sum_{i=-\infty}^{\infty} \delta_{T_i}^{\mu_i} ,$$

where $\ldots < T_{-1} < T_0 < 0 \leq T_1 < T_2 < \ldots$ form the point process itself, the sequence $\{\mu_i\}$ represents random marks, $\delta_{T_i}^{\mu_i}$ is the process with only one event at the point T_i and with mark μ_i. The joint distribution of $\{T_i, \mu_i\}$ is assumed to be given.

The subject of investigation is the asymptotic behaviour of

$$f(\Phi_\rho) = f\left(\sum_{i=-\infty}^{\infty} \delta_{\frac{T_i}{\rho}}^{\mu_i}\right)$$

as $\rho \to 0$ for given class of functionals f and, in particular, asymptotic expansions for $Ef(\Phi_\rho)$ in powers of ρ (about more general statement of the problem see Remark 2 in Section 1).

The first results in this direction were obtained apparently in [1] and later on in [2], [3] (see also more detailed bibliography there).

The paper [1] dealt only with Poisson stationary point processes in R. More general stationary processes in R were considered in [2]–[3] with essential use of corresponding thechniques like Campbell's formula etc. Stationary processes in more general state spaces, including R^d, were considered in [4]. As to conditions used in [2]–[4] one has to note that they assume some properties of processes and of functionalf which in fact are not necessary (stationarity, for special cases–monotonicity, etc...). These conditions require further investigation to find out what properties of the process, expressed in terms of its characteristics, are equivalent to them.

In section 1 we present a direct approach which does not require stationarity of the processes (though stationarity some times makes the verifiction of some conditions much easier) and uses transparent conditions which are necessary or close to them. In section 2 this approach is extended to computation of higher order derivatives for which we give the explicite formulas by the example of second derivatives. In section 3 we show that the same approach works for point processes in R^d, $d \geq 2$. Applications are given to the non stationary Poisson field.

1 One–dimentional case. First term of expansion

We shall need the following assumptions concerning the distribution of $\{T_i, \mu_i\}$ and functional f.

($\mathbf{A_1}$). *The distributions of T_0 and T_1 have bounded densities $p_0(t)$, $p_1(t)$ respectively; there exist limits $p_0(-0) = p_0(0)$, $p_1(+0) = p_1(0)$.*

($\mathbf{A_2}$). *The intensity measure*

$$\Lambda(B) = \sum_i P(T_i \in B)$$

has density

$$\lambda(t) = \frac{\Lambda(dt)}{dt} \ ;$$

there exist limits

$$\lambda(+0) = p_1(0), \ \lambda(-0) = p_0(0) \ .$$

($\mathbf{A_3}$). $P((T_0, T_1) \in (-\varepsilon, \varepsilon)) = o(\varepsilon)$ *as* $\varepsilon \to 0$.

Instead of (A_3), one can consider the condition of the form

($\widetilde{\mathbf{A_3}}$). $p_0(-\rho t)P(T_1 < \rho t / T_0 = -\rho t) \to 0$ *as* $\rho \to 0$ *for each fixed* t.

In fact condition (A_3) will be used in the sequel only to obtain convergence of the form $(\widetilde{A}_3)$ (see Lemma 2).

As special cases of the process $T = \{T_i\}$ under consideration one can have in mind

Example 1 $T = \{T_i\}$ is ordinary stationary point process, generated by a stationary sequence $\{\tau_i\}$ with $P(\tau_i > 0) = 1$.

Example 2 T is a Poisson process with given intensity measure $\Lambda(B)$ with density $\lambda(t) = \frac{\Lambda(dt)}{dt}$, continuous at point $t = 0$.

Example 3 T is constructed as follows: let $\{\tau_i\}$ be an arbitrary sequence of r.v., $P(\tau_i > 0) = 1,\ S_k = \tau_i + \ldots + \tau_k,\ \nu = \min\{k\ :\ S_k \geq L\}$ for some fixed $L \geq 0$. We put $T_1 = -L + S_\nu \geq 0,\ T_i = T_1 + \tau_{\nu+1} + \ldots + \tau_{\nu+i-1}$ for $i \geq 2;\ T_i = T_1 - \tau_\nu - \ldots - \tau_{\nu-i}$ for $i \leq 0$, so $T_0 = -L + S_{\nu-1} < 0$.

Conditions (A) are always fullfilled in Example 1. In this case $\lambda(t) = \lambda =$ const,

$$
\begin{aligned}
P(T_1 > t) &= P(T_0 < -t) = (E\tau_1)^{-1} \int_0^t P(\tau_1 > u)du \ , \\
p_1(t) &= p_0(-t) = (E\tau_1)^{-1} P(\tau_1 > t) \ , \\
\lambda &= (E\tau_1)^{-1} = p_1(0) = p_0(0) \ .
\end{aligned}
$$

In Example 2

$$p_1(t) = \lambda(t) e^{-\Lambda((0,t))} \ .$$

A similar identity holds for $p_0(t)$, so T_1 and T_0 have densities, $p_1(0) = p_1(0) = \lambda(0)$.

In Example 3, T_1 and T_0 have densities, continuous at point $t = 0$ if all sums $S_k - L$ have densities, continuous at this point.

In all three cases there exists density $\lambda(t)$ of intensity $(\lambda(t)dt = \sum P(T_i \in dt))$, continuous at point 0, $p_1(0) = p(0) = \lambda(0)$. The last property is fulfilled in many practically interesting models.

Consider now conditions (B) for functional f. Denote $\mathcal{F}_{s,r}$ the σ–algebra generated by $T_s, \ldots, T_r$; $\mathcal{F}_s = \mathcal{F}_{s,s}$.

$(\mathbf{B_1})$. *There exists a functional* $g \geq 0$, *such that for any* $0 \leq l \leq L,\ 0 \leq m \leq M$

$$\left| E\left[f\left(\sum_{i=-L}^{M} \delta_{T_i}^{\mu_i} \right) / \mathcal{F}_{-L,M} \right] - E\left[f\left(\sum_{i=-l}^{m} \delta_{T_i}^{\mu_i} \right) / \mathcal{F}_{-l,m} \right] \right| \leq \sum_{\substack{i\in[-L,M] \\ i\notin[-l,.m]}} E[g(\delta_{T_i}^{\mu_i})/\mathcal{F}_i] \ ,$$

where g satisfies some integrability conditions, which will be given in (B_2).

If one denotes the values of conditional expectations

$$E\left[f\left(\sum_{i=s}^{r} \delta_{\frac{T_i}{\rho}}^{\mu_i} \right) / \mathcal{F}_{s,r} \right] \ , \ E\left[g\left(\delta_{\frac{T_i}{\rho}}^{\mu_i} \right) / \mathcal{F}_i \right]$$

at point $T_i = t_i\rho, \ i = s, \ldots, r$, as

$$\begin{aligned} Ef\left(\sum_{i=s}^{r} \delta_{t_i}^{\mu_i} \right) &= f_{s,\ldots,r}(t_s, \ldots, t_r) \ , \\ Eg(\delta_{t_i}^{\mu_i}) &= g_i(t_i) \end{aligned}$$

respectively ($f_{s,\ldots,r}(t_s, \ldots, t_r)$ is defined for $t_s < \ldots < t_0 < 0 \leq t_1 < \ldots < t_r$), then condition (B_1) can be rewritten in the form

$$|f_{-L,\ldots,M}(t_{-L}, \ldots, t_M) - f_{-l,\ldots,m}(t_{-l}, \ldots, t_m)| \leq \sum_{\substack{i\in[-L,M] \\ i\notin[-l,m]}} g_i(t_i) \ .$$

In particular

$$|f_i(t_i) - f(\emptyset)| \leq g_i(t_i)$$

where $\emptyset$ is the "empty" process. We can put without loss of generality $f(\emptyset) = 0$.

($\mathbf{B_2}$). $g_i(t) \leq g(t)$, *where g is independent of ρ,*

$$\int_{-\infty}^{\infty} g(t)dt < \infty \ .$$

For all small enough ρ

$$\lambda(\rho t)g(t) \leq g^{(\lambda)}(t) \ , \ \int_{-\infty}^{\infty} g^{(\lambda)}(t)dt < \infty \ .$$

We can take here and later on without loss of generality that $g(t)$ to be monotone with respect to $|t|$.

If $\lambda(t)$ is bounded then the second condition in (B_2) always follows from the first one. According to (B_2) intensity density $\lambda(t)$ can be increasing as $|t| \to \infty$.

Note that if distributions of μ_i depend on T, then conditional expectations $f_{s,\ldots,r}(t_s,\ldots,t_r)$, $g_i(t_i)$, generally speaking, depend on ρ. We need in this case condition

(C). *There exist limit conditional distributions*

$$\lim_{\rho \to 0} P\left(f(\delta_{t_i}^{\mu_i}) < u / T_i = \rho t_i\right) \ , \ i = 0, 1 \ ,$$

in a sense of weak convergence.

We denote the random variables with this distributions and their expectations $f^0(\delta_{t_i}^{\mu_i})$ and $f_i^0(t_i) = \lim_{\rho \to 0} f_i(t_i)$ respectively. If μ_i are independent of T, then condition (C) is not needed, $f_i^0(t_i) = f_i(t_i)$.

Theorem 1 *Under conditions* $(A), (B), (C)$

$$Ef(\Phi_\rho) = a_1 \rho + o(\rho) \ , \tag{1}$$

where

$$a_1 = p_0(0) \int_{-\infty}^{0} f_0^0(t)dt + p_1(0) \int_0^{\infty} f_1^0(t)dt \ .$$

For any point $u > 0$ *of continuity of*

$$\pi(u) = p_0(0) \int_{-\infty}^{0} P(f^0(\delta_t^{\mu_0}) > u)dt + p_1(0) \int_0^{\infty} P(f^0(\delta_t^{\mu_i}) > u)dt \ ,$$

we have

$$P(f(\Phi_\rho) > u) = \rho \pi(u) + o(\rho) \ . \tag{2}$$

Similar formula holds for $P(f(\Phi_\rho) < -u), \ u > 0$.

Remark 1 If $\mu = \{\mu_i\}$ is a stationary sequence, independent of T, then $f_i(t) = Ef(\delta_t^{\mu_i})$ does not depend on i and can be denoted as $f(t)$. In this case for $p_0(0) = p_1(0) = \lambda(0)$ the first assertion of Theorem 1 can be rewritten in shorter form

$$Ef(\Phi_\rho) = \rho\lambda(0)\int_{-\infty}^{\infty} f(t)dt + o(\rho) \ .$$

Proof : According to condition (B)

$$\begin{aligned} Ef(\Phi_\rho) &= Ef\left(\delta_{T_0/\rho}^{\mu_i} + \delta_{T_1/\rho}^{\mu_1}\right) + R \ , \qquad (3) \\ |R| &\le \sum_{i\neq 0,1} Eg(T_i/\rho) \ . \end{aligned}$$

Let us estimate the main term. Denote

$$\Delta_{0,1}(t_0,t_1) = E[f(\delta_{t_0}^{\mu_0} + \delta_{t_1}^{\mu_1}) - f(\delta_{t_0}^{\mu_0}) - f(\delta_{t_1}^{\mu_1})] \ , \qquad (4)$$

where we understand $f\left(\sum_{i=r}^s \delta_{t_i}^{\mu_i}\right)$ as a r.v. with distribution which is conditional distribution of $f\left(\sum_{i=r}^s \delta_{T_i}^{\mu_i}\right)$ with respect to $(T_r,\dots,T_s)$ at the point $T_r = t_r,\dots,T_s = t_s$. Then

$$Ef\left(\delta_{T_0/\rho}^{\mu_0} + \delta_{T_1/\rho}^{\mu_1}\right) = E\Delta_{0,1}\left(\frac{T_0}{\rho},\frac{T_1}{\rho}\right) + Ef\left(\delta_{T_0/\rho}^{\mu_0}\right) + Ef\left(\delta_{T_1/\rho}^{\mu_1}\right) \ . \qquad (5)$$

It follows from (A) that $(t = \rho u)$

$$\begin{aligned} Ef\left(\delta_{T_1/\rho}^{\mu_1}\right) &= \int_0^\infty p_1(t)Ef(\delta_{t/\rho}^{\mu_i})dt = \qquad (6) \\ & \rho\int_0^\infty p_1(u\rho)Ef(\delta_u^{\mu_i})du = \rho\left(p_1(0)\int_0^\infty f_1^0(u)du + o(1)\right) \end{aligned}$$

as $\rho \to 0$ since $p_1(u\rho) \to p(0)$, $p_1(u\rho)f_1(u)$ has integrable majorant $cg(u)$. Similarly

$$Ef\left(\delta^{\mu_0}_{T_0/\rho}\right) = \rho\left(p_0(0)\int_{-\infty}^{0} f_0^0(u)du + o(1)\right) \ . \tag{7}$$

We shall need the following lemmas

Lemma 1 $|\Delta_{0,1}(t_0,t_1)| \leq 2\ min\ (g(t_0), g(t_1))$.

Proof : For $t_1 \geq |t_0|$ we have according to (B)

$$|\Delta_{0,1}(t_0,t_1)| \leq |Ef(\delta^{\mu_1}_{t_1})| + g_1(t_1) \leq 2g(t_1) \ .$$

Together with a similar inequality for $|t_0| > t_1$, this proves the lemma.

Lemma 2

$$a_0(\rho t) \equiv p_0(-\rho t)P(T_1 < \rho t / T_0 = -\rho t) \to 0 \ , \tag{8}$$

$$a_1(\rho t) \equiv p_1(\rho t)P(T_0 > -\rho t / T_1 = \rho t) \to 0 \ . \tag{9}$$

with respect to Lebesgue measure on $[0, H]$ *for any fixed* $H > 0$.

Proof : By (A_3)

$$\begin{aligned} o(\rho) &= P(T_1 < H\rho, T_0 > -H\rho) \geq P(T_1 < -T_0, T_0 > -H\rho) \\ &= \int_0^H P(T_0 \in -d(\rho t))P(T_1 < \rho t / T_0 = -\rho t) \\ &= \rho\int_0^H p_0(-\rho t)P(T_1 < \rho t / T_0 = -\rho t)dt \ . \end{aligned}$$

It implies

$$\int_0^H a_0(\rho t)dt \to 0 \ .$$

Since $a_0(\rho t) \geq 0$ it proves (8). The proof of (9) is the same.

If to use $(\widetilde{A}_3)$ then we won't need Lemma 2.

Lemma 3 $E\left|\Delta_{0,1}\left(\frac{T_0}{\rho}, \frac{T_1}{\rho}\right)\right| = o(\rho)$.

Proof : By Lemma 1

$$\begin{aligned} E\left|\Delta_{0,1}\left(\frac{T_0}{\rho},\frac{T_1}{\rho}\right)\right| &= \int_{-\infty}^{0}\int_{0}^{\infty} P(T_0\in dt_0, T_1\in dt_1)\left|\Delta_{0,1}\left(\frac{t_0}{\rho},\frac{t_1}{\rho}\right)\right| \\ &\le \int_{t_0=-\infty}^{0}\int_{t_1<-t_0} + \int_{t_1=0}^{\infty}\int_{t_0\ge -t_1} \qquad (10) \\ &\le 2\int_{t_0=-\infty}^{0} g\left(\frac{t_0}{\rho}\right) P(T_0\in dt_0, T_1<-t_0) \\ &\quad +2\int_{t_1=0}^{\infty} g\left(\frac{t_1}{\rho}\right) P(T_1\in dt_1, T_0\ge -t_1)\ . \end{aligned}$$

The first term here does not exceed

$$2\rho\int_{-\infty}^{0} g(t)p_0(t\rho)P(T_1<-\rho t/T_0=\rho t)dt = 2\rho\int_0^{\infty} a_0(\rho t)g(-t)dt = o(\rho)$$

(see Lemma 2). The same holds for the second term in (10). Lemma 3 is proved.

It follows from (5)–(7) and Lemma 3 that

$$Ef\left(\delta^{\mu_0}_{T_0/\rho}+\delta^{\mu_0}_{T_1/\rho}\right) = \rho\left[p_0(0)\int_{-\infty}^{0} f_0(u)du + p_1(0)\int_0^{\infty} f_1(u)du\right] + o(\rho)\ .$$

It remains to prove that $R=o(\rho)$ (see (3)). We have

$$\begin{aligned} \sum_{i=2}^{\infty} Eg(T_i/\rho) &= \int_0^{\infty}\sum_{i=2}^{\infty} P(T_i\in dt)g\left(\frac{t}{\rho}\right) \qquad (11) \\ &= \int_0^{\infty}(\lambda(t)-p_1(t))g\left(\frac{t}{\rho}\right)dt = \rho\int_0^{\infty} g(u)(\lambda(u\rho)-p_1(u\rho))du\ . \end{aligned}$$

By condition (A) $\lambda(u\rho)-p_1(u\rho)\to 0$ as $\rho\to 0$ for each u. Besides by (B_2)

$$\begin{aligned} 0\le\lambda(u\rho)-p_1(u\rho) &\le \lambda(u\rho)\ , \\ g(u)\lambda(u\rho) &\le g^{(\lambda)}(t) \end{aligned}$$

for all small enough ρ. It means that integral in (11) goes to 0 as $\rho \to \infty$. The same will be true for $\sum_{i=-\infty}^{-1} Eg\left(\frac{T_i}{\rho}\right)$. The first assertion of the theorem is proved.

It was proved in fact that

$$\begin{aligned} f(\Phi_\rho) &= \xi_0\left(\frac{T_0}{\rho}\right) + \xi_1\left(\frac{T_1}{\rho}\right) + \eta\,, \quad E|\eta| = o(\rho)\,, \\ \xi_i(t_i) &= f(\delta_{t_i}^{\mu_i})\,, \quad i = 0, 1\,, \end{aligned}$$

where μ_i have corresponding conditional distributions.

For any $\alpha > 0$ we have

$$P(f(\Phi_\rho) > x) \leq P(|\eta| > \alpha) + P\left(\xi_0\left(\frac{T_0}{\rho}\right) + \xi_1\left(\frac{T_1}{\rho}\right) > x - \alpha\right)\,,$$

where

$$P(|\eta| > \alpha) \leq \frac{E|\eta|}{\alpha} = o(\rho)\,,$$

$$P\left(\xi_0\left(\frac{T_0}{\rho}\right) + \xi_1\left(\frac{T_1}{\rho}\right) > x - \alpha\right)$$

$$= \int_{-\infty}^{0}\int_{0}^{\infty} P\left(\xi_0\left(\frac{t_0}{\rho}\right) + \xi_1\left(\frac{t_1}{\rho}\right) > x - \alpha\right) P(T_0 \in dt_0,\ T_1 \in dt_1) = \sum_{s=1}^{4}\int_{G_s}\,,$$

$$\begin{aligned} &G_1 = \{t_0 \geq -\rho H,\ t_1 \leq \rho H\}\,, \quad G_2 = \{t_0 \geq -\rho H,\ t_1 > \rho H\}\,, \\ &G_3 = \{t_0 < -\rho H,\ t_1 \leq \rho H\}\,, \quad G_4 = \{t_0 < -\rho H,\ t_1 > \rho H\} \end{aligned}$$

for some $H > 0$. For integrals $\int_{G_s}$ we have

$$\int_{G_1} \leq P(T_0 \geq -H\rho,\ T_1 \leq H\rho) = o(\rho)\,,$$

$$\begin{aligned}
\int_{G_4} &\leq \int_{G_4} \frac{g\left(\frac{t_0}{\rho}\right) + g\left(\frac{t_1}{\rho}\right)}{x-\alpha} P(T_0 \in dt_0,\ T_1 \in dt_1) = \\
& \frac{1}{x-\alpha}\left(\int_{-\infty}^{-H\rho} p_0(t_0) g\left(\frac{t_0}{\rho}\right) dt_0 + \int_{H\rho}^{\infty} p_1(t_1) g\left(\frac{t_1}{\rho}\right) dt_1\right) = \\
& \frac{\rho(p_0(0)+p_1(0))}{x-\alpha} \int_H^\infty g(u) du (1+o(1)) \ , \\
\int_{G_2} &\leq \int_{G_2} \left[\frac{g\left(\frac{t_1}{\rho}\right)}{\alpha} + P\left(\xi_0\left(\frac{t_0}{\rho}\right) > x - 2\alpha\right)\right] P(T_0 \in dt_0,\ T_1 \in dt_1) \\
&\leq \frac{\rho}{\alpha} p_1(0) \int_H^\infty g(u) du + \rho p_0(0) \int_{-\infty}^0 P(\xi_0^0(u) > x - 2\alpha) du + o(\rho) \ , \\
\int_{G_3} &\leq \rho p_1(0) \int_0^\infty P(\xi_1^0(u) > x - 2\alpha) du + o(\rho) \ ,
\end{aligned}$$

where $\xi_i^0(u)$ corresponds to the limit conditional distributed of μ_i when $|T_i| \to 0$. These inequalities imply

$$P(f(\Phi_\rho) > x) \leq \rho\pi(x-2\alpha) + o(\rho) + c\rho \int_H^\infty g(u) du \ .$$

Choosing H large enough and α small enough, we can estimate $\limsup_{\rho\to 0} \rho^{-1}$ $P(f(\Phi_\rho) > x)$ as close to $\pi(x)$ as we want. The same can be said about $\lim\mathrm{int}_{\rho\to 0}\rho^{-1}$ $P(f(\Phi_\rho) > x)$. The Theorem is proved.

Apparently, considerations used in the second part of Theorem 1 are equivalent to the checking of condition (B_1) for the functional $\bar{f}(\Phi_\rho) = I_{[f(\Phi_\rho)>x]}$.

Remark 2 The goal of this work is the investigation of asymptotic behaviour of $f(\Phi_\rho)$, where Φ_ρ is the sequence of dilation of marked point processes $\Phi_\rho = \{T_i(\rho),\ \mu_i(\rho)\}$, $\rho \to 0$. In Theorem 1 the simplest possible sequence was choosen, it has the form $T_i(\rho) = \frac{T_i}{\rho},\ \mu_i(\rho) = \mu_i$, where $\{T_i, \mu_i\}$ is given fixed sequence, independent of ρ. In this case it turned out that the answer about behaviour of $f(\Phi_\rho)$ is stated in terms of $Ef(\delta_u^{\theta_i})$ where the distribution of mark θ_i: $P(\theta_i < t) = \lim_{\rho\to 0} P(\mu_i < t/T_i = u\rho)$ does not depend on u, which defines the position of corresponding point of T. If μ_i depends on T this fact does not always look satisfactory.

Apparently more general statement of this problem is natural. It assumes the consideration of process $\{T_i(\rho),\ \mu_i(\rho)\}$ in "triangular scheme", where $T_i(\rho) \to_p \infty$ and parameter ρ can be identified say, with $\Lambda([-1,1]) = \sum P(T_i \in [-1,1]) = \rho$. It assumes also convergence

$$\frac{\partial^2}{\partial u dv} P(T_0(\rho) < u,\ T_1(\rho) < v / T_0(\rho) \ \geq \ -N,\ T_1(\rho) \leq N) \ \to N^{-2} \ ,$$

(it means asymptotical "uniformity" of distribution of $T_s(\rho),\ s = 0,1$), and the existence of limit distribution of $\mu_i(\rho)$ under condition $T_i(\rho) = u_i$ (it depends generally speaking on u_i). In this case, of course, we shall need more conditions which would provide "regular" behaviour of distributions of $T_i(\rho)$ and $\mu_i(\rho)$, allowing to obtain the result analogous to Theorem 1. This result would look similar but in this case it will be stated in terms of $Ef(\delta_{u_i}^{\theta_i})$ where distribution

$$P(\theta_i < t) = \lim_{\rho \to 0} P(\mu_i(\rho) < t / T_i(\rho) = u_i)$$

depends, generally speaking, on u_i. This dependence always takes place for instance in case when $\mu_i(\rho) = b(\gamma_i, T_i(\rho))$ for given function b and γ_i independent of T. In this case $\theta_i = b(\gamma_i, u_i)$.

2 Higher order terms of expansion

For the existence of two–terms expansion we shall need the following conditions in addition to $(A), (B)$:

$(\mathbf{A_1^{(2)}})$. *There exist bounded densities $p_{s,s+1}(u,v)$ of joint distributions of (T_s, T_{s+1}), $s = -1, 0, 1$, with limits $p_{-1,0}(-0,-0) = p_{-1,0}(0,0)$, $p_{0,1}(-0,+0) = p_{0,1}(0,0)$, $p_{1,2}(+0,+0) = p_{1,2}(0,0)$.*

The densities $p_0(t)$, $p_1(t)$ have bounded derivatives with limits $p_0'(-0) = p_0'(0)$, $p_1'(+0) = p_1'(0)$.

$(\mathbf{A_2^{(2)}})$. *The intensity measure density $\lambda(t)$ is differentiable from the right and from the left at the point $t=0$,*

$$\lambda'(+0) = p_i'(0) + p_{1,2}(0,0)\ ,\ \ \lambda'(-0) = p_0'(0) - p_{-1,0}(0,0)$$

(here $p_{1,2}(0,0)$ and $-p_{-1,0}(0,0)$ mean in fact derivatives $p_2'(0)$ and $p_{-1}'(0)$ of the densities of T_2 and T_{-1} respectively).

$(\mathbf{A_3^{(2)}})$.

$$\begin{aligned} P((T_{-1},T_1)\in(-\varepsilon,\varepsilon)) &= o(\varepsilon^2)\ , \\ P((T_0,T_2)\in(-\varepsilon,\varepsilon)) &= o(\varepsilon^2)\ . \end{aligned}$$

Instead of $(A_3^{(2)})$ one can consider close conditions of the form

$(\tilde{\mathbf{A}}_{\mathbf{3}}^{(2)})$.

$$P_{s,s+1}(\rho u_s,\rho u_{s+1})P(T_{s+2} < -\rho u_s / T_s = \rho u_s, T_{s+1} = \rho u_{s+1}) \to 0$$

as $\rho\to 0$ for $s=-1,0$ and for any fixed $u_s<u_{s+1}$.

In fact condition $(A_3^{(2)})$ will be used only for the proof of $(\tilde{A}_3^{(2)})$–type–convergence (see Lemma 6). Consider now condition $(B^{(2)})$, which includes (B_1) and condition

$(\mathbf{B_2^{(2)}})$.

$$\int_{-\infty}^{\infty} |t| g(t)dt < \infty\ .$$

For all small enough ρ, $|u\rho|\geq 1$,

$$\frac{\lambda(u\rho)}{|u\rho|}g(u) \leq \tilde{g}^{(\lambda)}(u),\ \int_{-\infty}^{\infty}|u|\,\tilde{g}^{(\lambda)}(u)du < \infty\ .$$

In Example 1 conditions $(A^{(2)})$ are fulfilled if the joint distribution of (τ_i,τ_{i+1}) has bounded "partial densities"

$$p^+(u,v)=\frac{\partial}{\partial v}P(\tau_1\geq u,\tau_2<v)\ ,\ p^-(u,v)=\frac{\partial}{\partial u}P(\tau_1<u,\tau_2\geq v)\ .$$

It follows from [4] that the joint distributions of (T_0,T_1) and (T_1,T_2) are equal respectively to

$$\begin{aligned} P(T_0<u,T_1>v) &= (E\tau)^{-1}\int_{v-u}^{\infty}P(\tau>t)dt\ , \quad u<0,\ v>0\ ; \\ P(T_1<u,T_2-T_1<v) &= (E\tau)^{-1}\int_0^u P(\tau_1\geq t,\ \tau_2<v)dt\ , \quad u>0,\ v>0\ . \end{aligned}$$

It gives us

$$\begin{aligned} p_{-1,0}(u,v) &= (E\tau)^{-1}p^-(v-u,-v)\ , \quad u<v<0\ , \\ p_{0,1}(u,v) &= (E\tau)^{-1}p(v-u)\ , \quad u<0, v>0\ , \\ p_{1,2}(u,v) &= (E\tau)^{-1}p^+(u,v-u)\ , \quad v>u>0\ , \\ p_0'(u) &= (E\tau)^{-1}p(-u)\ , \quad u<0\ , \quad p_1'(u)=(E\tau)^{-1}p(u)\ , \quad u>0\ . \end{aligned}$$

Conditions $A^{(2)}$ are fulfilled also for Example 2 if density $\lambda(t)$ is differentiable, there exist limits $\lambda'(\pm 0)$.

In Example 3 condition $A^{(2)}$ holds if $L>0, \{\tau_j\}$ are i.i.d. r.v. with differentiable density.

We shall need also condition

$(\mathbf{C^{(2)}})$. *In addition to (C) there exist limit conditional distributions*

$$\lim_{\rho\to 0}P(f(\delta_{u_s}^{\mu_s}+\delta_{u_{s+1}}^{\mu_{s+1}})<v/T_s=u_s\rho,\ T_{s+1}=u_{s+1}\rho)\ , \quad s=-1,0,1\ ,$$

in a sense of weak convergence.

Denote

$$f^{\circ}_{s,s+1}(u_s,u_{s+1})=\lim_{\rho\to 0}f_{s,s+1}(u_s,u_{s+1})\ .$$

Theorem 2 *Under conditions* $(A^{(2)}), (B^{(2)}), (C^{(2)})$

$$Ef(\Phi_\rho) = a_1\rho + a_2\rho^2 + o(\rho^2) \ ,$$

where a_1 *is given on Theorem 1,*

$$\begin{aligned} a_2 &= \sum_{s=-1}^{1} p_{s,s+1}(0,0) \int_{D_s} \Delta^0_{s,s+1}(u_s, u_{s+1}) du_s du_{s+1} \\ &\quad + p_{-1,0}(0,0) \int_{-\infty}^{0} |u| f^0_{-1}(u) du + p_{1,2}(0,0) \int_0^\infty u f^0_2(u) du \\ &\quad + p'_0(0) \int_{-\infty}^{0} u f^0_0(u) du + p'_1(0) \int_0^\infty u f^0_1(u) du \ ; \end{aligned}$$

$$\Delta^0_{s,s+1}(u_s, u_{s+1}) = f^0_{s.s+1}(u_s, u_{s+1}) - f^0_s(u_s) - f^0_{s+1}(u_{s+1}) \ , \quad s = -1, 0, 1 \ ;$$

$$\begin{aligned} D_{-1} &= \{(u_{-1}, u_0) \ : u_{-1} < u_0 < 0\} \ , \ D_0 = \{(u_0, u_1) \ : u_0 < 0, \ u_1 \geq 0\} \\ D_1 &= \{(u_1, u_2) \ : 0 \leq u_1 < u_2\} \ . \end{aligned}$$

Remark 3 In "continuous" case (see Remark 1) with stationary $\{\mu_j\}$, independent of T, coefficient a_2 can be written in shorter form:

$$a_2 = p(0,0) \int\int_{u<v} \Delta(u,v) du\, dv + p(0,0) \int |u| f(u) du + p'(0) \int u f(u) du \ ,$$

where $\Delta(u,v)$, $p(0,0)$, $p'(0)$ are common values of $\Delta_{s,s+1}(u,v)$, $p_{s,s+1}(0,0)$ $(s = -1, 0, 1)$ and $p'_0(0)$, $p'_1(0)$ respectively.

Proof : Some auxiliary assertions, which we will need, we put down in the general form, useful for the investigation of higher order terms of expansion as well. Define recursively for $u \geq 2$

$$\begin{aligned}
\Delta_{s,\dots,s+k}(t_s,\dots,t_{s+k}) \;&=\; f_{s,\dots,s+k}(t_s,\dots,t_{s+k}) - \\
&- \sum_{i=0}^{1} \Delta_{s+i,\dots,s+i+k-1}(t_{s+i},\dots,t_{s+i+k-1}) - \qquad (12)\\
&- \sum_{i=0}^{2} \Delta_{s+i,\dots,s+i+k-2}(t_{s+i},\dots,t_{s+i+k-2}) - \dots \\
&\dots - \sum_{i=0}^{k-1} \Delta_{s+i,s+i+1}(t_{s+i},t_{s+i+1}) - \sum_{i=0}^{k} f_{s+i}(t_{s+i}) \;,
\end{aligned}$$

where

$$\Delta_{s,s+1}(t_s,t_{s+1}) = f_{s,s+1}(t_s,t_{s+1}) - f_s(t_s) - f_{s+1}(t_{s+1})$$

(see also (4)).

Lemma 4

$$|\Delta_{s,\dots,s+k}(t_s,\dots,t_{s+k})| \le 2\min(g(t_s), g(t_{s+k})) \;.$$

Proof : Assume $t_{s+k} \ge |t_s|$ and denote $\nabla_{s,\dots,s+k}(t_s,\dots,t_{s+k}) = f_{s,\dots,s+k}$ $(t_s,\dots,t_{s+k}) - f_{s,\dots,s+k-1}(t_1,\dots,t_{s+k-1})$. Then putting in (12) the value of $\Delta_{s+1,\dots,s+k}$ and skipping for brevity arguments of f, Δ and ∇, we obtain

$$\begin{aligned}
&\Delta_{s,\dots,s+k} = f_{s,\dots,s+k} \pm f_{s,\dots,s+k-1} - f_{s+1,\dots,s+k} \\
&\pm f_{s+1,\dots,s+k-1} - \Delta_{s,\dots,s+k-1} - \Delta_{s,\dots,s+k-2} - \dots - \Delta_{s,s+1} - f_s = \\
&= \nabla_{s,\dots,s+k} - \nabla_{s+1,\dots,s+k} + f_{s,\dots,s+k-1} - f_{s+1,\dots,s+k-1} \\
&-\Delta_{s,\dots,s+k-1} - \dots - \Delta_{s,s+1} - f_s = \\
&= \nabla_{s,\dots,s+k} - \nabla_{s+1,\dots,s+k} + f_{s,\dots,s+k-1} - \Delta_{s+1,\dots,s+k-1} - \\
&- \sum_{i=0}^{1} \Delta_{s+1+i,\dots,s+i+k-2} - \sum_{i=0}^{2} \Delta_{s+1+i,\dots,s+i+k-3} - \dots \\
&- \sum_{i=0}^{k-3} \Delta_{s+1+i,s+2+i} - \sum_{i=0}^{k-2} f_{s+1+i} - \Delta_{s,\dots,s+k-1} - \Delta_{s,\dots,s+k-2} - \dots \\
&\dots - \Delta_{s,s+1} - f_s = \nabla_{s,\dots,s+k} - \nabla_{s+1,\dots,s+k} \;.
\end{aligned}$$

It remains to use condition (B_1), by which $|\nabla_{i,\ldots,s+k}| \le g(t_{s+k}),\ i=s,s+1$, so $|\Delta_{s,\ldots,s+k}| \le 2g(t_{s+k})$.

In case $|t_s| > t_{s+k}$ (it can be only if $s \le 0$) we obtain similarly $|\Delta_{s,\ldots,s+k}| \le 2g(t_s)$. Lemma is proved.

Lemma 5 *If $\int |t|^k g(t)dt < \infty$ then the function $\Delta_{s,\ldots,s+k}$ is integrable.*

Proof : Consider first the case when $s \ge 1$:

$$\begin{aligned}
& \int_{0<t_s<\ldots<t_{s+k}<\infty} |\Delta_{s,\ldots,s+k}(t_s,\ldots,t_{s+k})| dt_s \ldots dt_{s+k} \\
\le\ & 2\int_0^\infty g(t_{s+k}) \left(\int_{0<t_s<\ldots<t_{s+k}} dt_s \ldots dt_{s+k-1} \right) dt_{s+k} \\
=\ & 2\int_0^\infty \frac{(t_{s+k})^k}{k!} g(t_{s+k}) dt_{s+k} < \infty \ .
\end{aligned}$$

The same result will be true for $s+k \le 0$.
If $s \le 0,\ s+k \ge 1$ then we have to consider the integral on the set

$$-\infty < x_s < \ldots < x_{s+l} < 0 \le x_{s+l+1} < \ldots < x_{s+k} < \infty \ ,$$

$0 \le l \le k-1$, which can be estimated from above according to Lemma 4 by

$$\begin{aligned}
& 2\int_{-\infty}^0 \int_0^\infty \min(g(t_s), g(t_{s+k})) \frac{|t_s|^l}{l!} \frac{t_{s+k}^{k-l-1}}{(k-l-1)!} dt_s dt_{s+k} \\
=\ & 2\left(\int_{|t_s| \le t_{s+k}} + \int_{|t| > t_{s+k}} \right) \\
\le\ & 2\int_0^\infty \frac{t_{s+k}^k}{(l+1)!(k-l-1)!} g(t_{s+k}) dt_{s+k} + 2\int_{-\infty}^0 \frac{|t_s|^k}{l!(k-l)!} g(t_s) dt_s < \infty \ .
\end{aligned}$$

Lemma 5 is proved.

Now and then we shall use special values of s and k.

Lemma 6 *For $s = -1, 0$*

$$p_{s,s+1}(\rho u_s, \rho u_{s+1}) P(T_{s+2} < -\rho u_s / T_s = \rho u_s, T_{s+1} = \rho u_{s+1}) \to 0$$

and for $s = 0, 1$

$$p_{s,s+1}(\rho u_s, \rho u_{s+1}) P(T_{s-1} > -\rho u_{s+1} / T_s = \rho u_s, T_{s+1} = \rho u_{s+1}) \to 0$$

with respect to Lebesgue measure on $(u_s, u_{s+1}) \in [0, H]^2$ for any fixed $H > 0$.

The proof is based on condition $(A_3^{(2)})$ and is very similar to the proof of Lemma 2, so we skip it.

Note, that if we use condition $\tilde{A}_3^{(2)}$ then we won't need Lemma 6.

Lemma 7

$$E\Delta_{s,\ldots,r}\left(\frac{T_s}{\rho}, \ldots, \frac{T_r}{\rho}\right) = o(\rho^2)$$

for $s = -1,\ r = 1;\ s = 0,\ r = 2;\ s = -1,\ r = 2.$

Proof : For $s = -1,\ r = 1$ we have

$$\begin{aligned} E\Delta_{-1,0,1}\left(\frac{T_{-1}}{\rho}, \frac{T_0}{\rho}, \frac{T_1}{\rho}\right) &= \int_{-\infty < t_{-1} < t_0 < 0 \le t_1 < \infty} \Delta_{-1,0,1}\left(\frac{t_{-1}}{\rho}, \frac{t_0}{\rho}, \frac{t_1}{\rho}\right) \cdot \\ \cdot P(T_{-1} \in dt_{-1}, T_0 \in dt_0, T_1 \in dt_1) &= \int_{t_1 \ge |t_{-1}|} + \int_{|t_{-1}| > t_1} . \end{aligned} \tag{13}$$

The first integral here does not exceed

$$\begin{aligned} & 2\int_{t_1=0}^{\infty} \int_{t_0=-t_1}^{0} \int_{t_{-1}=-t_1}^{t_0} g\left(\frac{t_1}{\rho}\right) p_{0,1}(t_0, t_1) dt_0 dt_1 \cdot P(T_{-1} \in dt_{-1} / T_0 = t_0, T_1 = t_1) \\ \le\ & 2\rho^2 \int_{u_1=0}^{\infty} \int_{u_0=-u_1}^{0} g(u_1) p_{0,1}(u_0\rho, u_1\rho) du_0\, du_1 \cdot P(T_{-1} > -u_1\rho / T_0 = u_0\rho, T_1 = u_1\rho) \\ =\ & 2\rho^2 \int_{u_1=0}^{\infty} \int_{u_0=-u_1}^{0} g(u_1) a_{0,1}(u_0\rho, u_1\rho) du_0\, du_1 \ , \end{aligned} \tag{14}$$

where according to Lemma 6 (or condition $\tilde{A}_3^{(2)}$) $a_{0,1}(u_0\rho, u_1\rho) \to 0$ in such a way that integral in (14) is

$$o\left(\int_{u_1=0}^{\infty}\int_{u_0=-u_1}^{0} g(u_1)du_0\,du_1\right) = o\left(\int_0^{\infty} u_1 g(u_1)du_1\right) = o(1)\ .$$

The second integral in (13) can be estimated in the same way. It proves Lemma for $\Delta_{-1,0,1}$. The case $s=0,\ r=2$ can be considered similarly.

For $s=-1,\ r=2$ we have

$$E\Delta_{-1,\dots,2}\left(\frac{T_{-1}}{\rho},\dots,\frac{T_2}{\rho}\right) = \tag{15}$$

$$= \int\dots\int \Delta_{-1,\dots,2}\left(\frac{t_{-1}}{\rho},\dots,\frac{t_2}{\rho}\right) P(T_{-1}\in dt_{-1},\dots,T_2\in dt_2) = \int_{|t_{-1}|\le t_2} + \int_{|t_{-1}|>t_2}\ .$$

The first integral does not exceed

$$\begin{aligned} & 2\int_{t_2=0}^{\infty}\int_{t_1=0}^{t_2} g\left(\frac{t_2}{\rho}\right) p_{1,2}(t_1,t_2)dt_1dt_2P(T_{-1} > -t_2/T_1 = t_1, T_2 = t_2) \\ = \;& 2\rho^2\int_{u_2=0}^{\infty}\int_{u_1=0}^{u_2} g(u_2)p_{1,2}(u_1,\rho,u_2\rho)du_1du_2P(T_{-1} > -u_2\rho/T_1 = u_1,\rho, T_2 = u_2\rho)\ . \end{aligned}$$

The convergence

$$p_{1,2}(u,\rho,u_2\rho)P(T_1 > -u_2\rho/T_1 = u_1,\rho,T_2 = u_2\rho) \to 0$$

follows from condition $(A_3^{(2)})$ by the same considerations as in Lemma 6, because under this condition

$$P(T_{-1} > -\varepsilon, T_2 < \varepsilon) \le P(T_{-1} > \varepsilon, T_1 < \varepsilon) = o(\varepsilon^2)\ .$$

The same is true for the second integral in (15). Lemma 7 is proved.

Now we can pass over to the proof of Theorem 2. According to (B_1)

$$
\begin{aligned}
Ef(\Phi_p) &= Ef\left(\delta^{\mu_{-1}}_{T_{-1}/\rho} + \ldots + \delta^{\mu_2}_{T_2/\rho}\right) + R \ , \qquad (16)\\
|R| &\leq \sum_{\substack{i>2,\\ i<-1}} Eg\left(\frac{T_i}{\rho}\right) \ .
\end{aligned}
$$

The main term by (12) can be written as

$$
\begin{aligned}
&E\Delta_{-1,\ldots,2}\left(\frac{T_{-1}}{\rho},\ldots,\frac{T_2}{\rho}\right) + E\Delta_{-1,0,1}\left(\frac{T_{-1}}{\rho},\frac{T_0}{\rho},\frac{T_1}{\rho}\right) + \qquad (17)\\
&+E\Delta_{0,1,2}\left(\frac{T_0}{\rho},\frac{T_1}{\rho},\frac{T_2}{\rho}\right) + \sum_{s=-1}^{1} E\Delta_{s,s+1}\left(\frac{T_s}{\rho},\frac{T_{s+1}}{\rho}\right) + \sum_{s=-1}^{2} Ef_s\left(\frac{T_s}{\rho}\right) \ .
\end{aligned}
$$

The first three terms here are $o(\rho^2)$ by Lemma 7.
For $E\Delta_{s,s+1}\left(\frac{T_s}{\rho},\frac{T_{s+1}}{\rho}\right)$ $(s=-1,0,1)$ we have

$$
\begin{aligned}
&\rho^{-2}E\Delta_{s,s+1}\left(\frac{T_s}{\rho},\frac{T_{s+1}}{\rho}\right) = \rho^{-2}\int_{D_s}\Delta_{s,s+1}\left(\frac{t_s}{\rho},\frac{t_{s+1}}{\rho}\right)p_{s,s+1}(t_s,t_{s+1})dt_s dt_{s+1}\\
&= \int_{D_s}\Delta_{s,s+1}(u_s,u_{s+1})p_{s,s+1}(\rho u_s,\rho u_{s+1})du_s du_{s+1} \to\\
&\to p_{s,s+1}(0,0)\int_{D_s}\Delta_{s,s+1}(u_s,u_{s+1})du_s du_{s+1} \ ,
\end{aligned}
$$

where domains D_s are described in Theorem 2. Further

$$
\begin{aligned}
&\rho^{-2}Ef_{-1}\left(\frac{T_{-1}}{\rho}\right) = -\rho^{-2}\int_{t_{-1}=-\infty}^{0}\int_{t_0=t_{-1}}^{0} p_{-1,0}(t_{-1},t_0)f_{-1}\left(\frac{t_{-1}}{\rho}\right)dt_2 dt_{-1}\\
&= \int_{u_1=-\infty}^{0}\int_{u_0=u_1}^{0} f_{-1}(u_{-1})p_{-1,0}(u_{-1}\rho,u_0\rho)du_0 du_{-1} \to\\
&\to p_{-1,0}(0,0)\int_{u_1=-\infty}^{0}|u_{-1}|f_{-1}(u_{-1})du_{-1} \ .
\end{aligned}
$$

Similarly

$$\rho^2 E f_2\left(\frac{T_2}{\rho}\right) \to p_{1,2}(0,0)\int_{u_2=0}^{\infty} u_2 f_2(u_2)du_2 \ .$$

As we know from §1 the remaining two terms in the last sum in (17) have different asymptotic behaviour. Since

$$p_1(v) = p_1(0) + rp_1'(0) + o(v)$$

we have

$$\begin{aligned}
Ef_1\left(\frac{T_1}{\rho}\right) &= \int_0^\infty f_1\left(\frac{t_1}{\rho}\right) p_1(t_1)dt_1 \\
&= \rho p_1(0)\int_0^\infty f_1(u_1)du_1 + \rho^2\int_0^\infty u_1 f_1(u_1)\frac{p_1(\rho u_1) - p_1(0)}{\rho u_1}du_1 \\
&= \rho p_1(0)\int_0^\infty f_1(u_1)du_1 + \rho^2 p_1'(0)\int_0^\infty u_1 f_1(u_1)du_1 + o(\rho^2) \ .
\end{aligned}$$

Similarly

$$Ef_0\left(\frac{T_0}{\rho}\right) = \rho p_0(0)\int_{-\infty}^0 f_0(u_0)du_0 + \rho^2 p_0'(0)\int_{-\infty}^0 u_0 f_0(u_0)du_0 + o(\rho^2) \ .$$

It remains to estimate the value R in (16):

$$\begin{aligned}
\sum_{i\geq 3} Eg\left(\frac{T_i}{p}\right) &= \int_0^\infty \sum_{i\geq 3} P(T_i\in dt)g\left(\frac{t}{\rho}\right) \\
&= \rho^2\int_0^\infty \frac{\lambda(u\rho) - p_1(u\rho) - p_2(u\rho)}{u\rho}ug(u)du \ .
\end{aligned} \tag{18}$$

Here for $v\to 0$

$$\begin{aligned}
\lambda(v) &= \lambda(+0) + \lambda'(+0)\rho + o(v) \ , \\
p_1(v) &= p_1(0) + p_1'(0)v + o(v) \ , \\
p_2(v) &= \int_0^v p_{1,2}(u,v)du = vp_{1,2}(0,0) + o(v) \ .
\end{aligned}$$

Since by $(A_2^{(2)})$

$$\lambda(+0) = p_1(0),\ \lambda'(+0) = p_1'(0) + p_{1,2}(0) = p_1'(0) + p_2'(0)$$

then

$$\frac{1}{v}(\lambda(v) - p_1(v) - p_2(v)) \to 0$$

what together with $(B_2^{(2)})$ means that (18) is $o(\rho^2)$. The same is true for $\sum_{i\le -2} Eg(T_i/\rho)$.

Theorem 2 is proved.

¿From the proof of Theorem 2 it is easy to understand which form the k–th term of expansion will have and what conditions one has to assume to provide the existence of the k–terms expansion. These main conditions along with (B_1) will be: the finiteness of $\int t^{k-1}g(t)dt$ and existence of densities $p_{s,\ldots,s+k-1}$ of joint distribution of $T_s,\ldots,T_{s+k-1},\ s \ge -k+1,\ s+k-1 \le k$, and of derivatives of order l of $p_{s,\ldots,s+k-l-1},\ l = 1,\ldots,k-1$.

The form of the results and of the conditions will look simpler if to reduce the problem considered above to the same problem for point process on $[0,\infty]$. This reduction can be done by ordering $|T_i|$ and including sign T_i in mark μ_i. It will be one–to–one description of process with respect to the original one (we shall use this reduction in the next section). In this case in Theorem 2 we shall need only densities $p_1, p_{1,2}$ and integrals only on domain D_1.

3 Two–dimensional case

In this section Φ denotes a marked point process in R^2. The joint distribution of points $T_i \in R^2$ and marks $\mu_i,\ i = 1,2\ldots$ is given. Notations $\delta_{T_i}^{\mu_i},\ \Phi_\rho = \sum_{i=1}^\infty \delta_{\frac{T_i}{\rho}}^{\mu_i}$ have the same meaning.

In multidimentional case it will be convenient to use polar coordinates for the description of points T_i. Put

$$T_i = (\xi_i, \eta_i),\ U_i = |T_i|,\ \varphi_i = \operatorname{arc} tg\frac{\xi_i}{\eta_i}\ ,$$

so

$$\xi_i = U_i \cos\varphi_i,\ \eta_i = U_i \sin\varphi_i\ .$$

To be more definite we shall assume that numeration of points $T_i = (U_i, \varphi_i)$ corresponds to the order of values $U_i\ :\ U_1 \le U_2 \le U_3 \ldots$ If two points have the same norm, then we can order them with respect to coordate $\varphi \in [0, 2\pi]$, but it is simpler to assume that $0 < U_1 < U_2 < \ldots$ with probability 1.

We shall reduce the problem, concerning behaviour of $f(\Phi_\rho)$ in two dimensional case, to the same problem for onedimensional case by construction of a new marked point process Φ^* with points $U_i,\ i = 1, 2 \ldots$ and marks $\mu_i^* = (\mu_i, \varphi_i)$. It is obvious that if the distribution of (T_i, μ_i) is given then the distribution of (U_i, μ_i^*) is also given and vice versa. After that it remains to apply Theorems 1, 2 to the new process. The condition (C) in this case becomes more essential, it means the existence of limit conditional distribution of φ_1 under condition $|T_1| \to 0$:

$$\lim_{u_1 \to 0} P(\varphi_1 < \varphi / U_1 = u_1) \tag{19}$$

(in a suitable sense, to provide the convergence of expectations which we need). The most important problem will be the calculation of values of densities $p_1, p_{1,2}, \ldots$ and of their derivatives at point 0 (see Theorems 1, 2). The typical situation will be $p_1(0) = p_{1,2}(0,0) = 0,\ p_0'(0) > 0$, so the first term of asymptotical expansion will disappear. One can see it on the following example.

Example 4 Consider Poisson field $T = \{T_i\}$ with intensity density

$$\lambda(u, \varphi) = \frac{\sum P(T_i \in (du, d\varphi))}{u du d\varphi}\ .$$

In this case

$$P(U_1 \in du, \varphi_1 \in d\varphi) = \lambda(u, \varphi) u du d\varphi e^{-I(u)}\ ,\ I(u) = \int_0^\infty \int_0^{2\pi} \lambda(v, \psi) v dv d\psi\ ,$$

so (U_1, φ_1) has density equals

$$\lambda(u, \varphi) u e^{-I(u)}\ .$$

The density of U_1 is equal to

$$p_1(u) = ue^{-I(u)} \int_0^{2\pi} \lambda(u, \varphi) d\varphi \ .$$

It means that $p_1(0) = 0$ if $\int_0^{2\pi} \lambda(u, \varphi) d\varphi = o(u^{-1})$ and that

$$p_1'(0) = \int_0^{2\pi} \lambda(0, \varphi) d\varphi$$

if there exists

$$\lambda(0, \varphi) = \lim_{u \to 0} \lambda(u, \varphi) \ . \tag{20}$$

Conditional density of φ_1 given $U_1 = u$ will be equal to

$$\frac{\lambda(u, \varphi)}{\int_0^{2\pi} (u, \psi) d\psi} \ ,$$

so the limit distribution (19) under condition (20) will be equal to

$$\frac{\lambda(0, \varphi)}{\int_0^{2\pi} \lambda(0, \psi) d\psi} \ .$$

Similarly can be described the joint distribution of $(U_1, \varphi_1), (U_2, \varphi_2)$ with density

$$u_1 \lambda(u_1, \psi_1) u_2 \lambda(u_2, \psi_2) e^{-I(u_2)} \ ,$$

so the density $p_{1,2}$ of (U_1, U_2) will be equal to

$$u_1 u_2 e^{-I(u_2)} \int_0^{2\pi} \int_0^{2\pi} \lambda(u_1, \psi_1) d\psi_1 \lambda(u_2 \psi_2) d\psi_2$$

what gives us $p_{1,2}(0,0) = 0$ under condition (20).

Summing up what was said above and assuming for simplicity that μ_i are independent of T we obtain from Theorem 2

Corollary 1 *Under condition* $(B_1), (B^{(2)})$, (20)

$$Ef(\Phi_\rho) = a_2\rho^2 + o(\rho^2) \ ,$$

where

$$a_2 = \int_0^{2\pi} \lambda(0,\varphi)d\varphi \int_0^\infty \int_0^{2\pi} uEf(\delta^{\mu_i}_{(u,\varphi)})du\lambda(0,\varphi)d\varphi \ .$$

If μ_i depend on T then expectation in this formula has to be taken with respect to corresponding conditional distribution (see Theorem 2) given $U_1 = u,\ \varphi_1 = \varphi$.

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Unité de recherche INRIA Lorraine, Technopôle de Nancy-Brabois, Campus scientifique,
615 rue du Jardin Botanique, BP 101, 54600 VILLERS LÈS NANCY
Unité de recherche INRIA Rennes, Irisa, Campus universitaire de Beaulieu, 35042 RENNES Cedex
Unité de recherche INRIA Rhône-Alpes, 46 avenue Félix Viallet, 38031 GRENOBLE Cedex 1
Unité de recherche INRIA Rocquencourt, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex
Unité de recherche INRIA Sophia-Antipolis, 2004 route des Lucioles, BP 93, 06902 SOPHIA-ANTIPOLIS Cedex

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