# Perturbation Analysis of Functionals of Random Measures

François Baccelli, Maurice Klein, Sergueï Zouev

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

# *Perturbation Analysis of Functionals of Random Measures*

François BACCELLI , Maurice KLEIN , Serguei ZUYEV

**N° 2225**

Mars 1994

PROGRAMME 1

Architectures parallèles,

bases de données,

réseaux et systèmes distribués

*Rapport de recherche*

**1994**

ISSN 0249-6399

# Perturbation Analysis of Functionals of Random Measures

François BACCELLI* , Maurice KLEIN** , Serguei ZUYEV ***

Programme 1 — Architectures parallèles, bases de données, réseaux
et systèmes distribués

Projet MISTRAL

Rapport de recherche n ° 2225 — Mars 1994 — 26 pages

**Abstract:** We use the fact that the Palm measure of a stationary random measure is invariant to phase space change to generalize the light traffic formula initially obtained for stationary processes on a line to general spaces. This formula gives a first order expansion for the expectation of a functional of the random measure when its intensity vanishes. This generalization leads to new algorithms for estimating gradients of functionals of geometrical random processes.

AMS 1991 Subject Classification: 60G55, 60G57, 62M30, 68Q20

**Key-words:** Stationary Random measures, Stationary Point Processes, Palm Measure, Campbell measure, Light Traffic Analysis, Perturbation Analysis, Gradient Estimates in Computer Simulations, Stochastic Geometry, Voronoi cell

*(Résumé : tsvp)*

*INRIA, Sophia-Antipolis, France
**CNET, Issy Les Moulineaux, France
***INRIA, Sophia-Antipolis, France

Unité de recherche INRIA Sophia-Antipolis
2004 route des Lucioles, BP 93, 06902 SOPHIA-ANTIPOLIS Cedex (France)

# Analyse des Perturbations de Fonctionnelles de Mesures Aléatoires

**Résumé :** Nous utilisons le fait que la mesure de Palm d'une mesure aléatoire stationnaire est invariante par rapport au changement de l'espace des phases, pour généraliser la formule du trafic faible (qui a été obtenue initialement pour les processus ponctuels stationnaires unidimensionnels) aux espaces généraux. Cette formule donne un développement du premier ordre de l'espérance d'une fonctionnelle de la mesure aléatoire quand son intensité tend vers 0. Cette généralisation permet d'obtenir de nouveaux algorithmes pour estimer les gradients de fonctionnelles de processus géométriques aléatoires.

**Mots-clé :** Mesures stationnaires aléatoires, processus ponctuels aléatoires, mesure de Palm, mesure de Campbell, analyse de trafic faible, analyse des perturbations, estimations de gradients en simulations, géométrie stochastique, cellule de Voronoi.

# Introduction

Various gradient formulae are known for functionals of stationary point processes on the line. These formulae have been used to develop simulation-based low-variance gradient estimators of functionals of point processes, which are very useful for optimization purposes (see e.g. the book *Ho & Cao* (1991)).

The aim of the present paper is to generalize the formula obtained for such processes in *Baccelli & Brémaud* (1993)) to random measures on a general space.

Formulas of this type have first been proved under specific Poisson-like assumptions, by *Reiman & Simon* (1989) for Poisson point processes on the real line and by *Møller & Zuyev* (1994) for general Poisson point processes.

The method that we propose here is based on the theory of stationary point processes with respect to a general shift, and particularly the notion of Palm measure. The conditions ensuring the validity of the formula consist of natural uniform integrability properties which are easy to check in various applications. This approach could also be generalized to obtain further terms of the Taylor expansion following the method proposed by *Błaszczyszyn* (1993) for stationary point processes on the line (these further terms are then obtained as integrals with respect to reduced high-order moment measures).

The structure of the article is the following.

In the first section, we define the objects and notations which we use in the text.

The second section concentrates on Palm calculus and effects of change of the phase space under homomorphism transformations. We show that the Palm measure defined on the reference probability space is preserved by these transformations.

The third section deals with the limiting behavior of functionals of a random measure when its density vanishes. We consider several important cases, when the random measure is diffuse or simple counting, and for various functionals of the measure.

The fourth section focuses on perturbation caused to a functional of a random measure by a "light" noise process. Derivative with respect to intensity of this noise reflects sensitivity of the functional and can be expressed in a closed form.

We show how the perturbation analysis formula derived in Section 4 can be used to obtain gradient estimation algorithms generalizing the ones used for queueing systems. Finally, some geometrical applications of these algorithms are also discussed.

# 1 Preliminaries

Throughout the paper,

- $[E, \rho_E, +]$ denotes an Abelian $\sigma$-group, i. e. a locally compact, complete separable metric commutative group, with neutral element 0, and where $-a$ denotes the inverse of $a \in E$. We shall assume that the metric $\rho_E$ in $E$ is such that all closed bounded sets are compact. Moreover the mapping $(a_1, a_2) \mapsto a_1 - a_2,\ a_1,\ a_2 \in E$ is assumed to be continuous, which makes $E$ a topological group.
- $\mathcal{B}_E$ is the Borel $\sigma$-algebra of $E$;
- $\Lambda(\cdot)$ - is the Haar measure on $\mathcal{B}_E$.

We also consider a space

- $[F, \rho_F, \mathcal{B}_F]$, which is a locally compact complete separable metric space endowed with its Borel $\sigma$-algebra.

In what follows we use the first letters of the alphabet $a,\ b,\ c, \ldots$ to denote elements of the space $E$ and capital ones $A,\ B,\ C, \ldots$ to denote elements of $\mathcal{B}_E$, whereas for the space $F$ we use the letters $x,\ y,\ z$ for elements and $U,\ V$, etc. for measurable sets.

The spaces $E$ and $F$ are *phase spaces* for measures which are the principal object of our studies here. Let

- $\mathcal{M}_E,\ \mathcal{M}_F$ be the set of locally finite measures on $[E, \mathcal{B}_E]$ and $[F, \mathcal{B}_F]$ respectively;
- $\mathcal{N}_E^s,\ \mathcal{N}_F^s$ be the set of locally finite simple counting measures on $[E, \mathcal{B}_E]$ and $[F, \mathcal{B}_F]$, i.e. the set of measures such that their value on any singleton is either 0 or 1;
- $\Xi_E$ be $\sigma$-algebra of subsets of $\mathcal{M}_E$ generated by the sets $\{m \in \mathcal{M}_E\ :\ m(B) \leq c\},\ B \in \mathcal{B}_E,\ c \in \mathbb{R}\}$;
- $\Xi_F$ be the same as $\Xi_E$ with $E$ replaced by $F$;
- $\phi :\ E \mapsto F$ be a $[\mathcal{B}_E, \mathcal{B}_F]$-continuous bijection;
- $\Phi\ :\ \mathcal{M}_E \mapsto \mathcal{M}_F$ be the mapping defined as follows: its value at $m \in \mathcal{M}_E$ is a measure $m_\Phi = \Phi(m)$ on $\mathcal{B}_F$ assigning to a set $U \in \mathcal{B}_F$ the value $m_\Phi(U) = m(\phi^{-1}(U))$.

In particular, the image of the unit measure $\delta_a \in \mathcal{B}_E$ concentrated on a point $a \in E$ is the measure $\Phi(\delta_a) = \delta_{\phi(a)} \in \mathcal{B}_F$.

Finally let

- $M$ be a random measure on $E$, i. e. a measurable mapping from some probability space $(\Omega, \mathcal{F}, \mathbf{P})$ to $[\mathcal{M}_E, \Xi_E]$;
- $N$ be a simple point process on $E$, i. e. a counting random measure on $[E, \mathcal{B}_E]$;

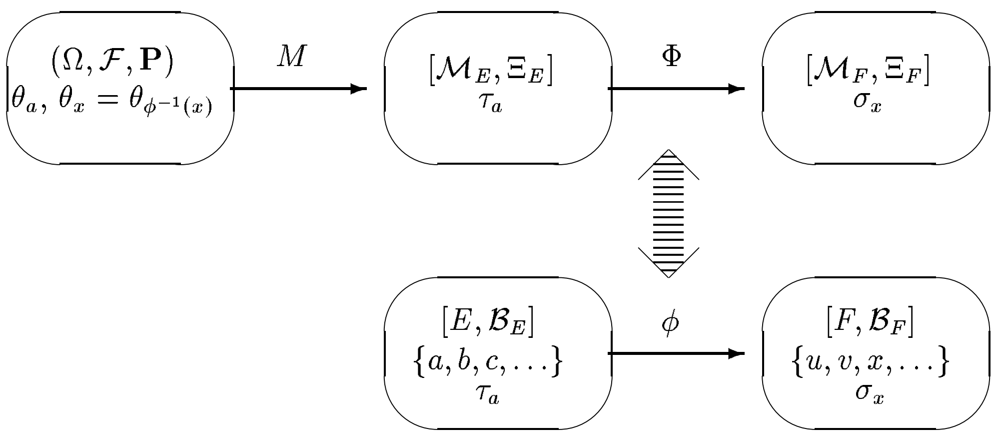


*Figure 1*

Let $[Y, \mathcal{G}]$ be some measurable space. By definition, *a measurable flow* indexed by elements of $E$ is a $[\mathcal{G}, \mathcal{G}]$-measurable family of mappings $\kappa_a : Y \mapsto Y$ satisfying the following properties:

1. $\kappa_a \circ \kappa_b = \kappa_{a+b}$;
2. $\kappa_0 =$ *identity mapping*;

and as a direct consequence:

3. for all $a \in E$ there exists $\kappa_a^{-1} = \kappa_{-a}$ such that $\kappa_a^{-1} \circ \kappa_a = \kappa_a \circ \kappa_a^{-1} = \kappa_0$.

The additive group induces a measurable flow $\tau_a$ on $E$ indexed by elements of $E$ defined by

- $\tau_a B = \{b + a : b \in B\}$

and a flow on $\mathcal{M}_E$ related to the flow on $E$ by

- $(\tau_a m)(B) = m(\tau_a B)$ for all $a \in E$ and $B \in \Xi_E$.

In particular, $\tau_a \delta_c = \delta_{c-a}$ for all $a, c \in E$.

There is no ambiguity, when we use the same letter $\tau$ for flows acting on different spaces: $E$ and $\mathcal{M}_E$. The operand of $\tau$ always allows us to distinguish between them.

The flow $\tau_a$ gives rise to the flows $\sigma_x$ indexed by elements of $F$ on $F$ and $\mathcal{M}_F$ defined by

- $\sigma_x(y) = \phi(\tau_{\phi^{-1}(x)}\, \phi^{-1}(y))$ for all $x, y \in F$;
- $(\sigma_x m)(U) = m(\sigma_x U)$ for all $x \in F$ and $U \in \mathcal{B}_F$.

The probability space $\Omega$ is also endowed with a flow $\theta_a$ indexed by elements of $E$ which satisfies the relations

- $M(\omega)(\tau_a B) = M(\theta_a \omega)(B)$ for all $\omega \in \Omega,\ a \in E$ and $B \in \mathcal{B}_E$.

Such a relationship between shifts is called *compatibility*; the shifts themselves are said to be *compatible*. As a consequence, we have

- $M_\Phi(\omega)(\sigma_x U) = M_\Phi(\theta_{\phi^{-1}(x)}\omega)(U)$ for all $\omega \in \Omega,\ x \in F$ and $U \in \mathcal{B}_F$.

Since $M(\omega)(\tau_a B) = (\tau_a M(\omega))(B)$, then also

- $M(\theta_a \omega) = \tau_a M(\omega)$ and
- $M_\Phi(\theta_{\phi^{-1}(x)}\omega) = \Phi(\tau_{\phi^{-1}(x)} M(\omega))$.

Figure 1 illustrates the objects defined above.

Finally, we suppose that the flow $\theta_a$ preserves the probability measure

- $\mathbf{P}(\theta_a \Sigma) = \mathbf{P}(\Sigma)$ for all $\Sigma \in \mathcal{F}$ and all $a \in E$ and therefore
- $\mathbf{P}(\theta_{\phi^{-1}(x)}\Sigma) = \mathbf{P}(\Sigma)$ for all $\Sigma \in \mathcal{F}$ and all $x \in F$

Thus the random measure $M$ in the phase space $E$ is *stationary* with respect to the flow $\tau_a$ that is by definition compatibility with the probability preserving flow $\theta_a$ (see e. g. *Neveu* (1976)). At the same time, the random measure $M_\Phi$ in $F$ is also stationary with respect to the flow $\sigma_x$ by the same argument.

# 2 Palm Calculus under Change of Space

The basic notions of Palm theory, which we use in the sequel, can be found in a number of books. We refer the reader to the classical work of *Neveu* (1976) or to more recent books like *Kerstan et. al.* (1982) and *Daley & Vere-Jones* (1988).

*The intensity measure* $\mu$ of a random measure $M$ is defined on $B \in \mathcal{B}_E$ by virtue of $\mu(B) = \mathbf{E}M(B) = \int M(\omega)(B)\mathbf{P}(d\omega)$ and is always assumed locally finite within this text. Since $M$ is stationary, then $\mu$ is invariant with respect to $\tau_\bullet$ and thus it is necessary equal to the Haar measure $\Lambda$ up to a multiplicative constant. Without loss of generality we may take $\Lambda = \mu$.

The intensity measure of the process $\Phi(M) \equiv M_\Phi$ is a measure $\mu_\Phi$ taking values $\mathbf{E}M_\Phi(U) = \mathbf{E}M(\phi^{-1}(U))$ on $U \in \mathcal{B}_F$. Since the latter is just $\Lambda(\phi^{-1}(U)) = \Phi(\Lambda)(U)$ then $\mu_\Phi$ coincides with $\Phi(\Lambda) = \Lambda_\Phi$ for short. Since $M_\Phi$ is stationary with respect to $\sigma_\bullet$, then $\Lambda_\Phi$ is invariant under the action of $\sigma_\bullet$.

**Theorem 1** *For all stationary measures $M : \Omega \mapsto \mathcal{M}_E$ there exists a unique positive locally finite measure $\widehat{\mathbf{P}}$ on $[\Omega, \mathcal{F}]$ such that for all $\mathcal{F} \otimes \mathcal{B}_E$-measurable functions $f : \Omega \times E \mapsto \mathbb{R}_+$*

$$\int_\Omega \mathbf{P}(d\omega) \int_E M(\omega,\, da)\, f(\theta_a\omega,\, a) = \int_\Omega \widehat{\mathbf{P}}(d\omega) \int_E \Lambda(da)\, f(\omega,\, a) \tag{1}$$

*or equivalently*

$$\int_\Omega \mathbf{P}(d\omega) \int_E M(\omega,\, da)\, f(\omega,\, a) = \int_\Omega \widehat{\mathbf{P}}(d\omega) \int_E \Lambda(da)\, f(\theta_{-a}\omega,\, a) \tag{2}$$

*The measure $\widehat{\mathbf{P}}$ is called the Palm measure of the stationary measure $M$. If the intensity measure $\Lambda$ is locally finite then $\widehat{\mathbf{P}}$ is a probability measure.*

(cf. *Mecke* (1967) or *Neveu* (1976), Théorème II. 4, Proposition II. 25).

The statements of the above theorem can be expressed in different forms according to the taste of the reader. Probabilists prefer to write expressions (1) and (2) in the form:

$$\mathbf{E} \int_E g(\tau_a M, a)\, M(da) = \int \mathbf{E}_0\, g(M, a)\, \Lambda(da)$$

$$\mathbf{E} \int_E g(M, a)\, M(da) = \int \mathbf{E}_0\, g(\tau_{-a} M, a)\, \Lambda(da)$$

for all non-negative $\Xi_E \otimes \mathcal{B}_E$-measurable functions $g : \mathcal{M}_E \times E \mapsto \mathbb{R}_+$, where $\mathbf{E}_0$ denotes expectation with respect to $\widehat{\mathbf{P}}$. The measure theoretic version is:

$$\Theta(\mathbf{P}.M) = \widehat{\mathbf{P}} \times \Lambda$$

$$\mathbf{P}.M = \Theta^{-1}(\widehat{\mathbf{P}} \times \Lambda)$$

where $\mathbf{P}.M = \mathbf{P}(d\omega)M(\omega, da)$ denotes semi-direct product of measures, $\times$ represents direct product and $\Theta$ is the flow on $\Omega \times E$ defined by $\Theta(\omega, a) = (\theta_a \omega, a)$. The measure $\mathbf{P}.M$ on $[\Omega \times E, \mathcal{F} \otimes \mathcal{B}_E]$ is called *the Campbell measure.*

We should note here various differences in terminology. Sometimes, the term Palm measure (respectively Campbell measure) is applied not to measure $\widehat{\mathbf{P}}$ (resp. to $\mathbf{P}.M$) itself, but to its image $\mathbf{P}_0$ by mapping $M$ on $\Xi_E$ (resp. by mapping $M \times identity_E$). However the main advantage of the measure $\widehat{\mathbf{P}}$ is that it is invariant by change of phase space as Assertion 1 below shows.

**Assertion 1** (Change of phase space formulae)

*For all $\mathcal{F} \otimes \mathcal{B}_F$-measurable functions $f : \Omega \times F \mapsto \mathbb{R}_+$*

$$\int_\Omega \mathbf{P}(d\omega) \int_F M_\Phi(\omega,\, dx)\, f(\theta_{\phi^{-1}(x)}\omega,\, x) = \int_\Omega \widehat{\mathbf{P}}(d\omega) \int_F \Lambda_\Phi(dx)\, f(\omega,\, x)$$

*where $\widehat{\mathbf{P}}$ is the Palm measure of $M$,*

$$\begin{aligned}\int_\Omega \mathbf{P}(d\omega) \int_F M_\Phi(\omega,\, dx)\, f(\omega,\, x) &= \int_\Omega \widehat{\mathbf{P}}(d\omega) \int_F \Lambda_\Phi(dx)\, f(\theta_{-\phi^{-1}(x)}\omega,\, x) \\ &= \int_\Omega \widehat{\mathbf{P}}(\theta_{\phi^{-1}(x)} d\omega) \int_F \Lambda_\Phi(dx)\, f(\omega,\, x) \end{aligned} \tag{3}$$

*As a direct consequence, the Palm measures $\widehat{\mathbf{P}}$ on $[\Omega, \mathcal{F}]$ for the measures $M : \Omega \mapsto \mathcal{M}_E$ and $M_\Phi : \Omega \mapsto \mathcal{M}_F$ coincide.*

*Proof.* The proof is based on the fact, that

$$\int_E f(x)\, \nu(dx) = \int_F f(\phi^{-1}(y))\, \nu_\Phi(dy)$$

for all bijections $\phi : E \mapsto F$ and measures $\nu_\Phi(\cdot) = \nu(\phi^{-1}(\cdot))$. Since this is trivially true for indicator functions $\mathbf{1}_B(x)$, $B \in \mathcal{B}_E$, then it holds true for all measurable functions $f$ by standard monotone class argument. Having changed similarly variable $a$ to $x = \phi(a)$ in (1) and (2), we obtain the desired formulae. The process $M_\Phi$ is stationary with respect to the flow $\theta_\bullet$, and thus $\widehat{\mathbf{P}}$ can be interpreted as the Palm measure of the process $M_\Phi$ as well.

**Remark 1** Denote the measure $\widehat{\mathbf{P}}(\theta_{\phi^{-1}(x)}\,\cdot)$ appearing in (3) by $\widehat{\mathbf{P}}_x$ and the corresponding expectation by $\mathbf{E}_x$. Relation (3) now becomes

$$\mathbf{E}\int_F g(M_\Phi, x)\, M_\Phi(dx) = \int_F \mathbf{E}_x\, g(M_\Phi, x)\, \Lambda_\Phi(dx) \tag{4}$$

for all non-negative $\Xi_F \otimes \mathcal{B}_F$-measurable functions $g:\ \mathcal{M}_F \times E \mapsto \mathbb{R}_+$.

The measure $\widehat{\mathbf{P}}_x$ can be alternatively defined for any random measure, not necessary stationary. It is the inverse image of the so called *local Palm measure* which is an appropriate version of Radon-Nikodym derivative at point $x$ of the Campbell measure with respect to the intensity measure (see e.g. *Daley & Vere-Jones* (1988), Chapter 12).

**Remark 2** Actually, the mapping $\phi$ establishes an isomorphism of groups $(E, +)$ and $(F, \dot{+})$, where the group operation $\dot{+}$ is defined by virtue of $x \dot{+} y = \phi(\phi^{-1}(x) + \phi^{-1}(y))$. The shift $\sigma_x$ can be represented as $\sigma_x(U) = U \dot{+} x$ and the measure $\Lambda_\Phi$ is just the Haar measure on the group $(F, \dot{+})$.

This construction despite of its apparent banality clearly demonstrates the difference between stationarity and homogeneity. As an example, consider a Poisson process on $\mathbb{R}$ driven by an intensity measure $\mu$ not equal to Lebesgue measure. Assume that the cad-lag function $f_\mu(x)$ defined by

$$f_\mu(x) = \begin{cases} \mu((0,x]), & \text{if } x > 0; \\ -\mu((x,0]), & \text{otherwise} \end{cases}$$

is strictly increasing inside the set carrying the support of $\mu$. Then given $f_\mu$, it is easy to construct a bijection $\phi$ between the phase space of the process and $\mathbb{R}$ (if $\mu(\mathbb{R})$ is infinite) or a circle (if $\mu(\mathbb{R})$ is finite) in such a way, that $\mu$ is the image of the Lebesgue measure. Since the first moment measure completely determines a Poisson process, then the original process is the image of a homogeneous stationary Poisson process on $\mathbb{R}$ if it is a.s. infinite or on a circle if it is a.s. finite. Thus the Poisson process is stationary, but with respect to the shift $\sigma_x(\bullet) = \bullet\ \dot{+} x$. However, it is by no means homogeneous!

# 3 Light Traffic Analysis Formulae

In this section we consider functionals of "light" random measures and point processes, in the sense that their intensities are close to zero. For such measures, Formula (3) can be expanded with respect to intensity, which provides a closed form expression.

We take as probability space $[\Omega, \mathcal{F}]$ the space $[\mathcal{M}_E, \Xi_E]$ itself and $M$ as identity mapping on $\mathcal{M}_E$. This is canonical space of the random measure $M$. Now there is no difference between the Palm measure $\widehat{\mathbf{P}}(d\omega)$ defined on $\mathcal{F}$ and the Palm measure $\mathbf{P}_0\,(dm)$ on $\Xi_E$, which is just the image of $\widehat{\mathbf{P}}$ by mapping $M$.

In this section $[F, \mathcal{B}_F, \dot{+}]$ will be a topological group group operation $\dot{+}$, isomorphic to the group $[E, \mathcal{B}_E, +]$. Let $\mathcal{A}$ be a family of isomorphisms of the groups $E$ and $F$ indexed by a scalar $\rho \in \mathbb{R}_+$. It means that each $\phi_\rho$ is a measurable bijection $E \mapsto F$ with the following property:

$$\phi_\rho(a+b) = \phi_\rho(a)\dot{+}\phi_\rho(b) \text{ for all } a, b \in E.$$

In particular, $\phi_\rho(0) = \dot{0}$ and $\phi_\rho(-a) = \dot{-}\phi_\rho(a)$, where $\dot{0}$ is the neutral element of $[F, \dot{+}]$ and $\dot{-}x$ is the $\dot{+}$-inverse of $x \in F$.

Moreover, we suppose that $\mathcal{A}$ forms an Abelian group with group operation $\circ$ isomorphic to the multiplicative group $\mathbb{R}_+$, so that

$$(\phi_{\rho_1} \circ \phi_{\rho_2})(\bullet) \stackrel{def}{=} \phi_{\rho_2}(\phi_1^{-1}(\phi_{\rho_1}(\bullet))) = \phi_{\rho_1\rho_2}(\bullet)$$

Thus we can identify the index $\rho$ with elements of the group $\mathcal{A}$ itself so that the inverse of a group element $\phi_\rho$ is $\phi_{\rho^{-1}}$ and $\phi_1$ is the neutral element of $\mathcal{A}$.

When the spaces $E$ and $F$ coincide, the group $\mathcal{A}$ reduces to a subgroup of the group of authomorphisms of $E$ with composition as the group operation $\circ$.

To each mapping $\phi_\rho$ from $E$ to $F$, there corresponds a mapping $\Phi_\rho:\ \mathcal{M}_E \mapsto \mathcal{M}_F$ by the rule:

$$\Phi_\rho(m)(U) = m(\phi_\rho^{-1}(U)) \text{ for all } U \in \mathcal{B}_F$$

When applied to a counting measure $m = \sum_i k_i \delta_{a_i}$, this gives:

$$\Phi_\rho(\sum_i \delta_{a_i}) = \sum_i k_i \delta_{\phi_\rho(a_i)}$$

In the previous section a bijective mapping was used to define a measurable flow $\sigma$ on the space $F$. Now the group structure of $F$ allows us to define it directly by:

$$\sigma_x(U) = U \dot{+} x,$$

while preserving the main property of the flows $\tau_\bullet$ and $\sigma_\bullet$, namely their compatibility with respect to any isomorphism $\phi_\rho$:

$$\phi_\rho(\tau_{\phi_\rho^{-1}(x)}\phi_\rho^{-1}(y)) = \phi_\rho(\phi_\rho^{-1}(y) + \phi_\rho^{-1}(x)) = y \dot{+} x = \sigma_x(y)$$

(compare with the definition of the flow $\sigma_\bullet$ in Section 2).

In the sequel instead of $\dot{+}$ we will simply write $+$ which we distinguish from the operation $+$ in $E$ by the operands it acts upon.

**Lemma 1** *For all $\rho > 0,\ a \in E,\ x \in F,$ and $m \in \mathcal{M}_E$ the following identities hold:*

$$\sigma_x \Phi_\rho(m) = \Phi_\rho(\tau_{\phi_\rho^{-1}(x)} m) \tag{5}$$
$$\Phi_\rho(\tau_a m) = \sigma_{\phi_\rho(a)} \Phi_\rho(m) \tag{6}$$

*Proof.* For all $U \in \mathcal{B}_F$ we have the following sequence of equalities:

$$\begin{aligned} \sigma_x \Phi_\rho(m)(U) &= \Phi_\rho(m)(\sigma_x U) = m(\phi_\rho^{-1}(\sigma_x U)) = m(\phi_\rho^{-1}(U) + \phi_\rho^{-1}(x)) \\ &= (\tau_{\phi_\rho^{-1}(x)} m)(\phi_\rho^{-1}(U)) = \Phi_\rho(\tau_{\phi_\rho^{-1}(x)} m) \end{aligned}$$

that is (5). Now (6) is immediate after the substitution $x = \phi_\rho(a)$.

□

We denote by $\Lambda_\rho$ the intensity measure of the random measure $M_\rho = \Phi_\rho(M)$. As it was explained in Section 2, if the probability measure $\mathbf{P}(dm)$ is stationary with respect to $\tau_\bullet$, then the random measures $M_\rho$ are stationary with respect to $\sigma_\bullet$ and hence their intensity measures are proportional to the Haar measure on $[F, \dot{+}]$, which we may choose to be equal to $\Lambda_1(dx)$. Thus there exist positive constants $\lambda_\rho$ such that $\Lambda_\rho(dx) = \lambda_\rho \Lambda_1(dx)$.

The results of this section are based on the following assertion.

**Assertion 2** *Given a functional $g:\ \mathcal{M}_F \times F \mapsto \mathbb{C}$, suppose that the following conditions hold:*

*(A)* *the intensity $\lambda_\rho$ vanishes as $\rho \downarrow 0$ and there exists a right derivative $\frac{d}{d\rho}\ \lambda_\rho\big|_{\rho=+0} = \lambda'_0$;*

*(B)* $\lim_{\rho\downarrow 0} g(\Phi_\rho(m), x) = 0$ $\ \mathbf{P}_0 \times \Lambda_1 -$ *a.e.;*

*(C)* *functionals $f_\rho(m, x) = g(\sigma_{-x}\Phi_\rho(m), x)\ :\ \mathcal{M}_E \times F \mapsto \mathbb{C}$ are uniformly $\mathbf{P}_0 \times \Lambda_1$-integrable;*

*($D_{\mathcal{M}}$)* *there exists a measurable function $f_0(m, x)$ (i.e. a $\mathbb{C}$-valued random process $F_0(x) = f_0(\cdot, x)$ on $F$) such that*

$$\lim_{\rho\downarrow 0} f_\rho(m, x) = f_0(m, x) \quad \mathbf{P}_0 \times \Lambda_1 - a.e.$$

*Then there exists*

$$\begin{aligned} &\frac{d}{d\rho}\bigg|_{\rho=+0} \int_{\mathcal{M}_E} \mathbf{P}(dm) \int_F g(\Phi_\rho(m), x)\, \Phi_\rho(m)(dx) \\ &= \lambda'_0 \int_F \Lambda_1(dx) \int_{\mathcal{M}_E} f_0(m, x)\, \mathbf{P}_0(dm) \end{aligned} \tag{7}$$

*or equivalently*

$$\left.\frac{d}{d\rho}\right|_{\rho=+0} \mathbf{E}\int_F g(M_\rho, x)\, M_\rho(dx) = \lambda_0' \int_F \mathbf{E}_0\, F_0(x)\, \Lambda_1(dx) \tag{8}$$

**Remark 3** Condition $(A)$ implies that the phase space $F$ is non-countable. Assume the contrary. Then the Haar measure $\Lambda_1$ of the singleton $\{0\}$ is not zero, otherwise $\Lambda_1$ is a zero measure and consequently the random measure is zero a.s. In what follows we exclude by convention this trivial case. Then

$$\Lambda_\rho(\{0\}) = \mathbf{E}\Phi_\rho(M)(\{0\}) = \mathbf{E}M(\phi_\rho(0)) = \mathbf{E}M(\{0\}) = const$$

and thus $\lambda_\rho = \Lambda_\rho(\{0\})/\Lambda_1(\{0\}) = 1$ does not vanish. As a consequence the Haar measure $\Lambda_1$ is always diffuse i.e. $\Lambda_1(\{x\}) = 0$ for any singleton $\{x\}$ if condition $(A)$ holds (cf. *Kerstan et al.* (1982), Statement 6.1.2).

What is important in Condition $(B)$ is the *existence* of the limit, not its value 0, since we can always consider a new function having subtracted the limit value from $g(\Phi_\rho(m), x)$. That will not affect the derivative and hence the validity of Formulae (7) and (8) for this new function.

*Proof.* Using standard decomposition techniques, it is sufficient to consider functions taking values in $\mathbb{R}_+$. Applying Formula (3) to the function $g(\Phi_\rho(m), x)$, we obtain

$$\begin{aligned} &\int_{\mathcal{M}_E} \mathbf{P}(dm) \int_F g(\Phi_\rho(m), x)\, \Phi_\rho(m)(dx) \\ = \quad &\lambda_\rho \int_F \Lambda_1(dx) \int_{\mathcal{M}_E} g(\Phi_\rho(\tau_{-\phi_\rho^{-1}(x)} m), x)\, \mathbf{P}_0(dm), \end{aligned}$$

which can be rewritten using (6) as

$$\begin{aligned} &\lambda_\rho \int_F \Lambda_1(dx) \int_{\mathcal{M}_E} g(\sigma_{-x}\Phi_\rho(m), x)\, \mathbf{P}_0(dm) \\ = \quad &\lambda_\rho \int_F \Lambda_1(dx) \int_{\mathcal{M}_E} f_\rho(m, x)\, \mathbf{P}_0(dm) \end{aligned} \tag{9}$$

Therefore by condition $(B)$

$$\left.\frac{d}{d\rho}\right|_{\rho=+0} \mathbf{E}\int_F g(M_\rho, x)\, M_\rho(dx) = \lim_{\rho\downarrow 0} \frac{\lambda_\rho}{\rho} \lim_{\rho\downarrow 0} \int_F f_\rho(m, x)\, \mathbf{P}_0(dm)\, \Lambda_1(dx)$$

and Formulae (7), (8) follow easily from conditions $(A)$, $(C)$, $(D_{\mathcal{M}})$.

□

We now consider particular instances of random measures, which are of interest for applications.

## 3.1 Absolutely Continuous Measures

Suppose that the random measure $M_1$ has a.s. continuous locally bounded density $M_1'(x)$ with respect to the measure $\Lambda_1(dx)$. Then using the representation of local Palm measures from *Daley & Vere-Jones* (1988), pp.456-457, we get the following expression for any integrable function $H : \mathcal{M}_F \mapsto \mathbb{C}$:

$$\mathbf{E}_0 H(M_1) = (\mathbf{E}M_1'(0))^{-1}\mathbf{E}\{H(M_1)\, M_1'(0)\}$$

Taking into account that $\mathbf{E}M_1'(0) = 1$, we obtain

**Corollary 1** *Assume that $M_1$ has a.s. continuous locally bounded density $M_1'(0)$ at 0 with respect to its intensity measure $\Lambda_1(dx)$ and conditions of Assertion 2 are fulfilled. Then*

$$\left.\frac{d}{d\rho}\right|_{\rho=+0} \mathbf{E}\int_F g(M_\rho, x)\, M_\rho(dx) = \lambda_0' \int_F \mathbf{E}\{F_0(x)\, M_1'(0)\}\, \Lambda_1(dx) \tag{10}$$

*In particular, if $F_0(x) = g(\emptyset, x)$, where $\emptyset$ is the zero-measure on $F$, then*

$$\left.\frac{d}{d\rho}\right|_{\rho=+0} \mathbf{E}\int_F g(M_\rho, x)\, M_\rho(dx) = \lambda_0' \int_F g(\emptyset, x)\, \Lambda_1(dx) \tag{11}$$

## 3.2 Simple Counting Measures

Another important case is when the random measure is a counting measure. We prefer to use the term *point process* instead of counting measure and to write $N_\rho$ rather than $M_\rho$. In addition we suppose that the point process is *simple* i.e. $N_1(\{x\})$ is either 0 or 1 almost surely for any singleton $\{x\}$. In doing so we do not loose in generality since one can always consider the set $X \times \mathbb{N}$ as a new phase space for a process with multiple points.

In the case of simple point processes, the Palm measure $\mathbf{P}_0$ is known to be concentrated on counting measures $n$ such that $n(\{0\}) = 1$ and hence $\Phi_\rho(n)(\{0\}) = 1$ (see e.g. *Daley & Vere-Jones* (1988), pp.469-470). Therefore we may consider the so-called *reduced* or *modified Palm measure* $\mathbf{P}_0^!$ defined for any $\Sigma \in \Xi_E$ by means of

$$\mathbf{P}_0^!(\Sigma) = (\Lambda(B))^{-1}\mathbf{E}\int_B \mathbf{1}_\Sigma(\tau_{-x}(N - \delta_x))\, N(dx) \tag{12}$$

for all measurable sets $B \in \mathcal{B}_E$ such that $0 < \Lambda(B) < \infty$ ($N$ is a canonical point process on $(\mathcal{N}_E^s, \Xi_E, \mathbf{P})$ and $\Lambda$ is the Haar measure on $[E, +]$).

**Assertion 3** *(Light traffic analysis formula) Given a function $g: \mathcal{N}_E^s \times F \mapsto \mathbb{C}$, suppose that conditions (A), (B), (C) of Assertion 2 and the following condition*

($D_{\mathcal{N}}$) *there exists a function $f_0(n,x)$ (i.e. a random process $F_0(x) = f_0(\cdot, x)$) such that*

$$\lim_{\rho \downarrow 0} g(\sigma_{-x}\Phi_\rho(n) + \delta_x, x) = f_0(n,x) \quad \mathbf{P}_0^! \times \Lambda_1 - a.e.$$

*are satisfied. Then there exists*

$$\left.\frac{d}{d\rho}\right|_{\rho=+0} \mathbf{E} \int_F g(N_\rho, x)\, N_\rho(dx) = \lambda_0' \int_F \mathbf{E}_0^!\, F_0(x)\, \Lambda_1(dx), \tag{13}$$

*where $\mathbf{E}_0^!$ is the expectation with respect to measure $\mathbf{P}_0^!$.*

*Proof.* By definition for any function $H: \mathcal{N}_E \mapsto \mathbb{C}$ we always have

$$\int_{\mathcal{N}_E^s} H(n)\mathbf{P}_0(dn) = \int_{\mathcal{N}_E^s} H(n + \delta_0)\mathbf{P}_0^!(dn),$$

so $\int_{\mathcal{M}_E} g(\sigma_{-x}\Phi_\rho(m), x)\, \mathbf{P}_0(dm)$ in the left hand side of (9) can be alternatively expressed as

$$\begin{aligned}
\int_{\mathcal{N}_E^s} g(\sigma_{-x}\Phi_\rho(n+\delta_0), x)\, \mathbf{P}_0^!(dn) &= \int_{\mathcal{N}_E^s} g(\sigma_{-x}\Phi_\rho(n) + \delta_x, x)\, \mathbf{P}_0^!(dn) \\
&= \mathbf{E}_0^! g(\sigma_{-x} N_\rho + \delta_x, x)
\end{aligned}$$

The rest follows the lines of the proof of Assertion 2.

□

Since the intensity measure $\lambda_\rho \Lambda_1$ of the point process $N_\rho$ vanishes as $\rho \downarrow 0$, one may expect that

$$\lim_{\rho \downarrow 0} g(\sigma_{-x}\Phi_\rho(n) + \delta_x, x) = g(\delta_x, x)$$

for almost all $n \in \mathcal{N}_E^s$, the situation which is actually often observed in practice.

This is summarized in the following corollary.

**Corollary 2** *Suppose that conditions of Assertion 3 hold for $f_0(n,x) = g(\delta_x, x)$. Then*

$$\left.\frac{d}{d\rho}\right|_{\rho=+0} \mathbf{E} \int_F g(N_\rho, x)\, N_\rho(dx) = \lambda_0' \int_F g(\delta_x, x)\, \Lambda_1(dx) \tag{14}$$

### 3.3 Functionals of the Form $G(N_\rho)$

Suppose we are given some measurable function $G: \mathcal{N}_F^s \mapsto \mathbb{C}$ and we are interested in the asymptotic behavior of $\mathbf{E}G(N_\rho)$. Functionals of the form $\int_F g(N_\rho, x) N_\rho(dx)$ studied before seem to be just a particular case of functionals of a general form $G(N_\rho)$, but fortunately, there exists a construction, first used in *Mecke* (1967), allowing one to reduce the latter to what precedes.

For this, introduce a function $h: F \times \mathcal{N}_F^s \mapsto \mathbb{R}$ such that $\int_F h(n,x) n(dx) = 1$ for any $n \in \mathcal{N}_F^s$. Then obviously $G(N_\rho) = \int_F h(N_\rho, x) G(N_\rho) N_\rho(dx)$ and we can use our previous results.

The simplest choice of such a function $h$ is as follows: let $x_1(n)$ denote a point of the support of $n$ such that $|x_j| \geq |x_1(n)|$ for all $x_j \in \operatorname{supp} n$, where $|x| = \rho_F(x, 0)$. If there are several such points in $\operatorname{supp} n$, $x_1(n)$ may be taken to be any of them. In other words, $x_1(n)$ is the point of $n$ which is the closest to 0. Now set

$$h(n,x) = \mathbf{1}_{\{x_1(n)\}}(x) \tag{15}$$

**Remark 4** Let $n = \delta_0 + \sum \delta_{x_i}$ be a simple counting measure with a point in 0. Since $\sigma_{-x} n = \delta_x + \sum \delta_{x_i + x}$, then $h(\sigma_{-x} n, x) = 1$ for all $x$ such that $|x| = |x - 0| < |x + x_i| = |x - (-x_i)|$ for all $i$. The set of those $x$ is known as the open *Voronoi cell with nucleus 0* of the point pattern $-n = \delta_0 + \sum \delta_{-x_i}$. By the same argument, $h(\sigma_{-x} n, x) = 0$ for all $x$ outside the closed Voronoi cell $V(-n)$ defined similarly by replacing $<$ by $\leq$ above. Thus $h(\sigma_{-x} n, x)$ is bounded from above by $\mathbf{1}_{V(-n)}(x)$; in addition, it coincides with this function with the possible exception of boundary points. Therefore, we have the following formula, which generalizes well known formulae on the real line which seems to appear first in *Kerstan et al.* (1974), p. 164:

$$\mathbf{E}\left[G(N)\right] = \lambda \mathbf{E}_0 \int_{V(-N)} G(\sigma_{-x} N) \Lambda(dx). \tag{16}$$

**Assertion 4** *Given a function $G: \mathcal{N}_F^s \mapsto \mathbb{C}$, suppose that the following conditions are satisfied:*

- *(A′) the function $|\phi_\rho(a)|$ is monotone in $\rho$ and tends to infinity for all $a \neq 0$. In addition, there exists a right derivative $\frac{d}{d\rho} \lambda_\rho\big|_{\rho=+0} = \lambda_0'$;*
- *(B′)* $\lim_{\rho \downarrow 0} G(\Phi_\rho(n)) = 0$ $\mathbf{P} - a.s.$
- *(C′) the functionals $f_\rho(n,x) = G(\sigma_{-x}\Phi_\rho(n)) \mathbf{1}_{V(-\Phi_\rho(n))}(x) : \mathcal{N}_E^s \times F \mapsto \mathbb{C}$ are uniformly $\mathbf{P}_0 \times \Lambda_1$-integrable;*

*($D'_{\mathcal{N}}$) there exists a measurable function $f_0(n,x)$ (i.e. a $\mathbb{C}$-valued random process $F_0(x) = f_0(\cdot,x)$) such that*

$$\lim_{\rho\downarrow 0} G(\sigma_{-x}\Phi_\rho(n) + \delta_x) = f_0(n,x) \quad \mathbf{P}^!_0 \times \Lambda_1 - a.e.$$

*Then there exists*

$$\left.\frac{d}{d\rho}\right|_{\rho=+0} \mathbf{E}G(N_\rho) = \lambda'_0 \int_F \mathbf{E}^!_0\, F_0(x)\, \Lambda_1(dx) \tag{17}$$

*Proof.* It is sufficient to show that the conditions of Assertion 3 are fulfilled for the function $h(\eta,x)G(\eta)$. First of all, condition $(A')$ implies condition $(A)$, since for all balls $B \subset F$ centered at 0 and $\mathbf{P}$-almost all $n$, $\Phi_\rho(n)(B)$ is monotone decreasing and equal to zero for all $\rho$ small enough. Therefore, by monotone convergence, there exists a limit of $\lambda_\rho = \mathbf{E}\Phi_\rho(n)(B)/\Lambda_1(B)$ equal to 0.

Condition $(B')$ implies condition (B) for the function $h(\eta,x)G(\eta)$ since $|h(\eta,x)| \le 1$, and $(C')$ is just equivalent to $(C)$.

Finally we have to show that $h_\rho f_\rho \to f_0 \;\; \mathbf{P}^!_0 \times \Lambda_1 - a.e.$, where $h_\rho(n,x) = h(\sigma_{-x}\Phi_\rho(n) + \delta_x, x) = h(\sigma_{-x}\Phi_\rho(n+\delta_0), x)$. From $(A')$, $\Phi_\rho(n)(U) = 0$ $P^!_0$ a.s., for all $\rho$ small enough. Thus by Remark 4, $h_\rho$ tends to 1 $\mathbf{P}^!_0 \times \Lambda_1 - a.e.$, and hence $(D'_{\mathcal{N}})$ implies $(D_{\mathcal{N}})$. Assertion 3 can now be applied to the function $g(\eta,x) = h(\eta,x)G(\eta)$, that completes the proof.

□

**Remark 5** As the considerations above show, the monotonicity in Condition $(A')$ can be replaced by the uniform integrability of the random variables $\Phi_\rho(N)(B) = N_\rho(B)$ for some measurable $B$ with $0 < \Lambda(B) < \infty$.

The usual situation is when $f_0(n,x) = G(\delta_x)$. In this case we have:

**Corollary 3** *Suppose that the conditions of Assertion 4 hold for the function $f_0(n,x) = G(\delta_x)$. Then*

$$\left.\frac{d}{d\rho}\right|_{\rho=+0} \mathbf{E}G(\Phi_\rho(m)) = \lambda'_0 \int_F G(\delta_x)\, \Lambda_1(dx) \tag{18}$$

## 3.4 Marked Point Processes

Mathematical background of the theory of marked point processes which we consider in this section can be found e.g. in *Kerstan et.al.*(1982), Chapters 6 and 11, or less formally in *Stoyan*(1987), Chapter 4.

Suppose that each point $a$ of the support of a simple counting measure $N$ can be represented as a pair $a = (a^*, k)$, where $a^*$ describes "position" and $k$ "mark" of the point $a^*$. Thus the phase space of marked process $N$ is given as a product $E = E^* \times K$ of *a position space* $E^*$ and *a mark space* $K$. $[E^*, \mathcal{B}_{E^*}, +]$ is supposed to be an Abelian $\sigma$-group and $[K, \mathcal{B}_K]$ a Polish space with its Borel $\sigma$-algebra.

A flow $\tau_a = \tau_{a^*}$ acts on the position space transforming a set $B \times L$ into $(B + a^*) \times L$, $B \in \mathcal{B}_{E^*},\ L \in \mathcal{B}_K$.

Suppose that the image space $F$ is also a product $F^* \times K$, where $[F^*, \mathcal{B}_{F^*}, \dot{+}]$ is an Abelian $\sigma$-group and $K$ is the same mark space. Let $\phi^*_\rho$ be a family of isomorphisms of the groups $E^*$ and $F^*$. Define a bijection $\phi_\rho :\ E \mapsto F$ as $\phi^* \times identity_K$.

Consider a marked point process $N$, which is stationary with respect to the flow $\tau_\bullet$. Then we can get the following representation for the intensity measure of the process $N_\rho$: $\Lambda_\rho(dx) = \lambda_\rho \Lambda_1^*(dx^*) Q(dk)$, where $\lambda_\rho$ is some positive constant, $\Lambda_1^*(dx^*)$ is a Haar measure on $[F, \dot{+}]$ and $Q(dk)$ is the so called *the Palm mark distribution* (cf. *Kerstan et al.* (1982), p.206). Substituting the last expression into (14) we get

**Corollary 4** *Suppose, that the conditions of Assertion 3 are fulfilled for the function* $f_0(n, x) = f_0(n, x^*, k) = g(\delta_{(x^*,k)}, x^*, k)$. *Then*

$$\begin{aligned} \left.\frac{d}{d\rho}\right|_{\rho=+0} \mathbf{E} \int_F g(N_\rho, x^*, k)\, N_\rho(d[x^*, k]) &= \lambda_0' \int_F g(\delta_{(x^*,k)}, x^*, k)\, Q(dk) \Lambda_1^*(dx^*) \\ &= \lambda_0' \int_F \mathbf{E}_0\, g(\delta_{(x^*,\kappa)}, x^*, \kappa)\, \Lambda_1^*(dx^*). \end{aligned} \tag{19}$$

Similarly (18) gives

**Corollary 5** *Suppose that the conditions of Assertion 4 are fulfilled for the function* $f_0(n, x) = f_0(n, x^*, k) = G(\delta_{(x^*,k)})$. *Then*

$$\begin{aligned} \left.\frac{d}{d\rho}\right|_{\rho=+0} \mathbf{E} G(N_\rho) &= \lambda_0' \int_F G(\delta_{(x^*,k)})\, Q(dk) \Lambda_1^*(dx^*) \\ &= \lambda_0' \int_F \mathbf{E}_0\, G(\delta_{(x^*,\kappa)})\, \Lambda_1^*(dx^*). \end{aligned} \tag{20}$$

This formula first appeared in *Baccelli* & *Brémaud* (1993) for stationary processes on the line $\mathbb{R}\,(= E = F)$ and dilations with coefficient $\rho^{-1}$ as transformations $\phi_\rho$.

# 4 Perturbation Analysis Formulae

In this section we study the behavior of the expectation of a random process $\Upsilon$ which is perturbed by a light "noise" process $\Phi_\rho$ independent of $\Upsilon$. Conditioning on $\Upsilon$, we can write the first term of the expansion for the expectation using light traffic formulae.

Let $\Upsilon$ be a random measure on $[F, \mathcal{M}_F]$ with distribution $\mathbf{P}^\Upsilon$, which is independent of the process $M_\rho$. The probability space of the random measure $\Upsilon$ will be assumed to be the canonical space $\mathcal{M}_F$, and we will take $\Upsilon(\gamma) = \gamma$ for all $\gamma \in \mathcal{M}_F$. $\mathbf{E}^\Upsilon$ will stand for the expectation with respect to $\mathbf{P}^\Upsilon$. Consider a function $G: \ \mathcal{M}_F \mapsto \mathbb{C}$ satisfying the following properties:

$(B'')$ there exists $\lim_{\rho\downarrow 0} \mathbf{E}^\Upsilon[G(\gamma + \Phi_\rho(n)) - G(\gamma)] = 0$ for $\mathbf{P}$-almost all $n$;

$(C'')$ the functionals $f_\rho(n,x) = \mathbf{E}_\Upsilon[G(\gamma + \sigma_{-x}\Phi_\rho(n)) - G(\gamma)]\mathbf{1}_{V(-\Phi_\rho(n))}(x): \ \mathcal{N}_E^s \times F \mapsto \mathbb{C}$ are uniformly $\mathbf{P}_0 \times \Lambda_1$-integrable;

$(D''_{\mathcal{N}})$ there exists

$$\lim_{\rho\downarrow 0} \mathbf{E}_\Upsilon[G(\gamma + \sigma_{-x}\Phi_\rho(n) + \delta_x) - G(\gamma)] = \mathbf{E}_\Upsilon[G(\gamma + \delta_x) - G(\gamma)] \ \mathbf{P}_0^! \times \Lambda_1 - a.e.$$

**Assertion 5** *(Perturbation analysis formula) Suppose condition $(A')$ of Assertion 4 and conditions $(B'')$, $(C'')$ and $(D''_{\mathcal{N}})$ above are fulfilled. Then*

$$\left.\frac{d}{d\rho}\right|_{\rho=+0} (\mathbf{E}^\Upsilon \times \mathbf{E})\, G(\Upsilon + N_\rho) = \lambda'_0\, \mathbf{E}^\Upsilon \int_F [G(\Upsilon + \delta_x) - G(\Upsilon)]\, \Lambda_1(dx) \tag{21}$$

## 4.1 Applications

Recall that a family $f_\rho(z)$ of functions on a measurable space $[Z, \nu]$ is uniformly integrable iff the following two conditions hold:

(I) For any $\varepsilon > 0$ there exists $\delta > 0$ such that for any measurable set $V$ we have

$$\int_V f_\rho(z) d\nu < \varepsilon \text{ for all } \rho \text{ provided } \nu(V) < \delta;$$

(II) For any $\varepsilon > 0$ there exists a measurable set $K_\varepsilon$ such that $\nu(K_\varepsilon) < \infty$ and

$$\int_{K_\varepsilon^c} f_\rho(z) d\nu < \varepsilon \text{ for all } \rho$$

(see e.g. *Dunford & Schwartz* (1988), p.150).

Both conditions are fulfilled if the family of functions is monotone or dominated by an integrable function - the most frequent cases in practice. Condition (I) is obviously valid if the functions $g(m, x)$ of Assertions 2 and 3 or the function $G(n)$ of Assertions 4 and 5 are bounded.

In the considered cases, the measure on the function space is the product of a probability measure and the locally finite intensity measure $\Lambda_1$ on $\mathcal{B}_F$. Therefore condition (II) holds whenever it is fulfilled for sets of the form $(Probability\ space) \times K_\varepsilon$, where $K_\varepsilon \in \mathcal{B}_F$ such that $\Lambda_1(K_\varepsilon) < \infty$.

**Example 1** *Finite Range Functions*

Suppose that the function $G : \mathcal{N}_F \mapsto \mathbb{C}$ only depends on the configuration of the random measure within a bounded set (possibly random). Specifically, we assume that there exists an almost surely finite non-negative random variable $r : \mathcal{N}_F \mapsto \mathbb{R}_+$ which posesses the following properties:

1. $r(n) \le r(n')$ for any $n,\ n' \in \mathcal{N}$ such that $\operatorname{supp} n' \subseteq \operatorname{supp} n$;
2. $r(n|_{B(n)}) = r(n)$, where $n|_{B(n)}$ denotes the restriction of $n$ to the set $B(n)$ — the closed ball of radius $r(n)$ centered at the origin (by definition $n|_B(U) = n(B \cap U)$ for any $U \in \mathcal{B}_F$).

We call $G$ *a finite range function* if

3. $G(n') = G(n),$ whenever $n|_{B(n)} = n'|_{B(n')}$.

Note that $n|_{B(n)} = n'|_{B(n')}$ and 2) yield $r(n) = r(n|_{B(n)}) = r(n'|_{B(n')}) = r(n')$ and hence $B(n) = B(n')$ for such $n,\ n'$.

As it can be shown, the second property is equivalent to say that the random ball $B(N)$ is *a stopping set*, and the third property means in fact that the function $G$ is measurable with respect to the stopping $\sigma$-algebra.

**Assertion 6** *Suppose condition (A′) of Assertion 4 is fulfilled. Suppose also that the finite range function* $G : \mathcal{N}_F \mapsto \mathbb{C}$ *is such that* $|G(n)| \le M$ *for some positive constant* $M$. *If in addition* $\mathbf{E}^\Upsilon \Lambda_1(B(\Upsilon)) < \infty$, *then*

$$\left.\frac{d}{d\rho}\right|_{\rho=+0} (\mathbf{E}^\Upsilon \times \mathbf{E})\, G(\Upsilon + N_\rho) = \lambda_0'\, \mathbf{E}^\Upsilon \int_{B(\Upsilon)} [G(\Upsilon + \delta_x) - G(\Upsilon)]\, \Lambda_1(dx) \qquad (22)$$

*Proof.* We show that the conditions of Assertion 5 are satisfied. Conditions $(B'')$ and $(D'')$ hold by bounded convergence. To show condition $(C'')$ it is sufficient to find a bound for the function

$$j_\rho(\gamma, n, x) = [G(\gamma + \sigma_{-x}\Phi_\rho(n)) - G(\gamma)]\, \mathbf{1}_{V(-\Phi_\rho(n))}(x)$$

which is uniform in $\rho$ and integrable with respect to $\mathbf{P}^\Upsilon \times \mathbf{P}_0 \times \Lambda_1$.

By properties 1) and 3) of the variable $r(N)$ this function differs from 0 only for $x$ such that $\sigma_{-x}\Phi_\rho(n)(B(\gamma)) > 0$. Note that if $n = \sum_i \delta_{x_i}$, then $\sigma_{-x}\Phi_\rho(n) = \sum_i \delta_{\phi_\rho(x_i)+x}$, so that $\sigma_{-x}\Phi_\rho(n)(B(\gamma)) > 0$ iff $x$ belongs to the set

$$\bigcup_i \{B(\gamma) - \phi_\rho(x_i)\}$$

Our last step consists in proving that

$$V(-\Phi_\rho(n)) \bigcap \bigcup_i \{B(\gamma) - \phi_\rho(x_i)\} \subseteq B(\gamma),$$

where $V(-\Phi_\rho(n))$ denotes the Voronoi cell of the point 0 of $-\Phi_\rho(n)$ (see Remark 4).

Since we consider condition $(C'')$ under the Palm measure $\mathbf{P}_0$, $n$ and hence $\Phi_\rho(n)$ have a point in 0, which will be denoted $x_0$. Let $C_i = V(-\Phi_\rho(n)) \bigcap \{B(\gamma) - \phi_\rho(x_i)\}$. Then $C_0 \subset B(\gamma)$ $\mathbf{P}_0 - a.s.$ If $x \in C_i$ for some $i \neq 0$, then $x$ belongs both to $V(-\Phi_\rho(n))$ and to the ball of radius $r(\gamma)$ and center $-\phi_\rho(x_i)$; so $|x| \le |x + \phi_\rho(x_i)|$ and $|x + \phi_\rho(x_i)| \le r(\gamma)$. Therefore $|x| \le r(\gamma)$ and consequently $x \in B(\gamma)$.

Thus we can bound $|j_\rho(\gamma, n, x)|$ from above by the function $2M\mathbf{1}_{B(\gamma)}(x)$ independent of $\rho$, which is $\mathbf{P}^\Upsilon \times \mathbf{P}_0 \times \Lambda_1$-integrable in view of $\mathbf{E}^\Upsilon \Lambda_1(B(\Upsilon)) < \infty$, and consequently, condition $(C'')$ also holds. Finally, we have used the finite range assumption on $G$ to replace the integral on $F$ in (21) by an integral on $B(\Upsilon)$, which leads to (22).

□

**Example 2** *Poisson Processes.*

Consider a bounded function $G(m)$ depending only on the restriction $m|_K$ of the measure $m$ on some compact set $K \in \mathcal{B}_F$ independent of $m$.

Let $\Upsilon$ and $N$ be two independent homogeneous Poisson processes in $\mathbb{R}^d$ of intensities $\lambda$ and 1 respectively. Consider a process $N_\rho$ which is the result of a dilation of the process $N$ with the center 0 and coefficient $\rho^{-1/d}$. Then the process $N_\rho$ is Poisson of intensity $\rho$ and $\Upsilon + N_\rho$ is Poisson of intensity $\lambda + \rho$. The derivative $\frac{d}{d\rho}\Big|_{\rho=+0}$ now corresponds to derivative $\frac{d}{d\lambda}$. Having applied now Assertion 6 to the function $G$, the perturbation analysis formula (22) becomes:

$$\frac{d}{d\lambda}\mathbf{E}G(N_\lambda) = \mathbf{E}\int_K [G(N_\lambda + \delta_x) - G(N_\lambda)]\, dx \tag{23}$$

This formula was proved in *Møller & Zuyev* (1994) in more general settings, where expressions for higher order derivatives of Poisson process are also given. If the function $G$ is the indicator function of some event, then we obtain the so called *Russo formula* for Poisson process, first proved in *Zuyev* (1992) (see also *Zuyev* (1993) for geometrical applications of this formula). In some sense it is similar to the formula first appeared in *Russo* (1981) for Bernoulli fields on a finite set, which expresses derivatives of probabilities of monotone events with respect to the Bernoulli measure parameter.

Formula (23) can be generalized to the class of infinitely divisible point processes, which are known to possess a Poisson cluster representation (see e.g. *Daley & Vere-Jones* (1988), chapter 8.4). Clusters here can be considered as independent marks and we can apply formula (19) to extend the perturbation analysis formula to this case (see *Møller & Zuyev* (1994)).

The following example focuses on statistical applications of the perturbation formulae to Voronoi tesselations. Basic definitions and a comprehensive bibliography on tesselations can be found in *Okabe et al.* (1992) and *Møller* (1994).

In the sequel, the state spaces are $E = F = \mathbb{R}^d$, shifts $\tau_x$ and $\sigma_x$ are translations of parameter $x$, $\phi_\rho$ are dilations with parameter $\rho^{-1/d}$. Such a choice of the dilation parameter gives $\Lambda_\rho = \rho\Lambda$, where $\Lambda$ is the Lebesgue measure in $\mathbb{R}^d$. Let $\Upsilon = \sum_{i=1}^\infty \delta_{x_i}$ denote a simple stationary and ergodic point process on $\mathbb{R}^d$, defined on the probability space $(\Omega, \mathcal{F})$, with finite intensity $\lambda$. We take for the distribution of the process $\Upsilon$ its Palm probability measure $\mathbf{P}_0^\Upsilon$ on $\Omega$.

For any point pattern $N = N(\omega) = \sum_{i=1} \delta_{x_i}$ on $\mathbb{R}^d$, let $V_i(N)$ denote the Voronoi cell associated with point $x_i$ and let $V(N)$ denote the (closed) cell which contains 0. This notation is consistent with that of Remark 4 in the particular case when $N(\{0\}) = 1$. Let $W(N)$ be the *Voronoi flower* (or *the fundamental region*) of the cell $V(N)$. By definition, the flower $W(N)$ is the set of locations $x$ in the space such that adding a point at this location changes $V(N)$, namely $V(N + \delta_x) \neq V(N)$. It is the union of balls having point 0 and $d$ points of $\operatorname{supp} N$ on their boundaries and no other such point inside. Finally, let $B(N)$ be the smallest ball of center 0 containing $W(N)$. Figure 2 shows a typical Voronoi cell (dashed) and its corresponding flower (pale dashed) in the two-dimensional case.

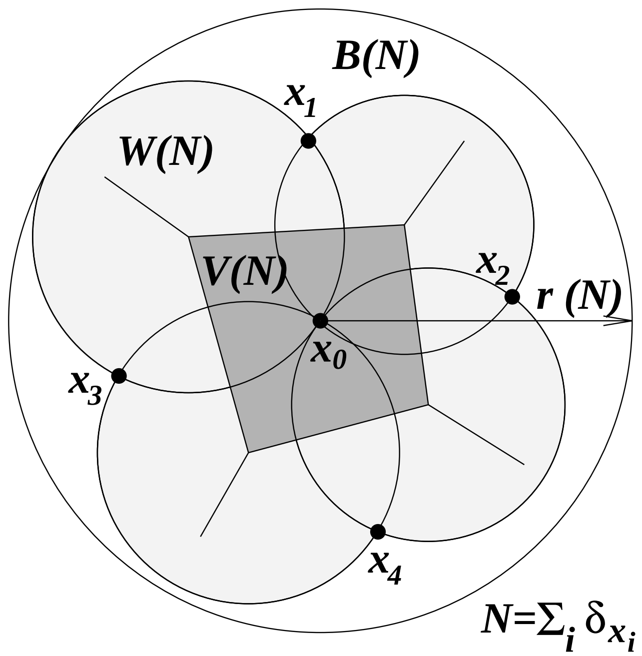


Figure 2

**Example 3** *Perturbation Estimators for Functionals of a Voronoi Cell.*

Let $v(V)$ be a certain geometric characteristic of the bounded polygon $V$, like for instance its volume, the number of its vertices etc. and let $G(V)$ be some bounded function of $v(V)$. This could for instance be:

- $G(V) = \exp(itv(V))$, where $t \in \mathbb{R}$;
- $G(V) = \exp(-sv(V))$, where $s \in \mathbb{R}_+$;
- $G(V) = \mathbf{1}_{[z,\infty)}(v(V))$, where $z \in \mathbb{R}_+$.

Let $N$ be another stationary point process, independent of $\Upsilon$, and of intensity 1. We are interested in estimating

$$D = \frac{d}{d\rho}\bigg|_{\rho=+0} (\mathbf{E}_0^{\Upsilon} \times \mathbf{E}^N)\, G(V(\Upsilon + N_\rho)), \tag{24}$$

where $N_\rho$ denotes the result of the $\mathbb{R}^d$-dilation with parameter $\rho^{-1/d}$ of $N$. As we saw in Example 2, for Poisson processes and with some modifications for infinitely divisible processes the quantity $D$ gives the derivative with respect to the intensity of the original process $\Upsilon$. However in practice one may expect that it also provides an approximation to the derivative for a wider class of point processes, not having "too strong" dependence between points.

Assume, that $\mathbf{E}_0^\Upsilon \Lambda(B(\Upsilon)) < \infty$. Then the conditions of Assertion 6 are satisfied and we can rewrite the RHS of (22) as

$$\begin{aligned} & \mathbf{E}_0^\Upsilon \left\{ \Lambda(W(\Upsilon)) \frac{1}{\Lambda(W(\Upsilon))} \int_{W(\Upsilon)} [G(V(\Upsilon + \delta_x)) - G(V(\Upsilon))]\, dx \right\} \\ = \;& \mathbf{E}_0 \left\{ \Lambda(W(\Upsilon)) \left[ G(V(\Upsilon + \delta_U)) - G(V(\Upsilon)) \right] \right\}, \end{aligned}$$

where $U$ is uniformly distributed in $W(\Upsilon)$ random variable and $\mathbf{E}_0$ is expectation with respect to the semi-direct product of the distribution $\mathbf{P}_0^\Upsilon(d\gamma)$ of $\Upsilon$ and the uniform probability measure on $W(\gamma)$.

Note here, that $\mathbf{E}_0$ can be regarded as the Palm distribution of another point process $\tilde{\Upsilon}$ say, constructed as follows: first we take a realization of the process $\Upsilon$ and then we add independent random variables $U_i$ in each flower $W_i(\Upsilon)$ associated with a point $x_i$ of $\operatorname{supp} \Upsilon$. More formally, $\tilde{\Upsilon}$ is a stationary cluster process in the phase space $\mathcal{X} = \mathbb{R}^d \times \mathbb{Z}_+ . (\mathbb{R}^d)^n$, with the center process $\tilde{\Upsilon}_c$ defined according to the rule: to each realization $\sum_i \delta_{x_i}$ of $\Upsilon$ there corresponds a realization $\sum_i \delta_{y_i}$ on $\mathcal{X}$, where $y_i = (x_i, n, x_{i_1}, \ldots, x_{i_n})$ describing the number $n$ and positions of those points $x_{i_1}, \ldots, x_{i_n} \in \operatorname{supp} \Upsilon$ contributing to the structure of $W_i(\Upsilon)$. The component process associated with a point $y_i$ reproduces the point $y_i$ itself and adds a new point $(U_i, 0)$ in $\mathcal{X}$ with $U_i$ uniformly distributed in $W(x_i; x_{i_1}, \ldots, x_{i_n}) = W_i(\Upsilon)$ (for the definition of a cluster process cf. *Daley* & *Vere-Jones* (1988), Chapter 8.2).

Remind that the process $\Upsilon$ is assumed to be ergodic, so is the center process $\tilde{\Upsilon}_c$ since they are defined on the same probability space. Now we use the result of *Westcott* (1971) (see also Statement 7.1.4 of *Kerstan et al.* (1982)) to conclude, that the cluster process $\tilde{\Upsilon}$ is also ergodic in its turn and hence a direct application of the pointwise ergodic theorem gives the following estimator for $D$:

$$\widehat{D}_R = \frac{1}{\lambda \Lambda(B_R)} \sum_{x_i \in \operatorname{supp} \Upsilon \cap B_R} \Lambda(W_i(\Upsilon)) \left[ G(V_i(\Upsilon + \delta_{U_i})) - G(V_i(\Upsilon)) \right], \tag{25}$$

where $B_R$ denotes deterministic ball of radius $R$ centered at 0. (cf. *Daley* & *Vere-Jones* (1988), Chapter 12.4 for ergodic theorems).

The geometrical algorithm for computing this estimator is exemplified in Figure 3.

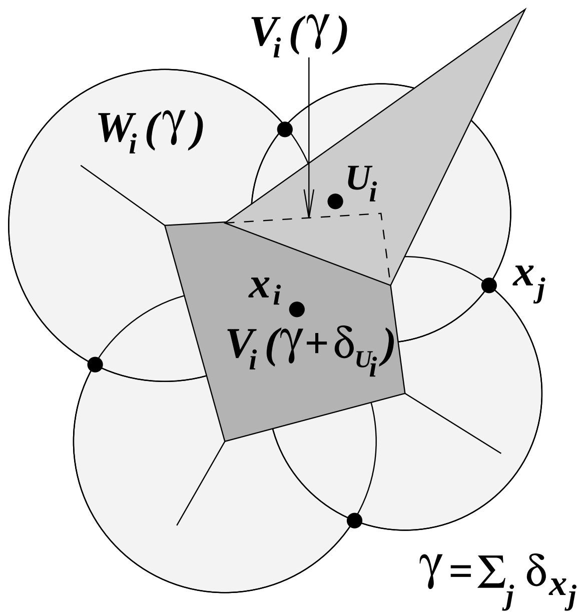


Figure 3

## Acknowledgements

The authors would like to thank Pr. A. A. Borovkov for useful discussions during the preparation of the present paper. In the special case of dilations in $\mathbb{R}^d$ further generalizations of the formulae obtained in the present paper were derived lately by *Borovkov* (1994), when relaxing the stationarity assumptions and replacing them by assumptions on the distributions of the points.

We are also grateful to Pr. V. Schmidt for his suggestions, which helped us to improve the presentation of the material in the paper.

# References


1. Baccelli, F. and Brémaud, P. (1993) Virtual Customers in Sensitivity and Light Traffic Analysis via Campbell's Formula for Point Processes. *Adv. Appl. Prob.*,**25**, 221-234

2. Baccelli, F. and Brémaud, P. (1994) *Elements of Queueing Theory.* Springer–Verlag, New-York.

3. Błaszczyszyn, B. (1993) Factorial Moment Expansion for Stochastic Systems. *Mathematical Institute, University of Wrocław, Preprint*

4. Borovkov, A.A. (1994) Asymptotic Expansions for Functionals of Dilation of Point Processes. *INRIA, Preprint.*

5. Daley, D. J. and Vere-Jones, D. (1988) *An Introduction to the Theory of Point Processes.* Springer–Verlag, New - York

6. Dunford, N. and Schwartz, J. (1988) *Linear Operators. Part I.* John Wiley & Sons, New-York

7. Ho, Yu-Chi and Cao, Xi-Ren (1991) *Perturbation Analysis of Discrete Event Dynamic Systems.* Kluwer Acad. Publ., Boston

8. Kerstan, J., Matthes, K. and Mecke, J. (1974) *Unbegrenzt teilbare Punktprozesse.* Akademie–Verlag, Berlin

9. Kerstan, J., Matthes, K. and Mecke, J. (1982) *Infinitely Divisible Point Processes.* Nauka, Moscow (in Russian)

10. Mecke, J. (1967) Stationäre zufällige Masse auf lokal-kompakten abelschen Gruppe. *Z.Wahrsch. verw. Geb.*, **9**, 36-58

11. Møller, J. (1994) *Lectures on Random Voronoi Tesselations.* Lecture Notes in Statistics, **87**, Springer–Verlag, New-York

12. Møller, J. and Zuyev, S. A. (1994) Gamma-type Results and Other Related Properties of Poisson Processes — *to appear*

13. Okabe, A., Boots, B. and Sugihara, K. (1992) *Spatial Tesselations. Concepts and Applications of Voronoi Diagrams.* John Wiley & Sons, New-York

14. Reiman, M. and Simon, B. (1989) Open Queueing Systems in Light Traffic. *Math. Oper. Research*, **14, 1**, 26-59

15. Russo, L. (1981) On the Critical Percolation Probabilities. *Z. Wahrsch. verw. Geb.*, **56**, 229-237

16. Stoyan, D. , Kendall, W. S. and Mecke, J. (1987) *Stochastic Geometry and its Applications.* J. Wiley & Sons, Chichester

17. Westcott, M. (1971) On Existence and Mixing Results for Cluster Point Processes. *J. Roy. Statist. Soc., Ser. B* **33**, 290-300

18. Zuyev, S. A. (1992) Estimates for Distributions of the Voronoi Polygon's Geometric Characteristics. *Random Structures and Algorithms,* **3**, 149-162

19. Zuyev, S. A. (1993) Russo's Formula for Poisson Point Fields and its Applications. *Discrete Math. Appl.*, **3**, 63-73

Unité de recherche INRIA Lorraine, Technopôle de Nancy-Brabois, Campus scientifique,
615 rue du Jardin Botanique, BP 101, 54600 VILLERS LÈS NANCY
Unité de recherche INRIA Rennes, Irisa, Campus universitaire de Beaulieu, 35042 RENNES Cedex
Unité de recherche INRIA Rhône-Alpes, 46 avenue Félix Viallet, 38031 GRENOBLE Cedex 1
Unité de recherche INRIA Rocquencourt, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex
Unité de recherche INRIA Sophia-Antipolis, 2004 route des Lucioles, BP 93, 06902 SOPHIA-ANTIPOLIS Cedex

Éditeur
INRIA, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)
ISSN 0249-6399