



# Grobner bases of toric ideals

Loïc Pottier

► **To cite this version:**

Loïc Pottier. Grobner bases of toric ideals. [Research Report] RR-2224, INRIA. 1994. <inria-00074446>

**HAL Id: inria-00074446**

**<https://hal.inria.fr/inria-00074446>**

Submitted on 24 May 2006

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

***Gröbner bases of toric ideals***

Loïc Pottier

**N° 2224**

Mars 1994

PROGRAMME 2

Calcul symbolique,  
programmation  
et génie logiciel ***rapport  
de recherche*****1994**





## Gröbner bases of toric ideals

Loïc Pottier

Programme 2 — Calcul symbolique, programmation et génie logiciel  
Projet SAFIR\*

Rapport de recherche n° 2224 — Mars 1994 — 12 pages

**Abstract:** We study here Gröbner bases of ideals which define toric varieties. We connect these ideals with the sub-lattices of  $Z^d$ , then deduce properties on their Gröbner bases, and give applications of these results. The main contributions of the report are a bound on the degree of the Gröbner bases, the fact that they contain Minkowski successive minima of a lattice (in particular shortest vector), and the algorithm (derived from Buchberger algorithm), which starts with ideal of polynomials with less variables than usual.

**Key-words:** Standard bases, Gröbner bases, toric varieties, successive minima

*(Résumé : tsvp)*

\*SAFIR est un projet commun à l'I.N.R.I.A. Sophia-Antipolis, à l'Université de Nice-Sophia Antipolis, et au C.N.R.S.

## Bases de Gröbner d'idéaux toriques

**Résumé :** Nous étudions ici les bases de Gröbner d'idéaux définissant des variétés toriques. Nous mettons en relation ces idéaux avec les sous-réseaux de  $Z^d$ , déduisons des propriétés de leurs bases de Gröbner, et donnons des applications de ces résultats. Les principales contributions de ce rapport sont une borne sur le degré des bases de Gröbner, le fait qu'elles contiennent les minima successifs de Minkowski d'un réseau (en particulier le plus court vecteur), et l'algorithme (dérivé de celui de Buchberger), qui prend en entrée un idéal de polynômes ayant moins de variables qu'usuellement.

**Mots-clé :** Bases standard, bases de Gröbner, variétés toriques, minima successifs

## 1 Toric varieties and toric ideals

We will consider here only toric varieties as algebraic varieties parametrized by monomials (see [14] or [6] for general studies of toric varieties):

$$X = \{(X_1, \dots, X_n) \in \mathbb{C}^n \mid X_1 = T^{a_1}, \dots, X_n = T^{a_n}, T_1, \dots, T_d \in \mathbb{C}\} \subset \mathbb{C}^n$$

where  $a_i$  are points in  $\mathbb{Z}^d$ .

The variety  $X$  is the zero-set of the ideal  $I$ , kernel of the map

$$\begin{aligned} \phi : k[X] &\longrightarrow k[T^{\pm 1}] \\ X_i &\longmapsto T^{a_i} \end{aligned}$$

Such a prime ideal is called now "toric ideal".

To the set  $\mathcal{A} = \{a_1, \dots, a_n\}$  we associate the integer lattice  $H = \{v \in \mathbb{Z}^n \mid \sum_i v_i a_i = 0\}$ , which is related to  $I$  by the

**Proposition 1**

$$X^\alpha - X^\beta \in I \iff \alpha - \beta \in H$$

As a consequence, we have:

**Proposition 2**

$$X^\alpha - X^\beta \in I \implies X^{(\alpha-\beta)^+} - X^{(\alpha-\beta)^-} \in I$$

where  $v^+$  denotes the vector  $v$  in which the negative coordinates are set to zero, and  $v^- = (-v)^+$ .

Basic properties of toric ideals can be found in [11]. We recall some of them:

**Proposition 3** [11]  *$I$  is generated by the binomials  $X^{v^+} - X^{v^-}$  where  $v \in H$ . Reduced Gröbner bases of  $I$  contains only binomials  $X^{v^+} - X^{v^-}$  where  $v \in H$ , and have degree lesser than  $n(n-d)a^d$  where  $a = \sup\{\|a_i\|\}$ .*

## 2 Properties of Gröbner bases of toric ideals

First we recall the definition of a Gröbner basis of an ideal.

Let  $<$  be a total order on monomials, compatible with multiplication (e.g. lexicographic order, reverse lexicographic order, etc). We denote  $in(f)$  the greatest monomial of a polynomial  $f$ . In the following, when a polynomial is written  $X^a + \dots$ , we suppose that  $X^a$  is the greatest monomial.

**Definition 1** A family  $F$  of polynomials of an ideal  $J$  is a Gröbner basis of  $J$  iff the set  $\text{in}(F) = \{\text{in}(f) | f \in F\}$  generates the ideal  $\text{in}(J) = \{\text{in}(f) | f \in J\}$ .

The basis is "reduced" if no monomial of its polynomials is a multiple of the greatest monomial of one of its polynomials.

## 2.1 Degree of toric Gröbner bases

In general the degree of Gröbner basis is doubly exponential in the number of variables [7]. But in the toric case this complexity falls to a simply exponential degree, as mentioned above ([11]). We prove now another simply exponential bound:

**Theorem 1** Let  $A$  be the  $d \times n$  matrix whose columns are the  $a_i$ s, with generic term  $a_{ij}$ . Then any reduced Gröbner basis of  $I$  has degree lesser than

$$\prod_i (1 + \sum_j |a_{ij}|)$$

Proof : Let  $H^+$  the set of vectors of  $H$  with positive coordinates. It is a monoid of finite type (by Gordan's lemma), generated by its minimal elements for the order  $v \leq w \iff \forall i v_i \leq w_i$ .

In [8] is proved that these minimal elements verify:

$$\sum_i v_i \leq \prod_i (1 + \sum_j |a_{ij}|)$$

More generally consider now the vectors of  $H$  with any given choice for the signs of their coordinates. They form a monoid to which the previous results can be applied (just change the sign of the corresponding columns of  $A$ ): they are finitely generated by minimal vectors verifying  $\sum_i |v_i| \leq \prod_i (1 + \sum_j |a_{ij}|)$

Suppose now that  $X^{\alpha^+} - X^{\alpha^-}$  is in a reduced Gröbner basis of  $I$ , with  $X^{\alpha^+} > X^{\alpha^-}$ . The vector  $\alpha$  belongs to one of the previous monoids. If it is not minimal, there we can write  $\alpha^+ = \beta^+ + \gamma^+$  and  $\alpha^- = \beta^- + \gamma^-$ , with  $\beta, \gamma \in H - \{0\}$ . Because  $X^{\alpha^+} > X^{\alpha^-}$ , we can suppose  $X^{\beta^+} > X^{\beta^-}$ . But  $X^{\beta^+} - X^{\beta^-} \in I$ , so  $\alpha^+ = \beta^+, \alpha^- = \beta^-$  because the Gröbner basis is reduced, and then  $\gamma = 0$ , which contradicts the hypotheses. So  $\alpha$  is minimal and then verify  $\sum_i \alpha_i \leq \prod_i (1 + \sum_j |a_{ij}|) \square$ .

## 2.2 Geometric characterization

We show here how any reduced Gröbner basis of the ideal  $I$  can be read on the lattice  $H$ , giving an explicit characterization which does not use Buchberger algorithm.

Let  $H_+$  the part of  $H$  constituted with vectors  $v$  of  $H$  such that  $X^{v^+} >_w X^{v^-}$  (for example, with lexicographic ordering, those vectors are vectors with first non zero coordinate which is positive).

**Theorem 2** *Let  $B$  be a part of  $H_+$  such that  $\forall v \in H_+, \exists w \in B, w^+ \preceq v^+$ . Then the family  $\{X^{v^+} - X^{v^-} \mid v \in B\}$  is a Gröbner basis of  $I$ .*

To prove this assertion, we will need some intermediate propositions.

First, a direct consequence of the definition of a Gröbner basis:

**Proposition 4**  *$F$  is a Gröbner basis of an ideal  $J$  iff every monomial of  $\text{in}(J)$  is divided by a monomial  $\text{in}(f)$  where  $f \in F$ .*

Let  $\mathcal{H}$  the family of polynomials  $\{X^{v^+} - X^{v^-} \mid v \in H_+\}$ .

**Proposition 5** *Every monomial of  $\text{in}(I)$  is divided by a monomial of  $\text{in}(\mathcal{H})$ .*

Proof : Let  $P$  a non zero polynomial of  $I$ . By definition of  $I$  and  $H_+$ ,  $\mathcal{H}$  generates  $I$ ,  $P$  can be written  $P = \sum_{i=1..p} a_i X^{\gamma_i} (X^{v_i^+} - X^{v_i^-})$  where the  $v_i$ s are in  $H_+$ .

The following lemma precises this:

**Lemma 1** *One can choose such an expression of  $P$  such that  $\text{in}(P) = X^{\gamma_j} X^{v_j^+}$  for a certain  $j$ .*

Proof : By induction on  $p$ , the number of terms of the expression of  $P$ .

For  $p = 1$  this is clear.

For  $p > 1$ , let  $Q = \sum_{i=1..p-1} a_i X^{\gamma_i} (X^{v_i^+} - X^{v_i^-})$ .

By induction hypothesis,  $\text{in}(Q) = X^{\gamma_j} X^{v_j^+}$  for a certain  $j < p$ .

If  $\text{in}(Q) < X^{\gamma_p} X^{v_p^+}$ , then  $\text{in}(P) = X^{\gamma_p} X^{v_p^+}$   $\square$

If  $\text{in}(Q) > X^{\gamma_p} X^{v_p^+}$ , then  $\text{in}(P) = \text{in}(Q) = X^{\gamma_j} X^{v_j^+}$   $\square$

If  $\text{in}(Q) = X^{\gamma_p} X^{v_p^+}$  and  $a_p + a_j \neq 0$ , then  $\text{in}(P) = X^{\gamma_p} X^{v_p^+}$   $\square$

Finally if  $\text{in}(Q) = X^{\gamma_p} X^{v_p^+} = X^{\gamma_j} X^{v_j^+}$  and  $a_p + a_j = 0$ , then  $P = a_p (X^{\gamma_j} X^{v_j^-} - X^{\gamma_p} X^{v_p^-}) + \sum_{i=1..p-1, i \neq j} a_i X^{\gamma_i} (X^{v_i^+} - X^{v_i^-})$ .



But  $a_p(X^{\gamma_j} X^{v_j^-} - X^{\gamma_p} X^{v_p^-})$  can be written  $\pm a_p X^\delta (X^{v^+} - X^{v^-})$  with  $v = \pm(v_p - v_j) \in H_+$ ; the expression of  $P$  has then  $p - 1$  terms, and we conclude by the induction hypothesis  $\square$

Now, we come back to the proposition: by the lemma,  $in(P) = X^{\gamma_j} X^{v_j^+}$ , we have  $X^{v_j^+} \in in(\mathcal{H})$ , and then  $in(P)$  is divided by a monomial of  $in(\mathcal{H})$   $\square$

We now prove the theorem: the definition of  $B$  shows that every monomial of  $in(\mathcal{H})$  is divided by a monomial  $X^{v^+}$  where  $v$  is in  $B$ . Then property 4 gives the conclusion  $\square$

**Definition 2** We call "Gröbner basis of  $H$ " (for the ordering  $<_w$  on monomials), the part  $Bs(H, <_w)$  of  $H_+$  whose elements are vectors  $v$  of  $H$  verifying:

- (i)  $\forall u \in H_+, u^+ \preceq v^+ \implies u^+ = v^+$
- (ii)  $\forall u \in H_+, u^+ \not\preceq v^-$

This definition is justified by the

**Theorem 3** The family  $\{X^{v^+} - X^{v^-} \mid v \in Bs(H, <_w)\}$  is the reduced Gröbner basis of  $I$  for the order  $<_w$  on monomials.

which is an easy consequence of the previous theorem.

### 3 Algorithms for toric Gröbner bases

#### 3.1 A good ideal to begin with

The usual techniques, used in [4] or [10], is to compute a Gröbner basis of the ideals  $J = (X_1 - T^{a_1}, \dots, X_n - T^{a_n}, T_1 T_1^{-1} - 1, T_m T_m^{-1} - 1)$ , or  $J = (X_1 - T^{a_1}, \dots, X_n - T^{a_n}, U T_1 \dots T_m X_1 \dots X_n - 1)$ , with Buchberger algorithm and an order eliminating  $U, T_j$  and  $T_j^{-1}$ , and then keep the binomials where  $U, T_j$  and  $T_j^{-1}$  do not appear, which form a Gröbner basis for  $I$ .

These methods use Gröbner bases computations with  $n + 2d$  or  $n + d + 1$  variables. We present here a method which uses only  $n + 1$  variables, and then is much more efficient in practice.

**Theorem 4** Let  $\{v_1, \dots, v_p\}$  be a basis of the lattice  $H$ . Let  $J$  be the ideal

$$(X^{v_1^+} - X^{v_1^-}, \dots, X^{v_p^+} - X^{v_p^-}, U X_1 \dots X_n - 1)$$

Let  $B$  be a Gröbner basis of  $J$  for an order eliminating the variable  $U$ . Then the binomials of  $B$  where  $U$  does not appear form a Gröbner basis of  $I$ .

Proof :

Consider the morphisms

$$\begin{array}{ccccc}
 k[X] & \xrightarrow{\phi_1} & k[X, U]/(UX_1 \dots X_n - 1) & \xrightarrow{\phi_2} & k[X^{\pm 1}] & \xrightarrow{\phi_3} & k[T^{\pm 1}] \\
 X_i & \mapsto & X_i & \mapsto & X_i & \mapsto & T_i^{a_i} \\
 & & U & \mapsto & X_1^{-1} \dots X_n^{-1} & & 
 \end{array}$$

We have  $I = \text{Ker}(\phi_3 \circ \phi_2 \circ \phi_1)$  ( $\phi_2$  is an isomorphism keeping the  $X_i$ s invariant).

Let  $J = \text{Ker}(\phi_3)$ . Then  $I = \phi_2^{-1}(J) \cap k[X]$ , and  $J = (X^v - 1)_{v \in H}$ .

Now, we prove that  $J = (X^{v_1} - 1, \dots, X^{v_p} - 1)$ , which gives the result, because that it implies  $J = (X^{v_1^+} - X^{v_1^-}, \dots, X^{v_p^+} - X^{v_p^-})$

Let  $v = \sum_j \alpha_j v_j$  a vector of  $H$ .

$$\begin{aligned}
 \text{We have } X^v - 1 &= X^{\sum_j \alpha_j v_j} - 1 \\
 &= (X^{\sum_j \alpha_j v_j} - X^{\alpha_1 v_1}) + (X^{\alpha_1 v_1} - 1) \\
 &= X^{\alpha_1 v_1} (X^{\sum_{j \geq 2} \alpha_j v_j} - 1) + (X^{\alpha_1 v_1} - 1)
 \end{aligned}$$

if  $a_1$  is nonnegative,  $(X^{v_1} - 1)$  divides  $X^{\alpha_1 v_1} - 1$ , otherwise  $(X^{-v_1} - 1)$  divides  $X^{\alpha_1 v_1} - 1$ .

Then we are led to the same problem for the vector  $\sum_{j \geq 2} \alpha_j v_j$ , and by induction we conclude  $X^v - 1 \in (X^{v_1} - 1, \dots, X^{v_p} - 1) \square$

The computation of a basis of  $H$  can be performed in polynomial time (by computing the Hermite normal form of the matrix  $A$ ) but coefficients of the  $v_i$  can be very large in regard with those of the matrix  $A$ . However, in practice the reduction of the number of variables in the binomials suffices to drastically reduce the time of the computations.

### 3.2 Improving Buchberger algorithm for the toric case

Buchberger algorithm [2] works with general polynomials. But the toric case is very specific:

**Remark 1** *Polynomials are only binomials, and reductions of binomials are again binomials.*

**Remark 2** *Binomials of the ideal  $I$  have no variable in common: if  $X^{a+c} - X^{b+c}$  belongs to  $I$  then  $X^a - X^b$  belongs to  $I$ . So we can perform these simplifications at any step of the algorithm. More, we can then represent (and implement) a binomial  $X^a - X^b$  of  $I$  by the vector  $a - b$  of  $H$ .*

**Remark 3** *Divisions of binomials correspond to additions or subtractions in  $H$ :*

- if  $X^{b^+}$  divides  $X^{a^+}$ , then  $X^{a^+} - X^{a^-}$  reduces to  $X^{(a-b)^+} - X^{(a-b)^-}$ . We denote this property on vectors by  $a \xrightarrow{+}_b a - b$ .
- if  $X^{b^+}$  divides  $X^{a^-}$ , then  $X^{a^+} - X^{a^-}$  reduces to  $X^{(a+b)^+} - X^{(a+b)^-}$ . We denote this property on vectors by  $a \xrightarrow{-}_b a + b$ .
- The  $S$ -polynomial (called also critical pair) of two binomials  $X^{a^+} - X^{a^-}$  and  $X^{b^+} - X^{b^-}$  corresponds to the vector  $a - b$ .

Then we have the

**Lemma 2** *Let  $B$  be a set of binomials.*

*If  $a_1 \xrightarrow{+} \dots \xrightarrow{+} a_r \xrightarrow{-} \dots \xrightarrow{-} a_l$  by binomials of  $B$ ,*

*if  $a_r$  is irreducible for any reduction  $\xrightarrow{+}$  by binomials of  $B$ , and  $a_l$  is irreducible for any reduction  $\xrightarrow{-}$  by binomials of  $B$ ,*

*then  $a_l$  is irreducible by the binomials of  $B$ .*

This lemma discards many unnecessary tests for division.

Each couple of binomial  $P_i$  and  $P_j$  gives a critical pair, ties to a monomial denoted  $lcm_{ij}$ , which is the lcm of the two leading monomials of  $in(P_i)$  and  $in(P_j)$ .

It is well-known [2] that a critical pair between  $P_i$  and  $P_j$  can be discarded in essentially two cases:

- $in(P_i)$  and  $in(P_j)$  are relatively prime (no variable in common).
- $lcm_{kl}$  strictly divides  $lcm_{ij}$ , i.e.  $lcm_{ij}$  is not minimal for the component-wise order.

## 4 Applications

### 4.1 Successive minima of a lattice

We show here how the Minkowski successive minima of the lattice  $H$  are given by a Gröbner basis of  $I$ .

Given a norm on  $\mathbb{R}^n$ , the successive minima  $\lambda_1, \dots, \lambda_p$  of a lattice  $H$  of dimension  $p$  are defined in the following way:  $\lambda_k$  is the radius of the smallest ball (for the chosen norm) containing  $k$  independent vectors of  $H$ .

In[3], general results on successive minima and lattices are given.

One can show that there exists a free family  $v_i$  of  $H$  such that  $\|v_i\| = \lambda_i$ .

The second fundamental theorem of Minkowski's geometry of numbers gives upper and lower bounds on  $\prod_i \lambda_i$  for the euclidian norm.

Here, we will use two norms:

the 1norm:  $\|x\|_1 = \sum_i |x_i|$

and the -1norm:  $\|x\|_{-1} = \sup(\|x^+\|_1, \|x^-\|_1)$

Some easy remarks will be useful:

**Remark 4** *Suppose that  $X^a - X^b$  reduces to 0 by binomials  $\{X^{a_1} - X^{b_1}, \dots, X^{a_q} - X^{b_q}\}$ . Then the vectors  $a - b, a_1 - b_1, \dots, a_q - b_q$  are not independant.*

We note  $lad(I)$  "the ladder of I", i.e. the minimal monomials (for the partial order of division of monomials) of  $in(I)$ .

**Remark 5** *Let  $X^a - X^b$  be a binomial of  $I$  such that  $X^a \in lad(I)$ . Then  $a = (a - b)^+$  and  $b = (a - b)^-$ .*

**Theorem 5 (Homogeneous case)** *Suppose that  $H$  is "homogeneous", i.e. contained in the hyperplane  $x_1 + \dots + x_n = 0$ .*

*Let  $\lambda_1, \dots, \lambda_p$  be the successive minima of  $H$  for the 1norm.*

*Let  $B$  be a Gröbner basis of  $I$  for  $\langle_{(1, \dots, 1)}$  (i.e. total degree order).*

*Then there exists independant vectors  $a_1, \dots, a_p$  of  $H$  such that  $\forall k, \|a_k\|_1 = \lambda_k$  and  $X^{a_k^+} - X^{a_k^-}$  is in  $B$ .*

*In other terms, a Gröbner basis of  $I$  for  $\langle_{(1, \dots, 1)}$  contains the successive minima of  $H$  for the 1norm.*

Proof : First remark that  $H$  is homogeneous, then the ideal  $I$  is also homogeneous, so for all binomial  $X^{v^+} - X^{v^-}$  of  $I$ , we have  $deg(X^{v^+}) = deg(X^{v^-}) = \|v\|_1/2$ .

We will build the vectors  $a_1, \dots, a_p$  by induction.

- Case  $k = 1$ :

Let  $a$  in  $H$  such that  $\|a\|_1 = \lambda_1$ . The binomial  $X^{a^+} - X^{a^-}$  is in  $I$ , then there exists a binomial  $X^{a_1^+} - X^{a_1^-}$  of  $B$  such that  $X^{a_1^+} \in lad(I)$ . Then  $X^{a_1^+}$  divides  $X^{a^+}$ , and  $deg(X^{a_1^+}) \leq deg(X^{a^+})$ . By definition of  $\lambda_1$ , we have then  $\|a_1\|_1 = \lambda_1$ , so  $a = a_1 \in B$   $\square$

- Case  $k > 1$ :

Let  $a_1, \dots, a_{k-1}$  giving the successive minima  $\lambda_1, \dots, \lambda_{k-1}$ , then there exists  $a$  in  $H$  such that  $\|a\|_1 = \lambda_k$ , and  $a$  independent of  $a_1, \dots, a_{k-1}$  (such an  $a$  exists).

Let  $B_k$  be the part of  $B$  formed by the binomials  $X^{b^+} - X^{b^-}$  where  $a_1, \dots, a_{k-1}, b$  are not independent.

If the binomial  $X^{a^+} - X^{a^-}$  reduces to 0 by division by binomials of  $B_k$ , by Remark 4,  $a_1, \dots, a_{k-1}, a$  are not independent. Then  $X^{a^+} - X^{a^-}$  reduced by  $B_k$  is a non zero binomial  $X^{a'^+} - X^{a'^-}$ , irreducible by  $B_k$ , with  $\|a'\|_1 = \|a\|_1 = \lambda_k$  because the binomials of  $B$  are homogeneous.

As  $X^{a'^+} - X^{a'^-} \in \mathcal{I}$ , there exists a binomial  $X^{a_k^+} - X^{a_k^-}$  of  $B - B_k$  such that  $X^{a_k^+} \in \text{lad}(I)$ ,  $X^{a_k^+}$  divides  $X^{a'^+}$ , and  $\text{deg}(X^{a_k^+}) \leq \text{deg}(X^{a'^+})$ ,  $\|a_k\|_1 \leq \lambda_k$ . But  $a_1, \dots, a_{k-1}, a_k$  are independent vectors, and then by definition of  $\lambda_k$ ,  $\|a_k\|_1 = \lambda_k$ ,  $a_k \in B$   $\square$

We can easily generalize the theorem to the non homogeneous case:

**Theorem 6 (General case)** *Suppose  $H$  is not necessarily homogeneous. A Gröbner basis of  $I$  for  $\langle_{(1, \dots, 1)}$  contains the successive minima of  $H$  for the  $-1$ norm.*

## 4.2 Delaunay triangulation

A triangulation of the set of points  $a_1, \dots, a_n$  of  $\mathbb{Z}^d$  is said "regular" iff there exists a vector  $w$  of  $\mathbb{R}^n$  such that this triangulation is obtained as the projection of the lower convex envelop of the points  $(a_1, w_1), \dots, (a_n, w_n)$  of  $\mathbb{R}^{d+1}$ .

A theorem of Sturmfels [10] gives relations between regular triangulations of  $a_1, \dots, a_n$  and the Gröbner bases of the toric ideal  $I$ .

Suppose that the points  $a_1, \dots, a_n$  are in an affine hyperplane and generate  $\mathbb{Z}^d$ . Let  $\Delta_w$  be the triangulation defined by  $w \in \mathbb{R}^n$ . Let  $I_\Delta$  be the ideal generated by the monomials  $X_{\sigma_1} \dots X_{\sigma_p}$  where  $\{\sigma_1, \dots, \sigma_p\}$  is not a face of  $\Delta_w$  (the Stanley-Reisner ideal of  $\Delta_w$ ).

**Theorem 7 ([10])**  $I_\Delta$  is the radical of  $\text{in}_w(I)$

The Delaunay triangulation is regular, as obtained with the vector  $(\|a_1\|^2, \dots, \|a_n\|^2)$ . Then, a consequence of the theorem is:

**Corollary 1** *Let  $w = (\|a_1\|^2, \dots, \|a_n\|^2)$ , then a Gröbner basis of  $I$  for  $\langle_w$  gives the Delaunay triangulation of  $a_1, \dots, a_n$ .*

The complexity of this computation depends directly from the degree of the Gröbner basis, bounded by  $(1+na)^d$  where  $a$  is the maximum coordinate of the  $a_i$ s. We cannot *a priori* compete with specialized algorithms for Delaunay triangulation (which have however a theoretical complexity of the same order), but we can expect to interpret them in the context of polynomials.

**Example:**

Let  $a_1 = (0, 0, 1), a_2 = (6, 0, 1), a_3 = (5, 2, 1), a_4 = (1, 4, 1), a_5 = (2, 2, 1), a_6 = (4, 3, 1)$  in the plane identified to  $\{z = 1\}$  in  $\mathbb{R}^3$ .

Computations with softwares Macaulay [1] or bastat [9], give a Gröbner basis for  $I$  with initial monomials

$\{X_1^2 X_6^6, X_1 X_2 X_6^6, X_1 X_3^2, X_1 X_3 X_4, X_2^2 X_6^6, X_3^6 X_4^2, X_2 X_4, X_3^5 X_4^3, X_3^4 X_4^2 X_5\}$

and then  $I_\Delta = (X_1 X_6, X_1 X_3, X_2 X_6, X_3 X_4, X_2 X_4)$ ,

the Delaunay triangulation is  $\Delta = \{\{1, 2, 5\}, \{1, 4, 5\}, \{2, 3, 5\}, \{3, 5, 6\}, \{4, 5, 6\}\}$ .

## References

- [1] D.Bayer,M.Stillman "Macaulay User's Manual".
- [2] B.Buchberger "Gröbner bases-an algorithmic method in polynomial ideal theory" Chapter 6 in N.K.Bose(ed.) "Multidimensional Systems theory", D.Reidel, 1985.
- [3] J.W.S.Cassels "An introduction to the geometry of numbers" Springer Verlag ed., 1959.
- [4] P.Conti,C.Traverso "Buchberger algorithm and integer programming", Proc. AAEC C-9, LNCS 539,1991,pp. 130-139.
- [5] D.Cox,J.Little,D.O'Shea "Ideal,Varieties and Algorithms", Springer Undergraduate Texts in Mathematics, 1992.
- [6] V.I.Danilov "The geometry of toric varieties" Russian Mathematical Surveys 33:2, 1978.
- [7] Mayr,Meyer "The complexity of the word problems for commutative semi-groups and polynomial ideals", Adv. Math. 46, pp 305-329, 1982.

- [8] L.Pottier "Bornes et algorithmes de calcul des générateurs des solutions de systèmes diophantiens linéaires", Comptes Rendus de l'Académie des Sciences, t.311, Série I, pp. 813-816, 1990.
- [9] L.Pottier "Bastat: calcul de bases standard d'idéaux toriques", documentation technique (ftp zenon.inria.fr file safir/93-03-LP-bastat.tar.Z).
- [10] B.Sturmfels "Gröbner bases of toric varieties", Tohoku Mathematical Journal 43, 1991, 249-261.
- [11] B.Sturmfels "Algorithms in invariant theory" to appear in RISC Series on "Symbolic computation", 1991.
- [14] B.Tessier 80 "Variétés toriques et polytopes" Séminaire Bourbaki, 33e année, n. 565, novembre 1980.



---

Unité de recherche INRIA Lorraine, Technôpole de Nancy-Brabois, Campus scientifique,  
615 rue de Jardin Botanique, BP 101, 54600 VILLERS LÈS NANCY  
Unité de recherche INRIA Rennes, IRISA, Campus universitaire de Beaulieu, 35042 RENNES Cedex  
Unité de recherche INRIA Rhône-Alpes, 46 avenue Félix Viallet, 38031 GRENOBLE Cedex 1  
Unité de recherche INRIA Rocquencourt, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex  
Unité de recherche INRIA Sophia-Antipolis, 2004 route des Lucioles, BP 93, 06902 SOPHIA-ANTIPOLIS Cedex

---

Éditeur  
INRIA, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)  
ISSN 0249-6399