# Variational analysis of a mixed finite element : finite volume scheme on general triangulation

Katherine Mer

INRIA
INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET AUTOMATIQUE

# *Variational Analysis of a Mixed Finite Element/ Finite volume Scheme on General Triangulations*

Katherine Mer

**N° 2213**

Mars 1994

PROGRAMME 6

Calcul scientifique,

modélisation

et logiciel numérique

Rapport de recherche

1994

ISSN 0249-6399

# Variational Analysis of a Mixed Finite Element/ Finite volume Scheme on General Triangulations

Katherine Mer *

Programme 6 — Calcul scientifique, modélisation et logiciel numérique
Projet Sinus

Rapport de recherche n° 2213 — Mars 1994 — 32 pages

**Abstract:** The accuracy of a mixed Finite Element/ Finite Volume scheme on unstructured triangular meshes is obtained by a variational error estimate analysis, for a scalar linear stationary convection-diffusion problem. The scheme studied here is composed of an equivalent centered Finite Volume/ Lagrange-Galerkin formulation, and a fourth-difference artificial dissipation term for the stabilization of first derivatives.

*(Résumé : tsvp)*

*Inria, BP 93, 06902 Sophia Antipolis Cedex, France & Université de Nice-Sophia Antipolis

Unité de recherche INRIA Sophia-Antipolis
2004 route des Lucioles, BP 93, 06902 SOPHIA-ANTIPOLIS Cedex (France)
Téléphone : (33) 93 65 77 77 – Télécopie : (33) 93 65 77 65

# Analyse Variationnelle d'une Méthode Mixte Eléments Finis/ Volumes Finis sur des Triangulations Générales

**Résumé :** Ce travail présente une analyse variationnelle de la précision d'une méthode mixte FEM/FVM pour l'équation de convection-diffusion linéaire stationnaire sur des maillages triangulaires non structurés. Le schéma étudié est composé d'une formulation centrée éléments finis/ volumes finis et d'un terme de viscosité artificielle d'ordre quatre sous forme différences finies pour la stabilisation des termes de convection.

# Introduction

With a centered collocated scheme that approaches first derivatives, one can use a stabilization model consisting of an artificial viscosity finite-difference term (cf. Jameson [18, 17, 26], Sweby [35], Tadmor [37] and Swanson and Turkel [34]) or a variational artificial viscosity term [21]. Otherwise, stabilization by upwinding techniques leads for example to the Baba-Tabata scheme in the scalar case [3], to Godunov-type methods for systems (cf. Godunov [13], Roe [32], Osher [29]) and their extension to second-order accuracy by M.U.S.C.L techniques due to Van Leer [23] (cf. also [2, 11, 4]). In the literature, one can also find Petrov-Galerkin methods like the streamline diffusion method, which has been introduced by Hughes and Brooks [16] and analyzed by Johnson and others [20, 27]. When stabilization by adding numerical dissipation terms is used, one should tune their coefficients to obtain satisfactory stability and accuracy properties (for practical examples, cf. [26, 34]). Conversely, the amount of diffusion that is introduced by upwinding techniques is fixed and not necessarily optimum, which may impose afterwards additional manipulations of the schemes. Upwind schemes have a good behavior near flow discontinuities, while centered schemes with additional dissipative terms are cost-effective.

We should distinguish two different types of stabilization: the monotone-like methods and the "anti-Gibbs" methods. The former are used to prevent over or undershoots near discontinuities while the latter provide global damping of oscillatory modes.

In this paper, we focus our study on a damping stabilization (anti-Gibbs) realized by adding artificial dissipation terms. These ones are of fourth-order and should not degrade the accuracy of the centered scheme. In this context, we will study, for the advection-diffusion equation, the influence on the accuracy of a fourth-difference dissipation term that we add to the Galerkin formulation. To obtain error estimates by a variational analysis, the scheme considered is interpreted in a finite-element framework. However, for a better guidance in the choice of the stabilization term, the scheme can also be interpreted as of finite-volume type.

Among the recent related works dealing with the convergence and accuracy properties of finite-volume schemes, we can mention the error estimates obtained by a variational approach; in such a context both discrete (e.g. [5, 22]) and continuous (e.g. [33]) error estimates are known. Concerning the finite-volume analysis for hyperbolic conservation laws in the unstructured case, we can refer to Vila [39] for an error estimate in $L^1(\Omega)$ and to Champier, Gallouët [7] and Cockburn *et al.* [10] for convergence results.

The main question considered in this paper is the following: does the Galerkin formulation stabilized by a fourth-difference artificial dissipation term enjoys the advantageous properties of the Galerkin scheme, which is high order accurate even for unstructured triangulations with highly varying element sizes. The answer is given for a Jameson-type dissipation term, that relies on a non consistent discretization of the Laplacian in the two dimensional computational space (cf. [26]). The numerical studies of this scheme for the Euler and Navier-Stokes equations show the importance of the value of the numerical dissipation coefficient: this one should be sufficiently small for a good accuracy and sufficiently high for good stability properties. The analysis made in this paper determines how should the coefficients of the artificial dissipation term be adapted to the mesh size in order to maintain the accuracy of the Lagrange-Galerkin scheme.

The paper is organized as follows:
In Section 1, the continuous model problem and the Finite Element discretization under study are presented. We introduce the usual Lagrange-Galerkin scheme for this problem. A perturbation term is added to the Galerkin formulation by using the finite-difference variational method, for which a discrete second order operator $\Delta^h$ is considered.
In Section 2, an error estimates analysis is performed with some assumptions on the second order discrete operator. The main result of the analysis states that the accuracy of the Galerkin scheme is maintained, i.e : first-order accuracy in $H^1(\Omega)$ and second-order accuracy in $L^2(\Omega)$, and first-order accuracy in $L^2(\Omega)$ in the convection-dominated case.
In Section 3, a finite-volume scheme composed of a vertex-centered formulation with barycentric cells and a fourth-difference dissipation term [26], is presented in the two dimensional unstructured-simplicial-mesh case. We then show that the above two formulations are equivalent, if some quadrature formulas are used and for a particular choice of the discrete operator $\Delta^h$. The properties of this last one, required to obtain the error estimates results of Section 2, are then established.

# 1 A Finite Element discretisation using a Finite Difference variational method

## 1.1 Continuous model problems

We consider a bounded polygonal domain $\Omega$ in $\mathbb{R}^n$, with boundary $\Gamma$. The function spaces which are used in this paper are the usual Sobolev spaces $W^{m,p}(\Omega)$ where $m$ is a non-negative integer and $p$ is a real number such that $1 \leq p \leq +\infty$. We shall use the following notations: The norm on $W^{m,p}(\Omega)$ is denoted by $\| \ \|_{m,p,\Omega}$. For $p$=2, $\| \ \|_{m,\Omega}$ holds for the norm of $H^m(\Omega)$ and $| \ |_{m,\Omega}$ its semi-norm. The norm of $L^2(\Omega)$ is denoted by $| \ |_{0,\Omega}$ and we denote by $( \ , \ )$ its inner product.
Let $\vec{V}$ be a given smooth velocity vector field, $f$ a given function in $L^2(\Omega)$ and $\mu$ a positive constant. We consider the following convection-diffusion boundary value problem:

$$\begin{cases} -\mu\Delta u + div\left(\vec{V}u\right) = f \quad \text{in } \Omega \\ u = 0 \quad \text{on } \Gamma \end{cases} \tag{1}$$

The weak form of the equation (1) with strong imposed boundary conditions, writes:

$$\begin{aligned} &\text{find } \ u \in H_0^1(\Omega) \ \text{ such that} \\ &\quad a(u,v) = l(v), \qquad\qquad \text{for all } v \in H_0^1(\Omega), \end{aligned} \tag{2}$$

where the bilinear form $a$ is defined by

$$a(u,v) = \mu \iint_{\Omega} \vec{\bigtriangledown} u \vec{\bigtriangledown} v d\vec{x} \ + \ \iint_{\Omega} div\left(\vec{V}u\right) v d\vec{x}, \tag{3}$$

and the linear form

$$l(v) = \iint_{\Omega} f v d\vec{x} \tag{4}$$

We make the following assumptions on the velocity vector field:

$$\begin{aligned} &\text{(i)} \quad \vec{V} \in \left(L^{\infty}(\Omega)\right)^n \\ &\text{(ii)} \quad div\vec{V} \geq \alpha_0 > 0 \ \text{ a.e. in } \Omega, \ \text{ for some positive integer } \alpha_0 \end{aligned} \tag{5}$$

Then $a(u,v)$ is a continuous $H_0^1(\Omega)-$elliptic bilinear form, and we then obtain the existence and uniqueness of the solution of (2) by the Lax-Milgram lemma.
Let suppose that the polygonal domain $\Omega$ has convex corners, then the problem (1) is regular in the sense ( cf. Grisvard [14] ):

$$\begin{cases} u \in H^2(\Omega), \\ \|u\|_{2,\Omega} \leq C|f|_{0,\Omega} \end{cases} \tag{6}$$

The solution of the adjoint problem is also regular.
In the case of a small diffusion coefficient, the solution $u$ is in general not globally smooth because it may change rapidly in a thin layer. Local error estimates can then be investigated ( see for example [20] [27] ). However, in the present paper, we present only global error estimates.

If we consider that the diffusion coefficient is small compared to $|\vec{V}|$, we shall note it $\epsilon$ instead of $\mu$.
We are also interested by the reduced problem with $\epsilon = 0$:

$$\begin{cases} div\left(\vec{V}u\right) = f \quad \text{in } \Omega \\ u = 0 \quad \text{on } \Gamma_- \end{cases} \tag{7}$$

where $\Gamma_-$ is the inflow boundary defined by $\Gamma_- = \left\{x \in \Gamma; \vec{n}(x).\vec{V}(x) < 0\right\}$, $\vec{n}$ being the outward unit normal to $\Gamma$. The solution in this case may be discontinuous across the characteristic curves.

## 1.2 The Lagrange-Galerkin formulation

Let $(\mathcal{T}_h)_h$ be a family of triangulations of $\Omega$ by $n$-simplices $T_k$, $k = 1,..,N_e$, which are open sets. The vertices of the $n$-simplices are called nodal points and are noted $P_i$, $i = 1,..,N+M$ where $N$ is the number of interior points and $M$ the number of points located on the boundary $\Gamma$; we note $\Sigma_h = \{P_i,\ i = 1,..,N+M\}$. For any $T \in \mathcal{T}_h$, let $h_T$ be the diameter of the smallest ball containing $T$, and $\rho_T$ the diameter of the largest ball contained in $T$. We define $h$ and $\rho$ by:

$$\begin{aligned} h &= \max\{h_{T_k},\ k = 1,..N_e\} \\ \rho &= \min\{\rho_{T_k},\ k = 1,..N_e\} \end{aligned}$$

The family of triangulations is supposed to be regular ([9], p.132), and then there exists a real positive number $\sigma$ such that:

$$\frac{h_T}{\rho_T} \leq \sigma\ ,\quad \forall T \in \bigcup_h \mathcal{T}_h.$$

The following obvious relation ([9],p.124) will be useful in §3.4:

$$\sigma_n \rho_T^n \leq mes(T) \leq \sigma_n h_T^n\ ,\quad \forall T \in \mathcal{T}_h$$

where $\sigma_n$ denotes the $dx$-measure of the unit sphere in $I\!R^n$.
We make also the following assumption, which is called the inverse assumption with a view to the obtaining of inverse estimates : There exists a real positive number $\nu$ such that

$$\frac{h}{h_T} \leq \nu\ ,\quad \forall T \in \bigcup_h \mathcal{T}_h.$$

Provided that the regular family of triangulations verifies the above assumption, the inverse estimates writes, in the particular case of $P_1$-finite elements ([9],p.140):

$$|v|_{1,T} \leq C h^{-1} |v|_{0,T} \quad \forall v \in P_1(T) \tag{8}$$

where $C = C(\sigma,\nu)$ and $T \in \mathcal{T}_h$.
Let note $\Lambda_i = \{P_k;\ P_k \text{ and } P_i \text{ are the vertices of the same } n\text{-simplex } T\}$ and $N_i = card(\Lambda_i)$. The elements of $\Lambda_i$ are said to be the neighbors of $P_i$. The minimal and maximal order of the family of triangulations are noted respectively $N_{min}$ and $N_{max}$:

$$N_{min} \leq N_i \leq N_{max}\ ,\quad \forall i = 1,..,N\ ,\ \forall h > 0$$

Let note $V = H^1(\Omega)$ and $V_0 = H_0^1(\Omega)$. We consider the following finite element approximation $V^h \subset V$ of the space $V$:

$$V^h = \left\{ v_h \in C^0(\overline{\Omega}); v_h \text{ is a linear polynomial in each } T \in \mathcal{T}_h \right\}$$

We note $V_0^h = \left\{ v_h \in V^h; v_h = 0 \text{ on } \Gamma \right\}$. The usual basis functions of $V_0^h$ are noted $\phi_{ih}$ and verify: $\phi_{ih}$ is linear on each $n$-simplex and $\phi_{ih}(P_j) = \delta_{ij}$ for $i, j = 1, ..N$.

For a given function $v$ defined on $\Sigma_h$, let note $\pi_h v$ the $V_0^h$-interpolant of $v$. Since $\pi_h$ leaves invariant all polynomials of one degree on each $n$-simplex of $\mathcal{T}_h$, we have the following interpolation error estimates [8]

$$\|v - \pi_h v\|_{m,\Omega} \leq h^{k+1-m} |v|_{k+1,\Omega}. \tag{9}$$

where $k \leq 1$ for a $P_1$-interpolation.
Using the interpolation inequality, we have also

$$|v - \pi_h v|_{m,\Gamma} \leq h^{k+1/2-m} |v|_{k+1,\Omega} \tag{10}$$

The most natural finite element formulation for equation (1) is the Galerkin formulation. It writes:

$$\begin{aligned} &\text{find } \; u_h \in V_0^h \; \text{ such that} \\ &a(u_h, v_h) = l(v_h), \qquad\qquad \text{for all } v_h \in V_0^h \end{aligned}$$

By assumptions (5), the above problem has a unique solution since $V_0^h$ is a finite-dimensional subspace of $V$. Using Cea's lemma and the Aubin-Nitsche lemma, one can prove that the Galerkin scheme for equation (1) is first order accurate in $H^1(\Omega)$ and second order accurate in $L^2(\Omega)$ if the exact solution $u$ is in $H^2(\Omega)$. Since we are also interested by the case $\epsilon << |\vec{V}|$ and the limit case $\epsilon = 0$, for which the above error estimates are no more valid, we can only prove that the Galerkin method is of order one in $L^2(\Omega)$ ( cf. §2.1.2 )( for super-convergence results, see for example Dupont [12] or Lesaint [24] ).

We shall in the next part present a modified scheme constructed by adding to the Galerkin formulation an artificial viscosity.

## 1.3 Finite Difference variational method on a singular perturbation problem

Our purpose here is to add a fourth-order artificial viscosity term in the variational form, to the classical Galerkin formulation. Since the space of discretization $V^h$ is not included in $\{v \in H^1(\Omega)/ \; \Delta v \in L^2(\Omega)\}$, we shall use the finite-difference variational method; this method consists in replacing the operators of a variational form by discrete operators.

We consider a discrete linear continuous operator $\Delta^h : C^0(\overline{\Omega}) \longrightarrow L^2(\Omega)$. The operator $\Delta^h$ is associated with the triangulation $\mathcal{T}^h$: $\Delta^h v$ is entirely defined by the

geometry of $\mathcal{T}^h$ and the values $v_i$ of $v$ at the nodal points of $\mathcal{T}^h$. In particular, for all function $v$ defined on $\Sigma_h$, $\Delta^h v$ will be a step function, constant on each element $C_i^h$ around the point $P_i$, of a dual mesh ( cf. §3.1 ). The value of $\Delta^h v$ on $C_i^h$ will be defined by the values of $v$ on the more or less neighboring nodal points of $P_i$. $\Delta^h v$ sounds like a finite-difference approximation of second derivatives of $v$, but it is not necessarily consistent (cf. §3.3(B)).
We now introduce a symmetrical bilinear form $b_h$ defined on $V^h \times V^h$ by :

$$b_h(u_h, v_h) = h^\alpha \iint_\Omega \Delta^h u_h \, \Delta^h v_h \, d\vec{x},$$

where $\alpha$ is a real positive constant.

To obtain the numerical scheme that we shall analyze, we add to the Galerkin formulation the bilinear form $b_h$. The resulting method is inspired by the finite-difference variational method and the singular perturbation problem which can be found both in [6]. It reads:

$$\begin{aligned} &\text{find } \; u_h \in V_0^h \; \text{ such that} \\ &\quad a_h(u_h, v_h) = l(v_h), \qquad\qquad \text{for all } v_h \in V_0^h, \end{aligned} \tag{11}$$

where $a_h$ is defined by :

$$a_h(u_h, v_h) = a(u_h, v_h) + b_h(u_h, v_h) \tag{12}$$

**Lemma 1** *Let define the following norm on $V^h$:*

$$\|v\|_{V^h} = \Big( \|v\|_{1,\Omega}^2 \, + \, |\Delta^h v|_{0,\Omega}^2 \Big)^{1/2} \tag{13}$$

*$V^h$ is an Hilbert space for this norm and if $\vec{V}$ satisfies (5), then the bilinear form $a_h$ verifies, for all $u_h$ and $v_h$ in $V_0^h$:*

$$\begin{aligned} |a_h(u_h, v_h)| \; &\leq (max(\mu, |V_i|_{0,\infty,\Omega} : i = 1, .., n) \; + \; h^\alpha) \, \|u_h\|_{V^h} \, \|v_h\|_{V^h}, \\ a_h(v_h, v_h) \; &\geq min(\mu \; C(\Omega), h^\alpha) \, \|v_h\|_{V^h}^2. \end{aligned}$$

*Proof* $\quad V^h$ is an Hilbert space for the norm (13) since $\Delta^h$ is linear continuous on $C^0(\overline{\Omega})$. The bilinear form $a_h$ is continuous since we have

$$\begin{aligned} |b_h(u_h, v_h)| \; &= h^\alpha \; | \iint_\Omega \Delta^h u_h \, \Delta^h v_h \, d\vec{x}| \\ &\leq h^\alpha \; |\Delta^h u_h|_{0,\Omega} \, |\Delta^h v_h|_{0,\Omega} \\ &\leq h^\alpha \; \|u_h\|_{V^h} \, \|v_h\|_{V^h}. \end{aligned}$$

Furthermore, the relation $\left(div(\vec{V}v_h), v_h\right) = \frac{1}{2}\left((div\vec{V})v_h, v_h\right)$ for all $v_h \in V_0^h$ implies that $a_h$ is $V_0^h$- elliptic:

$$\begin{aligned} a_h(v_h, v_h) &= \left(div(\vec{V}v_h), v_h\right) + \mu\, |\nabla v_h|^2_{0,\Omega} + h^\alpha |\Delta^h v_h|^2_{0,\Omega} \\ &\geq \mu\, |\nabla v_h|^2_{0,\Omega} + h^\alpha\, |\Delta^h v_h|^2_{0,\Omega} \\ &\geq \mu\, C(\Omega)\, \|v_h\|^2_{1,\Omega} + h^\alpha |\Delta^h v_h|^2_{0,\Omega} \\ &\geq \min\left(\mu\, C(\Omega), h^\alpha\right) \|v_h\|^2_{V^h}. \end{aligned}$$

By the Lax-Milgram lemma, we obtain the existence and uniqueness of a solution $u_h \in V_0^h$ of problem (11-12).

# 2 Finite Element analysis

## 2.1 Global error estimates

We shall in this section obtain error estimates for the scheme presented above. We consider two cases, which are the convection-diffusion case and the convection-dominated convection-diffusion case. The first analysis follows the usual progression of the finite-element error analysis for elliptic problems: extending Céa's lemma, we obtain error estimates in $H^1(\Omega)$; next, error estimates in $L^2(\Omega)$ are obtained by an extension of the Aubin-Nitsche lemma. For the convection-dominated case and the limit case where the diffusion coefficient $\epsilon = 0$, we obtain error estimates in $L^2(\Omega)$; we are here inspired by the techniques of demonstration of Johnson *et al.* [20] ( see also [27] ).

### 2.1.1 The convection-diffusion case

We shall prove here global error estimates, first in the $H^1(\Omega)-$norm and after in the $L^2(\Omega)$-norm, under some assumptions on the finite-difference operator $\Delta^h$, which are:

$$|\Delta^h v_h|_{0,\Omega} \leq C_1 h^{-1}\, |v_h|_{1,\Omega}\,, \quad \text{for all } v_h \in V_0^h \tag{14}$$

$$|\Delta^h \varphi|_{0,\Omega} \leq C_2 h^{-m}\, \|\varphi\|_{2,\Omega}\,, \quad \text{for all } \varphi \in H^2(\Omega) \tag{15}$$

These assumptions are used to estimate consistency errors of the form $|b_h(u_h, v_h)|$. The first assumption is inspired by the inverse estimate (8); then we shall call it a discrete inverse estimate. In the general case, inequality (15) is easily obtained with $m = 2$, but depending on the "quality" of the discrete operator $\Delta^h$, it may be obtained with $m = 1$ or $m = 0$, the case $m = 0$ being an optimal case. For a better understanding of this assertion, we refer to Section 3.4 where the above assumptions are demonstrated for a particular choice of the discrete operator $\Delta^h$, in the two dimensional case.

**(A) Global error estimates in** $H^1(\Omega)$. We have the following result:

**Theorem 1** *Suppose the assumptions (5) and (14) are verified. Let $u$ be the exact solution of (1) and $u_h$ be the solution of the approximate problem (11-12). Suppose that $u \in H^2(\Omega)$ and $n \leq 3$. If $\alpha \geq 3$ then*

$$\|u - u_h\|_{1,\Omega} \leq C\ h$$

*Proof* Let $u_h$ be the solution of the approximation problem (11-12). Then:

$$a_h(u_h, u_h) = l(u_h) = (f, u_h) \tag{16}$$

Estimating each term of the equality, we obtain:

$$\beta\|u_h\|_{1,\Omega}^2 \leq |f|_{0,\Omega}|u_h|_{0,\Omega} \leq |f|_{0,\Omega}\|u_h\|_{1,\Omega},$$

where $\beta = \mu C(\Omega)$. Finally, we have:

$$\|u_h\|_{1,\Omega} \leq C\ , \quad \text{where } C = \frac{|f|_{0,\Omega}}{\beta}. \tag{17}$$

Furthermore, the following stability result holds:

$$\sqrt{\frac{\alpha_0}{2}}|u_h|_{0,\Omega} + \sqrt{\epsilon}|\vec{\bigtriangledown} u_h|_{0,\Omega} \leq \frac{2\sqrt{2}}{\sqrt{\alpha_0}}|f|_{0,\Omega} \tag{18}$$

These estimates will be useful in the sequel.
We suppose that the exact solution $u$ is in $H^2(\Omega)$ ( cf. §1.1 ) and that the space dimension $n \leq 3$, so that we can define the $V_0^h$−interpolation of $u$, $\pi_h u$.
By substracting equations of problems (2) and (11-12), we have:

$$a(u - u_h, v_h) = b_h(u_h, v_h), \ \ \forall v_h \in V_0^h$$

which gives:

$$a(u - u_h, u - u_h) = a(u - u_h, u - \pi_h u)\ +\ b_h(u_h, \pi_h u - u_h)$$

We have:

$$\beta\|u - u_h\|_{1,\Omega}^2 \leq M\|u - u_h\|_{1,\Omega}\|u - \pi_h u\|_{1,\Omega}\ +\ |b_h(u_h, \pi_h u - u_h)|$$

Using assumption (14), we estimate the term $|b_h(u_h, \pi_h u - u_h)|$:

$$\begin{aligned} |b_h(u_h, \pi_h u - u_h)| &\leq h^\alpha\ |\Delta^h u_h|_{0,\Omega}|\Delta^h(\pi_h u - u_h)|_{0,\Omega} \\ &\leq C_1^2 h^{\alpha-2}\ \|u_h\|_{1,\Omega}\|\pi_h u - u_h\|_{1,\Omega} \\ &\leq C' h^{\alpha-2}\ \|\pi_h u - u_h\|_{1,\Omega} \end{aligned}$$

$$\leq C'h^{\alpha-2}\ \|\pi_h u - u\|_{1,\Omega}\ +\ C'h^{\alpha-2}\ \|u - u_h\|_{1,\Omega}$$

Using Young's inequality, we have finally:

$$\begin{aligned}\beta\|u-u_h\|^2_{1,\Omega}\ &\leq M\frac{\gamma_1}{2}\ \|u-u_h\|^2_{1,\Omega}\ +\ M\frac{1}{2\gamma_1}\ \|u-\pi_h u\|^2_{1,\Omega}\ +\ C'\ h^{\alpha-2}\ \|\pi_h u - u\|_{1,\Omega}\\ &+ C'\ h^{\alpha-2}\frac{\gamma_2}{2}\ \|u-u_h\|^2_{1,\Omega}\ +\ C'\ h^{\alpha-2}\ \frac{1}{2\gamma_2}\ ,\end{aligned}$$

where $\gamma_1$ and $\gamma_2$ are arbitrary positive constants.
We conclude by choosing appropriate values of $\gamma_1$ and $\gamma_2$ and using the interpolation error estimates (9).

**(B) Global error estimates in $L^2(\Omega)$.** The process to obtain error estimates in $L^2(\Omega)$ will be inspired by the proof of the Aubin-Nitsche lemma. The main point is an argument of duality [9].
We have the following theorem:

**Theorem 2** *Suppose the assumptions (5), (14) and (15) are verified. Let $u$ be the exact solution of (1) and $u_h$ be the solution of the approximate problem (11-12). Suppose that the polygonal domain $\Omega$ has convex corners, and $n \leq 3$. If $\alpha \geq 3 + m$ then*

$$|u - u_h|_{0,\Omega} \leq Ch^2$$

*Proof* Let note $H = L^2(\Omega)$ and $V_0 = H^1_0(\Omega)$. The definition of the $L^2(\Omega)$-norm of an element $v$ is

$$|v|_{0,\Omega} = \sup_{g\in L^2(\Omega)} \frac{|(g,v)|}{|g|_{0,\Omega}} \tag{19}$$

Let define the adjoint variational problem associated to a function $g \in H$:

$$\begin{aligned}&\text{find}\ \ \varphi_g \in V_0\ \ \text{such that}\\ &\quad a(v,\varphi_g) = (g,v), \qquad\qquad \text{for all } v \in V_0,\end{aligned}$$

Since we have $V \subset H$ with continuous injection and density, we can write:

$$\forall g \in H,\ \forall v \in V,\ (g,v) = g(v).$$

Using the Lax-Milgram lemma, we obtain that this problem has a unique solution, thanks to assumptions (5). Furthermore, thanks to the assumption on the domain $\Omega$, the solution $\varphi_g$ is in $H^2(\Omega)$ ( cf. §1.1. ) and we have

$$\|\varphi_g\|_{2,\Omega} \leq C|g|_{0,\Omega}.$$

For a given $g \in H$, we have

$$a(u-u_h,\varphi_g) = (g, u-u_h)$$

furthermore, for all $\varphi_h \in V_h$,

$$a(u-u_h,\varphi_h)=b_h(u_h,\varphi_h)$$

then

$$\begin{aligned}(g,u-u_h) &= a(u-u_h,\varphi_g)-a(u-u_h,\varphi_h)+b_h(u_h,\varphi_h)\\ &= a(u-u_h,\varphi_g-\varphi_h)+b_h(u_h,\varphi_h)\end{aligned}$$

Let choose $\varphi_h=\pi_h\varphi_g$, which is defined since $n\leq 3$, then

$$|(g,u-u_h)|\leq M\|u-u_h\|_{1,\Omega}\|\varphi_g-\pi_h\varphi_g\|_{1,\Omega}+|b_h(u_h,\pi_h\varphi_g)|$$

where $M=\max(\mu,|V_i|_{0,\infty,\Omega}:i=1,..,n)$.
Using the definition (19) of the $L^2(\Omega)$-norm, we obtain

$$|u-u_h|_{0,\Omega}\leq M\|u-u_h\|_{1,\Omega}\sup_{g\in L^2(\Omega)}\left\{\frac{1}{|g|_{0,\Omega}}\|\varphi_g-\pi_h\varphi_g\|_{1,\Omega}\right\}+\sup_{g\in L^2(\Omega)}\left\{\frac{1}{|g|_{0,\Omega}}|b_h(u_h,\pi_h\varphi_g)|\right\}$$

Using interpolation error and the regularity of $\varphi_g$, we have

$$|u-u_h|_{0,\Omega}\leq MCh\|u-u_h\|_{1,\Omega}+\sup_{g\in L^2(\Omega)}\left\{\frac{1}{|g|_{0,\Omega}}|b_h(u_h,\pi_h\varphi_g)|\right\}$$

By the two assumptions (14) and (15) and the regularity arguments, we have

$$\begin{aligned}|b_h(u_h,\pi_h\varphi_g)| &\leq C'h^{\alpha-1-m}\|\varphi_g\|_{2,\Omega}\\ &\leq C''h^{\alpha-1-m}|g|_{0,\Omega}\end{aligned}$$

We conclude by using the result of theorem 1. We have obtained in this section that our modified scheme has the same error estimate as the Galerkin scheme.

### 2.1.2 The convection-dominated case

An error analysis independent of the diffusion coefficient is interesting when this one is small compared to $|\vec{V}|$, so that the usual condition on the mesh size is difficult to recover. We will therefore give a result in the case where the diffusion coefficient $\epsilon$ verifies $\epsilon\leq 1$ ( the constant 1 could be replaced by any other constant ).

**Theorem 3** *Suppose that the assumption (14) is verified. Let $u$ be the exact solution of (1) with $\mu$ replaced by $\epsilon(\epsilon\leq 1)$, or the solution of (7). Let $u_h$ be the solution of the approximate problem (11-12). Suppose that $\vec{V}$ verifies (5), $u\in H^2(\Omega)$ and $n\leq 3$. If $\alpha\geq 5$, then*

$$|u-u_h|_{0,\Omega}\leq Ch$$

*where $C$ is a constant independent of $\epsilon$.*

*Proof* We have the equality:

$$a(u-u_h,v_h)=b_h(u_h,v_h),\quad \forall v_h\in V_0^h$$

*Remark :* For the Galerkin formulation, one has $a(u-u_h,v_h)=0$, for all $v_h\in V_0^h$. For this reason we will obtain the same error estimate than the one of the Galerkin method in this case.

Let define $e=u-u_h$ and $\eta_h=u-\pi_h u$. We have then

$$a(e,e)=a(e,\eta_h)+b_h(u_h,\pi_h u-u_h) \tag{20}$$

Since the difference $e$ is equal to zero on the boundary $\Gamma$ ( because we have strongly imposed boundary conditions ), we obtain a coercivity relation for the error $e$:

$$a(e,e)\geq |||e|||^2, \tag{21}$$

where we have defined the following norm on $H^1(\Omega)$

$$|||v|||=\left(\frac{\alpha_0}{2}|v|^2_{0,\Omega}+\epsilon|\vec{\nabla}v|^2_{0,\Omega}\right)^{1/2}$$

The right member of expression (20) is estimated as follows: Using the Young's inequality, we have

$$\begin{aligned}
a(e,\eta_h) &= \left(div\left(\vec{V}e\right),\eta_h\right)+\epsilon\left(\vec{\nabla}e,\vec{\nabla}\eta_h\right)\\
&= -\left(e,\vec{V}.\vec{\nabla}\eta_h\right)+\epsilon\left(\vec{\nabla}e,\vec{\nabla}\eta_h\right)\\
&\leq C_1\frac{\gamma}{2}|e|^2_{0,\Omega}+C_1\frac{1}{2\gamma}|\vec{\nabla}\eta_h|^2_{0,\Omega}+\epsilon\frac{\gamma'}{2}|\vec{\nabla}e|^2_{0,\Omega}+\epsilon\frac{1}{2\gamma'}|\vec{\nabla}\eta_h|^2_{0,\Omega}
\end{aligned}$$

Using the assumption (14) and an inverse estimate (8), followed by the $L^2$-stability (18), we have also

$$|b_h(u_h,\pi_h u-u_h)|\leq Ch^{\alpha-4}|\pi_h u-u_h|_{0,\Omega}\leq Ch^{\alpha-4}|\eta_h|_{0,\Omega}+\frac{\gamma''}{2}|e|^2_{0,\Omega}+C\frac{1}{2\gamma''}h^{2\alpha-8}$$

Using the two last estimates with appropriate coefficients $\gamma$, $\gamma'$ and $\gamma''$, and again the interpolation error estimates, we obtain

$$|||e|||^2\leq C\left((1+\epsilon)h^2+h^{2\alpha-8}+h^{\alpha-2}\right)$$

If $\alpha\geq 5$, we have then

$$|||e|||^2\leq Ch^2,$$

which provides the announced error estimate in $L^2(\Omega)$. For the limit case $\epsilon = 0$, we define the spaces $V_0 = \left\{v \in V; v_{/\Gamma_-} = 0\right\}$ and $V_0^h = \left\{v_h \in V^h; v_{h/\Gamma_-} = 0\right\}$. We have the coercivity relation (21) with

$$|||v||| = \left(\frac{\alpha_0}{2}|v|^2_{0,\Omega} + \frac{1}{2}\int_{\Gamma_+} \vec{V}.\vec{n}v^2 d\gamma\right)^{1/2}$$

where $\Gamma_+ = \Gamma \setminus \Gamma_-$.
Since we have now

$$a(e,\eta_h) \leq C_1\frac{\gamma}{2}|e|^2_{0,\Omega} + C_1\frac{1}{2\gamma}|\vec{\nabla}\eta_h|^2_{0,\Omega} + \frac{\gamma'}{2}\int_{\Gamma_+} \vec{V}.\vec{n}e^2 d\gamma + \frac{1}{2\gamma'}\int_{\Gamma_+} \vec{V}.\vec{n}\eta_h^2 d\gamma,$$

the same error estimate is obtained in this case using the estimates (9) (10) .

# 3 Presentation of a Finite Volume scheme

We present in this section a finite volume scheme for the convection-diffusion equation in the two dimensional case. It consists in a centered part obtained by a vertex-centered finite volume method with barycentric cells called the "Finite Volume Galerkin" scheme [2] [1], to which we add a fourth-difference artificial dissipative term which has been previously constructed by Jameson and others [19, 26, 25]. This scheme is then shown to be equivalent to the finite element scheme (11-12) studied in Sections 1,2, for which a numerical quadrature and a specific finite difference operator $\Delta^h$ are used. The properties of this operator used for the finite element analysis are then demonstrated in Section 3.4.

## 3.1 Finite Volume discretization

Given a triangulation $\mathcal{T}_h$ as described in chapter 1, we define a dual grid $\mathcal{C}_h$ of $\mathcal{T}_h$. We note $C_i^h, i = 1,..,N$ the open elements of $\mathcal{C}_h$. For the construction of $\mathcal{C}_h$, the following properties are required:

$$(i) \quad \bigcup_{i=1}^{N} \overline{C_i^h} = \Omega \ ,$$

$$(ii) \quad C_i^h \bigcap C_j^h = \emptyset \ \text{ for all } j \neq i \ ,$$

$$(iii) \quad C_i^h \text{ is closely connected ,}$$

$$(iv) \quad P_i \in C_i^h \ , \ i = 1,..,N.$$

In the finite element framework, $C_i^h$ is called a "lumped region" associated to $P_i$, and it is called a "control volume" in the finite volume framework.