Random walks in a quarter plane with zero drifts : transcience and recurrrence

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N° 2209
Mars 1994

PROGRAMME 1

Architectures parallèles,
bases de données,
réseaux et systèmes distribués

Rapport de recherche

1994

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March 8, 1994

Abstract

In this paper we continue the study of the classification problem for random walks in the quarter plane, with zero drifts in the interior of the domain. The necessary and sufficient conditions for these random walks to be ergodic were found earlier in [3]. Here we obtain necessary and sufficient conditions for transience by constructing suitable Lyapounov functions.

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Marches aléatoires dans le quart de plan avec dérives nulles : Transience and recurrence.

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25 février 1994

Résumé

Dans cet article, on poursuit l'étude de la classification des marches aléatoires dans le quart de plan, lorsque les dérives moyennes sont nulles à l'intérieur du domaine. Les conditions nécessaires et suffisantes d'ergodicité avaient été obtenues jadis dans [3]. On donne ici les conditions nécessaires et suffisantes pour la transience, en construisant des fonctions de Lyapounov ad hoc.

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1 Introduction and notation

We consider a discrete time homogeneous Markov chain $\mathcal{L} = \{\xi_\eta, n \geq 0\}$, irreducible and aperiodic, defined on the lattice $\mathbf{Z}_+^2 = \{(i,j) : i,j \geq 0 \ are\ integers\}$ in the positive quarter plane and satisfying the recursive equation

$$\xi_{n+1} = [\xi_n + \theta_{n+1}]^+,$$

where the distribution of θ_{n+1} depends only on the position of ξ_n in the following way ("maximal " space homogeneity)

$$p\{\theta_{n+1} = (i,j)/\xi_\eta = (k,l)\} = \begin{cases} r_{ij}, & for \quad k,l \geq 1, \\ q_{ij}, & for \quad k \geq 1, l = 0, \\ p_{ij}, & for \quad k = 0, l \geq 1, \\ r_{ij}^0, & for \quad k = l = 0. \end{cases}$$

Moreover we assume, for the one step transition probabilities, the following conditions :

A (*Lower boundedness*)

$$\begin{cases} r_{ij} = 0, & if \quad i < -1 \quad or \quad j < -1\,; \\ q_{ij} = 0, & if \quad i < -1 \quad or \quad j < 0\,; \\ p_{ij} = 0, & if \quad i < 0 \quad\ \ or \quad j < -1\,. \end{cases}$$

B (*Moment condition*)

$$\mathbf{E}[\| \theta_{n+1} \|^{2+\alpha} / \xi_n = (k,l)] \leq B < \infty, \ \ \forall (k,l) \in \mathbf{Z}_+^2, \tag{1.1}$$

where $\| z \|$, $z \in \mathbf{Z}_+^2$, denotes the euclidian norm and α is an arbitrary but strictly positive number.

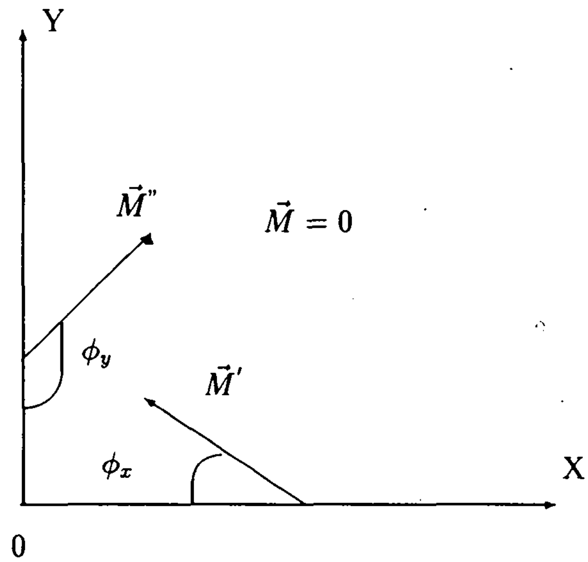

Figure. 1.

$\vec{\mathbf{M}} \equiv \vec{\mathbf{M}}(x,y) = (M_x, M_y)$ will denote the vector of one-step mean jumps (drifts) from the interior point (x,y) (i.e. $x,y \geq 1$), i.e

$$M_x = \sum_{i,j} i r_{ij} \quad \text{and} \quad M_y = \sum_{i,j} j r_{ij}.$$

Similarly, for points on the axes, we set

$$\begin{aligned}
\vec{\mathbf{M}}' &= \left(M_x', M_y'\right) = \left(\sum_{i,j} i q_{ij}, \sum_{i,j} j q_{ij}\right), \\
\vec{\mathbf{M}}'' &= \left(M_x'', M_y''\right) = \left(\sum_{i,j} i p_{ij}, \sum_{i,j} j p_{ij}\right), \\
\vec{\mathbf{M}}^0 &= \left(M_x^0, M_y^0\right) = \left(\sum_{i,j} i r_{ij}^0, \sum_{i,j} j r_{ij}^0\right).
\end{aligned} \tag{1.2}$$

Occasionally, we will write

$$\theta_{n+1} = (\theta_x, \theta_y), \quad \text{on} \quad \{\xi_n = (x,y) \geq 0\} \ .$$

Let also

$$\begin{aligned}
\lambda_x &= \sum_{i,j} i^2 r_{ij}, \\
\lambda_y &= \sum_{i,j} j^2 r_{ij}, \\
R &= \sum_{i,j} i j r_{ij}.
\end{aligned} \tag{1.3}$$

On figure 1, ϕ_x (resp. ϕ_y) is the angle between $\vec{\mathbf{M}}'$ and the negative x-axis (resp. between $\vec{\mathbf{M}}''$ and the negative y-axis), counter clockwise oriented .
In general, random walks in $\mathbf{Z}_+^N$ provide a useful representation of a number of queueing models, describing the evolution of specific computer or telecommunications networks. The zero-drift situation is also important, since it is directly related to diffusion approximations, which is an other way of modelling these systems. In dimension two, the classification problem was studied in [1,2,5] and completely solved except for the case $\vec{\mathbf{M}} = 0$. For the first time the case $\vec{\mathbf{M}} = 0$ was investigated in [3], where the authors found the necessary and sufficient conditions for ergodicity in terms of second moments and covariance of the one-step jumps inside $\mathbf{Z}_+^2$ and also in terms of ϕ_x and ϕ_y. This result was obtained for almost all admissible values of the parameters (λ_x, λ_y, R, ϕ_x, ϕ_y), except a set of measure zero. There was also shown the existence of a domain, in the parameter space, where the random walk is null recurrent. Up to now, the question of transience of these random walks when $\vec{\mathbf{M}} = 0$ was not considered : this is the subject of the present article, where we get the exact necessary and sufficient conditions for the random walk to be transient. In particular, this result does extend the range of parameters for which the random walk is null recurrent.

In section 2, we quote the main result (Theorem 2.1) and recall two classical theorems used to prove recurrence and transience. All technical lemmas are put together in section 3 and the proof of Theorem 2.1 is given in section 4.

2 Statement of main theorem

Theorem 2.1 *Let the following conditions take place :*

$$\vec{\mathbf{M}} = 0\,,\;\; M_x'' > 0 \text{ and } M_y' > 0. \tag{2.1}$$

Then the random walk is recurrent if

$$\lambda_x \cot\phi_y + \lambda_y \cot\phi_x + 2R \geq 0\,,$$

perhaps with the exception of the case

$$\lambda_x \cot\phi_y + R = \lambda_y \cot\phi_x + R = 0\,,$$

and transient if

$$\lambda_x \cot\phi_y + \lambda_y \cot\phi_x + 2R < 0.$$

Remark 1 *Under condition (2.1) of Theorem 2.1, we have*

$$\cot\phi_x = -\frac{M_x'}{M_y'}\,,\;\; \cot\phi_y = -\frac{M_y''}{M_x''}. \tag{2.2}$$

Remark 2 *The cases $M_y' = 0$ and $M_x'' = 0$ (absorption on the axes) will not be considered, since there are easier.*

Remark 3 *The case $\lambda_x \cot\phi_y + R = \lambda_y \cot\phi_x + R = 0$ can not be classified by our method. In this case the knowledge of the first moments on the axes is not sufficient to solve the classification problem and one should very likely expect necessary and sufficient conditions given also in terms of the second moments on the axes.*

Remark 4 *It will emerge from the proof that, up to some further unpleasant technical difficulties, one could solely assume the existence of moments of order $1 + \alpha$ on the axes.*

We recall now two known criteria, which will used in the proof of Theorem 2.1, for countable Markov chains to be respectively recurrent and transient.
Consider a time homogeneous Markov chain $\mathcal{L}$, with a countable state space $\mathcal{A} = \{\alpha_i, i \geq 0\}$. $\mathcal{L}$ is supposed to be irreducible and aperiodic. The position of the chain at time n is ξ_n.

Theorem 2.2 *The Markov chain $\mathcal{L}$ is recurrent, if and only if there exist a finite set A and a positive function $f(\alpha)$, $\alpha \in \mathcal{A}$, such that*

$$\mathbf{E}\left[f(\xi_{m+1}) - f(\xi_m) \mid \xi_m = \alpha_i\right] \leq 0, \text{ for all } \alpha_i \notin A,$$

and $f(\alpha_j) \to \infty$, when $j \to \infty$.

The proof of this theorem can be found in [6].

Theorem 2.3 *The Markov chain $\mathcal{L}$ is transient, if and only if there exists a set A and a positive function $f(\alpha)$, $\alpha \in \mathcal{A}$, such that the followihg inequalities are fulfilled*

$$\mathbf{E}\left[f(\xi_{m+1}) - f(\xi_m) \mid \xi_m = \alpha_i\right] \leq 0, \alpha_i \notin A\,,$$

$$f(\alpha_k) < \inf_{\alpha_j \in A} f(\alpha_j), \text{ for at least one } \alpha_k \notin A.$$

The proof of this theorem can be found in [4].

3 Construction of Lyapounov functions and technical lemmas

In classifying random walks by means of Lyapounov functions, it can (sometimes) be more comfortable to work with the *canonical* random walk $\lambda_x = \lambda_y$, $R = 0$. To this end, we introduce a new system of coordinates, according to the following linear substitution

$$\begin{aligned} u &= x + ky\,, \\ v &= a(ky - x)\,, \end{aligned}$$

with

$$a = \sqrt{\frac{\sqrt{\lambda_x\lambda_y} + R}{\sqrt{\lambda_x\lambda_y} - R}} \quad \text{and} \quad k = \sqrt{\frac{\lambda_x}{\lambda_y}}\,. \tag{3.1}$$

Then a transition from the point (x, y) to $(x + i, y + j)$, with probability r_{ij}, corresponds in the new coordinate system to a transition, with the same probability, from the point (u, v) to the point

$$(u + (i + kj), v + a(kj - i))\,.$$

In the (u, v) system, the Ox-axis (resp. Oy-axis) is transformed into the line of equation $v + au = 0$ (resp. $v - au = 0$). It is not difficult to see that

$$\begin{aligned} \lambda_u &= \sum_{i,j} r_{ij}(i + kj)^2 = 2\,(\lambda_x + kR), \\ \lambda_v &= \sum_{i,j} r_{ij}a^2(kj - i)^2 = 2\,(\lambda_x + kR), \\ R_{uv} &= \sum_{i,j} r_{ij}(i + kj)(kj - i) = 0\,. \end{aligned} \tag{3.2}$$

Remark 5 *All arguments will be made for the random walk in the new coordinate system and we shall use repeatedly the following simple identities*

$$\begin{aligned} \frac{a^2 - 1}{a^2 + 1} &= \frac{R}{\sqrt{\lambda_x\lambda_y}}\,, \\ a^2 &= \frac{\lambda_x + kR}{\lambda_x - kR}\,. \end{aligned}$$

In general, $\lambda_u = \lambda_v \geq 0$. To avoid degenerate (simpler) situations, we shall indeed assume $\lambda_u = \lambda_v > 0$.

A Lyapounov function for transience. In the case

$$\lambda_x \cot\phi_y + \lambda_y \cot\phi_x + 2R < 0\,,$$

we propose a function of the form

$$f_1(u, v) = \left(u^2 + v^2\right)^{-\delta}(1 + \mu\psi(u, v)) - \varepsilon u^{-2\delta}\,, \tag{3.3}$$

where $\psi(u,v)$ is the angle between the vector (u,v) and the positive v-axis (see figure 2). It turns out that, for suitably choosen constants $\mu > 0$, $\varepsilon > 0$ and $\delta > 0$, this function will satisfy the conditions of Theorem 2.2, so that the random walk will be transient.

A Lyapounov function for recurrence. The above function with $\delta < 0$ and $\varepsilon < 0$ could also be used to prove the recurrence when

$$\lambda_x \cot\phi_y + \lambda_y \cot\phi_x + 2R > 0\,,$$

but it would be impossible to consider the boundary case

$$\lambda_x \cot\phi_y + \lambda_y \cot\phi_x + 2R = 0\,.$$

Therefore, we introduce a slightly more complicated function

$$f_2(u,v) = \log\log(u^2+v^2) + \rho\frac{\log(\psi(u,v)+\theta_0)}{\log(u^2+v^2)}\,, \tag{3.4}$$

where ρ and θ_0 are constant to be properly fixed and $\psi(u,v)$ is the angle introduced above.

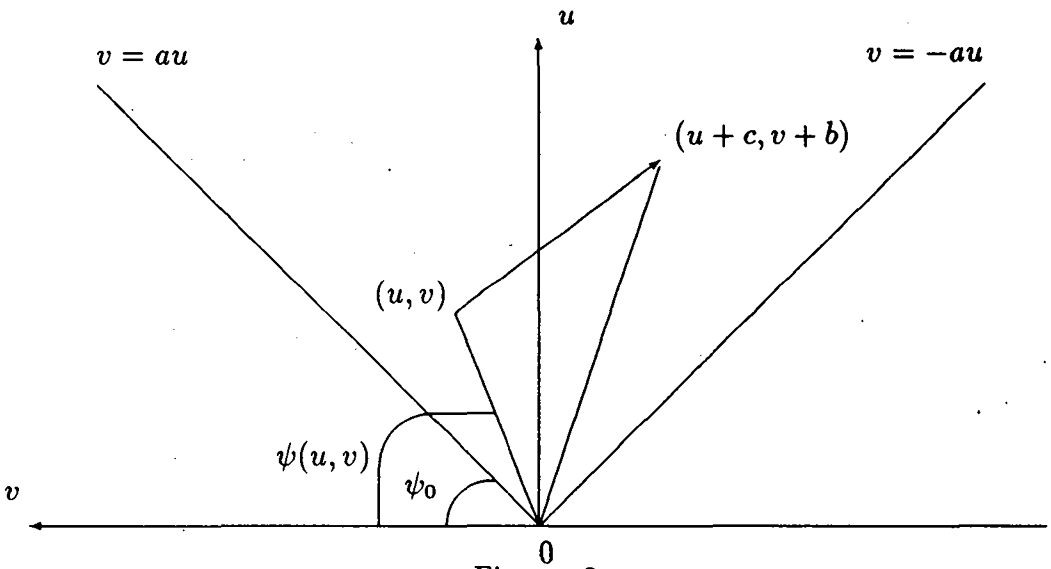

Figure. 2.

Lemma 3.1 *There exist constants M and $u_0(M)$ such that, for all $|u| > u_0(M)$, $c < M$ and $|b| < M$,*

$$\psi(u+c,v+b) - \psi(u,v) =$$
$$(bu-cv)\left(u^2+v^2\right)^{-1} + \left[cb\left(v^2-u^2\right) + uv\left(c^2-b^2\right)\right]\left(u^2+v^2\right)^{-2} + o\left(\left(u^2+v^2\right)^{-1}\right).$$

Proof. Since $c < M$ and $b < M$, it is possible to choose $u_0(M)$, such that

$$\psi(u+c,v+b) - \psi(u,v) \sim \sin\left(\psi(u+c,v+b) - \psi(u,v)\right),$$

when $|u| > u_0(M)$, since $\sin x = x + O(x^3)$, as $x \to 0$.
Then, using Taylor's formula $(1+x)^\alpha = 1 + \alpha x + O(x^2)$, we have

$$\begin{aligned}
&\psi(u+c, v+b) - \psi(u,v) \sim \sin\left(\psi(u+c,v+b) - \psi(u,v)\right) \\
&= \sin\psi(u+c,v+b)\cos\psi(u,v) - \cos\psi(u+c,v+b)\sin\psi(u,v) \\
&= \frac{bu - cv}{\sqrt{(u^2+v^2)\left[(u+c)^2+(v+b)^2\right]}} \\
&= \frac{bu-cv}{u^2+v^2}\left(1 - \frac{uc+vb}{u^2+v^2} + O\left((u^2+v^2)^{-1}\right)\right).
\end{aligned}$$

The Lemma is proved. ∎

Corollary 3.1 *Let c and b be bounded by some constant M. For the points on the line $v = -au$, Lemma 3.1 takes the form*

$$\psi(u+c, au+b) - \psi(u, au) = u^{-1}\frac{b+ac}{1+a^2} + o(u^{-1}).$$

Similarly, on the line $v = au$, we have

$$\psi(u+c, -au+b) - \psi(u, -au) = u^{-1}\frac{b-ac}{1+a^2} + o(u^{-1}).$$

Notation: In the sequel, we will write, for any function f defined on $\mathbf{R}^2$,

$$\begin{aligned}
\mathbf{E}\left[f(\xi_{n+1}) \mid \xi_n = (u,v)\right] - f(u,v) &= \mathbf{E}[f(u+\theta_x+k\theta_y), v + a(k\theta_y - \theta_x)] - f(u,v) \\
&\stackrel{\text{def}}{=} \mathbf{\Delta} f(u,v).
\end{aligned}$$

Lemma 3.2 *There exist constants N and $u_0(N)$ such that, for all interior points [i.e. $x, y \geq 1$ or, equivalently, $-au < v < au$, with $|u| > u_0(N)$], the following equalities hold*

i) $\mathbf{\Delta}(u^2+v^2)^{-\delta} = 4\delta^2(\lambda_x + kR)(u^2+v^2)^{-(1+\delta)} + o\left((u^2+v^2)^{-(1+\delta)}\right)$;

ii) $\mathbf{\Delta} u^{-2\delta} = 2\delta(2\delta+1)(\lambda_x + kR)\,u^{-2(\delta+1)} + o\left(u^{-2(\delta+1)}\right)$;

iii) $\mathbf{\Delta}\left[(u^2+v^2)^{-\delta}\psi(u,v)\right] = 4\delta^2(\lambda_x+kR)(u^2+v^2)^{-(1+\delta)}\psi(u,v) + o\left((u^2+v^2)^{-(1+\delta)}\right)$.

Proof. To prove these formulas, it suffices to apply Taylor's formula, together with the assumption $\vec{\mathbf{M}} = 0$, the conditions **A** and **B** of section 1 and the notation (1.3). We will not write all detailed and tedious calculations, but we show, for argument's sake, the proof of (iii) which is more delicate and, in some sense, includes i) and ii).
The original expression can be written

$$\begin{aligned}
&\mathbf{\Delta}\left[(u^2+v^2)^{-\delta}\psi(u,v)\right] = \\
&\quad \psi(u,v)\mathbf{\Delta}(u^2+v^2)^{-\delta} + \mathbf{E}\left[\left((u+c)^2+(v+b)^2\right)^{-\delta}\left(\psi(u+c,v+b) - \psi(u,v)\right)\right],
\end{aligned}$$

where we have put $c = (\theta_x + k\theta_y)$ and $b = a(k\theta_y - \theta_x)$.

• In the right-hand side member of the above equation, the first term brings the basic contribution, which can be computed from (i). This yields

$$\psi(u,v)\mathbf{\Delta}(u^2+v^2)^{-\delta} = 4\delta^2\left(\lambda_x + kR\right)\left(u^2+v^2\right)^{-(\delta+1)}\psi(u,v) + o\left((u^2+v^2)^{-(\delta+1)}\right).$$

• Let us denote by $A(u,v)$ the second term. It is necessary to prove that

$$A(u,v) = o\left((u^2+v^2)^{-(\delta+1)}\right).$$

To this end, we write $A(u,v)$ under the form

$$A(u,v) = A_1(u,v) + A_2(u,v),$$

with

$$\begin{cases} A_1(u,v) &= \mathbf{E}\left[((u+c)^2+(v+b)^2)^{-\delta}\left(\psi(u+c,v+b)-\psi(u,v)\right)\mathbf{1}_{\{|\theta_x+\theta_y|\le N\}}\right], \\ A_2(u,v) &= \mathbf{E}\left[((u+c)^2+(v+b)^2)^{-\delta}\left(\psi(u+c,v+b)-\psi(u,v)\right)\mathbf{1}_{\{|\theta_x+\theta_y|> N\}}\right], \end{cases}$$

where N is a constant chosen sufficiently large.
To estimate $A_1(u,v)$, we directly apply Lemma 3.1, after rewriting

$$A_1(u,v) = \left(u^2+v^2\right)^{-\delta}\mathbf{E}\left[\left(1+\frac{2(cu+bv)+c^2+b^2}{u^2+v^2}\right)^{-\delta}\left(\psi(u+c,v+b)-\psi(u,v)\right)\mathbf{1}_{\{|\theta_x+\theta_y|\le N\}}\right].$$

Then, taking N sucht that $u = O(N^{1+\beta})$, $0<\beta<\alpha/2$, where α comes in the moment condition of section 1, we have

$$A_1(u,v) = \left(u^2+v^2\right)^{-(\delta+1)}\mathbf{E}\left[(bu-cv)\mathbf{1}_{\{|\theta_x+\theta_y|\le N\}} + B(u,v)\right],$$

where

$$\mathbf{E}B(u,v) = (1+2\delta)\mathbf{E}\left[\frac{uv\,(b^2-c^2)+cb\,(v^2-u^2)}{u^2+v^2}\mathbf{1}_{\{|\theta_x+\theta_y|\le N\}}\right] + o(1).$$

Since $M_x = 0$, $M_y = 0$ and $\mathbf{E}\,(b^2-c^2) = 0$ (see 3.2), it becomes easy to see that

$$\mathbf{E}[(bu-cv) + B(u,v)\mathbf{1}_{\{|\theta_x+\theta_y|\le N\}}] = o(1),$$

so that $A_1(u,v) = o\left((u^2+v^2)^{-(\delta+1)}\right)$.
To estimate $A_2(u,v)$, we first note (see figure 2) that

$$|(\psi(u+c,v+b)-\psi(u,v))| \le (\pi - 2\psi_0).$$

Hence

$$|A_2(u,v)| \le (\pi-2\psi_0)\left(u^2+v^2\right)^{-(\delta+1)}\mathbf{E}\left[\left(\frac{(u+c)^2+(v+b)^2}{u^2+v^2}\right)^{-\delta}(u^2+v^2)\mathbf{1}_{\{|\theta_x+\theta_y|>N\}}\right].$$

From the lower boundedness of the jumps, it follows that

$$\left[\frac{(u+c)^2+(v+b)^2}{u^2+v^2}\right]^{-\delta} < C < \infty .$$

Now, for any random variable X such that $\mathbf{E}[|X|^r] < \infty$, we use the following simple result

$$\mathbf{E}[|X|^s \mathbf{1}_{\{X>z\}}] = o(|z^{s-r}|) \quad , \quad \forall\ 0 \le s \le r . \tag{3.5}$$

Hence, taking into account the moment condition **B**, we get

$$\mathbf{E}\left[(u^2+v^2)\mathbf{1}_{\{|\theta_x+\theta_y|>N\}}\right] = o(N^{-(2+\alpha)})(u^2+v^2) = o(N^{2\beta-\alpha}),$$

so that

$$A_2(u,v) = o\left(\left(u^2+v^2\right)^{-(\delta+1)}\right) .$$

The lemma is proved. ■

Notation : From now on, ψ_0 will denote the angle between the positive v – axis and the line $u = av$, as shown on figure 2.

The next lemma deals with the behaviour of the function f_1 on the lines $v = -au$ and $v = au$.

Lemma 3.3 *The following equalities hold :*

1. *On the line $v = -au$,*

$$\begin{aligned}\mathbf{\Delta} f_1(u,-au) &= -2\delta(1+\mu(\pi-\psi_0))\frac{a^{2\delta}}{(1+a^2)^\delta}\left(M_x' - M_y'\frac{R}{\lambda_y}\right)u^{-(2\delta+1)} \\ &\quad +2\delta\varepsilon u^{-(2\delta+1)}\left(kM_y' + M_x'\right) + \frac{2\mu k a^{(1+2\delta)}}{(1+a^2)^{\delta+1}}M_y' u^{-(2\delta+1)} + o\left(u^{-(2\delta+1)}\right) .\end{aligned}$$

2. *On the line $v = au$,*

$$\begin{aligned}\mathbf{\Delta} f_1(u,au) &= -2\delta\left(1+\mu\psi_0\right)\frac{a^{2\delta}}{(1+ka^2)^\delta}\left(M_y'' - M_x''\frac{R}{\lambda_x}\right)u^{-(2\delta+1)} \\ &\quad +2\delta\varepsilon u^{-(2\delta+1)}\left(kM_y'' + M_x''\right) - \frac{2\mu a^{(1+2\delta)}}{(1+a^2)^{\delta+1}}M_x'' u^{-(2\delta+1)} + o\left(u^{-(2\delta+1)}\right) .\end{aligned}$$

To prove this lemma it is sufficient to use Remark 4, notations (1.2) and Corollary 3.1. ■

Lemma 3.4 *For all interior points $-au < v < au$, with u^2+v^2 sufficiently large, there exist $\varepsilon > 0$ such that the following equalities take place*

1.

$$\begin{aligned}&\mathbf{\Delta}[\log\log\left(u^2+v^2\right)] = \\ &\quad -4(\lambda_x + kR)\left(u^2+v^2\right)^{-1}\log^{-2}(u^2+v^2) + o\left((u^2+v^2)^{-1-\varepsilon}\log^{-2}(u^2+v^2)\right) .\end{aligned}$$

2.

$$\Delta \left[\frac{\log(\psi(u,v)+\theta_0)}{\log(u^2+v^2)} \right] = o\left((u^2+v^2)^{-1-\varepsilon} \log^{-2}(u^2+v^2) \right) ,$$

for any $\theta_0 \geq 0$.

Proof. For the sake of brevity, we introduce the following symbols and quantities :

$$\begin{aligned} Q &\equiv Q(u,v) = u^2+v^2 = D^2 \ , \text{ where } D \text{ is large enough ;} \\ \Delta Q &= Q(u+c, v+b) - Q(u,v) = 2(uc+vb) + b^2 + c^2 ; \\ \mathcal{A} &= \{c + |b| \leq D^\gamma\} \ , \text{ for some fixed } 0 < \gamma < 1 \text{ to be suitably chosen later,} \end{aligned}$$

and $\mathcal{A}^c$ will denote the complementary set of $\mathcal{A}$.

Rewrite now $\Delta \log\log Q(u,v)$ as

$$\Delta \log\log Q(u,v) = B_1 + B_2 , \tag{3.6}$$

with

$$\begin{cases} B_1 &= \mathbf{E}\left[\log\log(Q+\Delta Q) \,;\mathcal{A}^c\right] , \\ B_2 &= \mathbf{E}\left[\log\log(Q+\Delta Q)\,;\mathcal{A}\right] . \end{cases}$$

In the course of the proof, two standard inequalities will play a crucial role : First (3.5) and also the c_s *inequality*

$$|x+y|^r \leq c_r|x|^r + c_r|y|^r \ , \ r > 0 , \tag{3.7}$$

where $c_r = 1$ or 2^{r-1}, according as $r \leq 1$ or $r \geq 1$.

- *Estimate of* B_1. Upon applying (3.7) on $\mathcal{A}$ for all large D, we have

$$\log\log(Q+\Delta Q) \leq Q^\beta + |\Delta Q|^\beta ,$$

where β is strictly positive but otherwise arbitrary. Therefore, by (3.5) and (1.1), it follows that

$$\begin{aligned} B_1 &= o\left(D^{\beta+\gamma(\beta-\alpha-2)}\right) + o\left(D^{\gamma(2\beta-\alpha-2)}\right) + o\left(D^{2\beta-\gamma(\alpha+2)}\right) , \\ &\doteq o\left(D^{2\beta-\gamma(\alpha+2)}\right) . \end{aligned}$$

Since it is always possible to choose $\beta > 0$ small enough and $0 < \gamma < 1$ to satisfy the inequality

$$2\beta - \gamma(2+\alpha) < -2 ,$$

or

$$\frac{2(1+\beta)}{2+\alpha} < \gamma < 1 , \tag{3.8}$$

as soon as $\beta < \alpha/2$, we have shown $B_1 = o(Q^{-1-\varepsilon_1})$, with $\varepsilon_1 > 0$.

• *Estimate of B_2*. Our goal is to show $B_1 = o(Q^{-1-\varepsilon_2})$, with $\varepsilon_2 > 0$. To this end, we will apply the expansion

$$\log\log(x+y) - \log\log x = \frac{y}{x\log x} - \frac{y^2(\log x + 1)}{2(x\log x)^2} + \frac{y^3[-\log z + 2(\log z + 1)^2]}{6(z\log z)^3},$$

where $z = x + \theta y$, $0 < \theta < 1$. Hence, since in the region $-au < v < au$

$$\mathbf{E}b = \mathbf{E}c = \mathbf{E}(bc) = \mathbf{E}(b^2 - c^2) = 0,$$

we obtain (skipping over intermediate computations)

$$B_2 = \frac{-4(\lambda_x + kR)}{Q\log^2 Q} + B_3 + B_4, \tag{3.9}$$

where

$$\begin{aligned} B_3 &= \frac{1}{Q\log Q}\mathbf{E}\left[\Delta Q - \frac{|\Delta Q^2|}{2Q} - \frac{(b^2+c^2)[4(uc+vb)+b^2+c^2]}{2Q\log Q}\,;\mathcal{A}^c\right], \\ B_4 &= o(Q^{-3}\log^{-1}Q)\mathbf{E}\left[|\Delta Q|^3\,;\mathcal{A}\right], \end{aligned}$$

after having used the fact that, $Q + \Delta Q = o(Q)$ on the event $\mathcal{A}$. Our next result is that B_3 and B_4 are of order $o(Q^{-1-\varepsilon_2})$, for some $\varepsilon_2 > 0$. This will be proved only for B_4, as it is the most difficult case. To estimate $\mathbf{E}\left[|\Delta Q|^3\,;\mathcal{A}\right]$, one first apply (3.7), so that

$$\mathbf{E}\left[|\Delta Q|^3\,;\mathcal{A}\right] \le K_1\mathbf{E}\left[|uc+vb|^3\,;\mathcal{A}\right] + K_2\mathbf{E}\left[(b^2+c^2)^3\,;\mathcal{A}\right], \tag{3.10}$$

where K_1 and K_2 are positive constants. Using (1.1) and the obvious decomposition, for any δ,

$$(x+y)^3 = (x+y)^{2-\delta}(x+y)^{1+\delta},$$

one concludes that the two terms in the right member of (3.10) are respectively of order

$$o\left(D^{3+\gamma(1-\alpha)}\right) \quad \text{and} \quad o\left(D^{\gamma(4-\alpha)}\right).$$

It follows that

$$\begin{aligned} B_4 &= o\left(D^{-3+\gamma(1-\alpha)}\right) + o\left(D^{-6+\gamma(4-\alpha)}\right) \\ &= o(Q^{-1-\varepsilon_2}), \end{aligned}$$

as soon as $\gamma < 1$. Similar arguments can be employed for B_3. Putting all these estimates in (3.9) and (3.6) produces an estimate for B_2 having the asserted form.

The proof of the assertion 2 of the lemma goes along the same lines (with the aid of lemma 3.1) and will be omitted. Lemma 3.4 is proved. ■

Lemma 3.5 *The following inequalities hold, for all $0 \le \theta_0 < \pi/2$ and u sufficiently large,*

1. *On the line $v = -au$,*

$$\Delta f_2(u,-au) = \left(M_x' - \frac{R}{\lambda_y}M_y' + \frac{\rho a k M_y'}{(1+a^2)(\pi-\psi_0-\theta_0)}\right)\frac{1}{u\log(u\sqrt{1+a^2})} + o\left(\frac{1}{u\log u}\right).$$

2. *On the line $v = au$,*

$$\Delta f_2(u,au) = \left(kM_y'' - \frac{kR}{\lambda_x}M_x'' - \frac{\rho a M_x''}{(1+a^2)(\psi_0+\theta_0)}\right)\frac{1}{u\log(u\sqrt{1+a^2})} + o\left(\frac{1}{u\log u}\right).$$

Proof. It mimics the method used in the preceding lemma. Details are omitted. One simply remarks that, from Lemma 3.1, the main contribution of the quantity

$$\Delta\left[\frac{\log(\psi(u,v)+\theta_0)}{\log(u^2+v^2)}\right]$$

is obtained from terms of the form

$$\frac{\Delta\psi(u,v)}{\log(u^2+v^2)}.$$

∎

4 Proof of Theorem 2.1

- *Condition for recurrence .*

For the random walk to be recurrent, it suffices to show the existence of a constant μ, rendering the function $f_2(u,v)$ positive and such that, outside some compact set, the following condition holds (see (3.3) and Theorem 2.2)

$$\Delta f_2(u,v) \leq 0. \tag{4.1}$$

This inequality for the interior points (i.e. $x, y \leq 1$ or $-au < v < au$) follows directly from Lemma 3.4, as we assumed $\lambda_x + kR > 0$.

Define

$$\begin{aligned} A_x &= \lambda_y \cot\phi_x + R, \\ A_y &= \lambda_x \cot\phi_y + R, \\ L &= \frac{a}{a^2+1} \quad \text{and} \quad \beta = \theta_0 + \psi_0. \end{aligned}$$

According to Lemma 3.5 and formulas (2.2), inequality (4.1) written for points on the lines $v = -au$ and $v = au$ imposes to satisfy the system

$$-A_x + \rho L_1 < 0, \tag{4.2}$$
$$-A_y + \rho L_2 < 0, \tag{4.3}$$

where we have set

$$L_1 = \frac{k\lambda_y}{\pi - \beta} \quad \text{and} \quad L_2 = \frac{\lambda_x}{k\beta}.$$

i) Case $A_x + A_y > 0$. Then it is always possible to choose ρ and $\theta_0 = 0$ (then $L_2 > L_1$, since $\psi_0 < \pi/2$) such that

$$A_x > \max(\rho L_1, \rho L_2) > \min(\rho L_1, \rho L_2) > -A_y,$$

which in turn implies (4.2) and (4.3). Thus the recurrence is proved.

ii) Case $A_x + A_y = 0$. Setting $A_x = A$, one still can satisfy (4.2) and (4.3), provided that

$$\rho L_1 < A < \rho L_2. \tag{4.4}$$

It is worth noting the following consequences from (4.4) :

$$A > 0 \;\Rightarrow\; 0 < \beta < \frac{\pi}{2} \text{ and } \rho > 0;$$
$$A < 0 \;\Rightarrow\; \frac{\pi}{2} < \beta < \pi \text{ and } \rho < 0.$$

The case $A_x = A_y \doteq 0$ cannot be captured by our analysis and remains open. The first part of Theorem 2.1 is proved, namely the random walk is recurrent if

$$\lambda_x \cot\phi_y + \lambda_y \cot\phi_x + 2R \geq 0.$$

- *Condition for transience* .

For the random walk to be transient it suffices to show the existence of constants $0 < \varepsilon < 1$, $0 < \delta < \delta_0$ and μ, rendering the function $f_1(u,v)$ and such that the following condition holds (see (3.3) and Theorem 2.3)

$$\boldsymbol{\Delta} f_1(u,v) \leq 0. \tag{4.5}$$

We consider first (4.5) on the axes. According to Lemma 3.3 and formulas (2.2) for points on the line $v = -au$, with u large enough, one can rewrite (4.5) as

$$\frac{a\mu}{(1+a^2)(1+\mu(\pi-\psi_0))} < \frac{\delta}{k}\left[-\frac{R}{\lambda_y} - \cot\phi_x + \frac{\varepsilon\gamma(\cot\phi_x - k)}{1+\mu(\pi-\psi_0)}\right], \tag{4.6}$$

where $\gamma = \dfrac{(1+a^2)^\delta}{a^2}$.

Similarly, for points on the line $v = au$, (4.5) is equivalent to

$$\frac{a\mu}{(1+a^2)(1+\mu\psi_0)} > \delta\left[\frac{kR}{\lambda_x} + k\cot\phi_y + \frac{\varepsilon\gamma(1-k\cot\phi_y)}{1+\mu\psi_0}\right]. \tag{4.7}$$

As $\pi - 2\psi_0 > 0$, we have, for any μ,

$$\frac{\mu}{1+\mu\psi_0} > \frac{\mu}{1+\mu(\pi-\psi_0)}.$$

Thus, for any fixed $|\mu|$ sufficiently small to ensure the positivity of $f_1(u,v)$, inequalities (4.6) and (4.7) will be satisfied if one can show the existence of positive ε and δ, such that

$$\frac{kR}{\lambda_x} + k\cot\phi_y + \frac{\varepsilon\gamma(1-k\cot\phi_y)}{1+\mu\psi_0} < R_1 < R_2 < \frac{1}{k}\left[-\frac{R}{\lambda_y} - \cot\phi_x + \frac{\varepsilon\gamma(\cot\phi_x - k)}{1+\mu(\pi-\psi_0)}\right], \quad (4.8)$$

where we have set

$$\begin{aligned} R_1 &= \frac{a\mu}{\delta(1+a^2)(1+\mu(\pi-\psi_0))}, \\ R_2 &= \frac{a\mu}{\delta(1+a^2)(1+\mu\psi_0)}. \end{aligned}$$

But (4.8) will hold if and only if it is possible at all to achieve the asserted inequality between the two extreme members of (4.8), which after reduction amounts exactly to

$$(1-\gamma\varepsilon)C < \gamma\varepsilon\left[\frac{\mu\psi_0(1-k\cot\phi_y)}{1+\mu\psi_0} + \frac{\mu(\pi-\psi_0)(\cot\phi_x - k)}{k[1+\mu(\pi-\psi_0)]} - 2\left(\frac{R}{\sqrt{\lambda_x\lambda_y}}+1\right)\right], \quad (4.9)$$

where we have put

$$C \equiv \frac{1}{\sqrt{\lambda_x\lambda_y}}(\lambda_x\cot\phi_y + \lambda_y\cot\phi_x + 2R)$$

and used the identity $k^2 = \lambda_x/\lambda_y$. Assume $C < 0$, according to the statement of Theorem 2.1. Then, for $|\mu|$ small enough, remarking that $|R| \le \sqrt{\lambda_x\lambda_y}$, the right-hand side member of (4.9) can be rendered greater than C for some $\varepsilon \ge 0$.

For points not on the axes, according to Lemma 3.2, the sought inequality (4.5) takes the form

$$2\delta(\lambda_x + kR)\left[\left(u^2+v^2\right)^{-(1+\delta)}(1+\mu\psi(u,v)) - \varepsilon(1+2\delta)u^{-2(1+\delta)}\right] \le 0,$$

which reduces to

$$\varepsilon + \frac{\varepsilon}{2\delta} \ge [1+\mu\psi(u,v)]\frac{u^{2(\delta+1)}}{(u^2+v^2)^{\delta+1}}. \quad (4.10)$$

It is now easy to see that, for any μ and any $\varepsilon > 0$, one can choose $\delta_0 > 0$ so that (4.10) be fulfilled, $\forall\ 0 < \delta < \delta_0$.

The proof of Theorem 2.1 is concluded. ■

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ISSN 0249 - 6399

RR_2209