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Abstract: In [BSBL93], the synthesis problem has been solved for a non-holonomic car-like robot only allowed to move forwards in a plane. Here we complete the study of this mobile robot by computing the SP-accessibility regions which are the sets of points reached from the origin configuration with an optimal path of length less than or equal to a given value. For that, we consider the problem of optimising the paths starting from an oriented point and arriving at another point with free orientation. After characterising these optimal paths, we compute the partition of the plane w.r.t. the types of the optimal paths, and finally we find the shapes of the SP-accessibility regions.

Key-words: Optimal control, Problem with variable endpoint, Synthesis problem, Bounded radius of curvature, Shortest paths

(Résumé : tsvp)

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Accessibilité par des Chemins Optimaux pour une Voiture sans Marche Arrière

Résumé : Nous nous intéressons au robot mobile de type voiture en mouvement dans le plan, uniquement en marche avant. Dans [BSBL93], a été résolu le problème de synthèse pour les chemins optimaux en longueur entre deux points orientés. Ici, nous voulons calculer la région d'accessibilité par des chemins optimaux d'un tel robot, c'est-à-dire le lieu des points atteints depuis la configuration origine par un chemin optimal de longueur inférieure ou égale à une constante donnée. Pour cela, nous avons résolu le même problème de synthèse mais pour les chemins optimaux entre un point orienté et un point où l'orientation n'est pas imposée, pour finalement en déduire la forme des régions d'accessibilité de ce robot.

Mots-clé : Commande optimale, Problème à extrémité libre, Synthèse, Courbure majorée, Plus courts chemins

1 Introduction

For any mobile system, we define the *SP-accessibility region* (SP for shortest path) $\mathcal{R}_d$ as the set of positions for which there exists a shortest path of length less than or equal to d reaching it from the origin configuration. In our particular case, the mobile system is a car-like robot only moving forwards in a plane, at a constant speed equal to 1. Assuming that there is no slipping of the wheels, the kinematic model of our robot is the following :

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ \kappa/\rho \end{pmatrix} \tag{1}$$

with (x, y) being the coordinates of a reference point on the robot, θ its orientation, ρ its minimum turning radius and κ the only control parameter verifying $|\kappa| \leq 1$. We call a *configuration* of the robot the triplet $\omega = (x, y, \theta)$. Hence its *configuration space* is $\Omega = \mathbb{R}^2 \times \mathcal{S}^1$, and our origin configuration is $\omega_0 = (0, 0, 0)$.

We would like to compute for any value of d the shape of the SP-accessibility region $\mathcal{R}_d$ for this robot.

This type of robot is often called Dubins robot, because its shortest paths between two configurations were first studied by L.E. Dubins in [Dub57]. He proved that they could be of at most six types. If we denote C an arc of circle of radius ρ (if it is described turning clockwise (*resp.* counterclockwise), it will be an R (*resp.* L) arc), and S a straight line segment, these types can be written as the following words :

- Family CCC : types RLR and LRL
- Family CSC : types RSR, LSL, RSL and LSR

In [Mel61], Z. A. Melzak determined the usual accessibility regions $R(t)$ in which the robot is after following a path (not necessarily optimal) of length t, for $t \leq \pi\rho$. We will also find these regions. Then H. G. Robertson [Rob70] proved that such a robot can reach a position in its initial turning circles (of radius ρ) only if the path length is strictly greater than $\pi\rho$. We will prove this result with another simpler method. E. J. Cockayne and G. W. C. Hall [CH75] completed the study of the regions $R(t)$, showing that all positions on the boundary of such a region are reached by a degenerate Dubins path either of type CS or CC. All these works do not care about optimality of the paths, yielding many differences between our result and these ones.

A question similar to the one above can be asked when backward motions are allowed. Such robots are called Reeds and Shepp robots, for J.A. Reeds and L.A. Shepp characterised the types of the optimal paths for this kind of mobile robot in [RS90]. They proved that they were made of at most five arcs of circle of radius ρ or line segments and at most two manoeuvres. P. Souères, J.-Y. Fourquet and J.-P. Laumond [SFL94] computed the SP-accessibility regions for this car-like robot.

We will follow the same method as in [SFL94] to study the regions $\mathcal{R}_d$ of Dubins robot. First we charaterise the types of the shortest paths from ω_0 to any position (x, y) with free final orientation. Then we compute the synthesis for these optimal paths, from which we can deduce the shapes of the SP-accessibility regions with respect to parameter d. We will point out many differences between our results and the ones obtained in [SFL94].

2 Optimal paths

Computing the SP-accessibility regions involves solving the following problem : finding optimal paths between a configuration and a point whose orientation is free. We will use a tool from optimal control theory to find the types of the optimal paths from ω_0 to any position of the plane. This tool is the Maximum Principle of Pontryagin [PBGM62]. It has already been used in [BCL91] and [ST91] in order to provide new proofs of the results of Dubins [Dub57] and of Reeds and Shepp [RS90] on the shortest paths between two configurations, respectively for our model of robot and the car-like robot also allowed to backup.

[PBGM62] also gives necessary conditions for the paths to be optimal when the final orientation is free. This problem is called the *problem with variable endpoint*. But first let us formulate the problem in terms of control theory. Our phase vector is (x, y, θ), already given in (1). The adjoint vector is denoted $\Psi(\psi_1, \psi_2, \psi_3)$. Then the Hamiltonian of our system and the adjoint system are the following :

$$\mathcal{H} = \psi_0 + \psi_1 \cos\theta + \psi_2 \sin\theta + \psi_3\,\kappa$$

$$\begin{cases} \dot\psi_1 = -\dfrac{\partial \mathcal{H}}{\partial x} = 0 \\[2ex] \dot\psi_2 = -\dfrac{\partial \mathcal{H}}{\partial y} = 0 \\[2ex] \dot\psi_3 = -\dfrac{\partial \mathcal{H}}{\partial \theta} = \psi_1 \sin\theta - \psi_2 \cos\theta = \psi_1 \dot y - \psi_2 \dot x \end{cases}$$

We deduce from the adjoint system that ψ_1 and ψ_2 are constant and $\psi_3 = \psi_1 y - \psi_2 x + \psi_{3_0}$, where ψ_{3_0} is constant equal to the value of ψ_3 at the initial configuration. For the problem with variable endpoint, the optimal paths must satisfy the Maximum Principle of Pontryagin and moreover a necessary optimality condition, called *transversality condition*. This condition says that, at the variable endpoint, the adjoint vector must be perpendicular to the hypersurface which the variable endpoint belongs to. In our case, this condition is the following :

$$\psi_3 = \psi_1 y - \psi_2 x + \psi_{3_0} = 0$$

which is the equation of a straight line the final point must lie on. With the notations used in [BCL91], this is the straight line denoted D_0, which plays an important part in optimal paths.

Lemma 2.1 [BCL91, Proposition 10]
In an optimal path, all line segments and inflection points lie on straight line D_0.

This lemma has the following consequences on the optimal types of paths described above :

- Family CSC

  These paths become CS. Indeed the transversality condition implies that the final endpoint must lie on D_0, and then by Lemma 2.1 both endpoints of the last arc must lie on D_0. Hence the last arc must be of length 0 or 2π. This last possibility is clearly not optimal.

- Family CCC

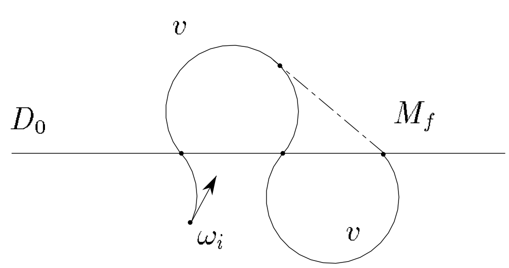

Figure 1 : Paths CC_vC_v are not optimal

Here, paths will be of shape CC_v with $v > \pi$. As previously, the transversality condition implies that the final endpoint must lie on D_0, and then Lemma 2.1 implies that both endpoints of the last arc lie on D_0. Thus the last arc must be of length 0 or equal to the intermediate arc length. This solution is not optimal because we can replace it by a shorter path CCS, where the line segment is tangent to the second arc and ends at the final point (see Figure 1). But this path is not optimal since it does not belong to any of the two families CSC nor CCC.

In conclusion, the types of the optimal paths for the problem with a variable final point are : RS, LS, RL and LR.

3 Synthesis

The synthesis for the optimisation problem with variable final endpoint consists in computing for each point (x, y) of the plane the type of the shortest path from ω_0 to (x, y). As in [BSBL93] we will solve this problem, first by computing the sets of points reached by each type of paths, and deciding which type is optimal in the intersections between those domains. Hence, we will obtain a partition of the plane into four regions, corresponding to the four types of optimal paths.

This study is simplified by the following property.

Proposition 3.1 *The partition of the plane admits the straight line (Ox) as an axis of symmetry.*

Proof : Consider point $M = (x, y)$, the shortest path γ from ω_0 to M and $M' = (x, -y)$ the point symmetric to M w.r.t. axis (Ox). Then path γ', the symmetric of γ w.r.t. axis (Ox), is necessarily the shortest path from ω_0 to M', since γ and γ' have the same (optimal) length. □

We will restrict our study to the half-plane Π above axis (Ox) of equation $y \geq 0$. Optimal paths leading to points in the other half-plane will be deduced by symmetry w.r.t. axis (Ox), replacing arcs R by arcs L and vice versa.

Definitions

- $\mathcal{C}_R$ is the circle of centre the point $\mathsf{C}_R(0, -\rho)$ and radius ρ.
- $\mathcal{C}_L$ is the circle of centre the point $\mathsf{C}_L(0, \rho)$ and radius ρ.

These two circles are tangent to the origin configuration ω_0 (see Figure 2). $\mathcal{C}_R$ is described clockwise when starting from ω_0, and $\mathcal{C}_L$ counterclockwise.

3.1 Domains for each type of path

We compute the domain of a given type of path, i.e. the set of points reached by a path of that type with the optimality conditions we know up to now, by integrating the kinematic equations of system (1) on each portion where the control parameter κ is constant. $\kappa = 0$ along a segment, $\kappa = -1$ along an arc R and $\kappa = 1$ along an arc L. In this section we do not restrict our study to half-plane Π.

▌ *CS* domains

It is sufficient to consider only type $L_u S_s$ (by Proposition 3.1, the domain of type RS will be deduced by symmetry of axis (Ox)). The only condition on the segment lengths is $u \in [0, 2\pi[$ and $s \geq 0$. Starting from ω_0, the first arc L of length ρu reaches the point :

$$M_1 \quad \begin{cases} \rho \sin u \\ \rho - \rho \cos u \end{cases}$$

Then after describing the line segment of length s, the path of type LS can reach all points :

$$M_{LS} \quad \begin{cases} \rho \sin u + s \cos u \\ \rho - \rho \cos u + s \sin u \end{cases} \qquad \text{with} \quad u \in [0, 2\pi[\quad \text{and} \quad s \in I\!R^+$$

For a given value of s and $u \in [0, 2\pi[$, point M_{LS} describes a circle of centre C_L and radius $\sqrt{s^2 + \rho^2}$. For a given value of u and $s \in I\!R^+$, it describes the half-line tangent to $\mathcal{C}_L$ and of orientation u. The domain is then the whole plane except the interior of circle $\mathcal{C}_L$ (see Figure 2). Iso-distance curves are clearly involutes[1] of circle $\mathcal{C}_L$ turning clockwise.

The domain corresponding to type RS is the symmetric of the domain we have just found. So it is the whole plane except the interior of circle $\mathcal{C}_R$. The iso-distance curves are then involutes of circle $\mathcal{C}_R$ turning counterclockwise.

[1] A practical construction of an involute of a circle is the following : it is the curve described by the endpoint of a string that one would wrap or unwrap around the circle.

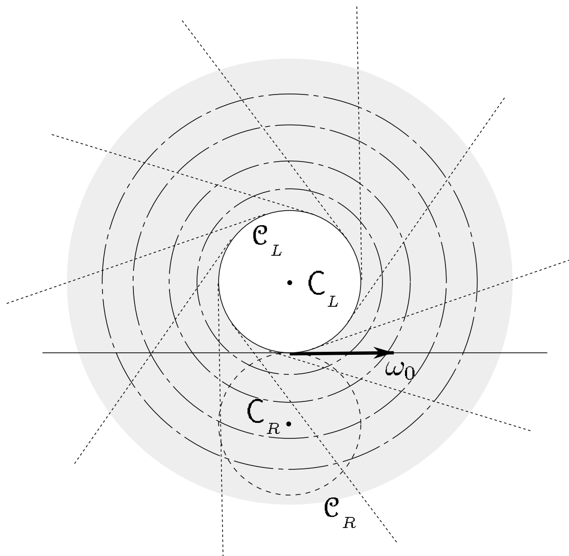

Figure 2 : Domain of type LS

▌ CC domains

Without loss of generality (see Proposition 3.1), we consider here paths of type $R_u L_v$. The conditions of optimality are $u \leq v$ and $v \in]\pi, 2\pi[$, respectively proved in [BSBL93] and [BCL91]. The first arc R of length u reaches the following point on circle $\mathcal{C}_R$:

$$M_1 \quad \begin{cases} 2\,\rho\ \sin u \\ -\rho + 2\,\rho\ \cos u \end{cases}$$

Then after the second arc of circle L of length v, the path can reach all points :

$$M_{RL} \quad \begin{cases} 2\,\rho\ \sin u - \rho\ \sin(u-v) \\ -\rho + 2\,\rho\ \cos u - \rho\ \cos(u-v) \end{cases} \qquad \text{with} \quad u \in [0, v[\quad \text{and} \quad v \in]\pi, 2\pi[$$

With a given value of u and v varying in $]\pi, 2\pi[$, point M_{RL} describes a half-circle of radius ρ and centre the point of coordinates $(2\,\rho\ \sin u, -\rho + 2\,\rho\ \cos u)$, which is itself on the circle of radius 2ρ and centre C_R. For a given value of v and u in $[0, v[$, point M_{RL} describes an arc of circle, of centre C_R and radius $\sqrt{5 - 4\cos v}$. This results in the domain shown in grey in Figure 3.

We state the following additional condition for a path RL to be optimal.

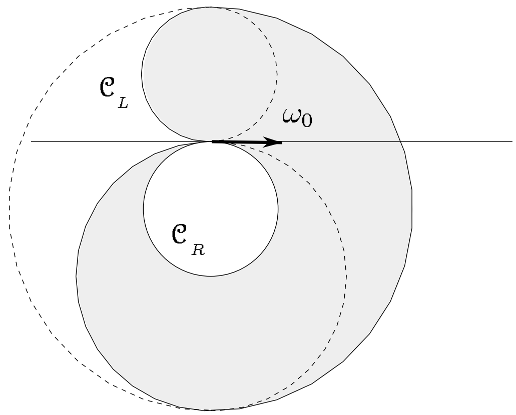

Figure 3 : Domain of type RL with $u \in [0, v[$ and $v \in]\pi, 2\pi[$

Proposition 3.2 *A path RL is not optimal if point M_{RL} lies outside circle $\mathcal{C}_L$.*

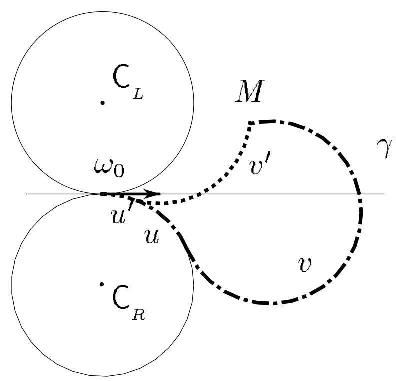

Figure 4 : Case when an RL path is not optimal

Proof : Suppose that point M_{RL} lies outside circle $\mathcal{C}_L$ and γ is the path of type $R_u L_v$ reaching this point, with $v > \pi$. Then there exists another circle tangent to $\mathcal{C}_R$ and passing through M_{RL} such that we can build a path γ' of type $R_{u'} L_{v'}$ shorter than γ (see Figure 4). Indeed, u' and v' are respectively shorter than u and v. But γ' can not be optimal, since $v' < \pi$. □

We can notice that this condition implies $u \in [0, \frac{\pi}{3}]$ and the domain corresponding to type RL is the interior of circle $\mathcal{C}_L$ (see Figure 5).

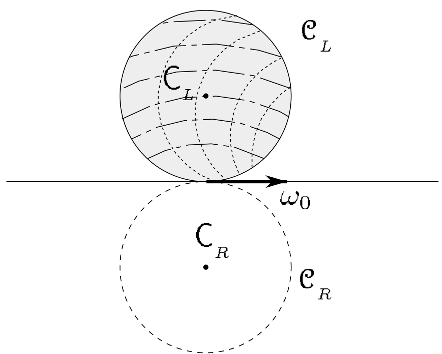

Figure 5 : Domain of type RL

Iso-distance curves for type RL are arcs of cardioids. Such a curve corresponding to a length $d > \rho\pi$ is obtained by following the point, which initial position is on $\mathcal{C}_L$ with polar angle d/ρ (the origin of polar angles on $\mathcal{C}_L$ being point $(0,0)$), lying on a circle of radius ρ which is rolling clockwise, without slipping, on circle $\mathcal{C}_R$.

Symmetrically, the domain for type LR is the interior of circle $\mathcal{C}_R$, and iso-distance curves are also arcs of cardioids built with a circle of radius ρ rolling counterclockwise on circle $\mathcal{C}_L$.

3.2 Intersections between domains

From previous results, half-plane Π is partitioned as follows : outside circle $\mathcal{C}_L$ types RL and LR can not be optimal, and the only candidates to optimlity are types RS and LS; inside circle $\mathcal{C}_L$ type LR can not be optimal and type LS does not exist, so we only have to consider types RL and RS. Thus the only intersections to study in half-plane Π are between RS and LS outside $\mathcal{C}_L$, and between RL and RS inside $\mathcal{C}_L$.

▌ Intersection between RS and LS

Proposition 3.3 *Paths of type RS reaching points in half-plane Π and lying outside $\mathcal{C}_L$ are always longer than paths of type LS reaching the same points.*

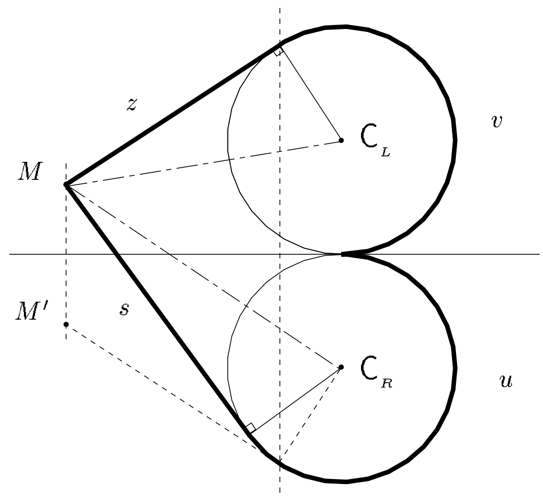

Figure 6 : Path RS is longer than path LS

Proof : We denote $R_u S_s$ the first path and $L_v S_z$ the second one, and M the point reached by those two paths, as shown in Figure 6. We will prove that $s > z$ and $u > v$.

- ▷ $s > z$

  We can see on Figure 6 that

$$s^2 = \|\overrightarrow{MC_R}\|^2 - \rho^2$$
$$z^2 = \|\overrightarrow{MC_L}\|^2 - \rho^2$$

  The bisector line of points C_R and C_L is axis (Ox). Hence line segment MC_R is always longer than line segment MC_L, and thus we have $s > z$.

- ▷ $u > v$

  Suppose first that $v \leq \pi$. Since M lies in Π, we have $u > \pi$, which implies $u > v$.

  Now suppose that $v > \pi$, as in Figure 6. Point M' is the symmetric of M w.r.t. axis (Ox). We define the path $R_{u'} S_{s'}$ reaching M' as the symmetric of path $L_v S_z$ leading to M. Hence $u' = v$. Since M is above M' on the same vertical line, angle u is indeed greater than angle u', thus greater than v.

As $s > z$ and $u > v$, path RS is longer than path LS, for any point lying in Π and outside $\mathcal{C}_L$. □

We can conclude that the optimal domain for type LS in half-plane Π is $\Pi \setminus \mathring{\mathcal{C}}_L$ ($\mathring{\mathcal{C}}_L$ denotes the interior of circle $\mathcal{C}_L$), and by symmetry the optimal domain for type RS in the other half-plane Π' (lying below axis (Ox)) is $\Pi' \setminus \mathring{\mathcal{C}}_R$.

Intersection beween RS and RL

We consider the paths $R_u S_s$ and $R_w L_v$ leading to the same point M lying inside circle $\mathcal{C}_L$. This implies that $u > \pi$ and $u > w$. As shown in Figure 7, to decide which path is the shortest, we have to compare the lengths of the sub-paths $R_{u-w} S_s$ and L_v.

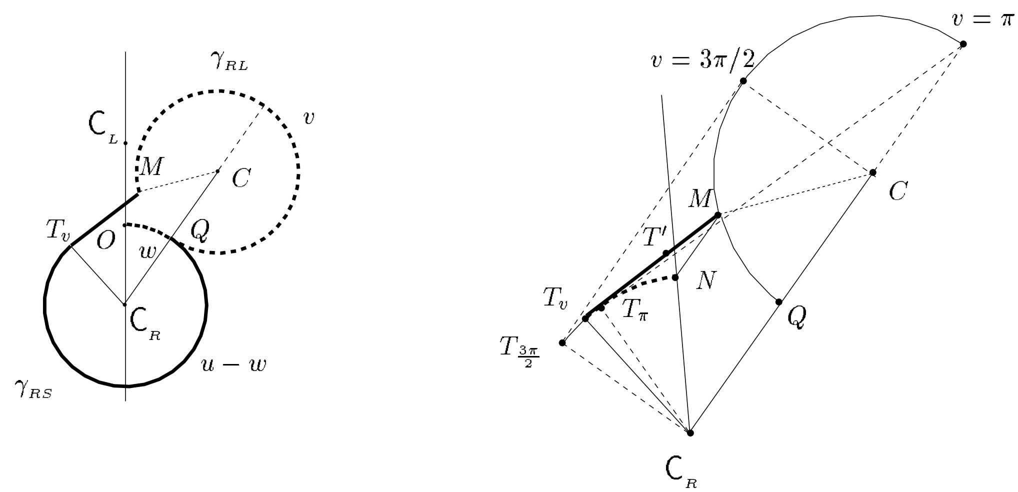

Figure 7 : Path RL is shorter than path RS (on the left, paths γ_{RS} and γ_{RL}, on the right a zoom in order to compare arc of circle $\overset{\frown}{T_v N}$ and line segment $T_v M$

We define the following points :

- T_v is the junction point between the arc of circle and the line segment of path RS.
- Q is the junction point between the two arcs of circle of path RL.

- N is the symmetric point to M w.r.t. the straight line tangent to the circles supporting the two arcs of path RL. Hence arc $\widehat{QN}$ described clockwise has length v.

Suppose that $\pi < v \leq 3\pi/2$. Then point T_v lies on the arc of circle $\mathcal{C}_R$ described clockwise from $T_{\frac{3\pi}{2}}$ to T_π, and point N lies on the arc $\widehat{QT_v}$ of circle $\mathcal{C}_R$ also described clockwise. Hence $w + v \leq u$ and path RL is shorter than path RS.

Now suppose $v > 3\pi/2$. Points T_v and N are as shown in Figure 7. The difference of length between paths RS and RL is equal to the difference of length between line segment T_vM and arc $\widehat{T_vN}$. The following lemma proves that path RL is shorter than path RS.

Lemma 3.1 *Line segment T_vM is longer than arc $\widehat{T_vN}$.*

> *Proof :* We consider the involute of circle $\mathcal{C}_R$ starting at point N and turning counterclockwise. The tangent to this curve at point N is the straight line (C_RN). We define T' as the image of T_v on the involute (see Figure 7) : T' lies on the line tangent to $\mathcal{C}_R$ at point T_v and the length of segment T_vT' is equal to the one of arc $\widehat{T_vN}$ described clockwise. Moreover, from the definition of the involute follows that point T' lies to the left of the straight line (C_RN). But for any value of v in $]3\pi/2, 2\pi[$, N lies between $T_{\frac{3\pi}{2}}$ and Q, and point M lies to the right of line (C_RN). Line segment T_vM is then longer than line segment T_vT', and consequently longer than arc $\widehat{T_vN}$. □

We have proved that a path RL leading to a point in half-plane Π is always shorter than the path RS leading to the same point. This property is a fortiori satisfied for a point lying inside circle $\mathcal{C}_L$. So the optimal domain for type RL is $\mathring{\mathcal{C}}_L$ and symmetrically, the optimal domain for type LS is $\mathring{\mathcal{C}}_R$.

3.3 Partition of the plane

From the results of the previous section, we can deduce the partition of half-plane Π : inside circle $\mathcal{C}_L$ and on the half-circle of $\mathcal{C}_L$ corresponding to strictly negative abscissas, paths of type RL are optimal, and on $\mathcal{C}_L$ or outside this circle, paths of type LS are optimal. Then with the symmetry w.r.t. axis (Ox), we can now give the partition of the whole plane :

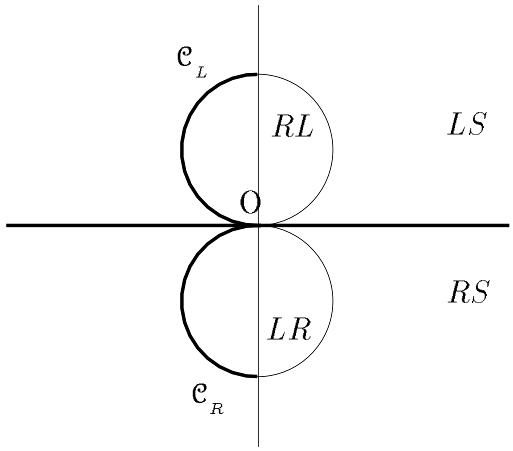

Figure 8 : Partition of the plane

- Paths of type RL are optimal in $\mathring{\mathcal{C}}_L$ and on the half-circle which is the portion of $\mathcal{C}_L$ satisfying $x < 0$.
- Paths of type LR are optimal in $\mathring{\mathcal{C}}_R$ and on the half-circle which is the portion of $\mathcal{C}_R$ satisfying $x < 0$.
- Paths of type LS are optimal in $\Pi \setminus \mathring{\mathcal{C}}_L$.
- Paths of type RS are optimal in $\Pi' \setminus \mathring{\mathcal{C}}_R$.

We can notice that there are two sets of points where two types of paths are both optimal. The first set is made of the two half-circles of $\mathcal{C}_L$ and $\mathcal{C}_R$ satisfying $x < 0$. The optimal paths reaching such points are in fact single arcs of circle either L or R, which are respectively degenerate forms of paths LS and RL, or RS and LR. The second set is axis (Ox), whose points are reached by optimal paths R_uS_s and L_uS_s, with $u = 0$ if $x \geq 0$.

Moreover, as $v > \pi$ for an optimal path of types RL_v or LR_v, we have proved again the result of H. G. Robertson [Rob70], namely that an optimal path reaching a point inside $\mathcal{C}_L$ or $\mathcal{C}_R$ must be of length strictly greater than $\rho\pi$.

4 SP-accessibility regions

We can also define the SP-accessibility region corresponding to a length d as the union of all iso-distance curves corresponding to lengths less than or equal to d. We already computed those iso-distance curves when we computed the domains for each type of path. They are shown in Figure 9.

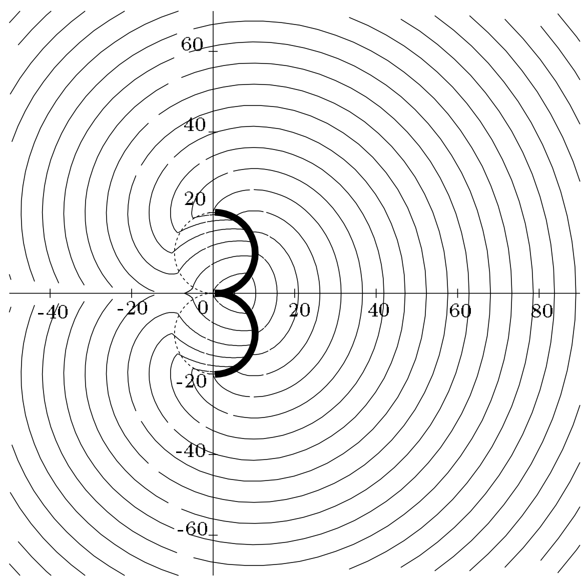

Figure 9 : Iso-distance curves for the problem with variable final endpoint

Since paths of type RL or LR have lengths strictly greater than $\rho\pi$, all iso-distance curves for lengths less than $\rho\pi$ only correspond to paths of type RS or LS. As a consequence, these iso-distance curves are not closed curves, but ends on circles $\mathcal{C}_L$ and $\mathcal{C}_R$ (see Figure 9). Thus, the two half-circles in thick lines are loci of discontinuity. We denote the union of these half-circles $\mathcal{D}$. Indeed, consider a point M with a positive abscissa on circle $\mathcal{C}_L$. It is reached by the optimal path $L_v S_0$ of length ρv, but also by path $R_u L_{2\pi-v}$ (which is not optimal), where $u = \frac{2}{3}(\pi - v)$, of length $\rho(\ 2\pi - v + \frac{2}{3}(\pi - v)\ )$ (see Figure 10). Then a point inside circle $\mathcal{C}_L$ very close to M will be optimally reached by a path of type RL whose length is very close to $\rho(\ 2\pi - v + \frac{2}{3}(\pi - v)\ )$, while a point outside circle $\mathcal{C}_L$ very close to M will be optimally reached by a path of type LS whose length will be very close to ρv.

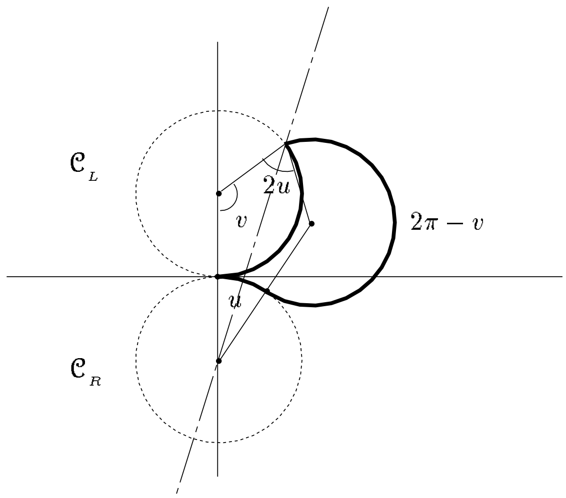

Figure 10 : Paths L and RL reaching a point with positive abscissa on circle $\mathcal{C}_L$

The shapes of the iso-distance curves are not as simple as in [SFL94] : they are not imbricated closed curves partly because of the discontinuity. In fact, $\mathcal{D}$ could be seen as a wall : a portion of an iso-distance curve, starting from a point $(x \geq 0, 0)$, can reach points behind $\mathcal{D}$ only if it corresponds to a sufficiently large value of d, in order to bypass $\mathcal{D}$.

This discontinuity has also as consequence that the boundary of some SP-accessibility regions are not only arcs of iso-distance curves but also arcs of $\mathcal{D}$. This is the main difference with the result of [CH75] : as the authors did not consider optimal paths, the boundary of their region $R(t)$ are always arcs of involutes or cardioids. But some of these arcs of cardioids correspond to non optimal paths CC reaching points that lie outside circles $\mathcal{C}_L$ and $\mathcal{C}_R$.

We give now the shapes of the SP-accessibility regions w.r.t. the possible values of length d (see Figure 11) :

- $d \in [0, \rho\,\pi]$

  The boundary of the SP-accessibility region is only made of arcs of involutes and of $\mathcal{D}$.

- $d \in]\rho\,\pi, \rho\,(\frac{3\pi}{2} + 1)[$

  Length d is strictly greater than $\rho\,\pi$ so the arcs of involutes are concatenated to arcs of cardioids inside circles $\mathcal{C}_L$ and $\mathcal{C}_R$, themselves concatenated to arcs of $\mathcal{D}$.

- $d = \rho\,(\frac{3\pi}{2} + 1)$

  The situation is the same as previously, but $\rho\,(\frac{3\pi}{2} + 1)$ is the value of d for which the involutes are tangent to the negative part of axis (Ox).

- $d \in]\rho\,(\frac{3\pi}{2} + 1), 2\rho\,\pi[$

  Arcs of involutes are secant to the negative part of axis (Ox) and thus, the boundary of the SP-accessibility region is made of two closed curves, one surrounding the other : the external curve is made of arcs of involutes, and the internal one is made of arcs of $\mathcal{D}$, cardioids and involutes.

- $d \in [2\rho\,\pi, d_{\max}[$

  Since $d \geq 2\rho\,\pi$ the internal curve is only made of arcs of cardioids and circles. The external curve is still made of arcs of involutes. $d_{\max}$ denotes the maximal length of an optimal path RL or LR.

- $d \geq d_{\max}$

  d is greater than the maximal length of an optimal path RL or LR, thus the external curve is the only one that remains.

5 Conclusion

The optimisation problem for the kinematic model of a car-like robot only allowed to move forwards, is an example of a non-linear system for which the synthesis problem is completely solved [BSBL93]. As a logic continuation, we have solved in this report the optimisation problem with variable endpoint : we found the types of optimal paths, computed the partition of the plane as the result of the synthesis problem, and studied the iso-distance curves in order to determine the SP-accessibility regions for this kind of mobile robot. Besides its theoretical interest, this result could find an application in path planning, as B. Mirtich and J. Canny already did in [MC92] for the mobile robot allowed to backup. This result has also applications in a completely different field : the theory of differential games, for the simulation of plane pursuits [Coc67].

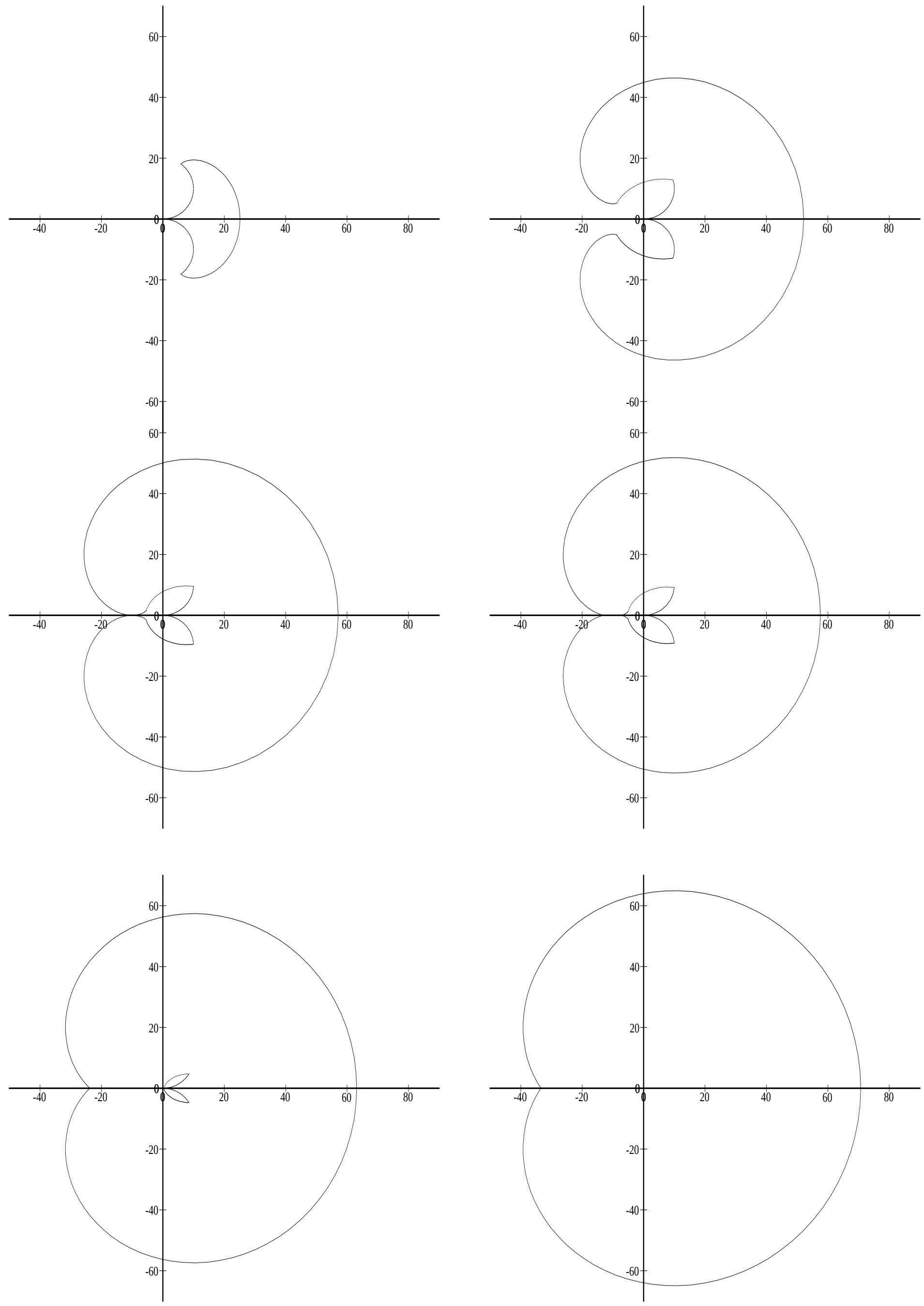

Figure 11 : SP-accessibility regions $\mathcal{R}_{\rho\frac{4\pi}{5}}$, $\mathcal{R}_{\rho\frac{3\pi+1}{2}}$, $\mathcal{R}_{\rho(\frac{3\pi}{2}+1)}$, $\mathcal{R}_{\rho\frac{11\pi}{6}}$, $\mathcal{R}_{\rho(2\pi+\frac{1}{100})}$ and $\mathcal{R}_{\rho\frac{9\pi}{4}}$

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