Convergence rate for the approximation of the limit law of weakly interacting particles 1: smooth interacting kernels

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Abstract: We consider a weakly interacting N-particles system, described by stochastic differential equations. According to the propagation of chaos theory, the corresponding empirical measure (μ_t^N) converge to a deterministic measure μ_t, when N goes to infinity.

We are interested in the computation of the limit law and its cumulative distribution function at a fixed date T. We construct an algorithm based on the time discretization of the N-system of stochastic differential equations on $[0,T]$.

In this first part, we prove the convergence of the method for smooth and bounded interacting kernels. For the computation of the cumulative distribution function, the rate of convergence is of order $O(\frac{1}{\sqrt{N}} + \sqrt{\Delta t})$, for the $L^1(\mathbb{R} \times \Omega)$ norm of the error, where N is the number of particles and Δt is the time step. As the limit law is smooth, we compute an approximation of the density by a regularization of the discrete time empirical measure. The rate of convergence is of order $O(\varepsilon^2 + \frac{1}{\varepsilon}(\frac{1}{\sqrt{N}} + \sqrt{\Delta t}))$, where ε is the regularization parameter.

In the second part [2], we will use the point of view of the propagation of chaos theory, in particular the interpretation of the underlying nonlinear PDE's as a limit equation for the law of the interacting particles, in order to develop a stochastic particle method for the one-dimensional Burgers equation. In this case, the interacting kernel of the particles system is discontinuous. As in the first part, we will give an accurate estimate of the $L^1(\mathbb{R} \times \Omega)$ norm of the error.

(Résumé : tsvp)

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Vitesse de convergence pour l'approximation de la loi limite de particules en interaction faible
1: Noyaux d'interaction réguliers

Résumé : On considère un système de N particules en interaction faible, décrit par un système d'équations différentielles stochastiques. D'après la théorie de la propagation du chaos, la mesure empirique des particules (μ_t^N) converge vers une mesure déterministe μ_t, quand N tend vers l'infini.

On s'intéresse au calcul de la loi limite et de la fonction de répartition associée à une date fixée T. On construit un algorithme de calcul, basé sur la discrétisation en temps du système d'équations différentielles stochastiques sur $[0, T]$.

Dans cette première partie, on prouve la convergence de la méthode pour des noyaux d'interaction réguliers et bornés. Pour le calcul de la fonction de répartition, la vitesse de convergence pour la norme $L^1(\mathbb{R} \times \Omega)$ de l'erreur, est d'ordre $O(\frac{1}{\sqrt{N}} + \sqrt{\Delta t})$, où N est le nombre de particules et Δt le pas de temps. La loi limite étant absolument continue par rapport à la mesure de Lebesgue, on calcule la densité en construisant une régularisation de la mesure empirique discrète. La vitesse de convergence est d'ordre $O(\varepsilon^2 + \frac{1}{\varepsilon}(\frac{1}{\sqrt{N}} + \sqrt{\Delta t}))$, où ε est le paramètre de régularisation.

Dans la seconde partie [2], on utilisera le point de vue de la théorie de la propagation du chaos, en particulier l'interprétation de l'EDP sous-jacente comme l'équation limite de la loi des particules en interaction, pour développer une méthode à particules aléatoires pour l'équation de Burgers uni-dimensionnelle. Dans ce cas, le noyau d'interaction du système de particules est discontinu. Comme dans la première partie, nous donnerons une estimation précise de la norme $L^1(\mathbb{R} \times \Omega)$ de l'erreur.

1. Introduction

We consider a system of N one-dimensional stochastic differential equations describing weakly interacting particles :

$$\left\{ \begin{array}{l} dX_t^{i,N} = \int_{\boldsymbol{R}} b(X_t^{i,N}, y)\mu_t^N(dy)\, dt \;+\; \int_{\boldsymbol{R}} s(X_t^{i,N}, y)\mu_t^N(dy) dw_t^i\;, \quad i = 1, \ldots, N, \\ \\ X_0^{i,N} = X_0^i, \end{array} \right. \tag{1.1}$$

where (w_t^1),..,(w_t^N) are independent one-dimensional Wiener processes, and μ_t^N is the random empirical measure :

$$\mu_t^N \;=\; \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^{i,N}}.$$

The functions b and s are called "interaction kernels".

The propagation of chaos theory provides convergence results for the process (μ_t^N) : in particular, under appropriate assumptions on the interaction kernels, it is proven that this empirical process converges in law to a deterministic measure μ_t, as $N \to \infty$, provided that $\mu_0^N \to \mu_0$ (see, e.g., Gärtner [6], Metivier [8], Meléard and Roelly [7], Sznitman [12]); besides, if $L(\mu)$ denotes the differential operator defined by

$$L(\mu)f(x) \;=\; \frac{1}{2}\left(\int_{\boldsymbol{R}} s(x,y)d\mu(y)\right)^2 f''(x) \;+\; \left(\int_{\boldsymbol{R}} b(x,y)d\mu(y)\right) f'(x),$$

the measure μ_t satisfies the McKean-Vlasov equation

$$\frac{d}{dt} < \mu_t, f > = < \mu_t, L(\mu_t) f > \tag{1.2}$$

with the initial condition egual to μ_0, for all real functions f of class $\mathcal{C}^\infty$ and with compact support. Such results are related to the nonlinear process (X_t) which describes the asymptotic individual behavior of the particles and which satisfies the stochastic differential equation

$$\left\{ \begin{array}{l} X_t \;=\; X_0 \;+\; \int_0^t \int_{\boldsymbol{R}} b(X_\theta, y)\mu_\theta(dy)dt \;+\; \int_0^t \int_{\boldsymbol{R}} s(X_\theta, y)\mu_\theta(dy)dW_\theta, \\ \\ \mu_t \;\text{ is the law of the random variable } X_t, \text{ for all } t \geq 0. \end{array} \right. \tag{1.3}$$

An example of results concerning this kind of processes is

Theorem 1.1 (Oelschläger, [10]) *If the kernels b and s are Lipschitz and bounded function and if s is strictly positive, then the SDE (1.3) has a unique strong solution.*

Applying Itô's formula, one deduces that the law of X_t satisfies the equation (1.2).

Using this probabilistic interpretation, we construct an approximation method for the distribution function μ_t, solution of (1.2) with a given probability measure μ_0 as initial condition, and its cumulative distribution function. The algorithm consists in simulating the particle system (1.1) for the computation of μ_t^N.

Let $V(t,.)$ denote the distribution function of μ_t; if $H(\cdot)$ denotes the Heaviside function, we have

$$V(t,x) = P[X_t < x] = \int_{-\infty}^{x} \mu_t(dy) = \mathbb{E} H(x - X_t).$$

In order to obtain a rate of convergence for the numerical computation of V, we will suppose that the following assumptions hold :

(H1) there exists a strictly positive constant s_0, such that $s(x,y) \geq s_0, \ \forall (x,y)$.

(H2) the kernels b and s are uniformly bounded functions of $\mathbb{R}^2$; b is globally Lipschitz and s has uniformly bounded first partial derivatives;

Under these assumptions, the measure μ_t has a smooth density and we note that, when s is constant (equal to σ, say), V solves the following nonlinear PDE obtained by formally integrating the McKean-Vlasov equation:

$$\frac{\partial V}{\partial t} = \frac{1}{2}\sigma^2 \Delta V \ - \ [\int_{\mathbb{R}} b(\cdot, y) \, \frac{\partial V}{\partial x}(t,y) dy \,] \, . \, \frac{\partial V}{\partial x}.$$

This latter fact is used in our analysis of the methodology applied to the Burgers equation, which requires specific developments because of the singularity of the corresponding interaction kernel b (see Bossy-Talay [2]).

We compute an approximation of the density by a regularisation of the discrete time empirical measure. A rate of convergence is obtained with a stronger assumption on the coefficients :

(H3) the kernel b is in $C_b^1(\mathbb{R}^2)$ and s is in $C_b^2(\mathbb{R}^2)$.

Remark 1.2. The hypothesis $(H1)$ could be somewhat relaxed: what is used in the proof is the existence of a density for the law of the process (z_t) defined below in (2.1), this density satisfying the exponential bounds (3.3).

2. Description of the algorithm and main results

2.1. Approximation of the initial condition. The algorithm starts with an approximation of the initial condition.

A time $t = 0$, $V(0, \cdot)$ is the distribution function of the law μ_0, denoted by $V_0(\cdot)$ in the sequel. Let $N \in \mathbb{N}$ be the number of particles. One chooses N points in $\mathbb{R}$, $(y_0^1, .., y_0^N)$, such that the piecewise constant function

$$\overline{V}_0(x) = \frac{1}{N} \sum_{i=1}^{N} H(x - y_0^i)$$

approximates V_0. Thus, at $t = 0$, one has N particles which define the empirical measure

$$\overline{\mu}_0 := \frac{1}{N} \sum_{i=1}^{N} \delta_{y_0^i}.$$

2.2. Motion of the particles. These particles are moved according to the dynamics of (X_t). We must solve two problems: the time discretization of (X_t), and the approximation of the coefficients of the SDE (1.3) at each time step, since these coefficients depend on the unknown law μ_t.

The existence and uniqueness of a strong solution (X_t) enables to consider the drift and the diffusion coefficients of (1.3) as functions β and σ depending only one time and space variables: more precisely, we define $\beta \ : \ [0, T] \times \mathbb{R} \longrightarrow \mathbb{R}$ by

$$\beta(t, x) := \int_{\mathbb{R}} b(x, y) \, \mu_t(dy),$$

and $\sigma \ : \ [0, T] \times \mathbb{R} \longrightarrow \mathbb{R}$ by

$$\sigma(t, x) := \int_{\mathbb{R}} s(x, y) \, \mu_t(dy).$$

We note that, under (H1), β and σ are Lipschitz in x, uniformly bounded in t and x; this ensures the existence and the uniqueness of the inhomogeneous Markov process solution to the stochastic differential equation

$$\left\{ \begin{array}{rcl} dz_t & = & \beta(t, z_t)dt \ + \ \sigma(t, z_t) \, dw_t, \\ \\ z_{t=0} & = & z_0. \end{array} \right. \tag{2.1}$$

When the law of z_0 is μ_0, the two processes (X_t) and (z_t) have the same law and

$$V(t,x) \;=\; \mathbb{E}H(x-X_t) \;=\; \mathbb{E}_{\mu_0}H(x-z_t) \;=\; \int_{\mathbb{R}} \mathbb{E}H(x-z_t(y))\,\mu_0(dy),$$

where $(z_t(y))$ is the solution to (2.1) starting at y at time 0. Note that (z_t) is a Markov process, whereas (X_t) is not: this is used in the sequel.

For any $t \in [0,T]$, a first approximation of $V(t,.)$ is given by

$$V(t,x) \;\simeq\; \mathbb{E}_{\overline{\mu}_0}H(x-z_t) \;=\; \frac{1}{N}\sum_{i=1}^{N} \mathbb{E}H(x-z_t(y_0^i)).$$

Consider N independent copies $(w_t^i)_{i=1}^N$ of the Brownian motion, and the family of independent processes $(z_t^i)_{(i=1,..,N)}$, solutions to

$$\begin{cases} dz_t^i \;=\; \beta(t,z_t^i)\,dt \;+\; \sigma(t,z_t^i)\,dw_t^i, \\ \\ z_0^i \;=\; y_0^i. \end{cases} \tag{2.2}$$

A new approximation of $V(t,.)$ is given by

$$V(t,x) \;\simeq\; \frac{1}{N}\sum_{i=1}^{N} H(x-z_t^i).$$

To simulate the motion of the (z_t^i)'s, we discretize in time: T being fixed, one chooses $\Delta t > 0$ and $K \in \mathbb{N}$ such that $T = K\Delta t$. The discretization times are denoted by $t_k = k\Delta t$, with $1 \le k \le K$.

Applying the Euler scheme to (2.2), one defines the independent discrete-time processes $(\overline{z}_{t_k}^i)$:

$$\begin{cases} \overline{z}_{t_{k+1}}^i = \overline{z}_{t_k}^i + \beta(t_k,\overline{z}_{t_k}^i)\,\Delta t \;+\; \sigma(t_k,\overline{z}_{t_k}^i)\,\Delta w_{k+1}^i, \\ \\ \overline{z}_0^i = y_0^i, \end{cases} \tag{2.3}$$

where $\Delta w_{k+1}^i = w_{t_{k+1}}^i - w_{t_k}^i$.
Thus, at time t_k (with $k = 1,..,K$), $V(t_k,.)$ can be approximated by

$$V(t_k,x) \;\simeq\; \frac{1}{N}\sum_{i=1}^{N} H(x-z_{t_k}^i).$$

It now remains to approximate $\beta(t_k,\cdot)$ and $\sigma(t_k,\cdot)$.

As we did for the initial condition, we approximate μ_{t_k} by the empirical measure $\overline{\mu}_{t_k}$ generated by the locations of the particles at time t_k. Denote the location of the particle number i at time t_k by $Y^i_{t_k}$ and define

$$\overline{\mu}_{t_k} = \frac{1}{N}\sum_{i=1}^{N}\delta_{Y^i_{t_k}}.$$

Then

$$\beta(t_k,x) = \int b(x,y)\mu_{t_k}(dy) \simeq \int b(x,y)\overline{\mu}_{t_k}(dy) = \frac{1}{N}\sum_{i=1}^{N} b(x,Y^i_{t_k}).$$

Analogously, we set

$$\sigma(t_k,x) \simeq \frac{1}{N}\sum_{i=1}^{N} s(x,Y^i_{t_k}).$$

Therefore, the motion of the particles involved in the algorithm is described by the family of discrete time processes $(Y^i_{t_k})_{i=1}^N$ satisfying

$$\begin{cases} Y^i_{t_{k+1}} = Y^i_{t_k} + \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} b(Y^i_{t_k},Y^j_{t_k})\,\Delta t + \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} s(Y^i_{t_k},Y^j_{t_k})\,\Delta w^i_{k+1}, \\ \\ Y^i_0 = y^i_0, \quad i=1,..,N. \end{cases} \tag{2.4}$$

Finally, the function $V(t_k,x)$ is approximated by

$$\overline{V}_{t_k}(x) = \frac{1}{N}\sum_{i=1}^{N} H(x - Y^i_{t_k}).$$

2.3. Main result.

We introduce an hypothesis on the initial law.

(H4) The initial law μ_0 has a strictly positive and continuous density u_0 satisfying: there exist strictly positive constants M, η and α such that

$$u_0(x) \leq \eta\,.\,exp\left(-\alpha\frac{x^2}{2}\right), \text{ for } |x| > M.$$

(If $\eta = 0$, μ_0 has a compact support).

A simple method to initialize the positions of the particles is to invert the distribution function V_0 :

$$y_0^i = \begin{cases} V_0^{-1}(\frac{i}{N}), & i = 1,..,N-1, \\ \\ V_0^{-1}(1 - \frac{1}{2N}), & i = N. \end{cases}$$

In the next section, we prove the following

Theorem 2.1. *Let $T > 0$ be fixed, let $\Delta t < 1$ be such that $T = \Delta t K$, $K \in \mathbb{N}$. Let $V(t_k, x)$ be the distribution function of μ_{t_k}. Let $\overline{V}_{t_k}(x)$ be the approximation corresponding to the above algorithm with N particles.*

Suppose (H1), (H2) and (H4). Then

$$\left\| V_0 - \overline{V}_0 \right\|_{L^1(\mathbb{R})} \leq \frac{C}{N} \sqrt{log(2N)},$$

where C depends on M, η and α.

Besides, there exist strictly positive constants L_1 and L_2, depending on s, b, u_0 and T, such that, $\forall\, k \in \{1,..,K\}$, one has

$$\mathbb{E} \left\| V(t_k,.) - \overline{V}_{t_k}(.) \right\|_{L^1(\mathbb{R})} \leq L_1 \left(\|V_0 - \overline{V}_0\|_{L^1(\mathbb{R})} + \frac{1}{\sqrt{N}} + \sqrt{\Delta t} \right),$$

and

$$Var \left(\left\| V(t_k,.) - \overline{V}_{t_k}(.) \right\|_{L^1(\mathbb{R})} \right) \leq L_2 \left(\|V_0 - \overline{V}_0\|^2_{L^1(\mathbb{R})} + \frac{1}{N} + \Delta t \right).$$

In order to obtain an approximation of the density μ_{t_k}, we construct a regularization by convolution of the discrete measure $\overline{\mu}_{t_k}$.

Let Φ_ε be the density of the gaussian law $N(0, \varepsilon^2)$ and set

$$\overline{\mu}^{\varepsilon}_{t_k} := \Phi_\varepsilon * \overline{\mu}_{t_k}.$$

We introduce a stronger hypothesis on the initial law μ_0 :

(H5) The initial law μ_0 has a strictly positive density u_0 in $C^2(\mathbb{R})$, satisfying: there exist strictly positive constants M, η and α such that

$$u_0(x),\ u_0'(x),\ u_0''(x) \leq \eta \,.\, exp\left(-\alpha \frac{x^2}{2}\right), \text{ for } |x| > M.$$

Theorem 2.2. *Let $T > 0$ be fixed, let $\Delta t < 1$ be such that $T = \Delta t K$, $K \in \mathbb{N}$.
Let $\overline{\mu}^{\varepsilon}_{t_k}$ be the approximation of μ_{t_k} corresponding to the above algorithm with N particles.*

Suppose (H1), (H3) and (H5). Then, there exist strictly positive constants L'_1 and L'_2, depending on s, b, u_0 and T, such that, $\forall\, k \in \{1, .., K\}$, one has

$$\mathbb{E} \left\| \mu_{t_k}(.) - \overline{\mu}^{\varepsilon}_{t_k}(.) \right\|_{L^1(\mathbb{R})} \leq L'_1 \left[\varepsilon^2 + \frac{1}{\varepsilon} \left(\|V_0 - \overline{V}_0\|_{L^1(\mathbb{R})} + \frac{1}{\sqrt{N}} + \sqrt{\Delta t} \right) \right],$$

and

$$Var \left(\left\| \mu_{t_k}(.) - \overline{\mu}^{\varepsilon}_{t_k}(.) \right\|_{L^1(\mathbb{R})} \right) \leq L'_2 \left[\varepsilon^4 + \frac{1}{\varepsilon^2} \left(\|V_0 - \overline{V}_0\|^2_{L^1(\mathbb{R})} + \frac{1}{N} + \Delta t \right) \right].$$

3. Proof

In all this section, C denotes any positive constant depending only on T and the functions b, s.

The proof of the theorem 2.1 follows from a decomposition of the error $V(t_k,.) - \overline{V}_{t_k}(\cdot)$ at each time t_k of the discretization. This decomposition represents the successive approximations of $V(t_k,.)$ that we introduced in the preceding section :

$$\begin{aligned} V(t_k,x) - \overline{V}_{t_k}(x) &= \mathbb{E}_{\mu_0} H(x - z_{t_k}) - \mathbb{E}_{\overline{\mu}_0} H(x - z_{t_k}) \\ &+ \mathbb{E}_{\overline{\mu}_0} H(x - z_{t_k}) - \frac{1}{N}\sum_{i=1}^{N} H(x - z^i_{t_k}) \\ &+ \frac{1}{N}\sum_{i=1}^{N} H(x - z^i_{t_k}) - \frac{1}{N}\sum_{i=1}^{N} H(x - \overline{z}^i_{t_k}) \\ &+ \frac{1}{N}\sum_{i=1}^{N} H(x - \overline{z}^i_{t_k}) - \frac{1}{N}\sum_{i=1}^{N} H(x - Y^i_{t_k}). \end{aligned} \tag{3.1}$$

The first term measures the approximation error of the measure μ_0. Introducing the processes (z^i_t), one makes appear a second term which essentially is a statistical error. Then one considers the discretization error induced by the Euler scheme. The last term corresponds to the approximation of the coefficients $\beta(t_k,\cdot)$ and $\sigma(t_k,\cdot)$ by means of the empirical measure $\overline{\mu}_{t_k}$, in other terms it measures the error induced by the substitution of the family of independent processes $(\overline{z}^i_{t_k})$ by the family of dependent processes $(Y^i_{t_k})$.

3.1. A preliminary remark.

The hypothesis (H2) implies that the functions $\sigma(\cdot,\cdot)$ and $\beta(\cdot,\cdot)$ are Lipschitz in x and Holder in t: for all $t \in [0,T]$ and for all $(x,y) \in \mathbb{R}^2$ one has

$$\begin{aligned} \mid \beta(t,x) - \beta(t,y) \mid &\le \int_{\mathbb{R}} \mid b(x,z) - b(y,z) \mid \mu_t(dz) \le L_b \mid x-y \mid \\ \text{and} & \\ \mid \sigma(t,x) - \sigma(t,y) \mid &\le \int_{\mathbb{R}} \mid s(x,z) - s(y,z) \mid \mu_t(dz) \le L_s \mid x-y \mid . \end{aligned} \tag{3.2}$$

On the other hand, as $\beta(t,x) = \mathbb{E} b(x,X_t)$, for all $\theta,\ t \in [0,T]$,

$$\mid \beta(\theta,x) - \beta(t,x) \mid \le \mathbb{E} \mid b(x,X_\theta) - b(x,X_t) \mid \le L_b\, \mathbb{E} \mid X_\theta - X_t \mid \le C|t-\theta|^{1/2}.$$

Besides, $0 < \sigma_* \le \sigma(t,x) \le \sigma^*,\ \forall (t,x) \in [0,T]\times\mathbb{R}$. Thus the transition probability of (z_t) has a smooth density denoted by $p(t,\theta;x,y)$. We denote by $\Gamma_t(\cdot,y) := p(t,0;\cdot,y)$

the density of $z_t(y)$. The processes (X_t) and (z_t) with the same initial law μ_0 being identical in law, the law μ_t has a density denoted by u_t, which is given by

$$u_t(x) \;=\; \int_{\mathbb{R}} p(t,0;x,y)\mu_0(dy) \;=\; \int_{\mathbb{R}} \Gamma_t(x,y)\mu_0(dy)\,, \quad \forall x \in \mathbb{R}\,,\ \forall t > 0.$$

These property of β and σ imply the following well-known estimate (cf. Friedman [5], p.139-150, or the chapter 1 of [4]): for any T, there exist strictly positive constants C_0 and C_1 such that, $\forall t \in [0,T]$, $\forall x,\ y$, $\forall \bar{\sigma} > \sigma^*$,

$$\begin{aligned} |\,\Gamma_t(x,y)\,| \;&\leq\; \frac{C_0}{\sqrt{t}}\, exp\left(-\frac{(x-y)^2}{2\bar{\sigma}^2 t}\right), \\ |\,\frac{\partial}{\partial y}\Gamma_t(x,y)\,| \;&\leq\; \frac{C_1}{t}\, exp\left(-\frac{(x-y)^2}{2\bar{\sigma}^2 t}\right). \end{aligned} \tag{3.3}$$

The four next subsections are devoted to estimates for each term of this error decomposition.

3.2. Error induced by the approximation of the initial condition.

This error is described by the

Lemma 3.1. *There exists a positive constant l_1, depending only on T, b and σ, such that, for any $t \in [0,T]$:*

$$\|\, \mathbb{E}_{\mu_0} H(x-z_t) \;-\; \mathbb{E}_{\overline{\mu}_0} H(x-z_t)\,\|_{L^1(\mathbb{R})} \;\leq\; l_1 \,\|\, V_0 \;-\; \overline{V}_0\,\|_{L^1(\mathbb{R})}.$$

Proof. We observe that

$$\begin{aligned} \mathbb{E}_{\overline{\mu}_0} H(x-z_t) \;&=\; \int_{\mathbb{R}} \mathbb{E} H(x-z_t(y))\,\overline{\mu}_0(dy) \;=\; \int_{\mathbb{R}} \mathbb{E} H(x-z_t(y))\, d\overline{V}_0(y) \\ &=\; \int_{-\infty}^{0} \mathbb{E} H(x-z_t(y))\, d\overline{V}_0(y) \;-\; \int_{0}^{+\infty} \mathbb{E} H(x-z_t(y))\, d(1-d\overline{V}_0(y)). \end{aligned}$$

The integration by parts formula for a Stieljes integral gives

$$\begin{aligned} \mathbb{E}_{\overline{\mu}_0} H(x-z_t) \;&=\; \mathbb{E} H(x-z_t(0)).\overline{V}_0(0) \;-\int_{-\infty}^{0} \frac{\partial}{\partial y}\mathbb{E} H(x-z_t(y)).\overline{V}_0(y)dy \\ &\quad + \mathbb{E} H(x-z_t(0)).(1-\overline{V}_0(0)) \;+\int_{0}^{+\infty} \frac{\partial}{\partial y}\mathbb{E} H(x-z_t(y)).(1-\overline{V}_0(y))dy. \end{aligned}$$

A similar computation for $\mathbb{E}_{\mu_0} H(x-z_t)$ gives

$$\mathbb{E}_{\mu_0} H(x-z_t) \; - \; \mathbb{E}_{\overline{\mu}_0} H(x-z_t) = \; - \int_{\mathbb{R}} \frac{\partial}{\partial y} \mathbb{E} H(x-z_t(y)) \, . \, (V_0(y) - \overline{V}_0(y)) \, dy,$$

so that

$$\|\mathbb{E}_{\mu_0} H(x-z_t) - \mathbb{E}_{\overline{\mu}_0} H(x-z_t)\|_{L^1(\mathbb{R})} \leq \int_{\mathbb{R}} \int_{\mathbb{R}} \mid \frac{\partial}{\partial y} \mathbb{E} H(x-z_t(y)) \mid . \mid V_0(y) - \overline{V}_0(y) \mid dy dx \; .$$

To end the proof, it remains to upper bound

$$\int_{\mathbb{R}} |\frac{\partial}{\partial y} \mathbb{E} H(x-z_t(y))| dx.$$

We note that

$$\mid \frac{\partial}{\partial y} \mathbb{E} H(x-z_t(y)) \mid = \mid \frac{\partial}{\partial y} P(z_t(y) < x) \mid = \mid \frac{\partial}{\partial y} \int_{-\infty}^{x} \Gamma_t(\alpha, y) d\alpha \mid \leq \int_{-\infty}^{x} \mid \frac{\partial}{\partial y} \Gamma_t(\alpha, y) \mid d\alpha,$$

where $\Gamma_t(., y)$ denote the density of the law of $z_t(y)$. From (3.3), we deduce that

$$\mid \frac{\partial}{\partial y} \mathbb{E} H(x-z_t(y)) \mid \leq \int_{-\infty}^{x} \frac{C_1}{t} \, exp \left(- \frac{(\alpha - y)^2}{2\bar{\sigma}^2 t} \right) \, d\alpha. \tag{3.4}$$

As well, one has

$$\begin{aligned} \mid \frac{\partial}{\partial y} \mathbb{E} H(x-z_t(y)) \mid \; &= \; \mid \frac{\partial}{\partial y}(1 - P(z_t(y) < x)) \mid = \mid \frac{\partial}{\partial y} P(z_t(y) > x) \mid \\ &\leq \; \int_{x}^{+\infty} \frac{C_1}{t} \, exp \left(- \frac{(\alpha - y)^2}{2\bar{\sigma}^2 t} \right) \, d\alpha. \end{aligned} \tag{3.5}$$

Thus, from (3.4), one gets

$$\begin{aligned} \int_{-\infty}^{y} \mid \frac{\partial}{\partial y} \mathbb{E} H(x-z_t(y)) \mid dx \; &\leq \; \int_{-\infty}^{y} \int_{-\infty}^{x} \frac{C_1}{t} \, exp \left(- \frac{(\alpha - y)^2}{2\bar{\sigma}^2 t} \right) d\alpha dx \\ &= \int_{-\infty}^{0} \int_{-\infty}^{x} \frac{C_1}{t} \, exp \left(- \frac{\alpha^2}{2\bar{\sigma}^2 t} \right) d\alpha \, dx, \end{aligned}$$

and from (3.5)

$$\int_{y}^{+\infty} \mid \frac{\partial}{\partial y} \mathbb{E} H(x-z_t(y)) \mid dx \; \leq \; \int_{0}^{+\infty} \int_{x}^{+\infty} \frac{C_1}{t} \, exp \left(- \frac{\alpha^2}{2\bar{\sigma}^2 t} \right) d\alpha dx.$$

We now use the following estimate, easy to prove. Let $g_\alpha(x)$ be the density of a gaussian law $N(0,\alpha)$, and let $G_\alpha(x)$ be its distribution function; then one has

$$\begin{aligned} &(i) \quad \forall\, x \le 0\,,\ G_\alpha(x) \;=\; \int_{-\infty}^{x} g_\alpha(y)dy \;\le\; \frac{1}{2}exp(-\frac{x^2}{2\alpha^2}), \\ &(ii) \quad \forall\, x \ge 0\,,\ (1-G_\alpha)\,(x) \;=\; \int_{x}^{+\infty} g_\alpha(y)dy \;\le\; \frac{1}{2}exp(-\frac{x^2}{2\alpha^2}), \end{aligned} \tag{3.6}$$

so that

$$\int_{\boldsymbol{R}} \mid \frac{\partial}{\partial y}\mathbb{E}H(x-z_t(y)) \mid dx \;\le\; \int_{\boldsymbol{R}} C_1\bar\sigma\frac{\sqrt{\pi}}{\sqrt{2t}}\, exp(-\frac{x^2}{2\bar\sigma^2 t}\,)dx \;=\; C_1\pi\bar\sigma^2$$

and

$$\|\, \mathbb{E}_{\mu_0}H(x-z_t) \;-\; \mathbb{E}_{\overline{\mu}_0}H(x-z_t)\,\|_{L^1(\boldsymbol{R})} \;\le\; C_1\pi\bar\sigma^2 \int_{\boldsymbol{R}} \mid V_0(y)-\overline{V}_0(y)\mid dy,$$

for any $\bar\sigma > \sigma^\star$, and with C_1 depending on T and $\sigma^\star$. □

3.3. The statistical error.

The statistical error is described by the

Lemma 3.2. *There exists a positive constant l_2 depending on T, b, σ and μ_0 such that, for all $t\in[0,T]$,*

$$(i) \quad \mathbb{E}\,\|\mathbb{E}_{\overline{\mu}_0}H(x-z_t) \;-\; \frac{1}{N}\sum_{i=1}^{N}\; H(x-z_t^i)\|_{L^1(\boldsymbol{R})} \;\le\; l_2\;\frac{1}{\sqrt{N}}$$

and

$$(ii) \quad \mathbb{E}\left(\|\mathbb{E}_{\overline{\mu}_0}H(x-z_t) \;-\; \frac{1}{N}\sum_{i=1}^{N}\; H(x-z_t^i)\|_{L^1(\boldsymbol{R})}\right)^2 \le\; (l_2)^2\;\frac{1}{N}.$$

Proof. By definition of the processes (z_t^i), one has

$$\mathbb{E}_{\overline{\mu}_0}H(x-z_t) \;=\; \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}H(x-z_t(y_0^i)) \;=\; \frac{1}{N}\sum_{i=1}^{N}\;\mathbb{E}H(x-z_t^i)).$$

Let us first prove the part (i). Set

$$\begin{aligned} A \;&:=\; \mathbb{E}\,\|\frac{1}{N}\sum_{i=1}^{N}\;\Big[\mathbb{E}H(x-z_t^i) \;-\; H(x-z_t^i)\,\Big]\,\|_{L^1(\boldsymbol{R})} \\ &\le\; \int_{\boldsymbol{R}}\sqrt{\mathbb{E}\mid\frac{1}{N}\sum_{i=1}^{N}\;[\mathbb{E}H(x-z_t^i) \;-\; H(x-z_t^i)]\mid^2}\; dx. \end{aligned}$$

The $(z_t^i)_{i=1}^N$ being independent, one gets

$$
\begin{aligned}
\mathbb{E}\left|\frac{1}{N}\sum_{i=1}^N\left[\mathbb{E}H(x-z_t^i)\;-\;H(x-z_t^i)\,\right]\right|^2 &= \frac{1}{N^2}\sum_{i=1}^N\mathbb{E}\left[\mathbb{E}H(x-z_t^i)\;-\;H(x-z_t^i)\,\right]^2\\
&= \frac{1}{N^2}\sum_{i=1}^N\mathbb{E}H(x-z_t^i).\mathbb{E}H(z_t^i-x),
\end{aligned}
$$

so that

$$
A\;\leq:\frac{1}{\sqrt{N}}\;\int_{\mathbb{R}}\sqrt{\frac{1}{N}\sum_{i=1}^N\mathbb{E}H(x-z_t^i).\mathbb{E}H(z_t^i-x)}\;\;dx.
$$

Let Γ_t^i denote the density of the law of z_t^i, then

$$
A\;\leq\;\frac{1}{\sqrt{N}}\;\int_{\mathbb{R}}\sqrt{\frac{1}{N}\sum_{i=1}^N\int_{-\infty}^x\Gamma_t^i(y)dy\;.\;\int_x^{+\infty}\Gamma_t^i(y)dy}\;\;dx.
$$

From (3.3), there exists a constant C_0 such that

$$
\Gamma_t^i(y)\;\leq\;\frac{C_0}{\sqrt{t}}\,exp\left(-\frac{(y-y_0^i)^2}{2\bar{\sigma}^2t}\right),\;\;\forall\,y\in\mathbb{R}\,,\;\;\forall\,\bar{\sigma}>\sigma^\star\,.
$$

Thus

$$
A\;\leq\;\frac{C_0}{\sqrt{N}}\;\int_{\mathbb{R}}\sqrt{\frac{1}{N}\sum_{i=1}^N\int_{-\infty}^{x-y_0^i}\frac{1}{\sqrt{t}}\,exp(-\frac{y^2}{2\bar{\sigma}^2t})dy\;.\;\int_{x-y_0^i}^{+\infty}\frac{1}{\sqrt{t}}\,exp(-\frac{y^2}{2\bar{\sigma}^2t})dy}\;\;dx.
$$

Let M be as in (H4). Decomposing the integral with respect to x in three parts (from $-\infty$ to $-M$, from $-M$ to M and from M to $+\infty$), we get

$$
\begin{aligned}
A\;\;\leq\;\;&\underbrace{\frac{C_0}{\sqrt{N}}\int_{-\infty}^{-M}\sqrt{\frac{1}{N}\sum_{i=1}^N\sqrt{\frac{2\pi}{t}}\bar{\sigma}\int_{-\infty}^{x-y_0^i}exp(-\frac{y^2}{2\bar{\sigma}^2t})dy}\;\;dx}_{I^-}\;+\;\frac{C_0}{\sqrt{N}}\,2M\,\sqrt{2\pi}\bar{\sigma}\\
&+\;\underbrace{\frac{C_0}{\sqrt{N}}\int_M^{+\infty}\sqrt{\frac{1}{N}\sum_{i=1}^N\sqrt{\frac{2\pi}{t}}\bar{\sigma}\int_{x-y_0^i}^{+\infty}exp(-\frac{y^2}{2\bar{\sigma}^2t})dy}\;\;dx}_{I^+}\;.
\end{aligned}
\tag{3.7}
$$

We will only treat I^- since I^+ can be treated by symmetry. By definition of the y_0^i's, it holds that

$$I^- \leq \frac{C_0}{\sqrt{N}} \int_{-\infty}^{-M} \sqrt{\frac{\bar{\sigma}}{N}\sqrt{\frac{2\pi}{t}} \left[\sum_{i=1}^{N-1} \int_{-\infty}^{x-V_0^{-1}(\frac{i}{N})} exp(-\frac{y^2}{2\bar{\sigma}^2 t})dy + \int_{-\infty}^{x-V_0^{-1}(1-\frac{1}{2N})} exp(-\frac{y^2}{2\bar{\sigma}^2 t})dy \right]} dx.$$

Let Ψ the function on $]0,1[$ defined by

$$\Psi(\theta) = \int_{-\infty}^{x-V_0^{-1}(\theta)} exp(-\frac{y^2}{2\bar{\sigma}^2 t})dy.$$

As V_0 is an increasing function, Ψ is a decreasing function and

$$\sum_{i=1}^{N-1} \frac{1}{2N} \Psi(\frac{i}{N}) \leq \sum_{i=1}^{N-1} \frac{1}{N} \Psi(\frac{i}{N}) \leq \int_0^{\frac{N}{N-1}} \Psi(\theta)d\theta,$$

$$\frac{1}{2N} \Psi(1-\frac{1}{2N}) \leq \int_{\frac{N}{N-1}}^1 \Psi(\theta)d\theta.$$

Thus,

$$\begin{aligned} I^- &\leq \frac{\sqrt{2}C_0}{\sqrt{N}} \int_{-\infty}^{-M} \sqrt{\sqrt{\frac{2\pi}{t}}\bar{\sigma} \int_0^1 \int_{-\infty}^{x-V_0^{-1}(\theta)} exp(-\frac{y^2}{2\bar{\sigma}^2 t})dy\, d\theta} \; dx \\ &\leq \frac{\sqrt{2}C_0}{\sqrt{N}} \int_{-\infty}^{-M} \sqrt{\sqrt{\frac{2\pi}{t}}\bar{\sigma} \int_{\mathbb{R}} exp\left(-\frac{(x-u)^2}{2\bar{\sigma}^2 t}\right) V_0(u)\, du} \; dx. \end{aligned}$$

From (H4) and (3.6), one deduces that

$$\mathbb{1}_{[u\leq -M]} V_0(u) + \mathbb{1}_{[u\geq M]}(1-V_0(u)) \leq \eta \sqrt{\frac{\pi}{2\alpha}}\, exp\left(-\alpha\frac{u^2}{2}\right), \tag{3.8}$$

and then it is easy to get

$$A \leq \frac{l_2}{\sqrt{N}}.$$

We now treat (ii).

$$\begin{aligned} \mathbb{E}\left\| \mathbb{E}_{\overline{\mu}_0} H(x-z_t) - \frac{1}{N}\sum_{i=1}^N H(x-z_t^i) \right\|_{L^1}^2 &= \mathbb{E}\left(\int_{\mathbb{R}} \left| \frac{1}{N}\sum_{i=1}^N \mathbb{E}H(x-z_t^i) - H(x-z_t^i) \right| dx \right)^2 \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{E}\left(\left| \frac{1}{N}\sum_{i=1}^N \mathbb{E}H(x_1-z_t^i) - H(x_1-z_t^i) \right| \right. \\ &\qquad \left. . \left| \frac{1}{N}\sum_{i=1}^N \mathbb{E}H(x_2-z_t^i) - H(x_2-z_t^i) \right| \right) dx_1 dx_2. \end{aligned}$$

It then remains to apply the Cauchy-Schwarz inequality and the result (i).□

3.4. The discretization error. The aim of this subsection is to prove the following lemma.

Lemma 3.3. *There exists a positive constant l_3, depending on T, b and σ, such that, $\forall\, k = 1,..,K$,*

$$(i) \quad \mathbb{E}\left\| \frac{1}{N}\sum_{i=1}^{N} H(x - z^i_{t_k}) \; - \; \frac{1}{N}\sum_{i=1}^{N} H(x - \bar{z}^i_{t_k}) \right\|_{L^1(\mathbb{R})} \;\leq\; l_3\,\sqrt{\Delta t}$$

and

$$(ii) \quad \mathbb{E}\left(\left\| \frac{1}{N}\sum_{i=1}^{N} H(x - z^i_{t_k}) \; - \; \frac{1}{N}\sum_{i=1}^{N} H(x - \bar{z}^i_{t_k}) \right\|_{L^1(\mathbb{R})} \right)^2 \;\leq\; (l_3)^2\,\Delta t.$$

Proof. Noting that

$$\forall\, a, b \in \mathbb{R}, \qquad \int_{\mathbb{R}} \mid H(x-a) - H(x-b) \mid dx \; = \mid a - b \mid, \tag{3.9}$$

one gets

$$\mathbb{E}\left\| \frac{1}{N}\sum_{i=1}^{N} H(x - z^i_{t_k}) \; - \; \frac{1}{N}\sum_{i=1}^{N} H(x - \bar{z}^i_{t_k}) \right\|_{L^1(\mathbb{R})} \;\leq\; \frac{1}{N}\sum_{i=1}^{N} \mid z^i_{t_k} \; - \; \bar{z}^i_{t_k} \mid$$

and

$$\mathbb{E}\left(\left\| \frac{1}{N}\sum_{i=1}^{N} H(x - z^i_{t_k}) \; - \; \frac{1}{N}\sum_{i=1}^{N} H(x - \bar{z}^i_{t_k}) \right\|_{L^1(\mathbb{R})} \right)^2 \;\leq\; \frac{1}{N}\sum_{i=1}^{N} \mathbb{E} \; \mid z^i_{t_k} \; - \; \bar{z}^i_{t_k} \mid^2 .$$

Set $\epsilon^i_k \;=\; \mathbb{E} \mid z^i_{t_k} - \bar{z}^i_{t_k} \mid^2$. The quadratic mean convergence rate of the Euler scheme for SDE's with coefficients which are Lipschitz functions in x and Holder of order $1/2$ in time is an easy generalization of Milshtein's result [9], see M. Bossy's thesis [1] for details.□

3.5. The dependency error. In this subsection, we study the error due to the substitution of the dependent Y^i's to the independent $\bar{z}^i$'s.

Lemma 3.4. *There exist positive constants l_4 and l_4', depending on σ, b and T, such that, for all $k = 1, .., K$,*

$$(i) \quad \mathbb{E}\left\| \frac{1}{N}\sum_{i=1}^{N} H(x-\bar{z}^i_{t_k}) - \frac{1}{N}\sum_{i=1}^{N} H(x-Y^i_{t_k})\right\|_{L^1(\mathbb{R})} \leq l_4\left(\sqrt{\Delta t} + \frac{1}{\sqrt{N}} + \|V_0 - \bar{V}_0\|_{L^1(\mathbb{R})}\right)$$

and

$$(ii) \quad \mathbb{E}\left(\|\frac{1}{N}\sum_{i=1}^{N} H(x-\bar{z}^i_{t_k}) - \frac{1}{N}\sum_{i=1}^{N} H(x-Y^i_{t_k})\|_{L^1(\mathbb{R})}\right)^2 \leq l_4'\left(\Delta t + \frac{1}{N} + \|V_0 - \bar{V}_0\|^2_{L^1(\mathbb{R})}\right).$$

Proof. From (3.9) it follows that

$$\|\frac{1}{N}\sum_{i=1}^{N} H(x-\bar{z}^i_{t_k}) - H(x-Y^i_{t_k})\|_{L^1(\mathbb{R})} \leq \frac{1}{N}\sum_{i=1}^{N} \mid \bar{z}^i_{t_k} - Y^i_{t_k} \mid .$$

Note that

$$\begin{aligned}
&\mathbb{E} \mid \bar{z}^i_{t_k} - Y^i_{t_k} \mid^2 \leq \\
&\qquad \mathbb{E} \mid \bar{z}^i_{t_{k-1}} - Y^i_{t_{k-1}} \mid^2 + \Delta t^2\, \mathbb{E} \mid \int b(\bar{z}^i_{t_{k-1}}, y) U_{t_{k-1}}(dy) - \frac{1}{N}\sum_{j=1}^{N} b(Y^i_{t_{k-1}}, Y^j_{t_{k-1}}) \mid^2 \\
&\qquad + \Delta t\, \mathbb{E} \mid \int s(\bar{z}^i_{t_{k-1}}, y) U_{t_{k-1}}(dy) - \frac{1}{N}\sum_{j=1}^{N} s(Y^i_{t_{k-1}}, Y^j_{t_{k-1}}) \mid^2 \\
&\qquad + 2\Delta t \mathbb{E}\left\{ \mid \bar{z}^i_{t_{k-1}} - Y^i_{t_{k-1}} \mid . \mid \int b(\bar{z}^i_{t_{k-1}}, y) U_{t_{k-1}}(dy) - \frac{1}{N}\sum_{j=1}^{N} b(Y^i_{t_{k-1}}, Y^j_{t_{k-1}}) \mid \right\}.
\end{aligned}$$

Set $E_k = \frac{1}{N}\sum_{i=1}^{N} \mathbb{E} \mid \bar{z}^i_{t_k} - Y^i_{t_k} \mid^2$; one has that

$$\begin{aligned}
E_k &\leq E_{k-1} + \Delta t^2 \frac{1}{N}\sum_{i=1}^{N} \mathbb{E} \mid \int b(\bar{z}^i_{t_{k-1}}, y)\mu_{t_{k-1}}(dy) - \frac{1}{N}\sum_{j=1}^{N} b(Y^i_{t_{k-1}}, Y^j_{t_{k-1}}) \mid^2 \\
&\quad + \Delta t \frac{1}{N}\sum_{i=1}^{N} \mathbb{E} \mid \int s(\bar{z}^i_{t_{k-1}}, y)\mu_{t_{k-1}}(dy) - \frac{1}{N}\sum_{j=1}^{N} s(Y^i_{t_{k-1}}, Y^j_{t_{k-1}}) \mid^2 \qquad (3.10) \\
&\quad + 2\,\Delta t \sqrt{E_{k-1}}\,.\sqrt{\frac{1}{N}\sum_{i=1}^{N} \mathbb{E} \mid \int b(\bar{z}^i_{t_{k-1}}, y)\mu_{t_{k-1}}(dy) - \frac{1}{N}\sum_{j=1}^{N} b(Y^i_{t_{k-1}}, Y^j_{t_{k-1}}) \mid^2} \\
:= &\ E_{k-1} + A_1 + A_2 + A_3.
\end{aligned}$$

A_1 is upper bounded by $C(\Delta t)^2$. Let us treat A_2:

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^{N} \mathbb{E}\mid \int s(\bar{z}^i_{t_{k-1}},y)\mu_{t_{k-1}}(dy) \;-\; \frac{1}{N}\;\sum_{j=1}^{N} s(Y^i_{t_{k-1}},Y^j_{t_{k-1}})\mid^2 \\ \leq \frac{2}{N}\sum_{i=1}^{N} \mathbb{E}\mid \int s(\bar{z}^i_{t_{k-1}},y)\mu_{t_{k-1}}(dy) \;-\; \frac{1}{N}\;\sum_{j=1}^{N} s(\bar{z}^i_{t_{k-1}},\bar{z}^j_{t_{k-1}})\mid^2 \\ + \frac{2}{N}\sum_{i=1}^{N} \mathbb{E}\mid \frac{1}{N}\;\sum_{j=1}^{N} s(\bar{z}^i_{t_{k-1}},\bar{z}^j_{t_{k-1}}) \;-\; s(Y^i_{t_{k-1}},Y^j_{t_{k-1}})\mid^2 .\end{aligned}$$

As s is Lipchitz, we observe that

$$\frac{2}{N}\sum_{i=1}^{N} \mathbb{E}\mid \frac{1}{N}\;\sum_{j=1}^{N} s(\bar{z}^i_{t_{k-1}},\bar{z}^j_{t_{k-1}}) \;-\; s(Y^i_{t_{k-1}},Y^j_{t_{k-1}})\mid^2 \leq CE_{k-1}.$$

Now set $\tilde{\mu}_{t_k} = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N}\delta_{\bar{z}^j_{t_k}}$ and $\tilde{V}_{t_k}(x) = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} H(x-\bar{z}^j_{t_k})$. Note that

$$\int_{\mathbb{R}} s(\bar{z}^i_{t_{k-1}},y)\mu_{t_{k-1}}(dy) \;-\; \frac{1}{N}\;\sum_{j=1}^{N} s(\bar{z}^i_{t_{k-1}},\bar{z}^j_{t_{k-1}}) \;=\; \int_{\mathbb{R}} s(\bar{z}^i_{t_{k-1}},y)\,[\mu_{t_{k-1}}(dy) \;-\; \tilde{\mu}_{t_{k-1}}(dy)],$$

so that, $s(\cdot,\cdot)$ being differentiable, one gets

$$\begin{aligned}\int_{\mathbb{R}} s(\bar{z}^i_{t_{k-1}},y)\,[\mu_{t_{k-1}}(dy) \;-\; \tilde{\mu}_{t_{k-1}}(dy)] \;&=\; \int_{\mathbb{R}} \frac{\partial s}{\partial y}(\bar{z}^i_{t_{k-1}},y)\,[V(t_{k-1},y) \;-\; \tilde{V}_{t_{k-1}}(y)]\,dy \\ &\leq\; L_s\,\|V(t_{k-1},x) \;-\; \tilde{V}_{t_{k-1}}(x)\|_{L^1(\mathbb{R})}.\end{aligned}$$

From (3.1) and the lemmas 3.1, 3.2 and 3.3, it follows that

$$\mathbb{E}\,(\,\|V(t_{k-1},x) \;-\; \tilde{V}_{t_{k-1}}(x)\|_{L^1(\mathbb{R})}\,)^2 \;\leq\; C(\|V_0-\overline{V}_0\|^2_{L^1(\mathbb{R})} \;+\; \frac{1}{N} \;+\; \Delta t),$$

from which

$$A_2 \leq C\,\Delta t\,(\|V_0-\overline{V}_0\|^2_{L^1(\mathbb{R})} \;+\; \frac{1}{N} \;+\; \Delta t) + C\Delta t E_{k-1}. \tag{3.11}$$

Now, consider A_3. We need a precise estimate on $\sqrt{A_1}$.

As $\mu_{t_{k-1}}$ is the law of $z_{t_{k-1}}$, one has $\int b(x,y)\mu_{t_{k-1}}(dy) \;=\; \mathbb{E}_{\mu_0} b(x,z_{t_{k-1}})$, and we set

$$\mathbb{E}_{\mu_0} b(x,z_{t_{k-1}})\Big|_{\bar{z}^i_{t_{k-1}}} \;:=\; \mathbb{E}_{\mu_0} b(x,z_{t_{k-1}})\Big|_{x=\bar{z}^i_{t_{k-1}}} \;:=\; \int b(\bar{z}^i_{t_{k-1}},y)\mu_{t_{k-1}}(dy).$$

Observe:

$$\begin{aligned}
\frac{1}{N}\sum_{i=1}^{N} \mathbb{E} \mid \int b(\bar{z}^i_{t_{k-1}}, y)\mu_{t_{k-1}}(dy) - \frac{1}{N}\sum_{j=1}^{N} b(Y^i_{t_{k-1}}, Y^j_{t_{k-1}}) \mid^2 \\
\leq \frac{2}{N}\sum_{i=1}^{N} \mathbb{E}\left| \mathbb{E}_{\mu_0} b(x, z_{t_{k-1}})\Big|_{\bar{z}^i_{t_{k-1}}} - \mathbb{E}_{\bar{\mu}_0} b(x, z_{t_{k-1}})\Big|_{\bar{z}^i_{t_{k-1}}} \right|^2 \\
+ \frac{2}{N}\sum_{i=1}^{N} \mathbb{E}\left| \mathbb{E}_{\bar{\mu}_0} b(x, z_{t_{k-1}})\Big|_{\bar{z}^i_{t_{k-1}}} - \frac{1}{N}\sum_{j=1}^{N} \mathbb{E} b(x, \bar{z}^j_{t_{k-1}})\Big|_{\bar{z}^i_{t_{k-1}}} \right|^2 \\
+ \frac{2}{N}\sum_{i=1}^{N} \mathbb{E}\left| \frac{1}{N}\sum_{j=1}^{N} \mathbb{E} b(x, \bar{z}^j_{t_{k-1}})\Big|_{\bar{z}^i_{t_{k-1}}} - \frac{1}{N}\sum_{j=1}^{N} b(\bar{z}^i_{t_{k-1}}, \bar{z}^j_{t_{k-1}}) \right|^2 \\
+ \frac{2}{N}\sum_{i=1}^{N} \mathbb{E}\left| \frac{1}{N}\sum_{j=1}^{N} b(\bar{z}^i_{t_{k-1}}, \bar{z}^j_{t_{k-1}}) - b(Y^i_{t_{k-1}}, Y^j_{t_{k-1}}) \right|^2 \\
:= \varepsilon^1_{k-1} + \varepsilon^2_{k-1} + \varepsilon^3_{k-1} + \varepsilon^4_{k-1}.
\end{aligned}$$

A previous computation shows that

$$\varepsilon^4_{k-1} \leq L_b^2 \, E_{k-1}. \tag{3.12}$$

On the other hand, it holds that

$$\begin{aligned}
\varepsilon^3_{k-1} &= \frac{2}{N}\sum_{i=1}^{N} \frac{1}{N^2}\sum_{j=1}^{N} \mathbb{E}\left(\mathbb{E} b(x, \bar{z}^j_{t_{k-1}})\Big|_{\bar{z}^i_{t_{k-1}}} - b(\bar{z}^i_{t_{k-1}}, \bar{z}^j_{t_{k-1}}) \right)^2 \\
&+ \frac{2}{N}\sum_{i=1}^{N} \frac{1}{N^2} \sum_{j,l=1; j\neq l}^{N} \mathbb{E}\left\{ \left[\mathbb{E} b(x, \bar{z}^j_{t_{k-1}})\Big|_{\bar{z}^i_{t_{k-1}}} - b(\bar{z}^i_{t_{k-1}}, \bar{z}^j_{t_{k-1}}) \right] \right. \\
&\qquad \left. \cdot \left[\mathbb{E} b(x, \bar{z}^l_{t_{k-1}})\Big|_{z^i_{t_{k-1}}} - b(\bar{z}^i_{t_{k-1}}, \bar{z}^l_{t_{k-1}}) \right] \right\}.
\end{aligned}$$

Since the $\bar{z}^i$'s are independent and b is Lipschitz, one deduces that

$$\varepsilon^3_{k-1} \leq \frac{8\,B^2}{N}. \tag{3.13}$$

Now, observe that

$$\varepsilon^2_{k-1} = \frac{2}{N}\sum_{i=1}^{N} \mathbb{E}\left| \frac{1}{N}\sum_{j=1}^{N} \mathbb{E} b(x, z_{t_{k-1}}(y_0^j))\Big|_{\bar{z}^i_{t_{k-1}}} - \frac{1}{N}\sum_{j=1}^{N} \mathbb{E} b(x, \bar{z}^j_{t_{k-1}})\Big|_{\bar{z}^i_{t_{k-1}}} \right|^2$$

$$= \frac{2}{N}\sum_{i=1}^{N} \mathbb{E}\left| \frac{1}{N}\sum_{j=1}^{N} \mathbb{E}(\, b(x, z^j_{t_{k-1}}) \; - \; b(x, \bar{z}^j_{t_{k-1}})\,)\Big|_{\bar{z}^i_{t_{k-1}}} \right|^2,$$

from which it comes that

$$\varepsilon^2_{k-1} \le 2L_b^2 \left\{ \frac{1}{N}\sum_{j=1}^{N} \mathbb{E}\mid z^j_{t_{k-1}} \; - \; \bar{z}^j_{t_{k-1}} \mid \right\}^2.$$

Applying the lemma 3.3, we conclude that

$$\varepsilon^2_{k-1} \le 2\, L_b^2\, l_3^2\, \Delta t. \tag{3.14}$$

It remains to treat ε^1_{k-1}. Remark that

$$\mathbb{E}_{\mu_0} b(x, z_{t_{k-1}})\Big|_{\bar{z}^i_{t_{k-1}}} \; - \; \mathbb{E}_{\bar{\mu}_0} b(x, z_{t_{k-1}})\Big|_{\bar{z}^i_{t_{k-1}}} \; = \; \left[\, \mathbb{E}_{\mu_0} b(x, z_{t_{k-1}}) \; - \; \mathbb{E}_{\bar{\mu}_0} b(x, z_{t_{k-1}}) \,\right] \Big|_{\bar{z}^i_{t_{k-1}}}$$

and that, for all x,

$$\mathbb{E}_{\mu_0} b(x, z_{t_{k-1}}) \; - \; \mathbb{E}_{\bar{\mu}_0} b(x, z_{t_{k-1}}) \; = \; \int_{\mathbb{R}} \mathbb{E} b(x, z_{t_{k-1}}(y))\,.[\, \mu_0(dy) \; - \; \bar{\mu}_0(dy)\,]\,.$$

Now integrate by parts and apply (3.3); it follows that, for all x, one has

$$\mathbb{E}_{\mu_0} b(x, z_{t_{k-1}}) \; - \; \mathbb{E}_{\overline{\mu}_0} b(x, z_{t_{k-1}}) \mid \le \frac{BC_1\sqrt{2\pi}}{\sqrt{t_{k-1}}} \bar{\sigma} \|V_0 \; - \; \bar{V}_0\|_{L^1(\mathbb{R})},$$

so that, for $k > 1$,

$$\varepsilon^1_{k-1} \le \frac{C}{t_{k-1}} \bar{\sigma}^2 \|V_0 \; - \; \bar{V}_0\|^2_{L^1(\mathbb{R})}. \tag{3.15}$$

Combining (3.12), (3.13), (3.14), (3.15), one gets

$$A_3 \; \le \; C\,\Delta t\,\sqrt{E_{k-1}}.\sqrt{\frac{\|V_0 - \bar{V}_0\|^2_{L^1(\mathbb{R})}}{t_{k-1}} \; + \; \Delta t \; + \; \frac{1}{N}} \; + \; C\,\Delta t\, E_{k-1}$$

Set

$$\delta := \|V_0 - \bar{V}_0\|^2_{L^1(\mathbb{R})} \; + \; \frac{1}{N} \; + \; \Delta t\;.$$

In view of this upper bound and (3.11), the inequality (3.10) becomes:

$$\begin{cases} E_k \le (1 + C\Delta t) E_{k-1} \; + \; C\,\Delta t\,(\delta \; + \; \Delta t) \; + \; C\,\Delta t \dfrac{\sqrt{E_{k-1}}}{\sqrt{t_{k-1}}}\sqrt{\delta}\;, \quad \text{for } k > 1, \\ E_1 \le C\Delta t. \end{cases}$$

Consider the sequence (γ_k) defined by

$$\begin{cases} \gamma_k \ := \ (1+C\Delta t)\gamma_{k-1} \ + \ C\,\Delta t\,(\delta \ + \ \Delta t) \ + \ C\,\Delta t \dfrac{\sqrt{E_{k-1}}}{\sqrt{t_{k-1}}}\sqrt{\delta}\ , \quad \text{for } k > 1, \\ \gamma_1 \ := \ C\Delta t. \end{cases}$$

Then, for all $k = 1, .., K$, $E_k \leq \gamma_k$. Suppose there exists an integer $q < K$ such that

$$\gamma_q \ \leq \ \delta \quad \text{ and } \quad \gamma_{q+1} \ \geq \ \delta.$$

As (γ_k) is increasing, it would hold that

$$\forall\, r \ \leq \ q\,, \qquad \gamma_r \ \leq \ \|V_0 \ - \ \bar{V}_0\|^2_{L^1(\mathbb{R})} \ + \ \frac{1}{N} \ + \ \Delta t,$$

$$\forall\, r \ \geq \ q+1\,, \qquad \gamma_r \ \geq \ \|V_0 \ - \ \bar{V}_0\|^2_{L^1(\mathbb{R})} \ + \ \frac{1}{N} \ + \ \Delta t.$$

Thus, one would have

$$\begin{cases} \gamma_k \ \leq \ \left(1 \ + \ C\,\Delta t \ + \ C\,\dfrac{\Delta t}{\sqrt{t_{k-1}}}\right)\gamma_{k-1} \ + \ C\,\Delta t\,(\delta \ + \ \Delta t), \quad k = q+1, .., K, \\ \gamma_q \ \leq \ \delta. \end{cases}$$

Noting that $\displaystyle\sum_{j=q}^{K-1} \frac{1}{\sqrt{j}} \ \leq \ \int_q^K \frac{1}{\sqrt{x}}\,dx \ = \ 2\,(\sqrt{K} \ - \ \sqrt{q})$, an iteration gives

$$\gamma_K \leq C\delta.$$

That ends the proof. $\square$

3.6. Proof of the convergence theorems. Having estimations for each terms of the decomposition of the error $V(t_k,.) - \overline{V}_{t_k}(\cdot)$, at each time t_k of the discretization, from lemmas 3.1, 3.2, 3.3 and 3.4 we get that $\forall\ k = 1, .., K$

$$\mathbb{E}\,\|\,V(t_k,x) \ - \ \overline{V}_{t_k}(x)\|_{L^1(\mathbb{R})} \ \leq \ L_1\,\|\,V_0 \ - \ \overline{V}_0\,\|_{L^1(\mathbb{R})} \ + \ L_2\,\frac{1}{\sqrt{N}} \ + \ L_3\,\sqrt{\Delta t}$$

and

$$Var(\|\,V(t_k,x) \ - \ \overline{V}_{t_k}(x)\|_{L^1(\mathbb{R})}\,) \ \leq \ L'_1\,\|\,V_0 \ - \ \overline{V}_0\,\|^2_{L^1(\mathbb{R})} \ + \ L'_2\,\frac{1}{N} \ + \ L'_3\,\Delta t$$

To complete the proof of the theorem 2.1, it remains to estimate the approximation error of V_0 by $\overline{V}_0$:

$$\begin{aligned}
\|V_0 - \overline{V}_0\|_{L^1(\mathbb{R})} &= \int_{\mathbb{R}} \mid V_0(x) - \overline{V}_0(x) \mid dx \\
&= \int_{-\infty}^{y_0^1} V_0(x)\, dx \,+\, \sum_{i=1}^{N-1} \int_{y_0^i}^{y_0^{i+1}} (\, V_0(x) - \frac{i}{N}\,)\, dx \,+\, \int_{y_0^N}^{+\infty} (1 - V_0(x))\, dx \\
&:= A \,+\, B \,+\, C.
\end{aligned}$$

As V_0 is an increasing function and by definition of the y_0^i, it holds that $B \leq \dfrac{1}{N}(y_0^N - y_0^1)$.

We will only treat A since C can be treated by symmetry.
Let M be as in (H4) and define the function ψ on $\mathbb{R}$ by :

$$\psi(x) \,=\, \eta\sqrt{\frac{\pi}{2\alpha}}\, exp(-\alpha\frac{x^2}{2}).$$

From (3.8), we have

$$V_0(x) \,\leq\, \psi(x)\,,\ \text{for } x \leq -M.$$

Suppose that $y_0^1 \leq -M$. Then

$$\begin{aligned}
A &= \int_{-\infty}^{-|\psi^{-1}(\frac{1}{N})|} V_0(x)\, dx \,+\, \int_{-|\psi^{-1}(\frac{1}{N})|}^{y_0^1} V_0(x)\, dx \\
&\leq \int_{-\infty}^{-|\psi^{-1}(\frac{1}{N})|} \psi(x)\, dx \,+\, \frac{1}{N}\,(\, y_0^1 + \mid \psi^{-1}(\frac{1}{N}\,) \mid).
\end{aligned}$$

From (3.6), one deduces that :

$$\int_{-\infty}^{-|\psi^{-1}(\frac{1}{N})|} \psi(x)\, dx \,\leq\, \sqrt{\frac{\pi}{2\alpha}}\,\frac{1}{N}, \tag{3.16}$$

so that

$$A \,\leq\, \frac{1}{N}\left(\sqrt{\frac{\pi}{2\alpha}} \,+\, y_0^1 \,+\, \sqrt{\mid \frac{2}{\alpha}\, log(\sqrt{\frac{2\alpha}{\pi}}\frac{1}{\eta}\frac{1}{N}) \mid} \right).$$

Now, if $y_0^1 \geq -M$

$$A \,=\, \int_{-\infty}^{-M} V_0(x)\, dx \,+\, \int_{-M}^{y_0^1} V_0(x)\, dx \,\leq\, \int_{-\infty}^{-M} \psi(x)\, dx \,+\, \frac{1}{N}\,(\, y_0^1 \,+\, M)\ .$$

As $-M \leq y_0^1 \leq - \mid \psi^{-1}(\frac{1}{N}) \mid$, it holds from (3.16) that, for any choice of y_0^1

$$A \leq \frac{1}{N}\left(\sqrt{\frac{\pi}{2\alpha}} + y_0^1 + (M \vee \sqrt{\mid \frac{2}{\alpha}\, log(\sqrt{\frac{2\alpha}{\pi}}\frac{1}{\eta}\frac{1}{N}) \mid)}\right)$$

and finally

$$\|V_0 - \overline{V}_0\|_{L^1(\boldsymbol{R})} \leq \frac{1}{N}\left(\sqrt{\frac{\pi}{2\alpha}} + 2\,(M \vee \sqrt{\mid -\frac{2}{\alpha}\, log(\sqrt{\frac{2\alpha}{\pi}}\frac{1}{\eta}\frac{1}{2N}) \mid)}\right).$$

Remark that if u_0 has a compact support (included in $[-M, M]$), then

$$\|V_0 - \overline{V}_0\|_{L^1(\boldsymbol{R})} \leq \frac{1}{N}\, 2M. \ \square$$

The proof of Theorem 2.2 is based on the following decomposition of the error :

$$\|\mu_{t_k} - \overline{\mu}^{\varepsilon}_{t_k}\|_{L^1(\boldsymbol{R})} \leq \|\mu_{t_k} - (\mu_{t_k} * \Phi_\varepsilon)\|_{L^1(\boldsymbol{R})} + \|\left(\mu_{t_k} - \overline{\mu}_{t_k}\right) * \Phi_\varepsilon\|_{L^1(\boldsymbol{R})}. \tag{3.17}$$

The first term of the right hand side corresponds to the rate of convergence of the regularization with a gaussian kernel. If the density μ_t is in the sobolev space $W^{2,1}(\mathbb{R})$ uniformly in time, the rate of convergence is given by the well-known estimate (cf. Raviart [11]) :

$$\|\mu_{t_k} - (\mu_{t_k} * \Phi_\varepsilon)\|_{L^1(\boldsymbol{R})} \leq C\,\varepsilon^2\,\|\mu_{t_k}\|_{2,1}. \tag{3.18}$$

Using the integration by part formula for a Stieljes integral, the second term of (3.17) becomes

$$\|\left(\mu_{t_k} - \overline{\mu}_{t_k}\right) * \Phi_\varepsilon\|_{L^1(\boldsymbol{R})} = \int_{\boldsymbol{R}} \left| \int_{\boldsymbol{R}} \Phi'_\varepsilon(x-y)\left(V(t_k,y) - \overline{V}_{t_k}(y)\right) dy \right| dx,$$

so that

$$\mathbb{E}\,\|\left(\mu_{t_k} - \overline{\mu}_{t_k}\right) * \Phi_\varepsilon\|_{L^1(\boldsymbol{R})} \leq \frac{2}{\sqrt{2\pi\varepsilon^2}}\,\mathbb{E}\,\|V(t_k,.) - \overline{V}_{t_k}(.)\|_{L^1(\boldsymbol{R})}. \tag{3.19}$$

The estimates of the Theorem 2.2 are obtained by combining (3.18) and (3.19).

It remains to show that the assumptions $(H3)$ and $(H5)$ ensure that the density μ_t is in $W^{2,1}(\mathbb{R})$ uniformly in $[0, T]$.

For a given function π in $C^2(I\!R)$, we define $u(t,x)$, for $(t,x)\in[0,T]\times I\!R$ by

$$u(t,x) \;:=\; exp(\pi(x))\;\mu_t(x)\;.$$

Then, u is solution of the linear parabolic equation :

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) \;=\; L(t)u(t,x),\;\; (t,x)\in[0,T]\times I\!R, \\ \\ u(0,x) \;=\; exp(\pi(x))\;\mu_0(x), \end{cases} \tag{3.20}$$

where

$$L(t) \;=\; \tilde{a}(t,x)\frac{\partial^2}{\partial x^2} \;+\; \tilde{b}(t,x)\frac{\partial}{\partial x} \;-\; \tilde{c}(t,x),$$

and

$$\tilde{a} \;:=\; \frac{1}{2}\sigma^2, \quad\; \tilde{b} \;:=\; \frac{\partial(\sigma^2)}{\partial x} \;-\; \beta \;-\; \pi',$$
$$\tilde{c} \;:=\; \frac{\partial\beta}{\partial x} \;-\; \frac{1}{2}\frac{\partial^2(\sigma^2)}{\partial x^2} \;+\; \pi'(\frac{\partial(\sigma^2)}{\partial x} \;-\; \beta) \;+\; \pi''\frac{1}{2}\sigma^2(\pi'' \;-\; (\pi')^2).$$

Choose $\pi(x)=ln(1+x^2)$, so that π' and π'' are bounded functions. Under the assumption $(H3)$, $\partial^p_{x^p}\sigma$ and $\partial^q_{x^q}\beta$ are bounded functions in $I\!R$, uniformly on $[0,T]$, for $p=0,1,2$ and $q=0,1$. Moreover, writing on the form $I\!Ek(x,X_t)$, it is easy to see that this functions are Hőlder in $[0,T]$, uniformly on $I\!R$, with exponent $\frac{1}{2}$. Thus, the coefficients of $L(t)$ satisfy the hypothesis of the following result (cf Cannarsa and Vespri, [3]):

if u_0 is in $H^2(I\!R)$ then problem (3.20) has
a unique solution $u\;\in\;C^1([0,t];L^2(I\!R))\cap C([0,T];H^2(I\!R))$.

The assumption $(H5)$ ensures that u_0 is in $H^2(I\!R)$. Then $exp(\pi)\;\mu_t$ is in $H^2(I\!R)$, uniformly on $[0,T]$ and we can easily deduce that μ_t is in $W^{2,1}(I\!R)$ uniformly on $[0,T]$. $\Box$

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