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On Semigroups of Matrices in the (max,+) Algebra

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Abstract: We show that the answer to the Burnside problem is positive for semigroups of matrices with entries in the $(\max,+)$-algebra (that is, the semiring $(\mathbb{R}\cup\{-\infty\},\max,+)$), and also for semigroups of $(\max,+)$-linear projective maps with rational entries. An application to the estimation of the Lyapunov exponent of certain products of random matrices is also discussed.

Key-words: Semigroups, Burnside Problem, $(\max,+)$ algebra, Lyapunov Exponents, Projective Space.

(Résumé : tsvp)

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Sur les semigroupes de matrices dans l'algèbre (max,+)

Résumé : Nous montrons que la réponse au problème de Burnside est positive pour les semigroupes de matrices à coefficients dans "l'algèbre $(\max, +)$" (c'est-à-dire le semianneau $(\mathbb{R} \cup \{-\infty\}, \max, +)$) ainsi que pour les semigroupes d'applications linéaires projectives à coefficients rationnels dans la même algèbre. On donne une application à l'estimation de l'exposant de Lyapunov de certains produits $(\max, +)$ de matrices aléatoires.

Mots-clé : Semigroupes, Problème de Burnside, algèbre $(\max, +)$, Exposants de Lyapunov, Espace projectif.

1 Introduction

In this paper, we give some results for semigroups of matrices with entries in the "$(\max, +)$-algebra", that is, the semiring $(\mathbb{R} \cup \{-\infty\}, \max, +)$, denoted $\mathbb{R}_{\max}$ in the sequel. This is a particular example of *idempotent semiring* (that is a semiring such that $a \oplus a = a$), also known as *dioid* [18, 19, 2]. This algebraic structure has been popularized by its applications to Graph Theory and Operations Research [18, 9]. Linear operators in this algebra are central in Hamilton-Jacobi theory and in the study of certain asymptotics (see [29] for a recent overview). The study of automata and semigroups of matrices with entries in the analogous "tropical" semiring $(\mathbb{N} \cup \{+\infty\}, \min, +)$ has been motivated by some decidability problems in language theory [28, 34, 26, 35, 36, 20, 21, 23, 24, 25]. From our point of view, the interest of the $(\max, +)$ algebra arose from the study of Discrete Event Dynamic Systems [2, 13], where sequences driven by $\mathbb{R}_{\max}$-linear equations represent synchronization and saturation phenomena. An account of the related $(\max, +)$-linear system theory can be found in [2, 8, 32]. Automata over the $(\max, +)$ algebra are a natural extension of this formalism and have noticeable applications to Discrete Event Systems [15, 14]. In particular, certain finiteness results for semigroups of matrices can be used to compute some asymptotic performance measures (mean-case and optimal-case Lyapunov exponents [15, 14], the latest being essentially equivalent to the classical nondeterministic complexity [37]). We also mention that the theory of $(\max, +)$ rational series in a single variable is well developed [13, 16, 25]. One basic theorem characterizes rational series by an ultimate periodicity property. This is intimately related with the cyclicity theorem for powers of irreducible matrices [2, 11], which states that such a matrix A satisfies $A^{n+c} = \lambda^c A^n$ for some $n \geq 0, c \geq 1$ and where λ is the "Perron root" of A. Some finiteness results presented here can be seen as an attempt to generalize this simple property to semigroups of matrices.

We first show that the answer to the Burnside problem for semigroups of matrices over $\mathbb{R}_{\max}$ is positive, which extends a theorem of Simon [34] for the tropical semiring. The main originality by comparison with Simon's proof consists in using the $(\max, +)$-spectral theory. A different proof based on an adaptation of a combinatorial argument of Straubing can be provided in another special class of dioids. Later, we consider semigroups of $(\max, +)$-linear projective maps. In a previous paper [15], we showed that under certain coarse irreducibility assumptions, finitely generated semigroups of linear projective maps with rational entries are finite. Here, we extend this result, showing that the answer to the Burnside problem is also positive for semigroups of linear projective maps with rational entries. The rationality assumption is important: we provide a counter example which is based on a kind of system of addition of irrational vectors. The decidability of the limitedness problem for rational series over $\mathbb{R}_{\max}$ is also obtained as an easy consequence of the decidability result of Hashiguchi [21, 20] for the same problem over the tropical semiring. We conclude by giving an application where these finiteness results allow a simple computation of the Lyapunov exponeint of some particular $(\max, +)$ automata.

2 Statement of the Results

Recall that a semigroup S is *torsion* if for all $s \in S$, there exists $n \in \mathbb{N}$ and $c \in \mathbb{N} \setminus \{0\}$ such that $s^{n+c} = s^n$. Consider the two following properties

1. S is finite
2. S is finitely generated and torsion.

Obviously, (1) $\Rightarrow$ (2). The well known "Burnside problem" consists in finding some classes of semigroups satisfying the converse implication. See [10] for a survey. The answer is positive for matrices with entries in commutative rings [30, 22, 38]. It is also positive in some more exotic semirings, such as the *tropical semiring*

$(\mathbb{N} \cup \{+\infty\}, \min, +)$ (Simon [34]), the "dual" –but non isomorphic– semiring $(\mathbb{N} \cup \{\pm\infty\}, \max, +)$ and the semiring of rational languages in a single letter $(Rat(a^*), \cup, .)$ (Mascle [28]). We show that this property also holds in the case of $\mathbb{R}_{\max}$:

2.0.1 Theorem *A finitely generated torsion semigroup $S \subset \mathbb{R}_{\max}^{n \times n}$ is finite.*

Moreover, this result admits an effective translation:

2.0.2 Theorem *It is decidable if a finitely generated semigroup $S \subset \mathbb{R}_{\max}^{n \times n}$ is torsion.*

Indeed, the proof provides an algorithm which after some reduction coincides with Simon's algorithm for the tropical semiring (whose complexity is essentially 3^{n^2}).

The proof of Theorem 2.0.1 uses the spectral theory of $\mathbb{R}_{\max}$ together with the linearity of the order. In spite of certain generalizations of the spectral theory to others dioids[1] [11, 16], the argument does not seem to extend easily. However, we mention another class of dioids for which the answer to the Burnside problem is also positive. We shall use the *natural order* of dioids, which can be defined by

$$a \leq b \iff a \oplus b = b \ .$$

2.0.3 Theorem *Let $\mathcal{D}$ be a commutative dioid such that*

$$\forall x, \quad \{y \in \mathcal{D} \mid \ y \leq x\} \text{ is finite } . \tag{1}$$

Then, a finitely generated torsion semigroup $S \subset \mathcal{D}^{n \times n}$ is finite.

The proof is an adaptation to the dioid's case of a combinatorial argument of Straubing [38]. This yields another proof of Mascle's finiteness result for matrices over the semiring $(\mathbb{N} \cup \{\pm\infty\}, \max, +)$ (but not for the tropical semiring, for which the algebraic order $\leq$ is the opposite of the standard one). Let us give another example of nontrivial dioid satisfying (1). Let $(M, +)$ be a commutative monoid. Then, the dioid $\mathcal{P}_f(M)$ of *finite* subsets of M, equipped with $\cup$ (as addition) and sum of subsets (as product) satisfies the condition of Theorem 2.0.3.

We next consider linear projective semigroups. We define the matrix projective space as the quotient of $\mathbb{R}_{\max}^{n \times n}$ by the parallelism relation

$$M \simeq M' \Leftrightarrow \exists \lambda \in \mathbb{R}_{\max} \setminus \{\varepsilon\}, \quad M = \lambda M'$$

(we use the notation ε for the zero element of semirings, in particular, in $\mathbb{R}_{\max}$, $\varepsilon = -\infty$). We write $\mathbb{P}\mathbb{R}_{\max}^{n \times n}$ for the quotient semigroup (of "linear projective maps"), and $\mathfrak{p}$ denotes the canonical morphism of multiplicative semigroups

$$\mathfrak{p} : \mathbb{R}_{\max}^{n \times n} \rightarrow \mathbb{P}\mathbb{R}_{\max}^{n \times n} \ .$$

Let us introduce the subsemiring of $\mathbb{R}_{\max}$: $\mathbb{Q}_{\max} \stackrel{def}{=} (\mathbb{Q} \cup \{-\infty\}, \max, +)$. We set $\mathbb{P}\mathbb{Q}_{\max}^{n \times n} \stackrel{def}{=} \mathfrak{p}\mathbb{Q}_{\max}^{n \times n}$ (this is the semigroup of linear projective maps with rational entries).

2.0.4 Theorem *A finitely generated torsion semigroup $S \subset \mathbb{P}\mathbb{Q}_{\max}^{n \times n}$ is finite.*

Finally, we extend to $\mathbb{R}_{\max}$ a theorem of Hashiguchi [21] for rational series over the semiring $(\mathbb{N} \cup \{+\infty\}, \min, +)$. See also [20], Simon [36] and Krob [24, 23] for a first extension to $(\mathbb{Z} \cup \{+\infty\}, \min, +)$.

[1] Recall that a dioid is a semiring whose addition is idempotent: $a \oplus a = a$.

2.0.5 Theorem *It is decidable if a rational series s with coefficients in $\mathbb{R}_{\max}$ is* limited*, that is, if the set of the values of the coefficients of s,*

$$S = \{(s|w) \mid \ w \in \Sigma^*\}$$

is finite.

3 Preliminary Results

3.1 Some Results from the $(\max, +)$ Matrix Theory

We next recall the definition and basic properties of the $(\max, +)$ spectral radius [17, 7, 2, 9]. First, the "norm" of a matrix is defined by

$$\|A\| = \bigoplus_{ij} A_{ij} = \sup_{ij} A_{ij} \quad .$$

3.1.1 Lemma *Let $A \in \mathbb{R}_{\max}^{n\times n}$. The following quantities are equal:*

1. $\sup\{r \in \mathbb{R}_{\max} \mid \ \exists u \in \mathbb{R}_{\max}^n \setminus \{\varepsilon\}, \ \ Au \geq ru\}$
2. $\sup\{r \in \mathbb{R}_{\max} \mid \ \exists u \in \mathbb{R}_{\max}^n \setminus \{\varepsilon\}, \ \ Au = ru\}$
3. $\bigoplus_{1\leq k\leq n} (trA^k)^{\frac{1}{k}} = \bigoplus_{1\leq k\leq n} \bigoplus_{i_1\ldots i_k} (A_{i_1 i_2} \ldots A_{i_k i_1})^{\frac{1}{k}}$
4. $\limsup_k \|A^k\|^{\frac{1}{k}}$.

This common value will be denoted by $\rho(A)$.

Of course, $a^{\frac{1}{k}}$ with the semiring notation of the $(\max, +)$-algebra means $\frac{a}{k}$ in the usual algebra. In the following, it should be clear from the context whichever algebra is used. However, we shall sometimes write $a^{\otimes \frac{1}{k}}$ to avoid ambiguities.

We begin with a Lemma which is almost obvious.

3.1.2 Lemma *For all $A \in \mathbb{R}_{\max}^{n\times n}$ and $k \geq 1$:*

$$\rho(A^k) = (\rho(A))^k \quad . \tag{2}$$

Proof $\rho(A^k) \leq (\rho(A))^k$ follows immediately from Lemma 3.1.1,3. The converse inequality follows from 3.1.1,2. For if $Au = ru$, then $A^k u = r^k u$, hence $\rho(A^k) \geq (\rho(A))^k$. ∎

The most useful result of the $(\max, +)$-matrix theory is perhaps the following *cyclicity* theorem which is the exact $(\max, +)$ counterpart of a well known asymptotic property for usual nonnegative matrices. Let us recall that the matrix M is *irreducible* if $\forall i, j, \exists k \geq 1, M_{ij}^k \neq \varepsilon$.

3.1.3 Theorem ([7, 2, 11]) *If $M \in \mathbb{R}_{\max}^{p\times p}$ is irreducible, then*

$$\exists N, c, \ \forall n \geq N, \quad M^{c+n} = (\rho(M))^c M^n \quad . \tag{3}$$

where $\rho(M)$ denotes the spectral radius of M.

It is very natural to look for generalizations of this cyclicity property to finitely generated semigroups of matrices. To this end, we observe that (3) rewrites as follows in the projective linear semigroup $\mathbb{P}\mathbb{R}_{\max}^{p\times p}$:

$$(\mathfrak{p}M)^n = (\mathfrak{p}M)^{n+c} \ .$$

That is, an irreducible linear projective map is torsion. This suggests to consider finitely generated projective linear semigroups. We just recall here a first result taken from [15, 14], that we shall use and generalize hereafter. Let $\Sigma = \{a_1, \dots, a_p\}$ be an alphabet of p letters, $\mu : \Sigma^+ \to \mathbb{R}_{\max}^{n\times n}$ a morphism (Σ^+ denotes the free semigroup on Σ, that is, the set of nonempty words equipped with the concatenation product). We consider the finitely generated semigroup $S = \mu(\Sigma^+)$. Alternatively, if $\mu(a_i) = A_i$, we shall write $S = \langle A_1, \dots, A_p\rangle$ for the semigroup generated by the matrices $A_1, \dots, A_p$. Obviously, each finitely generated semigroup S can be written $\mu(\Sigma^+)$ for some alphabet Σ and morphism μ. We say that the semigroup S is *primitive*[2] if there is an integer N such that for all word w,

$$|w| \geq N \Rightarrow \ \ \forall i,j \ \ \mu(w)_{ij} > \varepsilon \ \ , \tag{4}$$

where $|w|$ denotes the length of the word w. That is, we require every sufficiently long product of matrices to be without ε entries. When S admits a unique generator, this reduces to the notion of primitivity well known in the theory of nonnegative matrices. We say that a set S of matrices is projectively finite if $\mathfrak{p}S$ is finite.

3.1.4 Theorem *Let $A_1, \dots, A_p \in \mathbb{Q}_{\max}^{n\times n}$. If $\langle A_1, \dots, A_p\rangle$ is a primitive semigroup, then it is projectively finite.*

Contrarily to the cyclicity theorem 3.1.3 which is essentially relative to the case $p = 1$, we require the entries to be rational (or equal to ε). This restriction is important, as it will be shown in §7.1.

For the sake of completeness, we include the proof, which exploits some bounding arguments and some norm properties which will be more intensively used hereafter in the study of the Burnside problem for projective linear semigroups.

Proof Let q be the lcm of the denominators of the entries of the matrices. Since $x \mapsto x^q$ ($x^q = x \times q$ with classical notations) is an automorphism of $\mathbb{Q}_{\max}$ which maps all the entries to integers, we shall assume that $a_1, \dots, a_p \in \mathbb{Z}_{\max}^{n\times n}$. We have already defined the norm $\|A\| = \sup_{ij} A_{ij}$. We shall also need the following dual bound:

$$|m|_\wedge = \inf_{m_{ij}\neq\varepsilon} m_{ij} \tag{5}$$

(recall that $\inf\emptyset = +\infty$). Obviously,

$$\forall A, B \in \mathbb{R}_{\max}^{n\times n}, \quad \begin{cases} \|A\oplus B\| = \|A\| \oplus \|B\|, \\ |A\oplus B|_\wedge \geq \ |A|_\wedge \wedge \ |B|_\wedge \end{cases} \tag{6}$$

$$\forall A, B \in \mathbb{R}_{\max}^{n\times n}, \quad \begin{cases} \|AB\| \leq \|A\|\|B\|, \\ |AB|_\wedge \geq \ |A|_\wedge \, |B|_\wedge \ \ . \end{cases} \tag{7}$$

The proof relies on the following Lemma.

3.1.5 Lemma *Let $K \in \mathbb{N}$. The set S of matrices $m \in \mathbb{Z}_{\max}^{n\times n}$ such that*

$$\frac{\|m\|}{|m|_\wedge} \leq K$$

is projectively finite.

[2] We leave it to the reader to check that this notion is independent of the choice of Σ and μ.

Indeed, after normalization, we may assume that $\forall m \in S,\ \ |m|_\wedge = e$. Since there is at most $(K+2)^{n^2} - 1$ matrices $m \in \mathbb{Z}_{\max}^{n\times n}$ such that $e = |m|_\wedge$ and $\|m\| \leq K$, the Lemma is proven.

Let

$$\begin{aligned} \underline{a} &= \min(\ |a_1|_\wedge, , \ldots, |a_p|_\wedge\ ), \\ \overline{a} &= \max(\ \|a_1\|, \ldots, \|a_p\|\ )\ \ . \end{aligned}$$

The primitivity assumption implies that for $w \in \Sigma^*$ long enough, we have a factorization $w = sur$ with $|s|, |r| \leq N$ and $\mu(s), \mu(r), \mu(u) \succ \varepsilon$ (N is the "primitivity index" satisfying (4)). Then

$$\begin{aligned} \|\mu(w)\| &= \|\mu(sur)\| \leq \|\mu(s)\|\|\mu(u)\|\|\mu(r)\| \\ &\leq (e \oplus \overline{a})^{2N} \mu(u)_{ij} \end{aligned} \tag{8}$$

for some indices ij belonging to the argmax in $\|\mu(u)\| = \sup_{ij} \mu(u)_{ij}$. Moreover

$$\mu(sur)_{kl} \geq \mu(s)_{ki}\mu(u)_{ij}\mu(r)_{jl} \geq (\underline{a} \wedge e)^{2N} \mu(u)_{ij}\ \ .$$

This implies that

$$\frac{\|\mu(w)\|}{|\mu(w)|_\wedge} \leq \left(\frac{e \oplus \overline{a}}{e \wedge \underline{a}}\right)^{2N}\ \ .$$

It remains to apply Lemma 3.1.5 to conclude. ■

3.2 Preparation

We first recall or prove some lemmas of general interest. The first one is a well known combinatorial result due to Brown[5]. We say that a semigroup S is *locally finite* if any finitely generated subsemigroup of S is finite.

3.2.1 Lemma (Brown) *Let $\phi : S \to T$ be a morphism from a semigroup S to a locally finite semigroup T. Then S is locally finite iff for all idempotent $E \in T$, $\phi^{-1}(E)$ is locally finite.*

We now give some lemma specific to the dioid or $(\max, +)$ case. We first define a notion of reducibility of semigroups of matrices over dioids.

3.2.2 Definition *Let $\mathcal{D}$ be a dioid and S a subsemigroup of $\mathcal{D}^{n\times n}$. We say that S is* reducible *if there exists a proper partition $\{1, \ldots, n\} = I \cup J$ such that*

$$\forall s \in S,\ \forall i \in I,\ \forall j \in J, \quad s_{ij} = \varepsilon\ \ .$$

Let μ be a morphism $\Sigma^+ \to \mathcal{D}^{n\times n}$, and $S = \mu(\Sigma^+)$. It is easily checked that S is reducible iff there exists a constant permutation matrix P, two morphisms $\mu_1 : \Sigma^+ \to \mathcal{D}^{p\times p}, \mu_2 : \Sigma^+ \to \mathcal{D}^{q\times q}$ and a map $\phi : \Sigma^+ \to \mathcal{D}^{p\times q}$, (with $1 < p < n$), such that

$$\forall w \in \Sigma^+, \quad P\mu(w)P^{-1} = \begin{bmatrix} \mu_1(w) & \phi(w) \\ \varepsilon & \mu_2(w) \end{bmatrix}\ \ . \tag{9}$$

Moreover, this is clearly equivalent to saying that the matrix $M = \bigoplus_{a\in\Sigma} \mu(a)$ is reducible (in the usual sense of the Perron-Frobenius theory). The interest of irreducible semigroups arises from the following Lemma, which shows that, with respect to the Burnside problem, we may only consider irreducible semigroups.[3]

[3] Lemma 3.2.3 holds in a dioid (and not in an arbitrary semiring) because of the following property: "*in a dioid, the set of all possible sums of the elements of a finite set is finite*"

3.2.3 Lemma *Let $S = \mu(\Sigma^+)$ be a reducible semigroup satisfying (9). Then S is torsion (resp. finite) iff $\mu_1(\Sigma^+)$ and $\mu_2(\Sigma^+)$ are torsion (resp. finite).*

It is clear that the condition is necessary. Conversely, an easy induction shows that

$$\phi(w) = \bigoplus_{a\in\Sigma, uav=w} \mu_1(u)\phi(a)\mu_2(v) \tag{10}$$

hence $\phi(w)$ is a finite sum of elements of the finite set $\mu_1(\Sigma^+)\phi(\Sigma)\mu_2(\Sigma^+)$ and the condition is also sufficient. ■

We now prove some specific $(\max, +)$ lemma.

3.2.4 Lemma *Let $A \in \mathbb{R}_{\max}^{n\times n}$. The following assertions are equivalent:*

1. *A is torsion*
2. *For all irreducible bloc B of A, $\rho(B) = e$ or ε.*

3.2.5 Corollary *Let $A \in \mathbb{R}_{\max}^{n\times n}$. The following assertions are equivalent:*

1. *A is projectively torsion*
2. *There exists $\lambda \in \mathbb{R}_{\max}$, such that for all irreducible bloc B of A, $\rho(B) = \lambda$ or ε.*

Corollary 3.2.5 if obtained from Lemma 3.2.4 by noticing that A is projectively torsion iff $A^{n+c} = \lambda^c A^n$ for some $n, c > 0$, that is iff $\lambda^{-1}A$ is torsion.

Proof of Lemma 3.2.4. From Lemma 3.2.3, we may assume that A is irreducible. If $\rho(A) = \varepsilon$, then A is nilpotent, hence it is torsion. Otherwise, it follows from the cyclicity property (3) that A is torsion iff $\rho(A) = e$. ■

Consider the following characterization of the $(\max, +)$ spectral radius (cf. Lemma 3.1.1):

$$\rho(A) = \bigoplus_{k\geq 1} (trA^k)^{\frac{1}{k}} \quad . \tag{11}$$

We next generalize this property to semigroups of matrices.

3.2.6 Proposition *Let $\Sigma = \{a_1, \ldots, a_p\}$, and $S = \mu(\Sigma^+)$ for some morphism $\mu : \Sigma^+ \to \mathbb{R}_{\max}^{n\times n}$. Let $M = \mu(a_1) \oplus \ldots \oplus \mu(a_p)$. Then*

$$\rho(M) = \bigoplus_{w\in\Sigma^+} (tr\mu(w))^{\frac{1}{|w|}} = \bigoplus_{w\in\Sigma^+} (\rho[\mu(w)])^{\frac{1}{|w|}} \quad .$$

Moreover, the $\sup$ is attained in both summations[4].

Proof of Proposition 3.2.6. Since $\forall i, \mu(a_i) \leq M$, we have

$$\forall w \in \Sigma^+,\ \mu(w) \leq M^{|w|} \quad ,$$

then

$$\bigoplus_{w\in\Sigma^+} (tr\mu(w))^{\frac{1}{|w|}} \leq \bigoplus_{k\geq 1} (trM^k)^{\frac{1}{k}} = \rho(M) \tag{12}$$

[4] recall that $\bigoplus_i \alpha_i = \sup_i \alpha_i$.

by (11). Similarly,

$$\bigoplus_{w\in\Sigma^+} (\rho\mu(w))^{\frac{1}{|w|}} \le \bigoplus_{n\ge 1} (\rho(M^n))^{\frac{1}{n}} = \rho(M) \tag{13}$$

by Lemma 3.1.2. We now prove the converse inequalities. Let $i_1,\ldots,i_k \in \{1,\ldots,n\}$ such that

$$\rho(M)^k = M_{i_1i_2}\ldots M_{i_ki_1} \quad .$$

Since $M = \bigoplus_s \mu(a_s)$, each entry of M corresponds to some entry of one of the $\mu(a_s)$. Precisely, for each $l \in \{1,\ldots,k\}$, take $s_k \in \{1,\ldots,p\}$ such that $M_{i_li_{l+1}} = \mu(a_{s_l})_{i_li_{l+1}}$ (with the convention $k+1=1$). Let $w = a_{s_1}\ldots a_{s_k}$. Then

$$\begin{aligned} \rho(\mu(w)) &\ge tr\mu(w) \ge \mu(w)_{i_1i_1} \\ &\ge \mu(a_{s_1})_{i_1i_2}\ldots\mu(a_{s_k})_{i_ki_1} = \rho(M)^k \end{aligned}$$

which shows that the converses inequalities in (13) and (12) hold. ■

The boolean semiring, $\mathbb{B} = \{\varepsilon, e\}$, which can be seen as a subsemiring of $\mathbb{R}_{\max}$ (with $e=0, \varepsilon=-\infty$), will play an important role in the sequel.

3.2.7 Lemma *Let $A \in \mathbb{B}^{n\times n}$ be an idempotent matrix. Then, the irreducible blocs of A are either ε or equal to some matrix J_k, where J_k denotes the $k\times k$-matrix whose entries are all equal to e.*

Proof Let B be an irreducible $k\times k$-bloc. Since A is idempotent, we have $B = B^+$. If $\rho(B)=\varepsilon$, then B is reduced to a single ε element. Otherwise, $B^+ = J_k$. ■

The following Lemma is reminiscent of the theory of (classical) spectral radii of Hadamard products of nonnegative matrices (see Elsner, Johnson, Dias da Silva [12], Theorem 7).

3.2.8 Lemma *Let $M \in \mathbb{R}_{\max}^{n\times n}$ be irreducible. Then*

$$\rho(M) = \min_D \|DMD^{-1}\| \tag{14}$$

where D ranges the set of diagonal matrix with non ε diagonal entries.

Proof of Lemma 3.2.8. We get from 3.1.1,2 that $\rho(M) = \rho(DMD^{-1}) \le \|DMD^{-1}\|$ for all D. Assume by homogeneity that $\rho(M) = e$, and let u be an associated eigenvector, i.e. $Mu = u$. Set $D^{-1} = diag(u_1,\ldots,u_n)$. Then, $\|DMD^{-1}\| = \bigoplus_{ij} u_i^{-1}M_{ij}u_j = \bigoplus_i u_i^{-1}u_i = e$. Hence, the inf in (14) is equal to $e = \rho(M)$. ■

4 Finiteness Results for $(\max,+)$-Semigroups

4.1 It Is Decidable if a Finitely Generated (max,+) Semigroup of Matrices Is Torsion

We now prove Theorem 2.0.2, which is an extension of Simon's decidability result [34]. Let $S = \langle A_1,\ldots,A_p\rangle$ and set $M = \bigoplus_{i=1}^p A_i$.

4.1.1 Lemma *For S to be torsion, it is necessary that $\rho(M) = e$.*

Proof From Proposition 3.2.6, we get

$$\rho(M) = (\rho[\mu(w)])^{\frac{1}{|w|}}$$

for some $w \in \Sigma^+$. If S is torsion, then $\mu(w)$ is torsion, hence, $\rho(\mu(w)) = e$. ■

This allows considering only the following case.

4.1.2 (Canonical form) We shall assume that

$$\forall i, \quad \|A_i\| \leq \|M\| = e \ . \tag{15}$$

Indeed, provided $\rho(M) = e$, condition (15) becomes satisfied if we replace each A_k by DA_kD^{-1}, with $D_{ii} = M^*_{1i}$ as in Lemma 3.2.8.

Thus, we are reduced to a semigroup of matrices of $(\mathbb{R}^-_{\max})^{n\times n}$ ($\mathbb{R}^-_{\max}$ is the subdioid of $\mathbb{R}_{\max}$ comprising the elements ≤ 0). We next introduce some morphisms from $\mathbb{R}_{\max}$ and $\mathbb{R}^-_{\max}$ to simpler structures.

Let

$$\pi : \ \mathbb{R}_{\max} \to \mathbb{B}, \quad \pi(x) = \begin{cases} e & \text{if } x \neq \varepsilon \\ \varepsilon & \text{if } x = \varepsilon. \end{cases}$$

We naturally extend π to $\mathbb{R}^{n\times n}_{\max}$ (componentwise). In the case on *nonpositive* reals, there is another useful morphism

$$\psi : \ \mathbb{R}^-_{\max} \to \mathbb{B}, \quad \psi(x) = \begin{cases} e & \text{if } x = e \\ \varepsilon & \text{if } x < e \end{cases}$$

that we also extend to $(\mathbb{R}^-_{\max})^{n\times n}$ in a similar way. The product morphism is given by

$$\pi \times \psi : (\mathbb{R}^-_{\max})^{n\times n} \to \mathbb{B}^{n\times n} \times \mathbb{B}^{n\times n} \ .$$

Finally, we introduce the following map

$$\kappa : \ \mathbb{R}^-_{\max} \to \mathbb{N}^-, \quad \kappa(x) = \begin{cases} e & \text{if } x = e \\ -1 & \text{if } \varepsilon < x < e \\ \varepsilon & \text{if } x = \varepsilon. \end{cases}$$

It should be noted that in the case of the dioid $(\mathbb{N} \cup \{+\infty\}, \min, +)$, the map analogous to κ is the key of Simon's proof [34]. κ is not a morphism, but we have

$$\begin{aligned} \kappa(A \oplus B) &= \kappa(A) \oplus \kappa(B), \\ \kappa(\kappa(A) \otimes \kappa(B)) &= \kappa(A \otimes B) \end{aligned} \ . \tag{16}$$

Hence, κ is a morphism from $\mathbb{R}^-_{\max}$ to the three elements dioid $\kappa(\mathbb{R}^-_{\max}) = \{\varepsilon, -1, e\}$ equipped with the two following laws $\oplus_\kappa$ and $\otimes_\kappa$:

$$a \oplus_\kappa b \stackrel{def}{=} a \oplus b, \quad a \otimes_\kappa b \stackrel{def}{=} \kappa(a \otimes b) \ .$$

We naturally extend κ to $\mathbb{R}^{n\times n}_{\max}$.

We claim that:

4.1.3 Proposition *Let $A \in (\mathbb{R}^-_{\max})^{n\times n}$. Let $\imath$ denote the injection*[5] $\kappa((\mathbb{R}^-_{\max})^{n\times n}) \to (\mathbb{R}^-_{\max})^{n\times n}$. *The following assertions are equivalent.*

[5] It should be noted that $\imath$ is *not* a morphism. In particular, stating that $\imath\circ\kappa(A)$ is torsion (in $(\mathbb{R}^-_{\max})^{n\times n}$) has nothing to do with stating that $\kappa(A)$ is torsion in the dioid $\kappa((\mathbb{R}^-_{\max})^{n\times n})$, which is always the case since $\kappa((\mathbb{R}^-_{\max})^{n\times n})$ is finite.

1. *A is torsion*
2. *$\imath\circ\kappa(A)$ is torsion*
3. *for each non ε strongly connected component of A, there exists a circuit*[6] *of A composed only of arcs of weight e,*
4. *each non ε strongly connected component of $\pi(A)$ contains at least a circuit of $\psi(A)$.*

Proof of the Proposition. (4) $\Leftrightarrow$ (3) is obvious.

(1) $\Longleftrightarrow$ (3): This follows from Lemma 3.2.4 and the fact that since all the entries of A are $\leq e$, a circuit with weight e has all its entries equal to e.

(3) $\Longleftrightarrow$ (2). This is clear from Lemma 3.2.4, since the irreducible blocs and the entries equal to e are exactly the same for A and $\imath\circ\kappa(A)$. ■

This yields the following extension to $\mathbb{R}_{\max}$ of Simon's algorithm [34].

4.1.4 Algorithm *Compute the finite semigroup $\kappa(S)$ and for each $t \in \kappa(S)$, check that the matrix $A = i(t)$ satisfies property 4.1.3,3.*

There is an equivalent version which uses the maps π and ψ:

4.1.5 Algorithm *Compute the finite semigroup $\pi \times \psi(S)$ and check property 4.1.3,4 for each $t \in \pi \times \psi(S)$*

The Theorem is proved. ■

When the semigroup S is primitive, we obtain a particularly simple result:

4.1.6 Theorem *A* primitive *semigroup $S = \mu(\Sigma^+) \subset (\mathbb{R}^-_{\max})^{n\times n}$ is torsion iff there is no nilpotent matrix in $\psi\mu(\Sigma^+)$.*

Proof Obvious from 4.1.5 since for all $w \in \Sigma^+$, $\pi(\mu(w))$ admits a single strongly connected component: there is a circuit in $\psi\mu(w)$ iff $\rho(\psi\mu(w)) = e$, which is equivalent to $\psi\mu(w)$ being non nilpotent. ■

4.1.7 Example Let us consider the matrices

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \quad and \quad B = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

and define

$$\begin{aligned} \tilde{A} &= \rho(A)^{-1}A = A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \\ \tilde{B} &= \rho(B)^{-1}B = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \quad . \end{aligned}$$

We claim that $\tilde{S} \stackrel{def}{=} \langle \tilde{A}, \tilde{B} \rangle$ is torsion. In order to show that, we reduce $\tilde{S}$ to the canonical form 4.1.2. We have

$$M = \tilde{A} \oplus \tilde{B} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad ,$$

[6] A *path* of length k is a family of $k+1$ indices $(i_1, \ldots, i_{k+1})$. A *circuit* is a path such that $i_{k+1} = i_1$. An *arc* is a path of length 1. The weight of the path $(i_1, \ldots, i_{k+1})$ is equal to $A_{i_1 i_2} \ldots A_{i_k i_{k+1}}$

$\rho(M)=e=0$ and $M^*=M$. Let $D=diag(M^*_{11},M^*_{12})=diag(0,1)$ and set

$$A'=D\tilde{A}D^{-1}=\begin{bmatrix}0 & 0\\ 0 & -1\end{bmatrix},\quad B'=D\tilde{B}D^{-1}=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\ .$$

$\tilde{S}$ is torsion iff $\langle A',B'\rangle$ is torsion. According to theorem 4.1.6, we have to check that $\psi\langle A',B'\rangle$ has no nilpotent elements. But

$$\psi(A')=\begin{bmatrix}e & e\\ e & \varepsilon\end{bmatrix}\geq\psi(B')=\begin{bmatrix}\varepsilon & e\\ e & \varepsilon\end{bmatrix},$$

hence we may bound from below a product of k matrices $\psi(A')$ and $\psi(B')$ by $\psi(B')^k$. Since $\psi(B')$ is not nilpotent, this shows that $\psi\langle A',B'\rangle$ contains no nilpotent elements, hence $\tilde{S}$ is torsion.

4.1.8 Remark We conclude this section with a brief indication of complexity. Because $\psi(s)\leq\pi(s)$, only three cases are possible: (i) $\phi(s)=\pi(s)=\varepsilon$, then $\kappa(s)=\varepsilon$, (ii) $\phi(s)=\varepsilon$ and $\pi(s)=e$, then $\kappa(s)=-1$, (iii) $\phi(s)=\pi(s)=e$, then $\kappa(s)=e$.

Hence, the cardinality of $\pi\times\psi(S)$ and $\kappa(S)$ are equal and bounded above by 3^{n^2}, and the two algorithms are essentially equivalent.

4.2 A Finitely Generated Torsion Subsemigroup of $\mathbb{R}^{n\times n}_{\max}$ is Finite

We now prove Theorem 2.0.1. Since the condition (15) must be satisfied, we may assume that $S\subset(\mathbb{R}^{-}_{\max})^{n\times n}$.

4.2.1 Proposition *Let $S\subset(\mathbb{R}^{-}_{\max})^{n\times n}$ be a finitely generated semigroup. S is finite iff the non ε entries of the matrices of S are bounded.*

In other words, the finiteness of S is equivalent to the following property

$$\begin{aligned}&\exists K\in\mathbb{R}^{-},\quad\forall s\in S,\forall i,j,\\ &\qquad s_{ij}\neq\varepsilon\Rightarrow K\leq s_{ij}\ .\end{aligned}\tag{17}$$

Proof Condition (17) is clearly necessary. Conversely, assume that (17) holds. Let $A_1,\dots,A_p$ be generators of S, and let

$$\lambda=\sup\{(A_k)_{ij}\mid\ 1\leq k\leq p,\ 1\leq i,j\leq n,\ \ \varepsilon<(A_k)_{ij}<e\}\ .$$

Let $w=A_{s_1}\dots A_{s_k}\in\Sigma^k$. We have

$$\mu(w)_{ij}=(A_{s_1})_{ii_2}\dots(A_{s_k})_{i_{k-1}j}\tag{18}$$

for some indices $i_2,\dots,i_k$. We set $i_1=i,i_k=j$. Let $N(w)$ denote the maximal number of indices $l:1\leq l\leq k$ such that $(A_{s_l})_{i_li_{l+1}}<e$ in a factorization of type (18). Then $\mu(w)_{ij}\leq\lambda^{N(w)}$. This implies that $\lambda^{N(w)}\geq K$, which shows that $N(w)$ is bounded. Since the factors of (18) are either equal to e or less than e, this shows that $\mu(w)_{ij}$ can only take a finite number of values. The Proposition is proved. ■

To show that a finitely generated torsion semigroup $S\in(\mathbb{R}^{-}_{\max})^{n\times n}$ is finite, it remains to check that the non ε entries of the matrices of S are bounded. Due to Brown's Lemma, we may assume that $\pi(S)=\{E\}$ for some idempotent matrix E. According to Lemma 3.2.7, we shall assume that E has the following form

$$E=\begin{bmatrix}J_k & *\\ \varepsilon & J_l\end{bmatrix}$$

(the general case is obtained by an immediate induction). Thus, we have a decomposition of the form (9) with $\mu_1(\Sigma^+) = J_k$ and $\mu_2(\Sigma^+) = J_l$. Moreover, for all w, We have necessarily

$$\|\mu_1(w)\| = \|\mu_2(w)\| = e \tag{19}$$

(otherwise $\rho(\mu_i(w)) \leq \|\mu_i(w)\| < e$ contradicts the fact that $\mu(w)$ is torsion). The proof of 3.1.4 shows that there exists a finite real K such that

$$\forall i \in \{1,2\},\ \forall w \in \Sigma^+,\ \frac{\|\mu_1(w)\|}{|\mu_i(w)|_\wedge} \leq K \ .$$

It follows from (19) that $|\mu_i(w)|_\wedge \geq K^{-1}$. Moreover, Formula (10) shows that $|\phi(w)|_\wedge$ is bounded below. This implies that $|\mu(w)|_\wedge$ is bounded below and concludes the proof of Theorem 2.0.1.

4.3 Proof of Theorem 2.0.3

The proof relies on two lemmas. The first one gives a symmetrized version of a classical polynomial identity (valid in rings). Let $X_1, \ldots, X_p$ denote p non commuting indeterminates and define

$$S_p^+ = \bigoplus_{\sigma \ \ even} X_{\sigma(1)} \ldots X_{\sigma(p)}$$
$$S_p^- = \bigoplus_{\sigma \ \ odd} X_{\sigma(1)} \ldots X_{\sigma(p)}$$

where the sums are taken over the even and odd permutations of $\{1, \ldots, p\}$. The well known Amitsur-Levitski theorem (see e.g. [6]) states that the polynomial identity $S_{2n}^+ = S_{2n}^-$ holds in the matrix ring $R^{n\times n}$ (where R denotes a commutative ring). Such combinatorial identities extend to semirings:

4.3.1 Lemma *Let A be a commutative semiring. Then, the identity $S_{2n}^+ = S_{2n}^-$ holds in $A^{n\times n}$.*

This can be easily deduced from the classical Amitsur-Levitski theorem by a technique of Reutenauer and Straubing ([33], proof of Lemma 1,2), see also [13], Chapter 1, Proposition 2.1.5 and [16], Proposition 2.2.1.

The proof of the theorem simply consists in adapting the argument of Straubing [38] (which shows that a finitely generated torsion semigroup of matrices over a commutative ring is finite) that we reproduce completely here. Recall that a word w is r-*divided* if it admits a factorization $w = w_1 \ldots w_r$ such that for all permutation $\sigma \neq Id$,

$$w_{\sigma(1)} \ldots w_{\sigma(r)} < w$$

($<$ denotes the lexicographic order). Then, Shirshov's Lemma states that if $q \geq 2r$, there exists an integer $N(|\Sigma|, q, r)$ such that, for all word $w \in \Sigma^*$ such that $|w| \geq N(|\Sigma|, q, r)$, either w admits a factorization $w = uv^q s$ with $1 \leq |v| < r$, either w contains a r-divided factor (i.e. $w = uvs$ where v is r-divided).

Set $r = 2n, p = \max\{\#\langle\mu(w)\rangle + 1 \mid \ |w| < r\}$ ($\#\langle\mu(w)\rangle$ is the order of the semigroup $\{\mu(w), \mu(w^2), \ldots, \}$ which is finite since the theorem assumes that for all w, $\mu(w)$ is torsion), and $q = \max(p, 2r)$. We claim that

$$\forall w \in \Sigma^+, \quad \mu(w) \leq \bigoplus_{|v| < N(|\Sigma|, q, r)} \mu(v) \ . \tag{20}$$

We show this assertion by induction on w (with respect to the military order, i.e. "length first, lexicographic second"). Let w such that $|w| \geq N(|\Sigma|, q, r)$. If $w = uv^q s$, then $\mu(v^q) = \mu(v^t)$ for some $t < s$, hence

$\mu(w) = \mu(w')$ with $w' = uv^t s < w$, and by the induction hypothesis, we are done. Otherwise, we have $w = uvs$, where $v = v_1 \dots v_r$ is r-divided. The symmetrized polynomial identity (4.3.1) implies that

$$\mu(v) \leq \bigoplus_{\sigma \; odd} \mu(v_{\sigma(1)} \dots v_{\sigma(r)}) \ ,$$

therefore

$$\mu(w) \leq \bigoplus_{w'} \mu(w')$$

where the w' are strictly less than w. This concludes the proof of (20). Since $\mu(\Sigma^+)$ is bounded above, the assumption (1) implies that it is finite. ∎

5 The Burnside Problem for Projective Linear Semigroups

The proof of Theorem 3.1.4 suggests that the following quantity will play an important role.

5.0.2 Definition *The* projective width[7] *of a set $S \subset \mathbb{R}_{\max}^{n\times n}$ is by definition*

$$\Delta(S) = \sup_{s\in S} \frac{\|s\|}{|s|_\wedge} \ . \tag{21}$$

If $S = \mu(\Sigma^+)$ for some morphism μ, we shall write $\Delta(\mu)$ instead of $\Delta(S)$. Arguing as in the proof of Theorem 3.1.4, we can state:

5.0.3 Lemma *A semigroup $S \subset \mathbb{Q}_{\max}^{n\times n}$ is projectively finite iff its projective width $\Delta(S)$ is finite.*

Thus, we have to show that $\Delta(\mu)$ is finite. The key point in the proof of the Theorem is the following.

5.0.4 Lemma *Let $S = \mu(\Sigma^+)$ be a reducible projectively* torsion *semigroup, and take μ_1, μ_2, ϕ as in (9). Then the following assertions are equivalent:*

1. *$\Delta(\mu)$ is finite*
2. *$\Delta(\mu_1)$ and $\Delta(\mu_2)$ are finite.*

Let us assume that the Lemma 5.0.4 is proved. Then, using Brown's Lemma, with $T = \mathbb{B}^{n\times n}$ and π the canonical morphism, we may assume that $\pi(\mu(\Sigma^+)) = \{E\}$ for some boolean idempotent matrix $E \in \mathbb{B}^{n\times n}$ having the form of Lemma 3.2.7. Therefore, by Lemma 3.1.5, the projective width of all the irreducible components of $\mu(\Sigma^+)$ is finite, hence by 5.0.4, $\Delta(\mu) < +\infty$, which together with Lemma 5.0.3, gives Theorem 2.0.4.

It remains to show Lemma 5.0.4. (1) $\Rightarrow$ (2) being obvious, we only prove the converse implication. We have to bound the differents terms of the form s_{ij}/s_{kl} which appear in $\|s\|/\,|s|_\wedge$ in (21).

First, we claim that

$$\forall w,\ \frac{\|\mu_1(w)\|}{|\mu_2(w)|_\wedge} \leq \Delta(\mu_1)\Delta(\mu_2) \ . \tag{22}$$

[7] It might seem simpler to speak of "diameter" instead of "width". We do not use the term "diameter" because there is a natural metric on the projective space, namely the "Hilbert's projective metric" [4] δ defined by $\delta(A,B) = \sup\{\lambda/\mu \mid \ \mu A \leq B \leq \lambda A\}$. When A and B are two matrices with the same boolean image, the distance $\delta(A,B)$ coincides with the projective width $\Delta(\{A,B\})$. In other words, Δ coincides with the diameter associated with the metric δ, but only for a subclass of two-elements sets.

Indeed, we have

$$\frac{\|\mu_1(w)\|}{|\mu_2(w)|_\wedge} \leq \frac{\|\mu_1(w)\|}{|\mu_1(w)|_\wedge} \frac{|\mu_1(w)|_\wedge}{\|\mu_2(w)\|} \frac{\|\mu_2(w)\|}{|\mu_2(w)|_\wedge} \ .$$

Moreover, since $\mu(w)$ is torsion, by Corollary 3.2.5,

$$|\mu_1(w)|_\wedge \leq \rho(\mu_1(w)) = \rho(\mu_2(w)) \leq \|\mu_2(w)\| \ ,$$

which shows (22).

We next give a second bound. Let

$$K_1 = \sup_{a\in\Sigma} \frac{\|\phi(a)\|}{|\mu_1(a)|_\wedge}$$

We claim that

$$\forall w \in \Sigma^+, \quad \frac{\|\phi(w)\|}{|\mu_1(w)|_\wedge} \leq K_1 \Delta(\mu_1)^2 \Delta(\mu_2) \ . \tag{23}$$

Indeed, from formula (10) together with (6) we have $\|\phi(w)\| = \|\mu_1(u)\phi(a)\mu_2(v)\|$ for some factorization $w = uav$ (with $a \in \Sigma$), hence

$$\begin{aligned}
\frac{\|\phi(w)\|}{|\mu_1(w)|_\wedge} &\leq \frac{\|\mu_1(u)\|\|\phi(a)\|\|\mu_2(v)\|}{|\mu_1(w)|_\wedge} \\
&\leq \frac{\|\mu_1(u)\|\|\phi(a)\|\|\mu_2(v)\|}{|\mu_1(u)|_\wedge |\mu_1(a)|_\wedge |\mu_1(v)|_\wedge} \\
&\quad \text{(by (7))} \\
&\leq \Delta(\mu_1) K_1 \Delta(\mu_1) \Delta(\mu_2)
\end{aligned}$$

A dual argument shows that

$$\forall w \in \Sigma^+, \quad \frac{\|\mu_1(w)\|}{|\phi(w)|_\wedge} \leq K_1' \Delta(\mu_1)^2 \Delta(\mu_2) \tag{24}$$

where

$$K_1' = \sup_{a\in\Sigma} \frac{\|\mu_1(a)\|}{|\phi(a)|_\wedge} \ .$$

It remains to show that $\|\phi(w)\| / |\phi(w)|_\wedge$ is also bounded. We have, by (10),(6) and (7):

$$\frac{\|\phi(w)\|}{|\phi(w)|_\wedge} \quad \leq \quad \frac{\|\mu_1(u)\phi(a)\mu_2(v)\|}{|\mu_1(x)\phi(b)\mu_2(y)|_\wedge}$$

for some factorizations $w = uav = xby$ ($a, b \in \Sigma$). We may assume a factorization of the form $u = xbz$ or $u = x$ (if it is not the case, we factorize v in a dual way instead of u).

1/ If $u = xbz$, we have

$$\frac{\|\phi(w)\|}{|\phi(w)|_\wedge} \leq \\ \frac{\|\mu_1(x)\|\|\mu_1(b)\|\|\mu_1(z)\|\|\phi(a)\|\|\mu_2(v)\|}{|\mu_1(x)|_\wedge |\phi(b)|_\wedge |\mu_2(z)|_\wedge |\mu_2(a)|_\wedge |\mu_2(v)|_\wedge}$$

Set

$$K_1'' = \sup_{a,b\in\Sigma} \frac{\|\mu_1(b)\|\|\phi(a)\|}{|\phi(b)|_\wedge |\mu_2(a)|_\wedge}$$

We get from (22):

$$\frac{\|\phi(w)\|}{|\phi(w)|_\wedge} \quad \leq \quad K_1''\Delta(\mu_1)^2\Delta(\mu_2)^2$$

2/ When $u = x$, we set

$$K = \sup_{a\in\Sigma} \frac{\|\phi(a)\|}{|\phi(a)|_\wedge}$$

and obtain by a similar and simpler argument

$$\frac{\|\phi(w)\|}{|\phi(w)|_\wedge} \quad \leq \quad K\Delta(\mu_1)\Delta(\mu_2) \ .$$

Let us define K_2,K_2',K_2'' in a way similar to K_1,K_1',K_1''. Putting together the above bounds, we obtain

$$\Delta(\mu) \leq \Delta(\mu_1)\Delta(\mu_2)[K \oplus (K_1'' \oplus K_2'')\Delta(\mu_1)\Delta(\mu_2)\oplus \\ \oplus(K_1 \oplus K_1')\Delta(\mu_1) \oplus (K_2 \oplus K_2')\Delta(\mu_2)] \ .$$

The Lemma is proved. ■

6 Decidability of the Limitedness Problem

We now prove Theorem 2.0.5.

6.0.5 Lemma *Let s be the rational series with* trim *linear representation (α,μ,β) and let $M = \bigoplus_{a\in\Sigma}\mu(a)$. If s is limited, then $\rho(M) = e$ or ε.*

Proof 1/ Assume that $\varepsilon < \rho(M) < e$. It follows from

$$(s|w) \leq \|\alpha\|\|M^{|w|}\|\|\beta\|$$

and from Lemma 3.1.1,(4) that there exists some constant K such that $(s|w) \leq K\rho(M)^{|w|} \to -\infty$ as $|w| \to \infty$. Since the representation is trim, this implies that $(s|w)$ takes arbitrary small (i.e. near $\varepsilon = -\infty$) non ε values. Hence s is not limited.
2/ Assume that $\rho(M) > e$. From 3.1.1,3, there exists i and k such that $M_{ii}^k > e$. Moreover, we have $M_{ii}^k = \mu(w)_{ii}$ for some word w. Since the linear representation is trim, there exists some k, j and two words u, v such that

$$\alpha_j\mu(u)_{ji}\mu(w)_{ii}\mu(v)_{ik}\beta_k \neq \varepsilon \ .$$

Hence, for all $p \geq 1$,

$$(s|uw^pv) \geq K(M_{ii}^k)^p$$

for some $K \neq \varepsilon$. Since $\lim_{p\to\infty}(M_{ii}^k)^p = +\infty$, s is not limited. ■

Due to Lemma 6.0.5, we may perform the same reduction as in the decision algorithm for the torsion property (see 4.1.2). Therefore, we may assume that

$$\mu(\Sigma^+) \subset (\mathbb{R}^-_{\max})^{n\times n} \tag{25}$$

(but α and β can have positive entries).

The conclusion is an immediate consequence of the following Lemma.

6.0.6 Lemma *Assume that (25) holds, then s is limited iff the non ε values of $(s|w)$ are bounded below.*

That is, we require that

$$\exists K \neq \varepsilon, \quad (s|w) \neq \varepsilon \Rightarrow (s|w) \geq K \ . \tag{26}$$

Proof of Lemma 6.0.6. This is a straightforward variant of the proof of Proposition 4.2.1. ∎

It remains to show that the boundedness property (26) is decidable.

6.0.7 Lemma *Under the assumption (25), the boundedness property (26) holds for s iff it holds for the series s' given by the linear representation $(\pi\alpha, \kappa\mu, \pi\beta)$.*

Proof Some routine calculus show that there exists two non ε constants K and K' such that

$$K(s|w) \leq (s'|w) \leq K'(s|w) \ .$$

∎

Since the limitedness problem for the series s' is decidable (since the representation of s' lives in $(\mathbb{N}^- \cup \{-\infty\}, \max, +)$, which is isomorphic to the tropical semiring, this follows from Hashiguchi's theorem [21], see also [20, 35, 36, 24, 26]), Theorem 2.0.5 is proved.

7 Related Remarks

7.1 Counter Example

We show that the theorem of finiteness of primitive semigroups of projective linear maps with rational entries does not extend to the irrational case. Let

$$A = \begin{bmatrix} \sqrt{2} & \eta \\ \eta & e \end{bmatrix}, \quad B = \begin{bmatrix} e & \eta \\ \eta & 1 \end{bmatrix}$$

where η is a small parameter to be fixed soon and $\sqrt{2} = 1.414\ldots$. We claim that $\mathfrak{p}\langle A, B\rangle$ is infinite. In order to show that, we recall that linear projective maps can be identified to homographic functions –as in the conventional algebra–. More precisely, we set for $x \in \mathbb{R}$,

$$u(x) \stackrel{def}{=} \mathfrak{p} \begin{bmatrix} x & e \end{bmatrix} \ .$$

We have

$$u(x)A = u(h_A(x))$$

where the homographic map associated with A is

$$h_A(x) \stackrel{def}{=} \frac{\sqrt{2}x \oplus \eta}{\eta x \oplus e} \ .$$

Similarly, we have

$$h_B(x) \stackrel{def}{=} \frac{x \oplus \eta}{\eta x \oplus 1}, \quad u(x)B = u(h_B(x)) \ .$$

Clearly, it is enough to show that the semigroup $\langle h_A, h_B\rangle$ (equipped with the composition product) is infinite. We assume that $\eta < 0$, and note that

$$\eta \leq x \leq -\eta \Rightarrow h_A(x) = \sqrt{2} + x, \ h_B(x) = x - 1 \ .$$

Let us further assume that $2 \times \eta \leq -\sqrt{2} - 1$ (for instance, take $\eta = -(\sqrt{2}+1)/2$. Then, the diameter of $[\eta, -\eta]$ is greater than $\sqrt{2}+1$, hence, for all $x \in [\eta, -\eta]$, either $x + \sqrt{2} \in [\eta, -\eta]$, either $x - 1 \in [\eta, -\eta]$. Let us define a sequence $x_n \in [\eta, -\eta]$ by $x_0 = e$ and $x_n = h_A(x_{n-1})$ or $h_B(x_{n-1})$ (we choose h_A of h_B so that $x_n \in [\eta, -\eta]$, if both $h_A(x_{n-1}), h_B(x_{n-1}) \in [\eta, -\eta]$, then, the choice is arbitrary). This "walk" in $[\eta, -\eta]$ is visualized on Figure 1. Starting from x_0, we may choose $x_1 = x_0 + \sqrt{2}$ or $x_1 = x_0 - 1$. On the picture, $x_1 = h_A(x_0) = x_0 + \sqrt{2}$. Similarly, we have set $x_2 = h_A(x_1)$. Then, because $h_A(x_2) > -\eta$, the only possible choice is $x_3 = h_B(x_2) = x_2 - 1$. Let $p(A, k)$ (resp. $p(B, k)$) denote the number of choices of

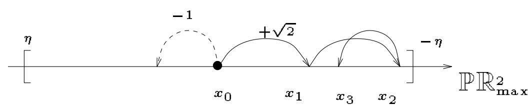

Figure 1: The x_n sequence

h_A (resp. h_B) up to step k (e.g., for the walk of the picture, $p(A,3) = 2, p(B,3) = 1$). We have

$$\forall k \geq 0,\ x_k = p(A,k) \times \sqrt{2} - p(B,k) \times 1 \ .$$

Moreover, $p(A,k) + p(B,k) = k$ and $p(A,k)$ and $p(B,k)$ are nondecreasing functions of k. Since $\sqrt{2}$ and 1 are linearly independent over $\mathbb{Q}$, all the $x_k, k = 1, 2, \ldots$ are distinct. Since the x_k are images of x_0 by some elements of $\langle h_A, h_B \rangle$, this shows that $\langle h_A, h_B \rangle$ (hence $\langle A, B \rangle$) is infinite.

7.2 An Upper Bound for the Lyapunov Exponent

We consider a random walk $w_k \in \Sigma^k$. That is, the word $w_k = a_1 \ldots a_k$ (with $a_i \in \Sigma$) occurs with the probability $p(w_k) = p(a_1) \ldots p(a_k)$ where the $p(a_i)$ are given nonnegative numbers such that $\sum_i p(a_i) = 1$. Let $\mu : \Sigma^* \to \mathbb{R}_{\max}^{p \times p}$ be a morphism. The maximal Lyapunov exponent of μ is defined as the limit

$$\begin{aligned} \lambda &= \lim_k \mathbb{E}\|\mu(w_k)\|^{\frac{1}{k}} \\ &= \lim_k \|\mu(w_k)\|^{\frac{1}{k}} \ \ a.s. \end{aligned}$$

(see [2],[15]). We just notice here that there is a subclass of matrices for which the Lyapunov exponent if immediately obtained due to the characterization of finite semigroups.

7.2.1 Theorem *Let $\mu : \Sigma^+ \to \mathbb{R}_{\max}^{n \times n}$ such that $\forall a,\ \rho(\mu(a)) \neq \varepsilon$ and define the morphism:*

$$\tilde{\mu} :\ \forall a \in \Sigma,\ \tilde{\mu}(a) \stackrel{def}{=} \rho(\mu(a))^{-1} \mu(a)$$

with the associated "worst case matrix"

$$M = \bigoplus_{a \in \Sigma} \tilde{\mu}(a) \ .$$

Then

$$\lambda \leq \rho(M) \mathbb{E} \rho(\mu(w_1)) = \rho(M) \left(\sum_{a \in \Sigma} \rho(\mu(a)) \times p(a) \right) \ . \tag{27}$$

Moreover, if $\tilde{\mu}(\Sigma^+)$ is torsion, then the equality holds in (27).

It should be noted that we have by construction

$$\forall a,\ \rho(M) \geq \rho(\tilde{\mu}(a)) = e$$

hence, we can decide if $\tilde{\mu}(\Sigma^+)$ is torsion by checking the non trivial inequality $\rho(M) \leq e$ and by using the algorithms 4.1.5,4.1.4.

7.2.2 Example The Lyapunov exponent of the semigroup $\langle A, B\rangle$ of Example 4.1.7 can be obtained almost without computation. Since the normalized semigroup $\tilde{\mu}(\Sigma^+)$ is torsion, we have

$$\iota = \rho(\mu(a)) \times p(a) + \rho(\mu(b)) \times p(b) = 1 \times p(b) = p(b)\ \ .$$

7.2.3 Example In the same vein, we can build many semigroups whose Lyapunov exponents are immediately obtained. Let

$$\mu(a) = \begin{bmatrix} 1 & * & * \\ * & * & * \\ * & * & * \end{bmatrix},\ \ \mu(b) = \begin{bmatrix} 2 & \# & \# \\ \# & \# & \# \\ \# & \# & \# \end{bmatrix}$$

where * stands for arbitrary finite real numbers ≤ 1 (we allow different values) and $\#$ stands for arbitrary finite real numbers ≤ 2. Let $E_{11} \in \mathbb{R}_{\max}^{n\times n} : (E_{11})_{ij} = \delta_{i=1,j=1}$ (Kronecker's δ). Then, $\psi\tilde{\mu}(a), \psi\tilde{\mu}(b) \geq E_{11}$, and since $E_{11}^2 = E_{11}$, $\psi\tilde{\mu}(w)$ is non nilpotent for all w. Hence the condition of Theorem 4.1.6 is satisfied, and we have

$$\iota = 1 \times p(a) + 2 \times p(b)\ \ .$$

Proof of Theorem 7.2.1. For simplicity we shall assume that the alphabet has only two letters ($\Sigma = \{a, b\}$), the extension to the general case being straightforward. We have

$$\mu(w) = \rho(\mu(a))^{|w|_a}\rho(\mu(b))^{|w|_b}\tilde{\mu}(w)\ \ , \tag{28}$$

hence

$$\|\mu(w)\|^{\frac{1}{|w|}} = \rho(\mu(a))^{\frac{|w|_a}{|w|}}\rho(\mu(b))^{\frac{|w|_b}{|w|}}\|\tilde{\mu}(w)\|^{\frac{1}{|w|}}\ \ . \tag{29}$$

We have

$$\lim_{|w|\to\infty} \rho(\mu(a))^{\frac{|w|_a}{|w|}}\rho(\mu(b))^{\frac{|w|_b}{|w|}} = \rho(\mu(a))^{p(a)}\rho(\mu(b))^{p(b)}\ a.s.$$

Setting $M = \bigoplus_{a\in\Sigma} \tilde{\mu}(a)$, we obtain

$$\tilde{\mu}(w) \leq M^{|w|} \simeq \rho(M)^{|w|}\ \ .$$

Then, it follows from (29) that

$$\lim_{|w|\to\infty} \|\mu(w)\|^{\frac{1}{|w|}} \leq \rho(\mu(a))^{p(a)}\rho(\mu(b))^{p(b)}\rho(M)\ a.s. \tag{30}$$

Moreover, if $\tilde{\mu}(\Sigma^+)$ is finite, then $\rho(M) = e = 0$, $\|\tilde{\mu}(w)\|^{\frac{1}{|w|}} \to 0$ as $|w| \to \infty$, hence, the equality holds in (30). ■

7.2.4 Remark A different upper bound for the Lyapunov exponent has been given previously by Baccelli and Konstantopoulos [3, 2] using large deviation estimates. The upper bound (27) is less general since it is only given for probability measures with finite support (it can be easily extended to measures with bounded support, but says nothing about the unbounded case). However, the equality cases for (27) (when the underlying normalized semigroup if finite) differ from the equality or accuracy cases for the large deviation bound.

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