A note on shortest paths in the plane subject to a constraint on the derivative of the curvature

Jean-Daniel Boissonnat, André Cerezo, Juliette Leblond

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET AUTOMATIQUE

A note on shortest paths in the plane subject to a constraint on the derivative of the curvature

Jean-Daniel Boissonnat André Cerezo Juliette Leblond

N° 2160

Janvier 1994

PROGRAMME 4

Robotique,

image

et vision

Rapport de recherche

1994

ISSN 0249-6399

A note on shortest paths in the plane subject to a constraint on the derivative of the curvature *

Jean-Daniel Boissonnat ** André Cerezo *** Juliette Leblond**

Programme 4 — Robotique, image et vision
Projet Prisme

Rapport de recherche n ° 2160 — Janvier 1994 — 14 pages

Abstract: We consider the class of C^2, piecewise C^3, planar paths joining two given configurations (position, orientation, and curvature) X_0 and X_f, and along which the derivative of the curvature (with respect to the arc length) remains bounded. We admit an infinite (countable) number of pieces, as long as the switching points do not accumulate more than a finite number of times. For generic X_0 and X_f, we prove that the path of minimal length satisfying the constraint is such that : either it contains no line segment, or it contains infinitely many arcs of clothoid. As a consequence, the number of C^3 arcs involved in a shortest path may not be uniformly bounded with respect to X_0 and X_f.

Key-words: motion planning, optimal control, constrained shortest paths in the plane.

(Résumé : tsvp)

*This research was partially supported by the ESPRIT III BRA Project 6546 (PROMotion).

**INRIA, Sophia-Antipolis
***Université de Nice, Parc Valrose, 06034 Nice Cedex, France

Unité de recherche INRIA Sophia-Antipolis
2004 route des Lucioles, BP 93, 06902 SOPHIA-ANTIPOLIS Cedex (France)
Téléphone : (33) 93 65 77 77 – Télécopie : (33) 93 65 77 65

Note sur les plus courts chemins dans le plan soumis à une contrainte sur la dérivée de la courbure

Résumé : On considère la classe des chemins C^2 dans le plan, C^3 par morceaux, joignant deux configurations données (position, tangente et courbure) X_0 et X_f, le long desquels la dérivée de la courbure (par rapport à l'abscisse curviligne) reste bornée. On admet un nombre infini (dénombrable) de morceaux, mais seulement un nombre fini de points d'accumulations pour les points de commutations. Pour des X_0 et X_f génériques, on prouve que le plus court chemin satisfaisant la contrainte est tel que : soit il ne contient pas de segment de droite, soit il contient aussi un nombre infini d'arcs de clothoide. En conséquence, le nombre de morceaux de classe C^3 constituant un plus court chemin n'est pas uniformément borné par rapport à X_0 et X_f.

Mots-clé : planification de trajectoires, commande optimale, plus courts chemins contraints dans le plan.

1 Introduction

Dubins [3] gave a characterization of the C^1 shortest paths of bounded curvature joining two given points in the plane with prescribed tangents. Reeds and Shepp [5] have solved an extended version of Dubins' problem where a rear gear is added, thus allowing cusps along the path. Both problems can be seen as rough models for the optimization of the motion of a car with or without rear gear. Looking at both Dubins' and Reeds and Shepp' problems from the point of view of optimal control theory gives much shorter proofs of their results, as was remarked by the authors in [1] and also by Sussman and Tang [6]. In this paper, we study here a natural generalization of Dubin's problem and look for the shortest C^2 path between two given points in the plane with prescribed tangents and curvature, with a bound on the derivative of the curvature (speed of the turning wheel). We will also make use of some results from optimal control theory.

We consider a class $\mathcal{C}$ of C^2 paths (in some oriented euclidean plane) joining two given configurations $X_0 = (M_0, I_0, \kappa_0)$ and $X_f = (M_f, I_f, \kappa_f)$, where M_0 (M_f) is a point of the plane, I_0 (I_f) the unit tangent vector and κ_0 (κ_f) the oriented curvature at point M_0 (M_f).

Definition 1 *A path belongs to class $\mathcal{C}$ if it satisfies the following two conditions.*

Regularity : *the path is a C^2 concatenation of an at most countable number of open C^3 arcs of finite length, and the set of endpoints of these arcs, also called the* switching points*, admits at most a finite number of accumulation points.*

Constraint : *the absolute value of the derivative of the curvature along the path, with respect to the arc length, is bounded from above by a given constant $B > 0$, at every point where it is defined.*

We look at such a path as being the trajectory of a point $M(t)$ moving from M_0 to M_f at constant speed 1, so that time and arc length coincide.
In some fixed orthonormal system of coordinates, the function $X(t) = (x(t), y(t), \alpha(t), \kappa(t)) \in \mathbb{R}^2 \times S^1 \times \mathbb{R}$ is thus well-defined and continuous everywhere

along a path of class $\mathcal{C}$. Here, $(x(t), y(t))$ are the coordinates of $M(t)$ in the plane, $\alpha(t)$ is the polar angle of the oriented tangent, and $\kappa(t)$ is the signed curvature of the path, $\kappa(t) > 0$, meaning that the car is turning left.
Between given initial and final configurations $X_0 = (x_0, y_0, \alpha_0, \kappa_0)$ and $X_f = (x_f, y_f, \alpha_f, \kappa_f)$, a path in class $\mathcal{C}$, if it exists, is entirely determined by the function $v(t) = \dot{\kappa}(t)$, defined and continuous everywhere, except at the switching points, by the following differential system :

$$\dot{X}(t) = \begin{cases} \dot{x}(t) = \cos\alpha(t) \\ \dot{y}(t) = \sin\alpha(t) \\ \dot{\alpha}(t) = \kappa(t) \\ \dot{\kappa}(t) = v(t) \end{cases} \tag{1}$$

If we add the boundary conditions $X(0) = X_0$, $X(f) = X_f$, and the constraint :

$$\forall t \in [0, T], \quad |v(t)| \leq B, \tag{2}$$

and if we search for a path of minimum length in class $\mathcal{C}$, we have turned the geometric problem into a classical question of optimal control theory where the functional :

$$J(v) = T = \int_0^T dt \tag{3}$$

is to be minimized among the set of control functions v satisfying (2).

2 Existence of an optimal solution

System (1) may be written as :

$$\dot{X} = F(X, v) = f(X) + v\, g(X)\,,$$

where the analytic vector fields f and g are given by :

$$f(X) = \begin{pmatrix} \cos\alpha \\ \sin\alpha \\ \kappa \\ 0 \end{pmatrix}, \; g(X) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

2.1 Complete controllability of the system

We first observe that the Lie algebra $\mathcal{L}(f,g)$ generated by f and g is, at each point, of dimension 4. Indeed, $\forall X \in \mathbb{R}^4$,

$$h(X) = [g,f](X) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \ , \ i(X) = [h,f](X) = \begin{pmatrix} -\sin\alpha \\ \cos\alpha \\ 0 \\ 0 \end{pmatrix} ,$$

and

$$\det \begin{bmatrix} \cos\alpha & 0 & 0 & -\sin\alpha \\ \sin\alpha & 0 & 0 & \cos\alpha \\ \kappa & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = -1 \, .$$

Moreover, the solutions of the associated autonomous system $\dot{X} = f(X)$ are circles (of radius $1/\kappa$), thus periodic. Hence, Bonnard's theorem [4, thm.III.4] applies, to establish complete controllability of (1) under the constraint (2). This means that any X_0 and X_f can always be joined by a path satisfying (1) and (2).

2.2 Existence of an optimal control

The existence of an optimal control for the problem (1), (2), (3), with given X_0 and X_f, is ensured by Fillipov's existence theorem (see [2, 5.1.ii] for example). Indeed, the hypotheses of the theorem are satisfied. The dynamic $F(X,v)$ and the cost $J(v)$ are smooth enough, the set $[-B,+B]$ of control is convex, and the initial and final configurations X_0 and X_f are fixed. Finally, one can easily check the existence of a constant C such that ${}^tX\,F(X,v) \leq C\,(|X|^2+1)$ for all $t \in [0,T]$, $X \in \mathbb{R}^2 \times S^1 \times \mathbb{R}$, $v \in [-B,+B]$.
Fillipov's theorem then asserts the existence of some $T^* > 0$ and of an optimal control $v^*(t)$ which is a measurable (thus locally integrable) function which satisfies (2) on $[0,T^*]$. The solution of (1) for $v = v^*$ is a path from X_0 to X_f which minimizes cost (3) under constraint (2).

3 Necessary conditions for a solution to be optimal

3.1 Pontryagin's Maximum Principle

We are going to apply Pontryagin's Maximum Principle as stated in [2, 5.1.i] in order to obtain necessary conditions for a solution (i.e. a measurable control v and a trajectory X) to minimize cost (3).
Let us denote by Ψ, ${}^t\Psi = (p, q, \beta, r)$, the adjoint state associated to X. For a minimum time problem, the Hamiltonian H is defined for every $t \in [0, T]$ by

$$H(t) = {}^t\,\Psi(t)\,F(X(t), v(t)) + e_0\,,$$

for some real constant e_0. This yields in the case of system (1) :

$$H(t) = p(t)\,\cos\alpha(t) + q(t)\,\sin\alpha(t) + \beta(t)\,\kappa(t) + r(t)\,v(t) + e_0\,. \tag{4}$$

The adjoint state Ψ is defined on $[0, T]$ as a solution to the adjoint system $\dot{\Psi} = -\frac{\partial H}{\partial X}$, which is here :

$$\dot{\Psi}(t) = \begin{cases} \dot{p}(t) = 0 \\ \dot{q}(t) = 0 \\ \dot{\beta}(t) = p(t)\,\sin\alpha(t) - q(t)\,\cos\alpha(t) = p\,\dot{y}(t) - q\,\dot{x}(t) \\ \dot{r}(t) = -\beta(t)\,. \end{cases} \tag{5}$$

In particular p and q are constant on $[0, T]$ and there exists $\lambda \geq 0$ and $\phi \in [0, 2\pi[$ such that, $\forall t \in [0, T]$:

$$\begin{cases} p(t) \equiv p = \lambda\,\cos\phi \\ q(t) \equiv q = \lambda\,\sin\phi \\ \dot{\beta}(t) = \lambda\,\sin(\alpha(t) - \phi) \\ \dot{r}(t) = -\beta(t)\,. \end{cases} \tag{6}$$

The Hamiltonian (4) can now be written as :

$$H(t, X, v, \Psi) = \lambda\,\cos(\alpha - \phi) + \beta\,\kappa + r\,v + e_0\,. \tag{7}$$

Let us define, for $t \in [0,T]$,

$$M(t) = M(t, X(t), \Psi(t)) = \min_{v \in [-B,+B]} H(t, X(t), v, \Psi(t))\,.$$

Maximum Principle of Pontryagin : *if $v(t)$ is a control which minimizes the cost and if $X(t)$ is the associated solution of (1) then there exists a non-zero corresponding solution $\Psi(t)$ of (6), and they satisfy the four conditions (i) to (iv) :*
(i) $\Psi(t)$ is an absolutely continuous solution of (6), $e_0 \geq 0$, and $(\Psi(t), e_0) \neq (0,0)$ for any $t \in [0,T]$,
(ii) for almost every $t \in [0,T]$,

$$H(t) = M(t)\,,$$

(iii) M is absolutely continuous in $[0,T]$ and, for almost every $t \in [0,T]$:

$$\frac{dM}{dt} = \frac{\partial H}{\partial t}(t, X(t), v(t), \Psi(t)) \equiv 0\,,$$

(iv) $H(T, X(T), v(T), \Psi(T)) = 0$.

A trajectory which satisfies (i), (ii), (iii), and (iv) above will be called *extremal*. In the sequel, we study extremal trajectories belonging to class $\mathcal{C}$, assuming that they exist.

3.2 Characterization of the arcs of the extremal solutions

From (iii), M is constant on $[0,T]$ and so is H from (ii). Thus, it follows from (iv) that

$$H(t) = 0\,,\ \forall t \in [0,T]\,. \tag{8}$$

From (ii) and (7), we deduce that :

$$r(t)\,v(t) \leq 0 \ \text{ for almost every } t \in [0,T]\,. \tag{9}$$

As X belongs to class $\mathcal{C}$, on each open C^3 portion of the trajectory, (ii) implies that $v(t) = \pm B$ with the sign of $-r$ if $r(t) \neq 0$ or, otherwise, that $\frac{\partial H}{\partial v} = r(t) =$

0. If $r(t) \equiv 0$ on some interval $[t_1, t_2] \subset [0, T]$, (6) implies that $\beta(t) \equiv 0$ and $\dot{\beta}(t) \equiv 0$. As α is continuous and $\lambda \neq 0$ (otherwise $p = q = \beta = 0$ and also $e_0 = 0$ since $H = 0$, which is forbiden by (i)), it follows that $\alpha(t) \equiv \phi$ (mod π). Of course then, $\kappa \equiv v \equiv 0$ on $[t_1, t_2]$. Hence, on each open C^3 portion of the path, $v(t) \in \{-B, +B, 0\}$, and since v has to be continuous on such a portion, it is of one of the three kinds :

1. $\mathrm{Cl}^+ : v(t) \equiv B,\ r(t) < 0$
2. $\mathrm{Cl}^- : v(t) \equiv -B,\ r(t) > 0$
3. $\mathrm{S} : v(t) \equiv 0,\ r(t) \equiv 0$

Arcs $\mathrm{Cl}^\pm$ are finite portions of *clothoids*. A clothoid, also known as a "Cornu spiral", is a curve along which the curvature κ depends linearly on the arc length (here equal to t) and varies continuously from $-\infty$ to $+\infty$. Hence, all clothoids Cl^+ (where $v(t) = B$) are translated and rotated copies of a unique clothoid Γ while all clothoids Cl^- (where $v(t) = -B$) are translated, rotated and reflected copies of Γ. Clothoids Cl^+ will be called *direct* clothoids and clothoids Cl^- will be called *indirect* clothoids. The canonical clothoid Γ is chosen as the one defined by the following equations :

$$\begin{aligned} x(t) &= \int_0^t \cos(\frac{B}{2}\tau^2)d\tau \\ y(t) &= \int_0^t \sin(\frac{B}{2}\tau^2)d\tau\ . \end{aligned}$$

Arcs S are line segments, all with the same orientation $\phi \pmod{\pi}$.
From the above discussion, we have :

Proposition 1 *Any extremal path in class $\mathcal{C}$ is the C^2 concatenation of line segments (with the same orientation) and of arcs of clothoids (with $\dot{\kappa} = \pm B$), all of finite length. The control function v is constant on each piece : $v = B$ on a direct clothoid Cl^+, $-B$ on an indirect one Cl^-, and 0 on a line segment S.*

In the sequel, we denote by "Cl" an arc of clothoid, by "S" an open line segment, and by "." a switching point. "Cl_μ" will further specify, when necessary, the length μ of the arc.

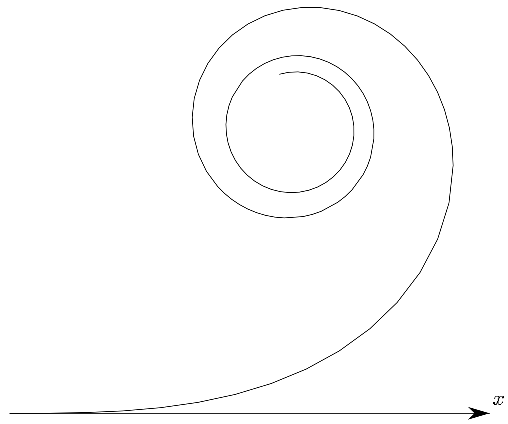

Figure 1: half of the clothoid Γ

In order to characterize the extremal paths, and, among them, the shortest ones, we consider the following problem : *how are these arcs Cl and S arranged together along an extremal trajectory of class $\mathcal{C}$?*
We provide in the next section a partial answer to this question.

3.3 Concatenation of arcs

Lemma 1 $r = 0$ *at any switching point (Cl.Cl, Cl.S or S.Cl).*

Proof : that $r = 0$ at a switching point Cl.S or S.Cl follows from the fact that $r \equiv 0$ on S and that r is continuous. At a switching point Cl.Cl, the sign of v changes and, by (9), also the sign of r. ∎

Lemma 2 *If* $\lambda = 0$*, the extremal path consists of one or two arcs and is of type Cl or Cl.Cl.*

Proof : if $\lambda = 0$, β is constant on $[0, T]$ by (6).
If $\beta = 0$, r is constant on $[0, T]$ by (6). Moreover, r cannot be identically 0, since, otherwise, $(\Psi, e_0) \equiv (0, 0)$, which is forbiden by the Maximum Principle *(i)*. Hence, it follows from Lemma 1 that the extremal path cannot contain a line segment nor a switching point and thus reduces to a single arc Cl.

If $\beta \neq 0$, $r(t)$ is a linear function of t by (6) and then vanishes at most once. Hence the extremal path is of type Cl or Cl.Cl, by Lemma 1. ■

Note that such paths are not *generic* : from any given initial configuration X_0 in $\mathbb{R}^2 \times S^1 \times \mathbb{R}$, the set of final configurations $\{X_f\}$ one can reach through such paths is only 1 or 2-dimensional.

Lemma 3 *If an extremal path contains a line segment S, $\lambda = e_0 > 0$.*

Proof : along a line segment $r \equiv \beta \equiv 0$ and $\alpha \equiv \phi \pmod{\pi}$. Hence, $H \equiv e_0 + \varepsilon\,\lambda = 0$, with $\varepsilon = \pm 1$. As $e_0 \geq 0$ and $\lambda > 0$ (from Lemma 2), we must have $\varepsilon = -1$ and $e_0 = \lambda$. ■

From the proof of the previous lemma, $\varepsilon = \cos(\alpha - \phi) = -1$ on S, and we have :

Corollary 1 *Along a line segment S, $\alpha \equiv \phi + \pi \pmod{2\,\pi}$.*

Lemma 4 *$\beta - p\,y + q\,x$ is constant along any extremal path. If $\lambda > 0$, for any given $c \in \mathbb{R}$, all the points of an extremal path where $\beta = c$ lie on the same straight line D_c, of direction $\phi \pmod{\pi}$.*

Proof : $\dot\beta = p\,\dot y - q\,\dot x$ from (5), and p and q are constant. Thus there exists a constant c_0 such that $py - qx = \beta + c_0$, which proves the first part of the lemma. If $\lambda \neq 0$, p and q cannot be both equal to 0 and $py - qx = c + c_0$ is the equation of a line of direction $\alpha = \phi \pmod{\pi}$. ■

As a consequence, we have :

Corollary 2 *Any line segment S of an extremal path is contained in D_0 and is run with $\alpha \equiv \phi + \pi \pmod{2\,\pi}$.*

Proof : since $\beta \equiv 0$ on S, it follows from Lemma 4 that S is contained in the line D_0 of direction ϕ. By Corollary 1, $\alpha \equiv \phi + \pi \pmod{2\,\pi}$. ■

Lemma 5 *If $\lambda > 0$, each open arc of clothoid Cl_μ with $\mu > 0$ of an extremal path, except possibly the initial and the final ones, intersects D_0 at least once.*

Proof : let Cl_μ be an arc of length μ of an extremal path which is not the initial nor the final arc. Both endpoints of such an intermediate arc are switching points. Let $]t_1, t_2[$ denote the time interval during which this intermediate arc Cl_μ is run. By Lemma 1, $r(t_1) = r(t_2) = 0$. As $t_2 - t_1 = \mu > 0$, there exists at least one $t \in]t_1, t_2[$, say t_3, such that $\dot{r}(t_3) = 0$ and thus, from (6), $\beta(t_3) = 0$. Finally, it follows from Lemma 4 that $M(t_3)$ belongs to D_0. ■

Observe that the hypothesis $\lambda > 0$ along an extremal path is true as soon as it contains either a line segment (Lemma 3) or more than only two arcs of clothoid (Lemma 2).

Lemma 6 *An extremal path contains no portion of type S.Cl_μ.Cl or of symmetric type Cl.Cl_μ.S with $\mu > 0$.*

Proof : assume that there exists such a portion S.Cl_μ.Cl and let $]t_1, t_2[$ denote the time interval during which Cl_μ is run, with $t_2 - t_1 = \mu > 0$. From Lemma 2, $\mathrm{S} \subset D_0$, and since the variables (x, y, α, κ) are continuous on $[t_1, t_2]$, Cl_μ is tangent to D_0 at $M(t_1)$ and $\kappa(t_1) = 0$. Hence, $M(t_1)$ is the inflexion point of the clothoid supporting Cl_μ and D_0 is the tangent to the clothoid at $M(t_1)$. This implies that $\mathrm{Cl}_\mu \setminus \{M(t_1)\}$ is entirely contained in an open half–plane delimited by D_0, see figure 1, which contradicts Lemma 5. ■

The last lemma is in fact superseded by the following, due to H.J. Sussmann.

Lemma 7 *An extremal path contains no portion of type S.Cl_μ with $\mu > 0$ (or Cl_μ.S).*

Proof : assume that there is a portion of type S.Cl_μ, with $\mu > 0$, in an extremal trajectory and let t_1 be the switching time between S and Cl_μ. From (6), (7), and (8), we obtain the following expressions of the four first derivatives of r (valid on S as well as on Cl_μ) :

$$\begin{cases} \ddot{r} &= -\lambda \sin(\alpha - \phi) \\ \dddot{r} &= -\lambda\, \kappa \cos(\alpha - \phi) \\ \ddddot{r} &= \lambda\, \kappa^2 \sin(\alpha - \phi) + (\beta\, \kappa + r\, v + e_0)\, v\,. \end{cases}$$

Hence, the adjoint variable r is of class C^3 in the neighborbood of t_1. Moreover, on S, $r = \dot{r} = 0$, $\alpha = \phi + \pi \pmod{2\pi}$, $\beta = 0$, $\kappa = 0$. From the above equations, we also have $\ddot{r} = \dddot{r} = 0$ on S, and, by continuity, at t_1. Moreover, $\ddddot{r}\,(t_1) = e_0\,v$. Thus, there exists an ε, $0 < \varepsilon \le \mu$, such that for $t \in [t_1, t_1 + \varepsilon[$ we have :

$$r(t) = e_0\,v(t)\,\frac{(t-t_1)^4}{4!} + o((t-t_1)^5)\,.$$

Now, from Lemma 3, $e_0 > 0$, so that r and v have the same sign on $[t_1, t_1 + \varepsilon[$ which contradicts (9). ∎

A consequence of Lemmas 6 and 7 is the :

Proposition 2 *If an extremal path of class $\mathcal{C}$ contains but is not reduced to a line segment, then it contains an infinite number of concatenated clothoid arcs which accumulate towards each endpoint of the segment which is a switching point.*

Proposition 2 together with the fact that a clothoid Cl is contained in a ball of bounded diameter D_{Cl} (depending on the parameter B) implies the following :

Proposition 3 *The number n of C^3 pieces contained in a generic extremal path cannot be uniformly bounded from above (with respect to X_0, X_f). However, if $d(M_0, M_f)$ denotes the euclidean distance in the plane between M_0 and M_f, we have that :*

$$n \ge \frac{d(M_0, M_f)}{D_{Cl}}\,.$$

Proof : either the shortest path contains (and is generically not reduced to) a line segment, and Proposition 2 implies that there are infinitely many arcs of clothoid, or it is made only with arcs of clothoid, the number of which clearly depends on (and increases with) the distance between X_0 and X_f. The bound from below is obvious. ∎

4 Conclusion

Note that it is not clear whether or not extremal trajectories described in Proposition 2, and, among them, the optimal ones, belongs to class $\mathcal{C}$: indeed, the set of switching points on an optimal trajectory (points where the control v is undefined) might even be uncountable. Moreover, we don't know yet if the statement of this proposition remains true whithout the assumption that the path contains a line segment. These points are under study.

However, Propositions 2 and 3 already indicate that the optimal control associated to problem (1), (2), (3) has a complex behaviour. Contrarily to what occurs for Dubins or Reeds and Shepp problems, for which every optimal trajectory contains at most a prescribed (finite) number of line segment and arcs of circles, the number of switching points is unbounded here and might be infinite.

Acknowledgments

The authors thank H.J. Sussmann for helpful comments concerning the questions studied here.

References

[1] J.-D. Boissonnat, A. Cérézo, and J. Leblond. Shortest paths of bounded curvature in the plane. In *Proc. 9th IEEE Internat. Conf. Robot. Autom.*, 1992.

[2] L. Cesari. Optimization theory and applications. Applications of Mathematics, No 17. Springer-Verlag, 1983.

[3] L. E. Dubins. On curves of minimal length with a constraint on average curvature and with prescribed initial and terminal positions and tangents. *Amer. J. Math.*, 79:497–516, 1957.

[4] C. Lobry. Contrôlabilité des systèmes non linéaires. In *Outils et modèles mathématiques pour l'automatique, l'analyse de systèmes et le traitement du signal*, 1: 187-214, CNRS, 1981.

[5] J. A. Reeds and L. A. Shepp. Optimal paths for a car that goes both forwards and backwards. *Pacific Journal of Mathematics*, 145(2), 1990.

[6] H. J. Sussmann and G. Tang. Shortest paths for the Reeds-Shepp car : a worked out example of the use of geometric techniques in nonlinear optimal control. Research Report SYCON-91-10, Rutgers University, New Brunswick, NJ, 1991.

Unité de recherche INRIA Lorraine, Technôpole de Nancy-Brabois, Campus scientifique,
615 rue de Jardin Botanique, BP 101, 54600 VILLERS LÈS NANCY
Unité de recherche INRIA Rennes, IRISA, Campus universitaire de Beaulieu, 35042 RENNES Cedex
Unité de recherche INRIA Rhône-Alpes, 46 avenue Félix Viallet, 38031 GRENOBLE Cedex 1
Unité de recherche INRIA Rocquencourt, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex
Unité de recherche INRIA Sophia-Antipolis, 2004 route des Lucioles, BP 93, 06902 SOPHIA-ANTIPOLIS Cedex

Éditeur
INRIA, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)
ISSN 0249-6399