On topology in multidimensional discrete

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Abstract: To study topology in digital binary images, different *discrete connectivities* have to be used for both the object and the background. We prove equivalences between discrete connectivities, as they are usually defined in a digital space, and connectivity in a continuous space. For that purpose, we introduce the notion of regularization of a set of voxels. It will permit us to define continuous sets from discrete ones. We discuss then the choice of a "good" couple of discrete connectivities for both the object and the background, and provide a characterization of such a valid pair. Finally, we use our best choice to demonstrate a separation theorem in dimension n for surfaces made of voxels.

(Résumé : tsvp)

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Topologie des espaces discrets multidimensionnels

Résumé : L'étude de la topologie des images binaires digitalisées nécessite l'utilisation de différentes *connexités discrètes* pour le fond et l'objet. Nous prouvons que les connexités discrètes usuellement définies dans un espace discret sont équivalentes à la connexité d'un espace continu. A cet effet, nous introduisons la notion de régularisation d'un ensemble de voxels. Cela nous permettra de définir des ensembles continus à partir d'ensembles discrets. Nous discuterons également le choix d'un "bon" couple de connexités discrètes pour le fond et l'objet, et nous établirons la caractérisation d'un tel couple. Finalement, nous utiliserons notre meilleur choix pour démontrer un théorème de séparation en dimension n pour des surfaces constituées de voxels.

1 Introduction

Multi-dimensional images may come from several fields, the most popular one being the medical field where three-dimensional (3-D) images are produced by X-ray Computed Tomography (CT), Magnetic Resonance Imaging (MRI), Position Emitting Tomography (PET), or Single Photon Emission Computed Tomography (SPECT), and four-dimensional (4-D) images may be a temporal sequence of 3-D images.

2-D images are well suited for the human perception because they may be displayed on a computer screen or printed on a sheet of paper. On the contrary, higher dimensional images are more complicated to visualize. 3-D images may be displayed by 2-D cross-sections but there is no volume rendering which will simplify their interpretation by physicians. For that purpose, which is particularly important in medical imaging, volume rendering or surface rendering algorithms have been widely proposed (a survey is done by Stytz and al. in [1]). These techniques need topological notions such as the connectivity of regions, the boundaries, the adjacency, etc.

The 2-D digital topology has been already widely studied (Kong and Rosenfeld did a recent survey in [2]), but its extension to higher dimensions is not obvious. A n-dimensional image is usually defined as a subset of $\mathbb{Z}^n$, that means we use cubic lattices. For that kind of images, it is well known that different *discrete connectivities* should be used for both the object and the background, else some topological paradoxes may appear (the Jordan theorem –or its converse– is no more verified) [2, 3]. Such a pair of connectivities is not hard to find in 2-D, because we dispose only of two usual discrete connectivities. It may become difficult in n-D because one may defined in a n-dimensional image (defined by a n-dimensional cubic lattice) n different discrete connectivities. Moreover, a mathematical framework is absolutely necessary because of the impossibility to check visually the validity of any result (as it is possible in 2-D).

In order to avoid this difficulty, one may use another lattices than the cubic one. For example, in a hexagonal lattice, each point has only one type of neighbor, then we do not have to define different connectivities for the object and the background. However, if we try to extend such a lattice to higher dimension, this difficulty may reappear. Ponce proposed a 3-D extension of the hexagonal lattice, which is the rhombododecahedronal one [4]: in this new lattice, each point has two different types of neighbor and different connectivities must be chosen.

We will prove here equivalences between discrete connectivities in a n-dimensional digital space (with a cubic lattice) and connectivity in the n-dimensional continuous space $\mathbb{R}^n$. Recent publications prove that this problem is still actual [5]. For that purpose, we use first a mathematical framework based on cellular complexes (already used by Kovalevsky [3] or Herman [6] in the image processing field, and by many others in the Computer Aided Design (CAD) field [7]), and we introduce the notion of *regularization* of a set of voxels. It permit us to define continuous sets from discrete ones.

Then, we investigate the case of binary images where different discrete connectivities have to be used for the object and the background. We discuss the choice of a "good" couple of discrete connectivities.

Finally, to illustrate this choice, we demonstrate a separation theorem for n-dimensional surfaces, for surfaces constituted by voxels.

Separation theorems are particularly important to validate surface or volume rendering algorithms.

The first way to define surfaces in to consider them as a set of voxel faces. This approach has been widely studied for visualization techniques. 3-D boundary tracking algorithms (where boundaries are defined by sets of voxel faces organized in graphs) were proposed by Artzy, Frieder and Herman [8], or by Gordon and Udupa [9]. Proofs on the topological properties of these algorithms have been difficult to establish and appears only several years later [10, 11]. Moreover, this problem is important enough to be the subject of theoretical publications.

For this point, the introduction of cellular complex has been useful. For instance, the theory of algebraic topology show that a set of faces defines 2-manifolds if the following conditions are satisfied:

- faces may intersect only in edges or vertices
- any edge belongs to only two faces
- faces around a vertex may be arranged in a cycle such that any pair of consecutive faces in the cycle intersect in an edge adjacent to the vertex.

This result has been proved again by Keskes and Faugeras in 1981 [12]. Rosenfeld, Kong and Wu studied such surfaces in 1991 [13] and Herman proposed recently a mathematical framework in order to deal with n-dimensional surfaces [14]. As there is a considerable work for this particular approach, our purpose is then not to develop it, but to study an alternative approach that is to consider surfaces as a set of voxels rather than a set of faces.

This second definition comes from the classical edge detection widely used in the computer vision field (see [15, 16, 17]): edges are either maxima of the gradient norm in the direction of the gradient or zero crossings of the laplacian. In that case, surfaces are constituted by voxels.

Morgenthaler and Rosenfeld proposed such a definition of a 3-D surface in 1981 [18]. Their article defined the notion of *simple surface* and gave a proof of the 3-D Jordan theorem for simple closed surfaces. However, the proposed definition of simple surface is not appropriate: a counter-example is given by Malandain, Bertrand and Ayache [19]. They proposed a new definition of simple surfaces, but did not use it to characterize Jordan surfaces.

2 Cellular complex

We recall here basic notions on cellular complexes.

2.1 Cellular decomposition

A cellular decomposition of a topological space K is defined as a partition of K in cells such that each element is homeomorphic to a numerical space. For all $n \geq 1$ and for all k $(0 \leq k \leq n)$ a k-dimensional cell is a subset of $\mathbb{R}^n$ homeomorphic to a k-dimensional open sphere.

When referring to a cellular decomposition of a space K, the following properties stand :

- K is a cellular complex.
- each element e of this partition is a cell of dimension k, or k-cell (a k-cell is called vertex for $k = 0$, edge for $k = 1$, face for $k = 2, \ldots$).
- K^k is the skeleton of dimension k (or k-skeleton) of K : it is the union of the elements of the partition of K which have a dimension less or equal to k.
- K is the union of subspaces K^k

A k-cell is *incident* to a k'-cell, if $k < k'$, if the k-cell is included in the closure[1] (according to the topology of K) of the k'-cell and if their intersection is empty.

A cellular complex K is finite if the set of its cells is finite, else it is infinite. A cellular complex K has a finite dimension k if it does not include cells of dimension $n > k$, else it has an infinite dimension. If K is a finite cellular complex, a k-cell is an open set of K^k, k-cells are then the connected components of $K^k \setminus K^{k-1}$.

2.2 Topology of cellular complexes

A cellular complex $K = (C, B, \dim)$ is an abstract set made of a subset C of cells with an antisymmetric, irreflexive and transitive relation B included in $C \times C$ called the bounding relation (or face relation) and a dimension function dim from C to the set $\mathbb{N}^+$ of positive integers such that $\dim(c_1) < \dim(c_2)$ for all pairs $(c_1, c_2) \in C$. The bounding relation B is a partial order in C. It indicates the ordered pairs (c_1, c_2) such that c_1 is said to bound c_2 which is denoted by $c_1 < c_2$. It means that c_1 is a part of the boundary of c_2.

The function dim defines the dimension of each cell. A cell c with $\dim(c) = k$ is called a k-cell. A complex is called k-dimensional or k-complex if the dimension of all its cells is less or equal to k.

[1] The closure of a set A is the smallest closed set including A. We denote it by $\overline{A}$.

The boundary of a subset $S \subset K$ relative to K is the set $\mathrm{Bd}(S, K)$ of cells c of K such that any open neighborhood of c contains cells both of S and its complement.

The open star of a cell $c \in K$, denoted by $\mathrm{St}(c, K)$ (or $\mathrm{St}(c)$ if there is no possible confusion), is the open set including c and all cells of K bounded by c. It should be noticed that open stars are the simplest open sets in a complex. Moreover, all open subsets are unions of some open stars.

A subcomplex $K' = (C', B', \dim')$ of a complex $K = (C, B, \dim)$ is a complex whose set C' is a subset of C and the relation B' is the restriction of B to $C' \times C'$. The bounding relation is the same as the one of K for each pair of cells of C'. Dimensions do not change: $\dim(c) = \dim'(c)$ for all $c \in C'$. To define a subcomplex K' of K, it is sufficient to define the corresponding subset C'.

3 Discrete connectivities

A n-dimensional image is usually defined as a subset of $\mathbb{Z}^n$, that means we use cubic lattices. In a binary image, each point may have only one of two values, say 0 and 1. The set of all points with a 1 value is called the *object* of the image, the set of all points with a 0 value is called the *background*. The main difficulty of the study of a n-dimensional binary image is that different connectivities must be used for the object and the background.

To study these discrete connectivities, we propose to consider a cellular decomposition of $\mathbb{R}^n$.

3.1 Cellular decomposition of $\mathbb{R}^n$

We denote C the set of parts of the n-dimensional space $\mathbb{R}^n$ such that $c \in C$ is equivalent to $c = I_1 \times \ldots \times I_n$, where I_i is a subset of $\mathbb{R}$ of the form $\{z_i\}$ or $]z_i, z_i + 1[$ with $z_i \in \mathbb{Z}$.

C is obviously a partition of $\mathbb{R}^n$. We denote by $\mathcal{P}(C)$, the set of parts of C.

Definition 3.1 *We call k-cell, an element $c \in C$ such that c has k I_i of the form $]z_i, z_i + 1[$ (and then $(n-k)$ I_i of the form $\{z_i\}$). The dimension of c is k.*

- n-cells are open sets of $\mathbb{R}^n$, they are the voxels of the image (voxel is here a generic term for designing the basic element of a digital image).
- 0-cells are closed sets of $\mathbb{R}^n$, they are the voxel vertices.
- k-cells, with $0 < k < n$, are neither open, nor closed.

Definition 3.2 *The closure of a* k*-cell $c \in C$ is defined by: $\overline{c} = J_1 \times \ldots \times J_n$ with:*

- *if I_i has the form $]z_i, z_i + 1[$, then* $$J_i = [z_i, z_i + 1] = \{z_i\} \cup]z_i, z_i + 1[\cup \{z_i + 1\}$$

- *if I_i has the form $\{z_i\}$, then $J_i = I_i$*

$\overline{c}$ is obviously the union of cells of C, we have $\overline{c} \in \mathcal{P}(C)$.

Definition 3.3 *We define the bounding relation B by $B(c', c)$ if and only if $c' \in \overline{c} \setminus \{c\}$. As $\overline{c} = \{c\} \cup \mathrm{Bd}(c)$, $c' \in \overline{c} \setminus \{c\}$ is equivalent to $c' \in \mathrm{Bd}(c)$.*

The subset C with the dim function and the bounding relation B define a cellular decomposition of $\mathbb{R}^n$. Every cell of C is a connected part of $\mathbb{R}^n$.

Proposition 3.4 *Let c be a k-cell of C, then*

- *$\forall c' \in C \;/\; c' \in \overline{c}$, c' is a ℓ-cell of C with $0 \leq \ell \leq k$*
- *let ℓ, $0 \leq \ell \leq k$, then $\overline{c}$ contains exactly $2^{(k-\ell)} C_k^{k-\ell}$ ℓ-cells, with $C_n^p = \frac{n!}{p!(n-p)!}$*

Proof: c contains $(n-k)$ I_i of the form $\{z_i\}$, if $c' \in C$ and $c' \in \overline{c}$, c' will contain these same $(n-k)$ I_i. The ℓ-cells of $\overline{c}$ are made by changing the remaining k intervals I_i of the form $]z_i, z_i+1[$: we have then $0 \leq \ell \leq k$.

To obtain a ℓ-cell, we have to modify $(k-\ell)$ intervals among k, there are $C_k^{k-\ell} = \frac{(k)!}{(k-\ell)!(\ell)!}$ different ways to do it. As any interval I_i of the form $]z_i, z_i+1[$ can be changed into $\{z_i\}$ or $\{z_i+1\}$, there are $2^{(k-\ell)}$ combinations for each possibility. ∎

For instance, if c is a k-cell of C with $k < n$, $\overline{c}$ contains:

- $2k$ $(k-1)$-cells
- 2^k 0-cells

Definition 3.5 *Similarly to the closure, the open star of a* k*-cell $c \in C$ is defined by:* $\mathrm{St}(c) = J_1 \times \ldots \times J_n$ *with:*

- *if I_i has the form $]z_i, z_i+1[$, then $J_i = I_i$*
- *if I_i has the form $\{z_i\}$, then*
$$J_i = \;]z_i-1, z_i+1[\;=\;]z_i-1, z_i[\cup \{z_i\} \cup]z_i, z_i+1[$$

Proposition 3.6 *Let c be a k-cell of C, then*

- *$\forall c' \in C \;/\; c' \in \mathrm{St}(c)$, c' is a ℓ-cell of C with $k \leq \ell \leq n$*
- *let ℓ, $k \leq \ell \leq n$, then $\mathrm{St}(c)$ contains exactly $2^{(\ell-k)} C_{n-k}^{\ell-k}$ ℓ-cells, with $C_n^p = \frac{n!}{p!(n-p)!}$*

The proof of proposition 3.6 is analogous to the one of proposition 3.4. For instance, if c is a k-cell of C with $k < n$, $\mathrm{St}(p)$ contains:

- $2(n-k)$ $(k+1)$-cells
- 2^{n-k} n-cells

From the above definitions, we obtain easily the following properties:

- $c \in \mathrm{St}(c)$
- $c \in \overline{c}$
- $\mathrm{St}(c) \cap \overline{c} = c$
- $\mathrm{St}(c)$ and $\overline{c}$ are connected subsets of $\mathbb{R}^n$
- $\forall c' \in C \;/\; c' \in \mathrm{St}(c)$, $\mathrm{St}(c') \subset \mathrm{St}(c)$
- $\forall c' \in C \;/\; c' \in \overline{c}$, $\overline{c'} \subset \overline{c}$

Proposition 3.7 *For all c and c' of C, we have $c' \in \mathrm{St}(c) \Longleftrightarrow c \in \overline{c'}$.*

Proof: We show first that $c' \in \mathrm{St}(c)$ implies $c \in \overline{c'}$.
As $c \in C$, we may express it by[2]: $c = \prod I_i^1 \times \prod I_i^2$ with I_i^1 of the form $]z_i, z_i+1[$ and I_i^2 of the form $\{z_i\}$.
We have then $\mathrm{St}(c) = \prod I_i^1 \times \prod I_i^3$ with I_i^3 of the form $]z_i-1, z_i+1[$. A cell c' of C included in $\mathrm{St}(c)$ will be denoted by $c' = \prod I_i^1 \times \prod I_i^4$, I_i^4 may have one of the three following form: $]z_i-1, z_i[$, $\{z_i\}$ or $]z_i, z_i+1[$.
We may express $\overline{c'}$ by $\overline{c'} = \prod J_i^1 \times \prod J_i^4$ with $J_i^1 = [z_i, z_i+1]$ and J_i^4 takes one of the three following form: $[z_i-1, z_i]$, $\{z_i\}$ or $[z_i, z_i+1]$.
It is then obvious that $I_i^1 \subset J_i^1$ and $I_i^2 \subset J_i^4$, hence c is included in $\overline{c'}$.
The proof of the converse is analogous. ∎

3.2 Definition of discrete connectivities

Using the above definitions and properties of the partition of $\mathbb{R}^n$, we will define discrete connectivities in an image.

Definition 3.8 *Two n-cells (or voxels) c_1 and c_2 are said k-connected (or k-adjacent) if there exists a k-cell c, $k < n$, such that $c_1 \in \mathrm{St}(c)$ and $c_2 \in \mathrm{St}(c)$.*

Proposition 3.9 *If two n-cells c_1 and c_2 are k-connected, then they are ℓ-connected for $0 \le \ell \le k$.*

[2] This notation is evidently abusive. In fact we should define an indices permutation that allows us to rewrite c with a reorganized order of intervals of $\mathbb{R}$. However, as it does not have an effect on the validity of proofs, we will use this reorganization each time it is necessary.

Proof: If two n-cells c_1 and c_2 are k-connected, there exists a k-cell c such that $c_1 \in \mathrm{St}(c)$ and $c_2 \in \mathrm{St}(c)$. For all ℓ, $0 \leq \ell \leq k$, $\overline{c}$ contains $2^{(k-\ell)} C_k^{k-\ell}$ ℓ-cells according to proposition 3.4: let c' be one of these ℓ-cells.

$c' \in \overline{c}$ implies $c \in \mathrm{St}(c')$, according to proposition 3.7. Then $\mathrm{St}(c) \subset \mathrm{St}(c')$ and we have $c_1 \in \mathrm{St}(c')$ and $c_2 \mathrm{St}(c')$. Hence c_1 and c_2 are ℓ-connected. ∎

Corollary 3.10 *Two n-cells c_1 and c_2 are k-connected if there exists a ℓ-cell c, $k \leq \ell < n$, such that $c_1 \in \mathrm{St}(c)$ and $c_2 \in \mathrm{St}(c)$.*

For instance, in a discrete space of dimension 3, the usual 26-connectivity, the 18-connectivity and the 6-connectivity correspond respectively to the 0-connectivity, 1-connectivity and 2-connectivity as defined above. The corollary 3.10 stands that two 6-connected voxels are 18-connected and 26-connected too, that is usually admitted.

Proposition 3.11 *A necessary and sufficient condition for two n-cells c_1 and c_2 to be k-connected is that $\overline{c_1} \cap \overline{c_2}$ contains a ℓ-cell with $k \leq \ell < n$.*

Proof: If a ℓ-cell c is included in the intersection $\overline{c_1} \cap \overline{c_2}$, then we have $c_1 \in \mathrm{St}(c)$ and $c_2 \in \mathrm{St}(c)$ according to proposition 3.7. c_1 and c_2 are ℓ-connected then k-connected (proposition 3.9). The condition is sufficient.

Moreover, if c_1 and c_2 are k-connected, there exists a k-cell c such that $c_1 \in \mathrm{St}(c)$ and $c_2 \in \mathrm{St}(c)$. According to proposition 3.7, we have $c \in \overline{c_1}$ and $c \in \overline{c_2}$, then q belongs to the intersection $\overline{c_1} \cap \overline{c_2}$. The condition is necessary. ∎

In fact, we are able to show that the intersection $\overline{c_1} \cap \overline{c_2}$ that is a closed set of $\mathbb{R}^n$ (as the intersection of two closed sets) is the closure of a single k-cell belonging to the intersection, and that there does not exist ℓ, $k < \ell$, such that c_1 and c_2 are ℓ-connected. This k-cell may define a “maximum” connectivity between c_1 and c_2. However, as we do not need this result and this notion of “maximum” connectivity, we did not show it.

These discrete k-connectivities, as defined above, give rise to the notion of k-connected components.

3.3 Subsets of a discrete image

A subset of a discrete image is a set of voxels, for instance the voxels with value 1 in a binary image i.e. the *object*. If we consider now these voxels as n-cells of $\mathbb{R}^n$, they are obviously separate connected components. This means that the number of connected components of the *object* is equal to its number of voxels.

To recover the notions of discrete connectivities we are studying, we have to introduce some regularization operations on this subset. These operations will reconnect the voxels to form connected components.

Definition 3.12 *A subset X of $\mathcal{P}(C)$ is a 0-regular set if it is equal to the closure of its interior* [3], *that is, $X = \overline{\mathring{X}}$. For any subset X, we denote by $R_0(X)$ its 0-regularization $\overline{\mathring{X}}$.*

Definition 3.13 *A subset X of $\mathcal{P}(C)$ is a $(n-1)$-regular set if it is equal to the interior of its closure, that is, $X = \mathring{\overline{X}}$. For any subset X, we denote by $R_{n-1}(X)$ its $(n-1)$-regularization $\mathring{\overline{X}}$.*

If these notions of regularization are new in the image processing field, they are already used in CAD.

Requicha uses *regularized* operations between objects in his geometric modeling system in order to manipulate *valid* solids: they suppress the "dangling" part of an object [7]. These *regularized* operations such as union, intersection, difference and complement are defined as the $(n-1)$-regularization of the result of the classical operations.

To deal with 3-manifolds with non-2-manifold borders (for instance, two parallelepipeds intersecting each other at one single edge or one single vertex), Arbab defines an other regularized union as the 0-regularization of the classical union [20].

If we consider a set of voxels X, we have the following results.

Proposition 3.14 $R_0(X) = \bigcup_{c n\text{-cell} \in X} \overline{c} = \bigcup_{c' \in X \cap C} \overline{c'}$. *$R_0(X)$ is the union of closures of voxels (n-cell) of X, and even the union of closures of its cells.*

Proof: Voxels (n-cell) are already open parts of $\mathbb{R}^n$, thus as X is a subset of voxels, we have $R_0(X) = \bigcup_{c n\text{-cell} \in X} \overline{c}$. For all cells c' of C included in $R_0(X)$, there exists a n-cell c such that $c' \in \overline{c}$, then we have $\overline{c'} \subset \overline{c}$ and $R_0(X) = \bigcup_{c' \in X \cap C} \overline{c'}$. ∎

Proposition 3.15 $R_{n-1}(X) = \bigcup_{c' \in X \cap C} \mathrm{St}(c')$. *$R_{n-1}(X)$ is the union of open stars of its cells. We may rewrite it by $R_{n-1}(X) = R_0(X) \setminus \mathrm{Bd}(R_0(X))$.*

Proof: $R_{n-1}(X)$ is the interior of $\overline{X}$, as $\overline{X} = R_0(X)$, $R_0(X)$ and $R_{n-1}(X)$ have the same boundary and we obtain $R_{n-1}(X) = R_0(X) \setminus \mathrm{Bd}(R_0(X))$.

According to the definition 3.5 of open stars, for any point of a k-cell, it is not hard to show that we may find an open neighborhood included in the open star of the k-cell. Thus, as a point belongs to the boundary if all its neighborhoods contain points both of $R_{n-1}(X)$ and its complement, the open star of

[3] The interior of a set A is the greatest open set included in A. We denote it by $\mathring{A}$.

its cell contains points both of $R_{n-1}(X)$ and its complement too. By contraposition, the open star of cells included in $R_{n-1}(X)$ are included in $R_{n-1}(X)$ too.■

In fact, a ℓ-cell belongs to the $(n-1)$-regularization of a set X of voxels if all the voxels of its open star belong to this set. It means that this ℓ-cell is completely surrounded by voxels of X.

As we will prove below, the connectivity of respectively $R_0(X)$ and $R_{n-1}(X)$ are equivalent to the 0-connectivity and the $(n-1)$-connectivity of X. In order to study other discrete k-connectivities $(0 < k < n-1)$, we introduce the notion of k-regularization. The k-regularization of a set of voxels X, denoted by $R_k(X)$ will be "less" connected than its 0-regularization but "more" connected than its $(n-1)$-regularization.

Definition 3.16 *A subset X of $\mathcal{P}(C)$ is a k-regular set $(0 < k < n-1)$ if it is equal to its 0-regularization minus the $(k-1)$-skeleton of its 0-regularization. For any subset X, we denote by $R_k(X)$ its k-regularization $R_0(X)\backslash(R_0(X))^{k-1}$.*

To obtain a k-regularization, we remove from the 0-regularization cells that have a dimension less than k.

According that the -1-skeleton (that is not defined) is an empty set, this definition stands for the 0-regularization too, but not for the $(n-1)$-regularization.

However, we prove in appendix A that the $(n-1)$-regularization of a set of voxels X according to definition 3.13 and the $(n-1)$-regularization of X that derives from definition 3.16 have the same number of connected components, and that these connected components contains the same voxels. Then these two definitions can be considered as equivalent to study sets of voxels.

3.4 Connectivity and discrete connectivities

In this part, we will prove the equivalence between the discrete k-connectivity for a set of voxels X and the connectivity of its k-regularization.

Proposition 3.17 *The k-connected components of a set of voxels X are the connected components of $R_k(X)$.*

Proof: We will first prove that, if two voxels of X belong to the same k-connected component of X, they belong to the same connected component of $R_k(X)$.

Two n-cells r_1 and r_2 of X belong to the same k-connected component of X if there exists a k-path[4] of voxels of X between r_1 and r_2. Let us denote this path by the serie r^i (for i from 0 to j) with :

- $r^0 = r_1$

[4] The notion of k-path derives from the notion of k-connectivity.

- $r^j = r_2$
- r^{i-1} is k-adjacent to r^i for i from 1 to j.

By definition, r^{i-1} is k-adjacent to r^i if there exists a ℓ_i-cell c^i, $k \leq \ell_i < n$, such that $r^{i-1} \in \mathrm{St}(c^i)$ and $r^i \in \mathrm{St}(c^i)$. This ℓ_i-cell belongs to $\overline{r^{i-1}}$, then it belongs to $R_0(X)$. As its dimension is greater or equal to k, it belongs to $R_k(X)$ too.

Let us consider $r^{i-1} \cup c^i \cup r^i$. It is a connected part of $\mathbb{R}^n$ (as $c^i \subset \overline{r^{i-1}}$, $r^{i-1} \cup c^i$ is connected). Then $\bigcup_{i=1\dots j} (r^{i-1} \cup c^i \cup r^i)$ is a connected part of $\mathbb{R}^n$ and is included in $R_k(X)$. Thus r_1 and r_2 belong to the same connected component of $R_k(X)$.

The proof of the converse is analogous. ∎

4 Topology of a n-dimensional binary image

4.1 Choice of discrete connectivities

Let us consider an n-dimensional digital image I_d as a block of n-cells of $\mathbb{R}^n$. In order to avoid any trouble with the image border, we consider its counterpart I_c in the continuous space defined by $I_c = \overset{\circ}{\overline{I_d}}$: it is the smallest connected open set of $\mathbb{R}^n$ that contain all n-cells. In the following, we consider the topology of $\mathbb{R}^n$ in I_c (open sets of I_c are the intersection of open sets of $\mathbb{R}^n$ with I_c).

The binarization of I_d defines a partition of I_d in two parts, one is usually called the *object*, denoted by X_d, and the other the *background*, denoted by B_d. By definition, we have $I_d = X_d \cup B_d$.

How to choose "good" discrete connectivities for both X_d and B_d ? In fact, the validity of a choice will depend on the counterpart of X_d and B_d in the continuous space, denoted respectively by X_c and B_c.

If we consider the k_X-connectivity for X_d and k_B-connectivity for B_d, the continuous counterparts of X_d and B_d are respectively $X_c = R_{k_X}(X_d)$ and $B_c = R_{k_B}(B_d)$.

Proposition 4.1 *To be valid, a pair of discrete connectivities must contain the* $(n-1)$*-connectivity.*

Proof: A couple of discrete connectivities is valid if open stars of cells of $X_c \cap B_c$ contain at the most three n-cell.

If $X_c \cap B_c$ includes a cell c with an open star containing more than three n-cell, i.e. four n-cells, they may be two n-cells of X_d and two of B_d. Then two parts of X_c may be connected together by the same cell than two parts of B_d, that may cause a connectivity paradox.

As the open star of a k-cell contains 2^{n-k} n-cells, thus $X_c \cap B_c$ should contain only $(n-1)$-cells.

From definitions of the previous section, we can consider the 0-regularization of X (5) as $R_0(X) = \overset{\circ}{\overline{X}} \cup \mathrm{Bd}(\overset{\circ}{\overline{X}})$, its k_X-regularization will be $R_{k_X}(X) = R_0(X) \setminus (R_0(X))^{k_X-1}$.

It is obvious that $\overset{\circ}{\overline{X}} \cap \overset{\circ}{\overline{B}} = \emptyset$, thus we have

$$X_c \cap B_c = \left(\mathrm{Bd}(\overset{\circ}{\overline{X}}) \setminus (\mathrm{Bd}(\overset{\circ}{\overline{X}}))^{k_X-1}\right) \cap \left(\mathrm{Bd}(\overset{\circ}{\overline{B}}) \setminus (\mathrm{Bd}(\overset{\circ}{\overline{B}}))^{k_B-1}\right)$$

It implies that the ℓ-cells included in the intersection $X_c \cap B_c$ verify $\max(k_X, k_B) \leq \ell \leq n-1$. Thus we must have $\max(k_X, k_B) = n-1$. ■

If one connectivity of a valid pair of discrete connectivities must be the $(n-1)$-connectivity, there is no condition on the other. This result is already implicitly known for 3-D images: in [2], two couples of connectivities are proposed to study a 3-D image, i.e. the (6,26)-connectivities and the (6,18)-connectivities; in fact, even the (6,6)-connectivities can be chosen, but this last couple does not bring information enough to be useful.

In fact, we prefer use the couple of $(n-1,0)$-connectivities.

Let us consider the $(n-1)$-connectivity for B_d. A "good" couple of discrete connectivities will permit us to use results of the continuous topology. Then the following equality has to be true:

$$R_{n-1}(B_d) \cup R_{k_X}(X_d) = I_c \tag{1}$$

As we have seen in appendix A, we have two equivalent definitions of the $(n-1)$-regularization. If we use the second one (deriving from definition 3.16), the equality 1 can not be satisfied. However, if we consider the first one (definition 3.13), we may obtain $R^1_{n-1}(B_d) \cup R_{k_X}(X_d) = I_c$ for all possible sets of voxels X if and only if $k_X = 0$.

4.2 Conclusion

We presented the choice of a couple of discrete connectivities for a n-dimensional digital image in a cubic lattice. We conclude that one of them must be the $(n-1)$-connectivity to avoid any topological paradoxes. In theory, the choice of the other one is free. However we prefer to use the 0-connectivity. So the union of regularizations of the two parts of the binary image will be the continuous counterpart of the digital image. Such a choice may be particularly important when thin objects, as vessels, are under study: in 3-D, if the 0-connectivity is not chosen for the object, the vessel may be disconnected.

[5] We do not precise X_d or X_c because their regularization will be the same.

We have to be fully aware that this result depends in fact on our definition of a discrete connectivity (definition 3.8), that implies an order relation for the set of discrete connectivities (see proposition 3.9).

In fact, we can slightly modify the definition of the discrete connectivity to not have this relation order:

Two n-cells (or voxels) c_1 and c_2 are said k-connected (or k-adjacent) if

- *there exists a k-cell c, $k < n$, such that $c_1 \in \mathrm{St}(c)$ and $c_2 \in \mathrm{St}(c)$,*
- *there exists no ℓ-cell c', $k < \ell$, such that $c_1 \in \mathrm{St}(c')$ and $c_2 \in \mathrm{St}(c')$.*

In this case, regularization operations have to be redefined, and the discussion about the choice of a good couple of discrete connectivities may be done like above.

However, as this new definition may be inconsistent with our intuitive perception, we do not study its consequences thoroughly.

5 Jordan surfaces in discrete spaces

We will use the choice of discrete connectivities we made (i.e. the $(n-1)$-connectivity for the background, the 0-connectivity for the object), in order to extend their work in n-D. Before to discuss characterization of Jordan surfaces, we recall some notions of topological characterization of voxels.

5.1 Topological characterization of voxels

Malandain and al. propose a method of topological classification of each object voxel in a binary image of dimension 3 [19]. For that purpose, they use two numbers of connected components in a small neighborhood of each point:

C^* is the number of 26-connected components of the object in a 26-neighborhood minus the central point (i.e. the point under consideration);

$\overline{C}$ is the number of 6-connected components of the background in a 18-neighborhood.

The first number, C^*, is designed to characterize curves (objects of geometrical dimension 1) because $C^* \geq 2$ means that locally the object is divided in two or more parts by the deletion of the central point.

The second one, $\overline{C}$, is designed to characterize surfaces (objects of geometrical dimension 2) because $\overline{C} \geq 2$ means that locally the object divides the background in two or more parts.

By the combined use of this two numbers, all points of a binary image of dimension 3 may be topologically classified in different types of point including: interior points (points inside volumes –objects of geometrical dimension 3–), surface points, curve points, isolated points (objects of geometrical dimension 0), simple points and other types that combine the previous ones.

For higher dimensions than 3, this approach is no more complete. In fact, using the n-dimensional extension of the two numbers, we will be able to characterize only objects of geometrical dimension 0, 1, $n-1$ and n, but not the objects of geometrical dimension from 2 to $n-2$.

However, a part of this work, concerning the surfaces, can be easily extended to higher dimensions, in order to define hyper-surfaces (objects of geometrical dimension $n-1$) and Jordan surfaces.

5.2 Hyper-surfaces made of voxels

6-connectivity in 3-D, as 4-connectivity in 2-D, may be defined by the metric D_1. Let x and y be two points of $\mathbb{R}^n$ defined by their coordinates $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$, then $D_1(x, y) = \sum_{i=1}^{n} |y_i - x_i|$.

In a n-dimensional image (with a cubic lattice) where points have integer coordinates, two points x and y are said to be $(n-1)$-connected (which is the extension of the 6-connectivity in higher dimensions than 3, as shown above) if $D_1(x, y) = 1$.

In the same way, the 0-connectivity can be defined by the metric D_∞, $D_\infty(x, y) = \max_{i=1,\ldots,n} |y_i - x_i|$.

We characterize a hyper-surface point in dimension n by the fact that it locally divides the background in two parts. We have then to compute the number of connected components of the background in a small neighborhood surrounding an object point.

For that purpose, we naturally use the $(n-1)$-connectivity for the background. The neighborhood is the ball of radius 2 for the D_1 metric, denoted by $V_1^2(x) = \{y \ / \ D_1(x, y) \leq 2\}$. In fact, we can only use the restriction of this neighborhood to the ball of radius 1 for the D_∞ metric, $V_\infty^1(x) = \{y \ / \ D_\infty(x, y) \leq 1\}$. This last neighborhood, $V_1^2(x) \cap V_\infty^1(x)$, is the generalization of the 18-neighborhood in dimension 3 to higher dimensions.

Let us denote by $\overline{C}$ the number of $(n-1)$-connected components of the background $(n-1)$-adjacent to x in $V_{\overline{C}}(x) = V_1^2(x) \cap V_\infty^1(x)$.

Definition 5.1 *An object point is said to be a hyper-surface point if* $\overline{C} = 2$.

Let us point out that this definition is right because we use 0-connectivity for the object and $(n-1)$-connectivity for the background, then the union of the continuous counterparts of them are the continuous counterpart of the digital image. If there is not the case, the continuous counterpart of the surface will not divide its complement in two parts.

As in the 3-D case, an object made of hyper-surface points is not necessarily a simple hyper-surface, i.e. a hyper-surface without junctions (see [19]): in fact, some junction points may be misclassified and considered as hyper-surface points. In order to check this particular point, we extend in n-D the same scheme than in 3-D.

First we will define the notion of simple hyper-surface. Let x be a hyper-surface point. We call B_x and C_x the two $(n-1)$-connected components of the background $(n-1)$-adjacent to x in $V_{\overline{C}}(x)$. We say that two hyper-surface points x and y belong to the same simple hyper-surface if there is a 0-path $x_0, x_1, \ldots, x_\ell$ of hyper-surface points such as $x_0 = x$, $x_\ell = y$ and for $i \in [0, \ldots, \ell - 1]$ the number of $(n-1)$-connected components of $(B_i \cup C_i \cup B_{i+1} \cup C_{i+1})$ is two. This relation simply means that if two points belong to a simple hyper-surface, there are a path from one to another in the hyper-surface and two "adjacent" paths in the background. It is easy to check that this relation is reflexive, symmetric and transitive, thus it is an equivalence relation. A simple hyper-surface is an equivalence class of this relation.

This notion of simple hyper-surface is not sufficient enough for defining Jordan surfaces. For instance, a Klein bottle which is a self-intersecting surface, may be a simple hyper-surface.

We introduce now the notion of simple hyper-surface point.

Definition 5.2 *An object point x is said to be a simple hyper-surface point if*

i) it is a hyper-surface point: $\overline{C} = 2$

ii) all its object neighbors y_i are hyper-surface points

iii) the number of connected components of $\bigcup_i (B_{y_i} \cup C_{y_i}) \cup B_x \cup C_x$ is two

Proposition 5.3 *An object made of simple hyper-surface points divides the background in two parts.*

Proof: Let us consider an object X made of hyper-surface points, and let us consider the subset B' of the background B which is *(n-1)*-adjacent to the object :

$$B' = \{y \in B \ / \ \exists x \in X, x \text{ and } y \text{ are } (n\text{-}1)\text{-adjacent } \}$$

In order to show that the object X divides the background in two parts, it is sufficient to show that the number of connected components of B' is two.

Let x_0 be a simple hyper-surface point, it is obvious that x_0 and all its neighbors $x_{1,i}$ belong to the same simple surface. Then, neighbors $x_{2,i}$ of points $x_{1,i}$ have the same property. Thus we obtain that the number of connected components of $\bigcup_{j=1,\ldots,n} (B_{x_{j,i}} \cup C_{x_{j,i}})$ is two.

According that we have a finite number of points in X, which is always the case in practice, there exists N such as $\bigcup_{j=1,\ldots,N} x_{j,i} = X$, we have $B' = \bigcup_{j=1,\ldots,N} (B_{x_{j,i}} \cup C_{x_{j,i}})$ and the number of connected components of B' is two.

$\blacksquare$

Let us point out that an object made of simple hyper-surface points is always a simple hyper-surface, but the reciprocal is false.

Please notice that our separation theorem is weaker than the Jordan-Brouwer separation theorem (or generalized $[C^\infty]$ Jordan curve theorem as proposed in [21, page 591]). An object made of simple hyper-surface points is not necessary "homeomorphic" to a sphere of $\mathbb{R}^n$, it can be the surface of a torus for instance.

However, we do not really need to verify this homeomorphism to a sphere in our applications (like surface detection algorithms) because only the property of separation is important. As a matter of fact, the Jordan surfaces are usually defined in the literature by this only last property (see [12, 14, 18]). Thus objects made of simple hyper-surface points will be called Jordan hyper-surfaces in the following.

As one can see in definition 5.2, our characterization of Jordan hyper-surface is completely local. It brings two main advantages:

1. we do not have to check anything on the whole image;
2. the computation of the characterization can be easily parallelized.

On the other hand, object points will be correctly classified in hyper-surface or simple hyper-surface points only if the object is already thin. If thinning in 2-D or in -3D has already been studied (see [22, 23, 24]), it has not be done in n-D. In fact, thinning algorithms need a characterization of the deletable (or simple) points like the ones given in [2, 19, 25] for the 3-D case, which has not been done in the n-D case.

If we consider the characterization proposed in [19, 25] which is $C^* = \overline{C} = 1$ with the previous notations, it stands no more in the n-D case because points belonging to objects of geometrical dimension from 2 to $n-2$ will verify this condition without being deletable.

6 Conclusion

Using the concept of cellular complexes, we provided a cellular decomposition of $\mathbb{R}^n$ that give a mathematical framework to study digital images. We defined the notion of *discrete connectivity* and we introduced the notion of regularization. This last notion permit us to define the continuous counterpart of a set of voxels and to prove equivalences between discrete connectivities and connectivity of the continuous space. Using these equivalences, we provide a characterization of a "good" couple of discrete connectivities both the object and the background in a digital binary image. We assume that the best pair of discrete connectivities is the $(n-1)$-connectivity associated with the 0-connectivity.

However, these results depends on the definition of discrete connectivities we provide (definition 3.8), that implies an order relation for the set of discrete connectivities. We used this one because it seems to us the more appropriate and natural. To study discrete connectivities thoroughly, one may use a more general definition, as the one we provide in the final discussion on the choice of discrete connectivities.

Finally, we use our choice of discrete connectivities to characterize Jordan hyper-surfaces constituted by voxels. It generalizes a part of the topological classification in $\mathbb{R}^3$ [19]. Its greatest advantage is that it is purely local and allows parallel computations. Further work implies the definition of simple points and thinning algorithms in n-D, that appears necessary to apply efficiently the above characterization of Jordan hyper-surfaces.

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A Comparison between the two definitions of $(n-1)$-regularization

Let us denote by $R^1_{n-1}(X)$ the $(n-1)$-regularization of a set of voxels X according to definition 3.13, i.e. $R^1_{n-1}(X) = \overline{\overset{\circ}{X}}$, and by $R^2_{n-1}(X)$ the $(n-1)$-regularization of X which derives from definition 3.16, i.e. $R^2_{n-1}(X) = R_0(X) \setminus (R_0(X))^{n-2}$.

$R^2_{n-1}(X)$ is made of n-cells and $(n-1)$-cells while $R^1_{n-1}(X)$ may contain all types of cells.

Let us now consider $R^1_{n-1}(X)$ minus its $(n-2)$-skeleton. We will prove that $R^1_{n-1}(X)$ minus its $(n-2)$-skeleton has the same number of connected components, and that these connected components contains the same voxels, than $R^1_{n-1}(X)$.

To prove this, we need the following proposition.

Proposition A.1 *For all cells c of C, the open star of c minus its $(n-2)$-skeleton is a connected open part of $\mathbb{R}^n$.*

Proof: By analogy with definition 3.16, we will denote $\mathrm{St}(c) \setminus (\mathrm{St}(c))^{n-2} = R^2_{n-1}(\mathrm{St}(c))$.

Let $c \in C$ and $c' \in C$ such that $c' \in R^2_{n-1}(\mathrm{St}(c))$. By definition, c' is either a n-cell or a *(n-1)*-cell. According to proposition 3.6, $\mathrm{St}(c')$ may contain only n-cells or *(n-1)*-cells. We have $\mathrm{St}(c') \subset R^2_{n-1}(\mathrm{St}(c))$ where we deduce that $\bigcup_{c' \in R^2_{n-1}(\mathrm{St}(c))} \mathrm{St}(c') \subset R^2_{n-1}(\mathrm{St}(c))$.

The equality $R^2_{n-1}(\mathrm{St}(c)) = \bigcup_{c' \in R^2_{n-1}(\mathrm{St}(c))} c'$ stands obviously, according that $c' \in \mathrm{St}(c')$, we obtain the following equality,

$$R^2_{n-1}(\mathrm{St}(c)) = \bigcup_{c' \in R^2_{n-1}(\mathrm{St}(c))} \mathrm{St}(c')$$

$R^2_{n-1}(\mathrm{St}(c))$ is the union of open sets of $\mathbb{R}^n$: it is an open set of $\mathbb{R}^n$.

To prove that $R^2_{n-1}(\mathrm{St}(c))$ is connected, we only have to show that for every two cells of $R^2_{n-1}(\mathrm{St}(c))$ we may find an open set which contains these two cells and is included in $R^2_{n-1}(\mathrm{St}(c))$. It is already sufficient to show that property for n-cells only. In fact, if we consider a *(n-1)*-cell c', $\mathrm{St}(c')$ is a connected open set including c' and two n-cells, we search then an open set as defined above with one of these two n-cells, and by union with $\mathrm{St}(c')$ we have the property for c'.

Let r_1 and r_2 be two n-cells of $R^2_{n-1}(\mathrm{St}(c))$. c is a k-cell, we express it by $c = I_1 \times \ldots \times I_n$, k I_i have the form $]z_i, z_i+1[$ and $(n-k)$ the form $\{z_i\}$. r_1 and r_2 may be written with the same formulation, by definition they are included in $\mathrm{St}(c)$: they contain the same k intervals I_i of the form $]z_i, z_i+1[$ as c. They may only differ from c by the $(n-k)$ remaining intervals I_i. They will be different from each other by ℓ intervals, $\ell \leq n-k$. Using our abusive notation, we reorganize the intervals to obtain: $r_1 = I'_1 \times \ldots \times I'_\ell \times I_{\ell+1} \times \ldots \times I_n$ and $r_2 = I''_1 \times \ldots \times I''_\ell \times I_{\ell+1} \times \ldots \times I_n$, with $I'_i \neq I''_i$ for i from 1 to ℓ. By definition of n-cells, we have $I'_i =]z'_i, z'_i+1[$ and $I''_i =]z''_i, z''_i+1[$. As these intervals come from $I_i = \{z_i\}$, we have too:

- either $z'_i + 1 = z''_i \ \ (= z_i)$
- or $z'_i = z''_i + 1 \ \ (= z_i)$

We can then build a series of n-cells r^i for i from 0 to ℓ and a series of *(n-1)*-cells c^i for i from 1 to ℓ, in $R^2_{n-1}(\mathrm{St}(c))$ such as:

- $r^0 = r_1$
- $r^i = I''_1 \times \ldots \times I''_i \times I'_{i+1} \times \ldots \times I'_\ell \times I_{\ell+1} \times \ldots \times I_n$ for i from 1 to $\ell - 1$
- $r^\ell = r_2$
- c^i is the *(n-1)*-cell such that $r^{i-1} \in \mathrm{St}(c^i)$ and $r^i \in \mathrm{St}(c^i)$, which may express by $c^i = I''_1 \times \ldots \times I''_{i-1} \times I_i \times I'_{i+1} \times \ldots \times I'_\ell \times I_{\ell+1} \times \ldots \times I_n$ with $I_i = \{z_i\}$.

Let us consider the set $D = \left(\bigcup_{i=0\ldots\ell} r^i \right) \cup \left(\bigcup_{i=1\ldots\ell} c^i \right)$. As series are obviously included in $R^2_{n-1}(\mathrm{St}(c))$, so does D.

$\forall\ c^i$, $\mathrm{St}(c^i) \subset D$ and $\forall\ r^i$, $r^i \in \mathrm{St}((c^{i+1}))$ and/or $r^i \in \mathrm{St}(c^i)$ then D may be written by $D = \bigcup_{i=1\ldots l} \mathrm{St}(c^i)$, it is an open set. Moreover, D is connected because for i from 1 to $\ell - 1$, $\mathrm{St}(c^i)$ and $\mathrm{St}(c^{i+1})$ are connected open sets, they have a non-empty intersection r^i, hence their union is a connected open set. ∎

As $R^2_{n-1}(\mathrm{St}(c))$ is a connected open set for all cells c, we can easily prove the following proposition.

Proposition A.2 *$R^1_{n-1}(X)$ minus its $(n-2)$-skeleton has the same number of connected components, and that these connected components contains the same voxels, as $R^1_{n-1}(X)$.*

Proof: $R^1_{n-1}(X)$ minus its $(n-2)$-skeleton and $R^1_{n-1}(X)$ will have different connected components if we find two cells c_1 and c_2 of $R^1_{n-1}(X)$ such that $\mathrm{St}(c_1) \cap \mathrm{St}(c_2) \neq$ '$\emptyset$ and $R^2_{n-1}(\mathrm{St}(c_1)) \cap R^2_{n-1}(\mathrm{St}(c_2)) = \emptyset$.

Let us consider two cells c_1 and c_2 such that $\mathrm{St}(c_1) \cap \mathrm{St}(c_2) \neq \emptyset$. Let $c \in \mathrm{St}(c_1) \cap \mathrm{St}(c_2)$, then $\mathrm{St}(c) \subset \mathrm{St}(c_1) \cap \mathrm{St}(c_2)$ according to the properties of the open star. $\mathrm{St}(c)$ contains 2^{n-k} n-cells if c is a k-cell. These n-cells belong to $R^2_{n-1}(\mathrm{St}(c_1))$ and $R^2_{n-1}(\mathrm{St}(c_2))$ too. Thus $R^2_{n-1}(\mathrm{St}(c_1)) \cap R^2_{n-1}(\mathrm{St}(c_2)) \neq \emptyset$.

As we can not find two cells c_1 and c_2 as described above, $R^1_{n-1}(X)$ minus its $(n-2)$-skeleton has the same connected components as $R^1_{n-1}(X)$. ∎

We just have now to prove the equivalence between $R^1_{n-1}(X)$ minus its $(n-2)$-skeleton and $R^2_{n-1}(X)$ in order to prove the equivalence between $R^1_{n-1}(X)$ and $R^2_{n-1}(X)$.

Proposition A.3 *$R^1_{n-1}(X)$ minus its $(n-2)$-skeleton has the same number of connected components, and that these connected components contains the same voxels, as $R^2_{n-1}(X)$.*

Proof: We already know that $R^1_{n-1}(X)$ is the interior of $R_0(X)$, thus $R_0(X) = R^1_{n-1}(X) \cup \mathrm{Bd}(R_0(X))$. We have then

$$R^2_{n-1}(X) = \left(R^1_{n-1}(X) \setminus \left(R^1_{n-1}(X)\right)^{n-2}\right) \cup \left(\mathrm{Bd}(R_0(X)) \setminus (\mathrm{Bd}(R_0(X)))^{n-2}\right)$$

It comes that the difference between $R^1_{n-1}(X)$ minus its $(n-2)$-skeleton and $R^2_{n-1}(X)$ is the set of $(n-1)$-cells of $\mathrm{Bd}(R_0(X))$.

The open star of one of these $(n-1)$-cells contains the $(n-1)$-cell itself, one n-cell in X and an other n-cell in the complement of X. Then this $(n-1)$-cell can not connect two disjointed connected components of $R^1_{n-1}(X)$ minus its $(n-2)$-skeleton.

$R^1_{n-1}(X)$ minus its $(n-2)$-skeleton has the same connected components as $R^2_{n-1}(X)$. ∎

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