A Graphical Approach of the Spectral Theory in the (max,+) Algebra

Jean Mairesse

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

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Abstract: We study matrices in the $(Max, +)$ algebra. We introduce a new tool for describing the deterministic spectral behaviour of matrices of size 3×3. It consists in a graphical representation of eigenvectors and domains of attraction in the projective space. It appears to be very helpful in understanding some of the phenomena occurring in this algebra.

Key-words: Discrete event systems, event graphs, (Max,+) algebra, spectral theory, projective space.

(Résumé : tsvp)

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Unité de recherche INRIA Sophia-Antipolis
2004 route des Lucioles, BP 93, 06902 SOPHIA-ANTIPOLIS Cedex (France)
Téléphone : (33) 93 65 77 77 – Télécopie : (33) 93 65 77 65

Une Représentation Graphique pour les Matrices dans l'Algèbre (Max,+)

Résumé : L'étude porte sur les matrices dans l'algèbre (Max,+). On introduit un nouvel outil de description du comportement spectral des matrices déterministes de dimension 3. Il s'agit d'une représentation graphique des vecteurs propres et de leur domaine d'attraction dans l'espace projectif. Cette représentation apparait très utile pour comprendre certains phénomènes intervenant dans l'algèbre (Max,+).

1 Introduction

Many communication or manufacturing networks can be represented by Discrete Events Dynamic Systems (DEDS). Recent research has dealt with the problem of finding a unified framework to study DEDS. Petri Networks, and more precisely Event Graphs (EG), are an example of such a formalism. They model phenomena such as synchronization or blocking. These networks have an easy algebraic interpretation in a non conventional algebra. More precisely, it is possible to show that a timed Event Graph can be represented as a linear recursive equation in the (Max,+) algebra, of the following kind:

$$y_{n+1} = A \otimes y_n \ ,$$

where y_{n+1} and y_n are $\mathbb{R}^J$-valued vectors and A is a matrix of size $J \times J$. The matrix-vector product has to be interpreted in the (Max,+) algebra. For a timed Event Graph, the dimension J is the number of transitions. The vector y_n consists of the dates of the nth firing of the transitions. For more insights on all modelling aspects, the reader is referred to [1] or [2].

The spectral theory of matrices in the (Max,+) algebra is now well known. It can be tracked back to [8] or, for the Russian school, to [10]. One of the main differences with the classical spectral theory is that there is a unique eigenvalue for irreducible matrices. As a consequence, the main interest and difficulty in the (Max,+) algebra is to study eigenvectors associated with the unique eigenvalue. For a timed EG, the eigenvalue is exactly the mean cycle time (inverse of the throughput rate). On the other hand, eigenvectors are associated with quantities such that number of tokens in a place, waiting times or idle times. Multiple eigenvectors will mean multiple regimes for these quantities.

In this paper, we present the classical spectral results under a new light. We develop a tool for describing the spectral behaviour of matrices of size 3×3. It consists in a graphical representation of asymptotic regimes in the projective space This representation enables us to get an intuition of the spectral behaviour of larger matrices. It appears also very useful in order to understand some phenomena occurring in this algebra.

The paper is organized as follows. Sections **2** and **3** review some basic results on the (Max,+) algebra and its spectral theory respectively. In Section **3**, we present also a complete spectral analysis of matrices of size 3 with the help of the graphical representation mentioned before. Section **4** is devoted to three examples of utilizations of this graphical representation. In the first one, we illustrate an algorithm of Braker and Olsder to compute eigenvectors of matrices. In the second, we give a "visual" example of a projectively infinite semigroup of matrices. The last example shows how the graphical representation can be used for stochastic models.

2 The (Max,+) Algebra

Definition 2.1 ((Max,+) algebra) *We consider the semi-field (improperly called algebra) $(\mathbb{R}^\star, \oplus, \otimes)$, where $\mathbb{R}^\star = \mathbb{R} \cup \{-\infty\}$. The law $\oplus$ is "Max" and $\otimes$ is the usual addition. We set $\varepsilon = -\infty$ and $e = 0$. The element ε is neutral for the operation $\oplus$ and absorbing for $\otimes$. The element e is neutral for $\otimes$. The law $\oplus$ is idempotent, i.e. $a \oplus a = a$. $(\mathbb{R}^\star, \oplus, \otimes)$ is an idempotent semiring, called a dioid. It is moreover a commutative dioid. We shall write it $\mathbb{R}_{Max}$.*

In the rest of the paper, the notations "+,×" will stand for the usual addition and multiplication. Nevertheless, we will write ab for $a \otimes b$ whenever there is no possible confusion.

We define the product spaces $\mathbb{R}^J_{Max}$, $\mathbb{R}^{J\times J}_{Max}$. We define the product of a vector by a scalar: $a \in \mathbb{R}_{Max}, u \in \mathbb{R}^J_{Max}$, $(a \otimes u)_i = a \otimes u_i$.
Matrix product is defined in the following way. Let $A, B \in \mathbb{R}^{J\times J}_{Max}$,

$$(A \otimes B)_{ij} = Max_k(A_{ik} + B_{kj}) = \bigoplus_k A_{ik} \otimes B_{kj} \ .$$

Matrix-vector or scalar-matrix products are defined in a similar way.

We are interested in an eigenvalue problem in $\mathbb{R}_{Max}$, similar to the one of the traditional algebra. We want to find non trivial solutions to the equation:

$$A \otimes x = \lambda \otimes x \ ,$$

where $A \in \mathbb{R}^{J\times J}$ is an irreducible (see definition 3.2) matrix, x is a column vector (the "eigenvector") and λ is a real constant (the "eigenvalue"). We also define periodic solutions of the eigenvalue problem.

Definition 2.2 *A periodic solution of period d is a set of vectors $\{x_1, \ldots, x_d\}$ of $\mathbb{R}^J$ verifying $Ax_i = \lambda x_{i+1}$, $i = 1, \ldots, d-1$ and $Ax_d = \lambda^d x_1$.*

Remark A periodic solution of period d for A implies the existence of d eigenvectors for A^d.

First of all, let us introduce the graphical representation that we are going to use extensively.

Definition 2.3 ($\mathbb{P}\mathbb{R}^J_{Max}$) *The projective space $\mathbb{P}\mathbb{R}^J_{Max}$ is defined as the quotient of $\mathbb{R}^J_{Max}$ by the parallelism relation:*

$$u, v \in \mathbb{R}^J \quad u \simeq v \iff \exists a \in \mathbb{R}_{Max} \setminus \{\varepsilon\} \text{ such that } u = a \otimes v \ .$$

Let π be the canonical projection of $\mathbb{R}^J_{Max}$ into $\mathbb{P}\mathbb{R}^J_{Max}$.

In the rest of the paper, we will concentrate on aperiodic matrices (see definition 3.2). In this case, a matrix A, which is a linear operator of $(\mathbb{R}^J_{Max}, \oplus, \otimes)$, can be restricted to an operator of $(\mathbb{R}^J, \oplus, \otimes)$ (i.e. if u is a vector whose coordinates are all different from ε, then Au has the same property). As a consequence, we will consider only vectors in $\mathbb{R}^J$ and their projection in $\mathbb{P}\mathbb{R}^J$, where $\mathbb{P}\mathbb{R}^J$ is defined exactly in the same way as $\mathbb{P}\mathbb{R}^J_{Max}$.

The canonical projection π of $\mathbb{R}^J$ into $\mathbb{P}\mathbb{R}^J$ can be interpreted geometrically. It is nothing else than the orthogonal projection on the hyperspace orthogonal to the vector $\mathbb{1} = (1, \ldots, 1)'$. The projective space $\mathbb{P}\mathbb{R}^J$ is isomorphic to $\mathbb{R}^{J-1}$. Let us consider a deterministic matrix $A \in \mathbb{R}^{J\times J}_{Max}$ and the $\mathbb{R}_{Max}$ eigenvalue problem $Ax = \lambda x$. For matrices of size 2 or 3, a graphical representation is possible in $\mathbb{R} \simeq \mathbb{P}\mathbb{R}^2$ and $\mathbb{R}^2 \simeq \mathbb{P}\mathbb{R}^3$ respectively. We represent eigenvectors and periodic regimes modulo the parallelism relation. Let us illustrate this.
Figure 1 corresponds to the matrix

$$A = \begin{pmatrix} 2 & e \\ 1 & 2 \end{pmatrix} .$$

As we will see in a moment, the spectral theory tells us that there is a strip (an interval in $\mathbb{P}\mathbb{R}^2$) of eigenvectors and no periodic regimes of period greater than 1. The line D is the hyperspace orthogonal to the first bisecting line. We consider the orthogonal projection of the picture on D.

From now on, we will consider mostly matrices of size 3 whose spectral behaviour is much richer and can be graphically represented in $\mathbb{R}^2 \simeq \mathbb{P}\mathbb{R}^3$.

Let us introduce a distance on $\mathbb{P}\mathbb{R}^J_{Max}$ which we are going to call the projective distance.

Definition 2.4 *We consider $x, y \in \mathbb{P}\mathbb{R}^J$. Let $u, v \in \mathbb{R}^J$ be two representatives of x and y respectively, i.e. $\pi(u) = x$ and $\pi(v) = y$.*

$$d(x,y) = d(u,v) = \bigoplus_i (u_i - v_i) \oplus \bigoplus_i (v_i - u_i) .$$

It is easy to check that $d(x, y)$ does not depend on the representatives u and v. It is also easy to check that it is a distance in $\mathbb{P}\mathbb{R}^J$. It is nothing else than the $\mathcal{L}_\infty$ norm on the projective space $\mathbb{P}\mathbb{R}^J$. We write either $d(x, y)$ or $d(u, v)$ with a little abuse of notation. We have the following property.

Proposition 2.1 *Let A be an irreducible matrix of size J. Let u, v be two vectors of $\mathbb{R}^J_{Max}$. We have:*

$$d(Au, Av) \leq d(u, v) .$$

There is no simple criterion to get a strict inequality.

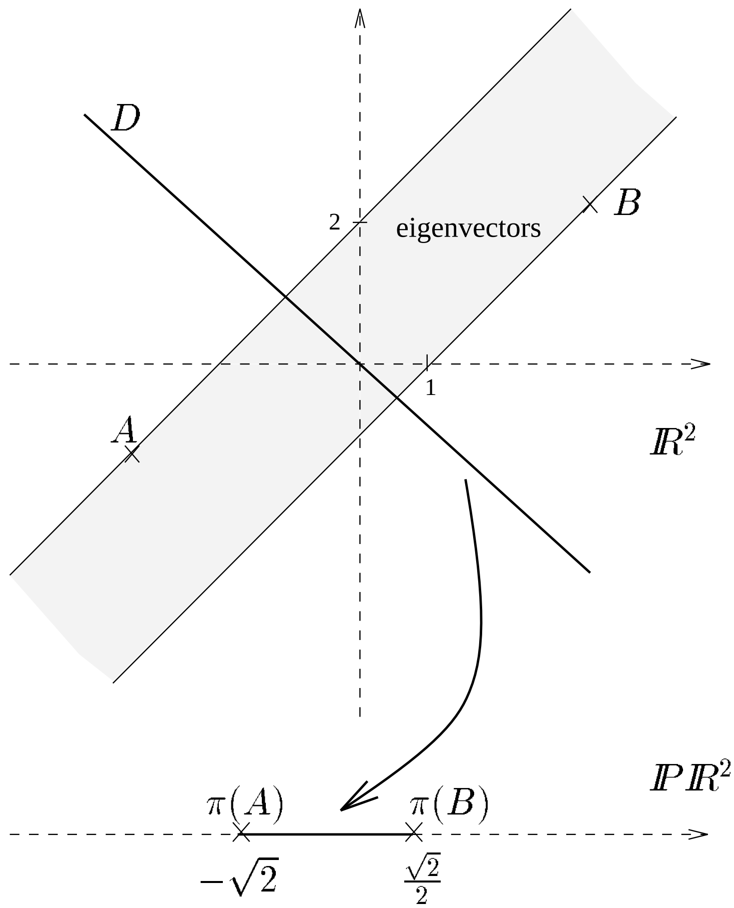

Figure 1: Dimension 2. A scs2-cyc1 matrix.

Let us represent the unit ball of the projective distance in $I\!PI\!R^3$.
The regular hexagon in Figure 2 is the section of the unit square (i.e. the unit ball of $I\!R^3$ for the $\mathcal{L}_\infty$ norm) by the projection plane. The three represented axes are the orthogonal projection of the basis of $I\!R^3$. The represented points are:

$$A = \pi \begin{pmatrix} 1 \\ e \\ e \end{pmatrix}, \; B = \pi \begin{pmatrix} e \\ 1 \\ e \end{pmatrix}, \; C = \pi \begin{pmatrix} e \\ e \\ 1 \end{pmatrix}, \; D = \pi \begin{pmatrix} 0.2 \\ 0.6 \\ 0.8 \end{pmatrix} .$$

The practical way of representing a point X of $I\!PI\!R^3$ is to choose a vector ($\in I\!R^3$) in the parallelism class of X and to draw it in the three axes obtained by projection of the orthonormal basis of $I\!R^3$ (it is easy to check that the point we obtain does not depend on the representative in the parallelism class). The point D of Figure 2 illustrates this, we have drawn two constructions: one corresponding to $(0.2,\ 0.6,\ 0.8)$ and the other one to $(0.8,\ 1.2,\ 1.4) = 0.6 \otimes (0.2,\ 0.6,\ 0.8)$.

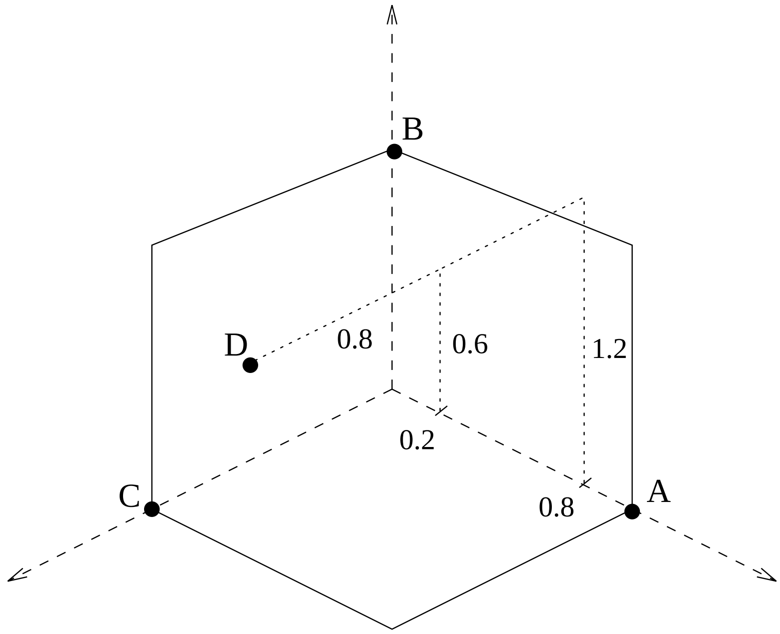

Figure 2: Unit ball of the projective distance

3 An Illustrated Spectral Theory

We are now ready to review the $\mathbb{R}_{Max}$ spectral theory of irreducible matrices. The results we are going to present are now classical. A complete treatment can be found in [2]. For the spectral theory of reducible matrices, the reference is [6]. The analog theory for non finite dimensions is exposed in [4]. However, the idea of illustrating the spectral behaviour by graphical representations in the projective space is new.

3.1 General Presentation

From now on, we consider only irreducible matrices in $\mathbb{R}_{Max}^{J\times J}$. We recall that we want to find non trivial solutions to the equation $Ax = \lambda x$. Let us recall some definitions adapted to the $\mathbb{R}_{Max}$ algebra.

Definition 3.1 *The communication graph of a square matrix A is a directed graph having a number of nodes equal to the size of A. This graph contains an arc from i to j iff $A_{ji} \neq \varepsilon$. The valuation of this arc is A_{ji}.*

Definition 3.2 *A square matrix A is irreducible if: $\forall i, j \;\; \exists m \geq 0 \;\;|\;\; (A^m)_{ij} > \varepsilon$ (or equivalently if its (communication) graph is strongly connected).*
A square matrix A is aperiodic if: $\exists m \geq 0, \;\; \forall i, j \;|\; (A^m)_{ij} > \varepsilon$.

Definition 3.3 *For each circuit* $\zeta = \{t_1, t_2, \cdots, t_j, t_{j+1} = t_1\}$, *we define the average weight by:*

$$p(\zeta) = \frac{A_{t_1 t_j} \otimes \cdots \otimes A_{t_3 t_2} \otimes A_{t_2 t_1}}{j} ,$$

(here the division is the conventional one).

Theorem 3.1 *There is a unique (non* ε*) eigenvalue,* λ*. It satisfies the relation*

$$\lambda = \max_{\zeta} \; p(\zeta) ,$$

where ζ *describes all the circuits of (the communication graph of)* A*. We call also* λ *the* **Lyapunov exponent** *of* A*.*

There might be several eigenvectors. An eigenvector has all its coordinates different from ε (due to the irreducibility assumption). A linear combination (in $\mathbb{R}^J_{Max}$) of eigenvectors is an eigenvector, i.e. if u_1 and u_2 are eigenvectors and $\alpha_1, \alpha_2 \in \mathbb{R}$, then $(\alpha_1 \otimes u_1) \oplus (\alpha_2 \otimes u_2)$ is also an eigenvector.

In particular, if u is an eigenvector and $\alpha \in \mathbb{R}$, then $\alpha \otimes u$ is also one. This was the motivation for the introduction of the projective space $\mathbb{P}\mathbb{R}^J_{Max}$ (see definition 2.3). We will in general study the image by the canonical projection ($\pi : \mathbb{R}^J \to \mathbb{P}\mathbb{R}^J$) of the set of eigenvectors (or periodic regimes) of a matrix.

Let us illustrate what the "linear combination of two vectors" means in $\mathbb{P}\mathbb{R}^J_{Max}$. We consider examples of dimension 3. Let $u = (u_1, u_2, u_3)'$ and $v = (v_1, v_2, v_3)'$ be two vectors of $\mathbb{R}^3$. Let $\lambda, \mu \in \mathbb{R}^3$.

$$\pi(\lambda \otimes \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \oplus \mu \otimes \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}) = \pi(\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \oplus (\mu - \lambda) \otimes \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}) .$$

The symbol $\bigwedge$ denotes the minimum of a finite set. We denote by $\bigwedge\bigvee$ the intermediate value of a set of three values. Let us suppose for example that we have,

$$\bigwedge_{i=1,2,3} (u_i - v_i) = u_1 - v_1, \quad \bigwedge\bigvee_{i=1,2,3} (u_i - v_i) = u_2 - v_2, \quad \bigoplus_{i=1,2,3} (u_i - v_i) = u_3 - v_3 .$$

Depending on the value of $\alpha = \mu - \lambda$, there are four possible cases.

1. If $\alpha \leq \bigwedge(u_i - v_i)$, then $\pi(u \oplus \alpha v) = \pi(u)$.
2. If $\bigwedge(u_i - v_i) \leq \alpha \leq \bigwedge\bigvee(u_i - v_i)$, then $\pi(u \oplus \alpha v) = \pi(\alpha v_1, u_2, u_3)'$.
3. If $\bigwedge\bigvee(u_i - v_i) \leq \alpha \leq \bigoplus(u_i - v_i)$, then $\pi(u \oplus \alpha v) = \pi(\alpha v_1, \alpha v_2, u_3)'$.

4. If $\bigoplus(u_i - v_i) \leq \alpha$, then $\pi(u \oplus \alpha v) = \pi(v)$.

This particular example corresponds to the case of points C $(\pi(u))$ and A $(\pi(v))$ in Figure 2. The broken segment between C and A in Figure 2 is the set of linear combinations of the two points.
When two values are equal in $\{u_i - v_i,\ i = 1, 2, 3\}$, the picture is degenerate.

We are now ready to understand the form of sets of eigenvectors, knowing that linear combination of eigenvectors are eigenvectors. We represent (in $\mathbb{R}^2 \simeq \mathbb{P}\mathbb{R}^3$) the image by π of the set of eigenvectors.

-
$$M = \begin{pmatrix} 1 & e & e \\ e & 1 & e \\ e & e & 1 \end{pmatrix}.$$

The picture is exactly the same one as in Figure 2. The points A, B and C are the images by π of the columns of M which are eigenvectors (easy to check). The regular hexagon represented is the convex hull (in $\mathbb{R}_{Max}$) of these three points. It is the image by π of the set of eigenvectors of M.

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$$M = \begin{pmatrix} 1 & e & e \\ e & 1 & e \\ e & e & -1 \end{pmatrix}.$$

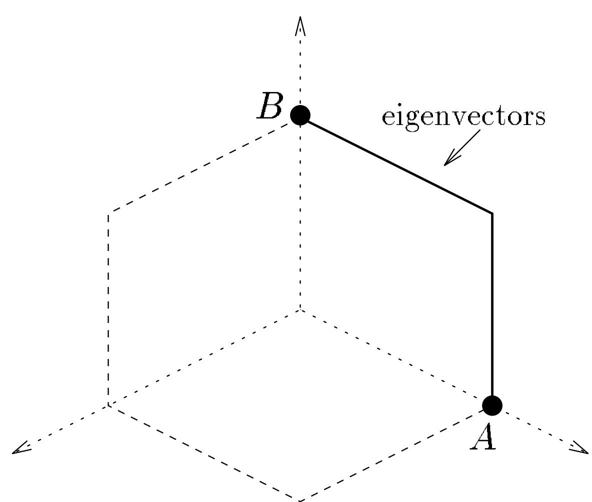

Figure 3: Set of eigenvectors

This case corresponds to Figure 3. The points A and B are (the image by π of) the two first columns of M. The broken segment between them is the set of eigenvectors of M. We obtain these eigenvectors as linear combination of A and B.

Let us recall some other definitions adapted to the $\mathbb{R}_{Max}$ algebra.

Definition 3.4 *We normalize a matrix by dividing (in $\mathbb{R}_{Max}$ i.e. by subtracting in the conventional algebra) all its entries by its eigenvalue.*

A normalized matrix has e as eigenvalue. The eigenvectors are unchanged.

The eigenvalue of a matrix A gives the asymptotic growth rate of A^k/k (see Theorem 3.3 for a more precise statement). As a consequence, we will call problems related with the eigenvalue, first order problems. On the other hand eigenvectors are related with the problem of computing differences such that $A^{k+1}u - A^k u$. We call them second order problems (see [1] or [9]). In the rest of the paper, we will concentrate on second order results.
Eigenvectors and periodic regimes are invariant by a translation of all the entries of a matrix by the same real constant. In the rest of the paper, we will write the matrix we want to study in a positive form (i.e. with all terms $> e$) or in a normalized form depending on which one seems more convenient.

Definition 3.5 *For a matrix A, with eigenvalue λ, we define:*

Critical circuit *A circuit ζ of A is said to be critical if its average weight is maximal, i.e. if $p(\zeta) = \lambda$.*

Critical graph *It consists of the nodes and arcs of A belonging to a critical circuit.*

Cyclicity *The cyclicity of a strongly connected graph (i.e. of an irreducible matrix) is the greatest common divisor of the lengths of all the circuits. The cyclicity of a general graph is the least common multiple of the cyclicities of its strongly connected subgraphs.*

The knowledge of the critical graph of a matrix accounts for much of its spectral behaviour. More precisely, to study the spectral behaviour of a matrix A, it is enough to know:

- The number of strongly connected subgraphs (**s.c.s.**) of its critical graph.
- The cyclicity of its critical graph.

In the following, a matrix whose critical graph is composed of j s.c.s. and whose cyclicity is k will be noted **scs**j**-cyc**k.

The two fundamental theorems that we are going to present now justify the previous assertion. For a normalized matrix A of size J, we define:

$$A^+ = A \oplus A^2 \oplus \cdots \oplus A^J .$$

We check that $A^+ \oplus A^{J+1} = A^+$.

Theorem 3.2 *Let A be a normalized matrix. Every eigenvector of A writes as a linear combination of columns of A^+. More accurately, we have:*

1. *Column $A^+_{i.}$, i belonging to the critical graph, is an eigenvector.*
2. *$\pi(A^+_{i.})$ and $\pi(A^+_{j.})$ are different iff i and j belong to two different s.c.s. of the critical graph.*

Let p be the number of s.c.s. of the critical graph of A ($p \leq J$). The previous theorem states that there are p extremal eigenvectors. Then $p-1$ is the "dimension" of (the image of) the set of eigenvectors of A in $\mathbb{P}\mathbb{R}^J$. This set is polyhedral. The faces of this set are hyperplanes. These hyperplanes have a finite number of possible directions. We consider the natural basis of $(\mathbb{R}^J, +, \times)$:

$$(\mathbf{e_1}, \ldots, \mathbf{e_J}) \text{ with } \mathbf{e_i} = (0, \ldots, 0, 1, 0, \ldots, 0)' ,$$

the term 1 of $\mathbf{e_i}$ being in the i^{th} place. We choose $J-2$ vectors of this basis. We take their canonical projection. The hyperplane of $\mathbb{R}^{J-1} \simeq \mathbb{P}\mathbb{R}^J$ defined by these $J-2$ independent vectors is a possible direction for a face of the set of eigenvectors. We conclude that there are C_J^{J-2} possible directions for these hyperplanes.

For example in $\mathbb{P}\mathbb{R}^3$, there are $C_3^1 = 3$ possible directions for the lines delimiting the set of eigenvectors which are $\pi(1,0,0)', \pi(0,1,0)'$ and $\pi(0,0,1)'$. The lines will then be of the form

$$D : \pi\left(\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \lambda \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) .$$

An example of this has already been given in Figure 2.

A corollary of Theorem 3.3 will be of particular use for us:
An irreducible matrix has a unique eigenvector (up to a multiplicative ($\otimes$) constant) iff its critical graph has a unique s.c.s.

In $\mathbb{R}^{Max}$, every irreducible matrix is cyclic in the sense of the next theorem.

Theorem 3.3 *For an irreducible matrix A of size J and whose eigenvalue is λ, there exist integers d and M such that:*

$$\forall m \geq M, \quad A^{m+d} = \lambda^d \otimes A^m ,$$

($\lambda^d = \lambda^{\otimes d} = d \times \lambda$). Furthermore the smallest d verifying the property is equal to the cyclicity of the critical graph of A. From now on, we will call it the cyclicity of A.

The good interpretation is that there exists an initial transient regime for the powers of a matrix A. After the transient regime, the sequence $\{A^n\}$ becomes periodic (more rigorously, it is the sequence $\{\pi(A^n)\}$ which becomes periodic).

Sometimes, we will be interested only in the stationary regime, we will then consider directly A^M. On the other hand, we will sometimes consider the transient regime of a matrix.

If d is the cyclicity of A then A^d is of cyclicity one. A cyclicity greater than one will provide periodic regimes of period greater than one for the eigenvalue problem.

Proposition 3.1 *An irreducible matrix has a unique eigenvector and no periodic regimes of period greater than one, iff its critical graph has a unique s.c.s. and its cyclicity is one, i.e. iff it is a* **scs1-cyc1** *matrix.*

Another easy consequence of Theorem 3.3 is the following where $d(.,.)$ is the projective distance and $u, v \in I\!R^J$.

$$\forall m \geq M, \; d(A^m u, A^m v) = d(A^M u, A^M v)\,.$$

3.2 Change of Basis

The results presented here are commonly known by people working on the subject. A matrix A of $I\!R_{Max}^{J\times J}$ can be considered as a linear (in a (Max,+) sense) operator on $I\!R_{Max}^J$. We want to have a formula of change of basis for the matrix associated with a given linear operator. We are only interested in permutations of the coordinates and translation of the origin.

Definition 3.6 *A matrix P is called a matrix of permutation if there is one and only one term equal to e in each line and column, the other terms being equal to ε. Let $\mathcal{G}_J$ be the group of permutations of $\{1,\ldots,J\}$. We consider $\sigma \in \mathcal{G}_J$. The matrix of permutation associated with σ is P defined by:*

$$P_{\sigma(i),i} = e, \; P_{ji} = \varepsilon, \; \forall j \neq \sigma(i)\,.$$

Lemma 3.1 *Let A be a $J \times J$ matrix and let $\hat{A}$ be the matrix associated with the same endomorphism in a new basis obtained from the original one by a permutation σ of the coordinates. Matrix P is the permutation matrix associated with σ and P^{-1} the one associated with σ^{-1}. We have*

$$\hat{A} = P^{-1} \otimes A \otimes P\,.$$

In the conventional algebra, translating the origin transforms a linear equation into an affine equation. In the $I\!R_{Max}$ algebra, on the other side, a linear operator remains linear.

We consider a matrix A and we note $\tilde{A}$ the matrix associated with the same endomorphism in a new basis. We obtain the new basis from the original one by a translation of the origin of the projective space. Let $u \in I\!R^J$ be (a representative of) the new origin written in the old basis. Here is the formula of change of basis.

Definition 3.7 *The Hadamard product of two matrices, symbolized by* $\circ$*, is given by the relation:*

$$(A \circ B)_{ij} = A_{ij} \otimes B_{ij} .$$

Lemma 3.2 *Let A be a $J \times J$ matrix. Let u be a column vector of size J. We define the matrix D, $D_{ij} = u_j - u_i$. This matrix can also be written as $D = \frac{u}{u}$ where $\frac{\cdot}{\cdot}$ is the left residuation in the $I\!R_{Max}$ algebra (see chapter 4 of [2]). With respect to the new origin u, the matrix A writes:*

$$\tilde{A} = A \circ D .$$

This can also be written, by analogy with the conventional algebra:

$$\tilde{A} = P^{-1} \otimes A \otimes P, \text{ where } P = \begin{pmatrix} u_1 & \varepsilon & \varepsilon \\ \varepsilon & \ddots & \varepsilon \\ \varepsilon & \varepsilon & u_J \end{pmatrix} .$$

Proof Let $v = (v_1, \cdots, v_J)'$ be a vector written in the old basis and let $\tilde{v} = (\tilde{v}_1, \cdots, \tilde{v}_J)'$ be this same vector in the new basis. We have $\tilde{v}_i + u_i = v_i$. We set $Av = w$ and $w = (w_1, \cdots, w_J)'$ and $(\tilde{w}_1, \cdots, \tilde{w}_J)'$ in the new and the old basis respectively.

$$\begin{aligned}(\tilde{A}\tilde{v})_i = (P^{-1} \otimes A \otimes P\tilde{v})_i &= (P^{-1} \otimes Av)_i \\ &= (P^{-1}w)_i = \tilde{w}_i\end{aligned}$$

■

An illustration of such a change of origin is provided by Figure 19.

It might be interesting to get another intuition on what a change of origin means. We present now an interpretation suggested by the modelling of Stochastic Event Graphs. Let us consider the communication graph associated with a positive and irreducible matrix $A \in I\!R_{Max}^{J \times J}$. We consider that there is a clock associated with each node of A. Let u be a vector of $I\!R^J$. We interpret u_i as a date of occurrence of a first event at node i. Then $(Au)_j$ is interpreted as the date of occurrence of the second event at node j. In this framework, a "change of origin" is just a change of the origin of time for some or all of the daters. It does not modify of course the evolution of the system.

Let us prove a very useful lemma. This lemma together with the previous one enables us to determine in which cases critical terms are greater than non-critical ones.

Lemma 3.3 *We consider a matrix A, irreducible, of size J, and u an eigenvector of A. Let P be the matrix of change of the origin associated with u. We define $\tilde{A} = P^{-1} \otimes A \otimes P$. We have the following property:*

$$\forall i, j \in 1, \cdots, J, \quad \tilde{A}_{ij} \leq \lambda \,,$$

and $\forall p, q$ such that (p, q) belongs to the critical graph, we have $\tilde{A}_{qp} = \lambda$, where λ is the Lyapunov exponent of A.

Proof We set $\mathbf{e} = (e, \ldots, e)'$.

$$\tilde{A}\mathbf{e} = \left(\bigoplus_k \tilde{A}_{1k}, \ldots, \bigoplus_k \tilde{A}_{Jk} \right) \,.$$

But using the fact that $\mathbf{e}$ is an eigenvector of $\tilde{A}$ ($P^{-1}AP\mathbf{e} = P^{-1}Au = P^{-1}\lambda u = \lambda\mathbf{e}$) and the definition of the Lyapunov exponent, we get:

$$\tilde{A}\mathbf{e} = (\lambda, \cdots, \lambda)' \,.$$

Then $\forall i, \ \bigoplus_k \tilde{A}_{ik} = \lambda$, which proves the first part of the lemma. Let us suppose there exist p, q such that (p, q) belongs to the critical graph and $\tilde{A}_{qp} < \lambda$. There is a critical circuit involving the arc (p, q). Using the first part of the theorem and $\tilde{A}_{qp} < \lambda$, we conclude that the mean weight of this critical circuit is strictly smaller than λ, which is a contradiction. ■

3.3 Spectral Theory in Dimension 3

We are now ready to have a closer look at aperiodic matrices of size 3. We are going to present an exhaustive inventory of the possible spectral behaviours. Using theorems 3.2 and 3.3, we show that there are only six possible cases, which can be sorted in four categories.

- scs1-cyc1
- scs3-cyc1 and scs1-cyc3.
- scs2-cyc1 and scs1-cyc2.
- scs2-cyc2

We are going to study them one after the other in the simplest case when all non-critical terms are equal. Then we will observe that the general behaviour is stable under small perturbations of non-critical terms. We will show precisely how these perturbations modify the behaviour. By this way, we will have described all possible aperiodic matrices of size 3.

In order for the reader to be convinced that all the cases are treated, we propose a classifying algorithm which, given a specific matrix, associates to it a paragraph and one or several figures of the paper.

We consider an irreducible matrix $A \in \mathbb{R}_{Max}^{J \times J}$.

Algorithm

1. Check if A is aperiodic.
2. Normalize matrix A.
3. Find an eigenvector of A.
4. Write A in a new basis.
5. Determine the critical graph of A.
6. Compute the projective size of A.
7. Check non critical terms of A. Final classification.

Let us detail the different stages.

Stage 1 *Check if A is aperiodic.*

Compute the powers of A until there is an m such that $\forall i, j,\ (A^m)_{ij} > \varepsilon$. If such an m does not exist then matrix A is not aperiodic and does not fit into our framework.
It is decidable whether A is aperiodic or not. In fact one only has to compute the powers of A until A^M where M is such that $\exists k \mid A^{M+k} = \lambda^k A^M$ (M is finite by Theorem 3.3).

Stage 2 *Normalize matrix A.*

Compute the smallest integer, denoted M, such that:

$$\exists d \in \mathbb{N},\ \exists \lambda \in \mathbb{R} \ \mid \ \forall m \geq M,\ A^{m+d} = \lambda^d \otimes A^m \ .$$

The eigenvalue of A is λ. Normalize matrix A. For simplicity, we will keep the original notation, i.e. $A := A - \lambda$.

Stage 3 *Find an eigenvector of A.*

Here is a general algorithm, used for example in [4], to compute an eigenvector of A. An alternative algorithm, valid for some kind of matrices only, is proposed in Section **4**, Example 1.

- Consider an i_0 such that $\exists k \mid (A^k)_{i_0 i_0} = e$. By Theorem 3.1, this condition is verified if and only if i_0 belongs to the critical graph of A.
- Compute:

$$u = \bigoplus_{k \geq 0} A^k \delta_{i_0}, \quad \text{where} \ (\delta_{i_0})_{i_o} = e, \ (\delta_{i_0})_j = \varepsilon, \ j \neq i_0 .$$

  Then u is an eigenvector of A.

Let us prove rapidly this last assertion. We denote by E the matrix defined by $E_{ii} = e, \ E_{ij} = \varepsilon, \ i \neq j$. We have:

$$u = \bigoplus_{k \geq 0} A^k \delta_{i_0} = E\delta_{i_0} \oplus \bigoplus_{k \geq 1} A^k \delta_{i_0} = E\delta_{i_0} \oplus Au .$$

We deduce that $Au \geq u$. Suppose we have $Au \neq u$, then we must have $(Au)_{i_0} > u_{i_0}$ as we clearly have $(Au)_j = u_j, \forall j \neq i_0$. Then we obtain $(Au)_{i_0} > e$ which implies in turn that $\exists k \mid (A^k)_{i_0 i_0} > e$. This contradicts the fact that e is the eigenvalue of A. ■

Stage 4 *Write A in a new basis.*

Consider the linear operator associated with A. Consider a new basis obtained from the original one by a translation of the origin. The new origin is the eigenvector of A calculated before. Write the operator associated with A in this new basis. For simplicity, we keep the notation A for the operator in the new basis. By Lemma 3.2, we have:

$$A := P^{-1}AP, \quad \text{where} \ P = \begin{pmatrix} u_1 & \varepsilon & \varepsilon \\ \varepsilon & \ddots & \varepsilon \\ \varepsilon & \varepsilon & u_J \end{pmatrix} .$$

By Lemma 3.3, all critical terms of A are now equal to e and all non-critical terms are less or equal to e. We recall that A_{ij} is a critical term if the arc (i,j) belongs to the critical graph.

Stage 5 *Determine the critical graph of A.*

Determine the critical graph. It suffices to draw the (communication) graph of terms equal to e in A and to keep only the circuits of this graph.
Compute the number of strongly connected subgraphs (j) and the cyclicity (k) of the critical graph. The paragraph corresponding to the general spectral behaviour of A is **scsj-cyck**.

Stage 6 *Compute the projective size of A.*

Consider A^M the stationary version of A as defined previously. We define critical columns (resp. lines) as columns (resp. lines) of A^M containing a critical term. We denote $\mathcal{C} = \{(i,j) \mid (i,j) \text{ belongs to a critical line or a critical column } \}$. Set:

$$\alpha = \bigwedge_{(i,j)\in\mathcal{C}} A^M_{ij} \ .$$

We call α the projective size[1] of A. If $\alpha = 1$, we are exactly in the frame of the examples and of the figures considered below. If $\alpha \neq 1$, the correct figure is obtained from the ones drawn below by an homothetic transformation of center $\mathbf{e} = (e,e,e)'$ and of ratio α.

Stage 7 *Check non critical terms of A. Final classification.*

Consider the couples $(i,j) \in \mathcal{C}$ which does not belong to the critical graph. If they are all equal to α, then the figures corresponding to matrix A are given in the first table. If these couples are not all equal to α, the figures get modified. One has now to report to the figures of table 2.

Remark In the scs2-cyc1 and scs1-cyc2 cases, the figures correspond to the situation where the critical columns are 1 and 2. If this is not the case of matrix A, consider a new basis obtained from the original one by a permutation of the coordinates (see Lemma 3.1). In the same way, in the scs2-cyc2 case, when the cycle of length 2 is not over coordinates (1,2), write A in a new basis obtained by permutation of the coordinates.

Table **1.**

Type of A	*Figure n°*
scs3-cyc1	4
scs1-cyc3	6
scs2-cyc1	8
scs1-cyc2	8
scs2-cyc2	11

, Table **2.**

Type of A	*Figure n°*
scs3-cyc1	5
scs1-cyc3	7
scs2-cyc1	9
scs1-cyc2	9,10
scs2-cyc2	12

.

Remark If A is scs1-cyc1, there is no figure as the spectral behaviour is trivial.

Let us consider the six possible spectral behaviour one after the other. For each case, we are going to draw the set of eigenvectors and periodic regimes, in $I\!P I\!R^3 \simeq I\!R^2$.

We will also represent the domains of attraction of the different eigenvectors and periodic regimes. For a matrix A, we call domain of attraction of an eigenvector (or of a periodic regime) the set of initial conditions $\{x_0\}$ such that $\pi(A^n x_0)$ converges to that eigenvector (or periodic regime). By Theorem 3.3, this convergence occurs in finite time.

[1] In the scs3-cyc1 and scs2-cyc1 cases, it is exactly equal to the projective radius of A. We recall that the projective diameter of A is defined by $\mathtt{D}(A) = \sup_{u,v\in R^J} d(Au, Av)$.

- **scs1-cyc1**

Let A be a scs1-cyc1 matrix. We denote by v the unique eigenvector of A. Proposition 3.1 together with theorem 3.3 gives:

$$\forall u_0 \in \mathbb{R}^J, \quad \pi(A^m u_0) \xrightarrow{m} \pi(v)\,.$$

The convergence occurs in finite time (Theorem 3.3). The domain of attraction of $\pi(v)$ is $\mathbb{PR}^J$ and the initial condition u_0 is forgotten. This case is of special importance for stochastic models (see [9]).

- **scs3-cyc1** and **scs1-cyc3**

If A is a scs1-cyc3 matrix, then A^3 is a scs3-cyc1 matrix (but the converse is false!). For example,

$$A = \begin{pmatrix} . & . & e \\ e & . & . \\ . & e & . \end{pmatrix}, \quad B = A^3 = \begin{pmatrix} e & . & . \\ . & e & . \\ . & . & e \end{pmatrix},$$

where (.) stands for -1. We consider first the scs3-cyc1 case.

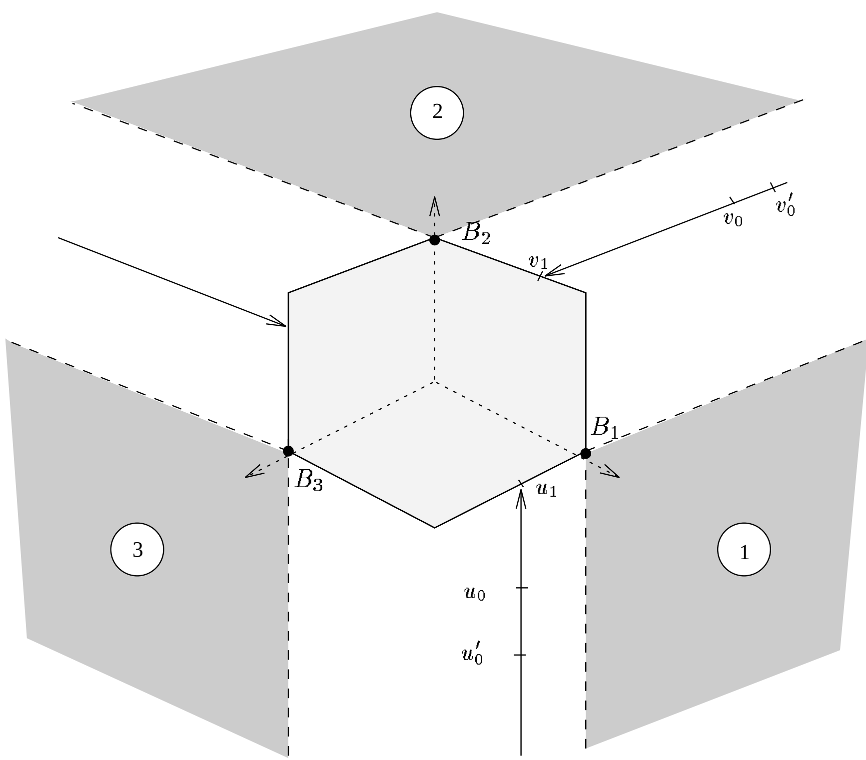

Figure 4: scs3-cyc1, domains of attraction.

There are three independent eigenvectors and no periodic regime of period greater than one (Theorem 3.2). Let us consider more specifically the matrix B. B is a normalized matrix and we check that it is stationary (i.e. $B^2 = B$). We have represented in Figure 4 the set of eigenvectors and the domains of attraction. Theorem 3.2 helps us understand this picture. Here we have $B^+ = B$. Then the three columns B_1, B_2 and B_3 of B are the extremal eigenvectors. These extremal eigenvectors (or more precisely their image by π) are represented by a black dot ($\bullet$). The set of eigenvectors is the $\mathbb{R}_{Max}$ convex hull of these three eigenvectors. It is filled in light gray.
If the initial condition x_0 is in the dark gray zone number i, then the limit value (of $\pi(B^n x_0)$) is $\pi(B_i)$. If the initial condition is in one of the white strips, then the limit value is the nearest point for the projective distance (and of course this limit is attained in one step as $B^2 = B$). For example, for initial conditions u_0 or u'_0 (resp. v_0, v'_0) the limit value is u_1 (resp. v_1).

We will now consider what happens if we modify the non-critical terms of the matrix B. We consider three different examples to illustrate it.

$$C = \begin{pmatrix} e & . & . \\ -0.5 & e & . \\ . & . & e \end{pmatrix}, \; D = \begin{pmatrix} e & -0.6 & . \\ . & e & 0.6 \\ . & . & e \end{pmatrix}, \; E = \begin{pmatrix} e & . & -0.5 \\ -0.2 & e & . \\ -0.2 & -0.5 & e \end{pmatrix},$$

where $(.) = -1$. We represent the set of eigenvectors and the limits between domains of attraction.

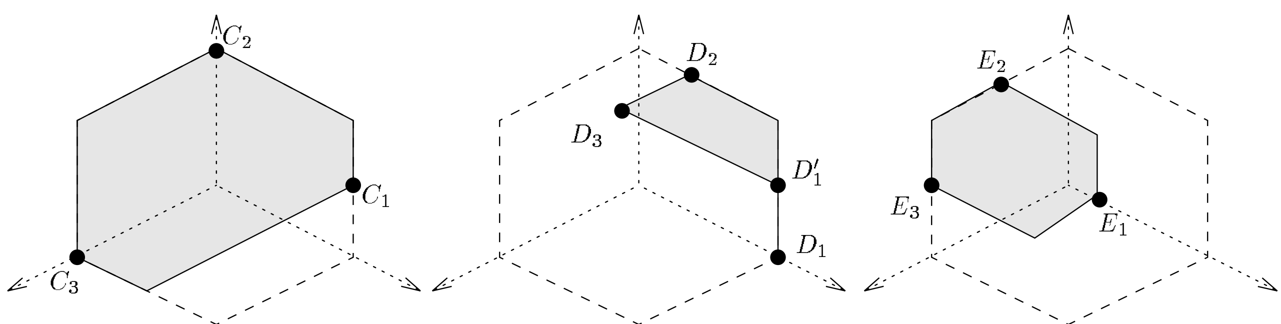

Figure 5: scs3-cyc1, three possible forms for the set of eigenvectors.

We can represent these sets very rapidly, using the procedure described at the end of Section **2**. Let us consider the matrix C first. We represent the three columns of C, $\pi(C_1)$, $\pi(C_2)$ and $\pi(C_3)$. The convex hull of these three points is the set of eigenvectors of C (note that $C = C^2$). For the matrix E, the interpretation is the same.

We consider now the matrix D. The difference with the two previous examples is that D is not stationary. The convex hull of the columns of D, $\pi(D_1)$, $\pi(D_2)$ and $\pi(D_3)$, is the image of D ($D(\mathbb{R}^3)$). It is different from the set of eigenvectors which is, here, the interior of this convex hull. Another way to obtain the set of eigenvectors is to consider the stationary version of D (i.e. D^2, as we have $D^3 = 1 \otimes D^2$). The set of eigenvectors is the convex hull of the columns of D^2, $\pi(D_1')$, $\pi(D_2)$ and $\pi(D_3)$.

Now we consider the case of scs1-cyc3 matrices. There is only one eigenvector but there are periodic regimes of period 3. The set of periodic regimes of period 3 of a scs1-cyc3 matrix M is equal to the set of eigenvectors of M^3. Let us consider more specifically the matrix A defined previously.

$$A = \begin{pmatrix} . & . & e \\ e & . & . \\ . & e & . \end{pmatrix}, \; (.) = -1 \; .$$

The previous remark provides us with the set Π of periodic regimes of A. To go further, we want to characterize, given an initial condition u in the hexagon Π, the periodic regime $\{u, Au, A^2u\}$.

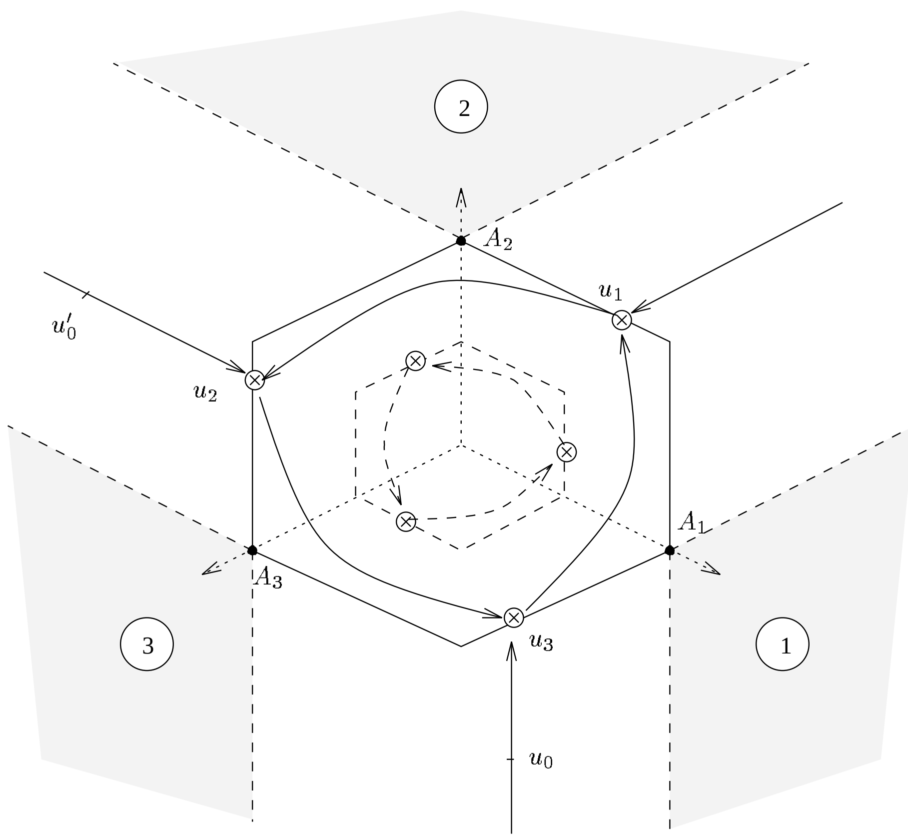

Figure 6: scs1-cyc3, periodic regimes.

It is easy to check that the unique eigenvector of A is $\mathbf{e} = (e, e, e)'$. We consider $u \in \Pi$, $u \neq \mathbf{e}$, Theorem 3.3 shows us that $\{u, Au, A^2u\}$ is a periodic regime. It implies that $A^3u = u$

and $d(A^3u, \mathbf{e}) = d(u, \mathbf{e})$. By Proposition 2.1, we have

$$d(A^3u, \mathbf{e}) \leq d(A^2u, \mathbf{e}) \leq d(Au, \mathbf{e}) \leq d(u, \mathbf{e})\,.$$

We conclude that:

$$d(A^2u, \mathbf{e}) = d(Au, \mathbf{e}) = d(u, \mathbf{e})\,.$$

The points of a periodic regime are at a constant distance (for the projective distance) of the unique eigenvector $\mathbf{e}$. The symmetry does the trick as the figure constituted by the three points $\{u, Au, A^2u\}$ must be invariant by a permutation of the three projective axes. The direction of rotation depends on the critical cycle, whether it is $(1,2,3)$ or $(1,3,2)$. For example A and A^2 have opposite directions of rotation.

In Figure 6, we have represented two periodic regimes of A (the columns of A, $\pi(A_1), \pi(A_2)$, $\pi(A_3)$ constitute a third one). The direction of rotation is counter-clockwise. We have also represented the domains of attraction. If the initial condition is in one of the gray zones then the stationary periodic regime is $\pi(A_1), \pi(A_2)$ or $\pi(A_3)$. If the initial condition is in one of the white strips, the limit regime consists of three points on the boundary of the hexagon. We have represented an example. It corresponds to initial conditions along one of the three large arrows. For example for an initial condition u_0 or u_0', the limit regime is $\{u_1, u_2, u_3\}$. More precisely, we have:

$$\pi(Au_0) = \pi(u_1),\ \pi(A^2u_0) = \pi(u_2),\ \pi(A^3u_0) = \pi(u_3),\ \pi(A^4u_0) = \pi(u_1), \ldots\,,$$

$$\pi(Au_0') = \pi(u_3),\ \pi(A^2u_0) = \pi(u_1), \ldots\,.$$

If the initial condition u belongs to Π, the stationary periodic regime is $\{u, Au, A^2u\}$ of course. We have also drawn an example of such a regime.

What happens if we perturb non-critical terms? To describe it, it will be useful to define the notion of subdiagonals.

Definition 3.8 *Let M be a matrix of size J. We call i^{th} subdiagonal of M the terms $\{M_{i1}, M_{i+1,2}, \ldots, M_{i+J-i,1+J-i}, M_{1,2+J-i}, \ldots, M_{i-1,J}\} = \{M_{i-1+k,k\ [J]}, \forall k\}$. The first subdiagonal is the diagonal of the matrix!*

For the matrix A above, the critical subdiagonal is the second one. If we perturb a non-critical term (i.e. a term of the first or third subdiagonal), after a transient regime, the whole subdiagonal will be equal to this term. Let us consider an example.

$$A' = \begin{pmatrix} a & b_1 & e \\ e & . & . \\ b_2 & e & . \end{pmatrix} \longrightarrow (A')^4 = \begin{pmatrix} a & b & e \\ e & a & b \\ b & e & a \end{pmatrix},\ (A')^5 = \begin{pmatrix} b & e & a \\ a & b & e \\ e & a & b \end{pmatrix},\ldots\,,$$

with $(.) = -1,\ -1 < a, b_1, b_2 < e,\ b = b_1 \oplus b_2$[1]. This provides us with specific pictures for the sets of periodic regimes. When we increase continuously a non-critical term, this set evolves in the same manner as the diaphragm of a camera. Let us illustrate it in Figure 7.

$$F = \begin{pmatrix} -0.8 & . & e \\ e & -0.8 & . \\ . & e & -0.8 \end{pmatrix},\ G = \begin{pmatrix} -0.5 & . & e \\ e & -0.5 & . \\ . & e & -0.5 \end{pmatrix},$$

$$H = \begin{pmatrix} -0.2 & . & e \\ e & -0.2 & . \\ . & e & -0.2 \end{pmatrix},\ (.) = -1\,.$$

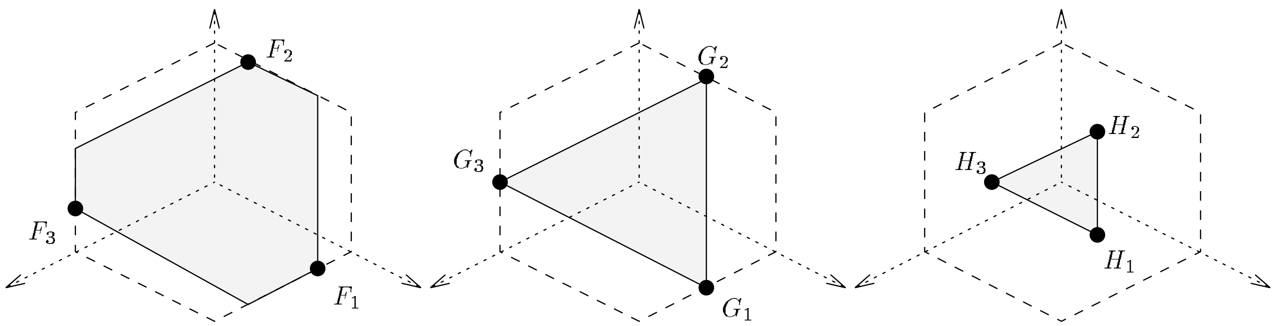

Figure 7: scs1-cyc3, three possible forms for the set of eigenvectors.

When the terms of the diagonal become equal to e, then we have a scs1-cyc1 matrix with $\mathbf{e} = (e, \ldots, e)'$ as unique eigenvector. If the terms of the diagonal are greater than e, then we get a scs3-cyc1 matrix where we find the same kind of pictures as in Figure 4, sets which have now to be interpreted in terms of eigenvectors.

Remark In the cases we have been dealing with so far, domains of attraction had a very easy algebraic characterization. In fact for an initial condition u the limit value was the "nearest" (for the projective distance) eigenvector or periodic regime. This is not a general result as we will see in the scs1-cyc2 case.

- **scs2 – cyc1** and **scs1 – cyc2**.

In the same way as previously, if A is a scs1-cyc2 matrix then A^2 is a scs2-cyc1 matrix, the converse being false. For example,

$$A = \begin{pmatrix} . & e & . \\ e & . & . \\ . & . & -2 \end{pmatrix},\ B = A^2 = \begin{pmatrix} e & . & . \\ . & e & . \\ . & . & -2 \end{pmatrix},\ (.) = -1\,,$$

[1]The size of matrix A is here $inf\ (a, b)$.

Let us consider first the scs2-cyc1 case and more precisely the matrix B. The general results of spectral theory tell us that there are two extremal eigenvectors (the first two columns of B as $B^+ = B$) and no periodic regime of period greater than 1. We have already represented the set of eigenvectors of B, in Figure **3**. We will represent it again together with the domains of attraction of the different eigenvectors in Figure 8.

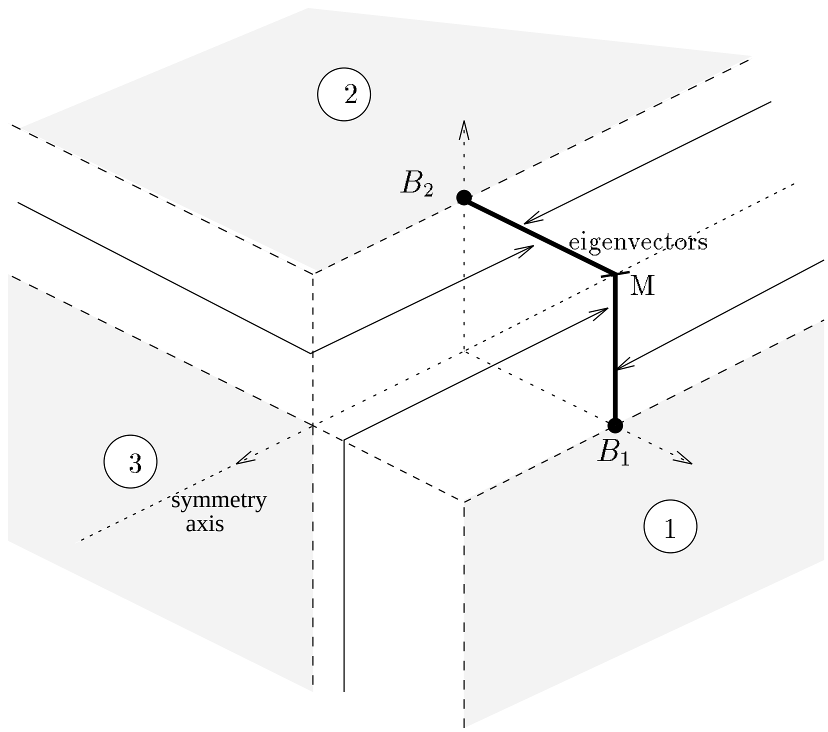

Figure 8: scs2-cyc1 (or scs1-cyc2), domains of attraction.

There is a symmetry axis for the whole figure (corresponding to the fact that matrix B is unchanged by a permutation of the first two columns). The extremal eigenvectors are symbolized by a dot, $\bullet$. In opposition with the scs3-cyc1 case, no eigenvector has a domain of attraction restricted to itself. If the initial condition x_0 is in the gray zone 1 or 2, the limit value of $\pi(B^k x_0)$ will be B_1 or B_2 respectively. If it is in zone 3, then the limit value will be M. When the initial condition is in one of the white strips, the limit value is given by the arrow.

The picture remains the same for the matrix A which is scs1-cyc2. There is only one eigenvector which is M. The bold "line" between B_1 and B_2 is the set of periodic regime of period 2. Two points of this set belong to the same periodic regime if they are "symmetric" with respect to M. For an initial condition in zone 3, the limit regime is the eigenvector M. For an initial condition in zones 1 or 2, the limit regime is $\{B_1, B_2\}$ and so on.

We now want to analyze what happens if we modify non-critical terms. We have to distinguish between modifications of terms belonging to critical columns (columns 1 and 2 here) and of terms belonging to non-critical columns. If we modify a term belonging to a critical column, the set of eigenvectors (obtained as the convex hull of critical columns) will also be modified. On the other hand, it is possible that a modification of a term of the non-critical column does not affect the set of eigenvectors but only the domains of attraction. Let us illustrate this idea.

$$C = \begin{pmatrix} e & . & -0.5 \\ . & e & . \\ . & . & -2 \end{pmatrix}, \; D = \begin{pmatrix} e & . & 0.5 \\ . & e & . \\ . & . & -2 \end{pmatrix}, \; (.) = -1 \; .$$

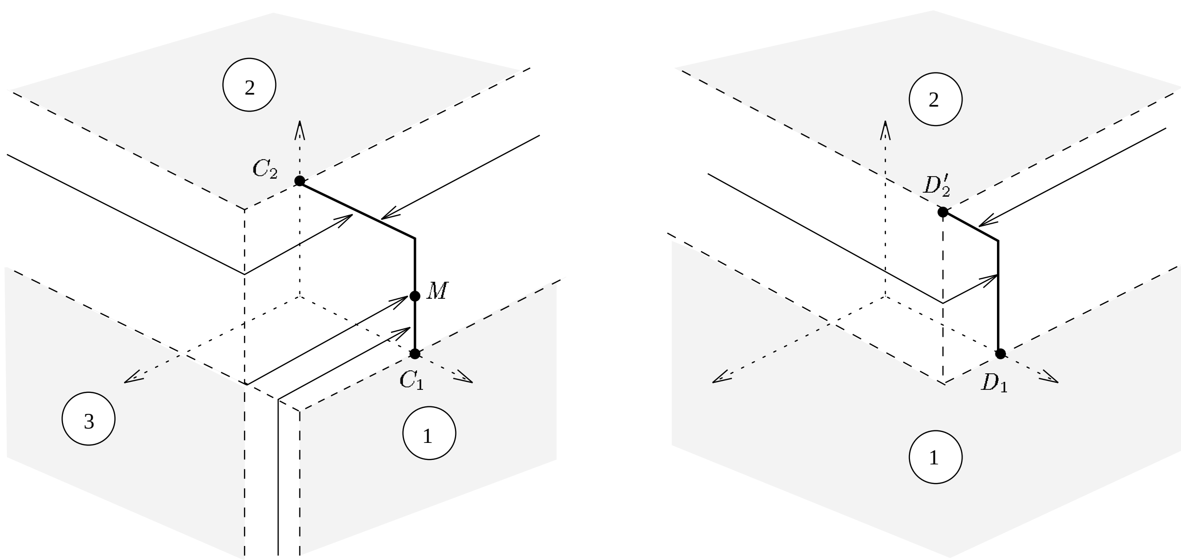

Figure 9: scs2-cyc1, other examples, matrices C and D.

For matrix C, the set of eigenvectors is not modified, but the domains of attraction are. The picture of Figure 9 has to be interpreted in the same way as previously. The gray zone 1 and 2 are the domains of attraction of C_1 and C_2 respectively. If the initial condition u_0 is in zone 3, the limit value of $\pi(B^k u_0)$ will be M.

For matrix D, the domains of attraction and the set of eigenvectors are modified. In fact, the stationary regime of D is:

$$D^2 = \begin{pmatrix} e & -0.5 & 0.5 \\ . & e & -0.5 \\ . & . & -0.5 \end{pmatrix}, \; (.) = -1 \; .$$

For matrix D^2, a term of a critical column has been modified. It is reflected by a corresponding modification of the set of eigenvectors. The points represented, D_1 and D_2'

are the critical columns of matrix D^2. In this example, zones 1 and 3 have melted. They constitute the domain of attraction of D_1.

Let us now consider what happens when we modify non-critical terms of a scs1-cyc2 matrix, E. The analysis made before remains valid. The set of periodic regimes of E is exactly the set of eigenvectors of the scs2-cyc1 matrix E^2. One interesting point to notice is that there might be no symmetry axis (see Figure 10) although all the stationary regimes are periodic of period 2. Figure 10 provides an example of this behaviour. It corresponds to the matrix:

$$E = \begin{pmatrix} . & e & . \\ e & . & . \\ -0.5 & . & -1.5 \end{pmatrix}, \ (.) = -1 \ .$$

Let E'_1 and E'_2 be the two extremal points of the set of periodic regimes (i.e. the critical columns of E^m, the stationary version of E). The unique eigenvector, M is the point of the set of periodic regimes equidistant (for the projective distance) from E'_1 and E'_2. A periodic regime consists of two points equidistant from point M.

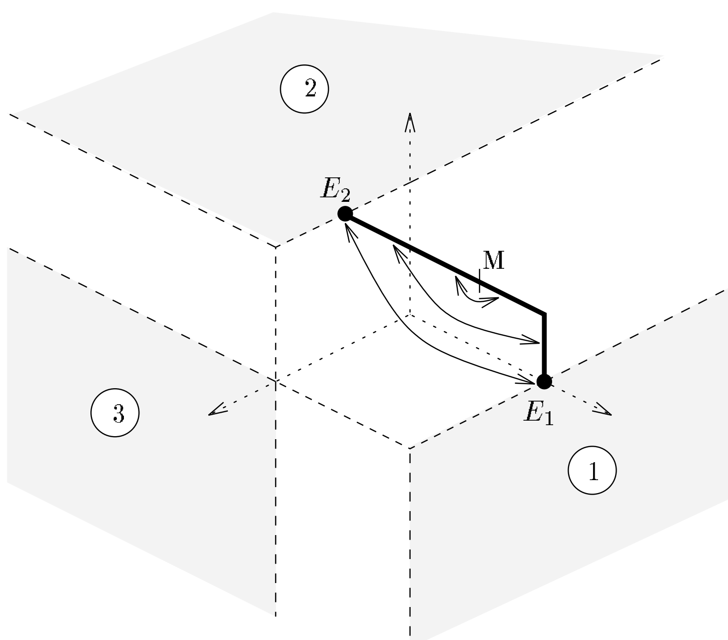

Figure 10: scs1-cyc2, matrix E, a non-symmetrical example.

The interpretation of the domains of attraction is the same as previously.

- **scs2-cyc2**

The canonical example of such a matrix is:

$$A = \begin{pmatrix} . & e & . \\ e & . & . \\ . & . & e \end{pmatrix}, \; (.) = -1 \, .$$

There are two extremal eigenvectors and also periodic regimes of period 2. If a matrix N is scs2-cyc2 then the matrix N^2 is scs3-cyc1. Then to find the set of eigenvectors and periodic regimes of a scs2-cyc2 matrix N, one only has to determine the set of eigenvectors of N^2 (see paragraph **scs3-cyc1** and **scs1-cyc3**).

Let us represent graphically eigenvectors, periodic regimes of period 2 and domains of attraction of matrix A in Figure 11.

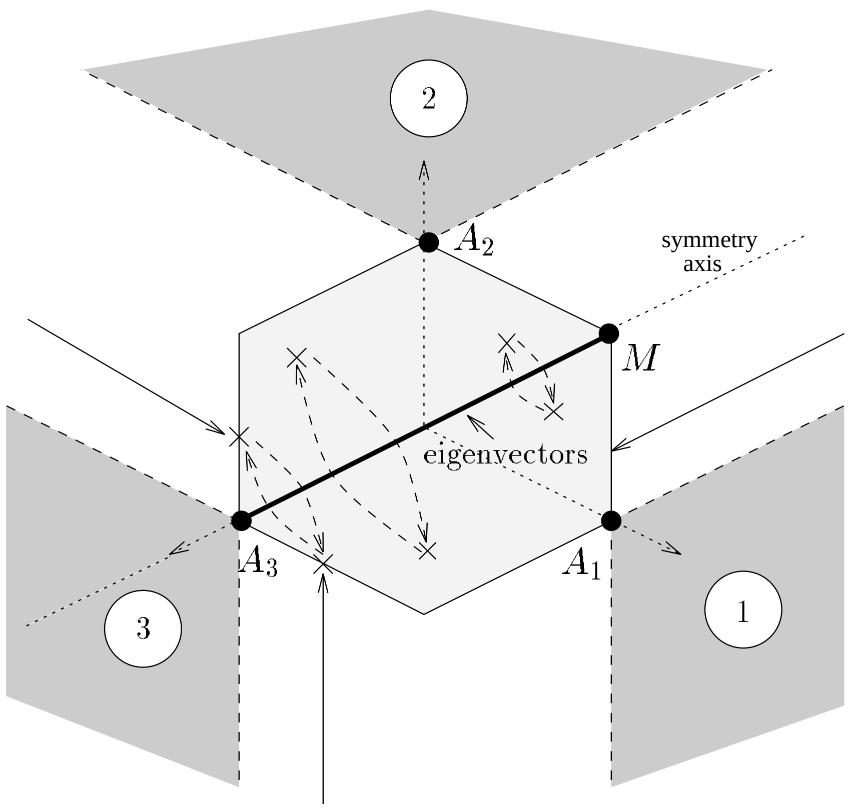

Figure 11: scs2-cyc2, domains of attraction.

There is a symmetry axis for the whole figure (matrix A is unchanged by a permutation of the first two coordinates). The set of eigenvectors (the interval $[M, A_3]$) splits the set of periodic regimes in two equal parts. The two points of a periodic regime of period 2 are symmetric with respect to the set of eigenvectors. The analysis of domains of attraction is analog to the one of cases scs3-cyc1 and scs1-cyc3. If the initial condition belongs to the zones 1, 2 or 3, the limit value will be either the periodic regime $\{A_1, A_2\}$ or A_3. If the initial condition belongs to one of the three white strips, the limit regime is a periodic regime of period 2, corresponding to the "nearest" point on the hexagon and its symmetrical point.

We have now to analyze what happens if we modify non-critical terms. The cases we have already considered are enough to understand what is going to happen. We will represent two characteristic examples.

$$B = \begin{pmatrix} a & e & . \\ e & . & . \\ b & . & e \end{pmatrix}, \; C = \begin{pmatrix} . & e & c \\ e & . & d \\ . & . & e \end{pmatrix}, \; (.) = -1 \; .$$

The reals a, b, c and d must satisfy the following constraints (in order for our matrices B and C to be scs2-cyc2):

$$-1 \leq a < e, \; -1 \leq b < 1, \; -1 \leq c < 1, \; -1 \leq d < 1 \; .$$

The stationary regimes of the matrices are:

$$B^3 = \begin{pmatrix} a & e & . \\ e & a & . \\ b & b & e \end{pmatrix}, \; C^3 = \begin{pmatrix} . & e & c \oplus d \\ e & . & c \oplus d \\ . & . & e \end{pmatrix}, \; (.) = -1 \; .$$

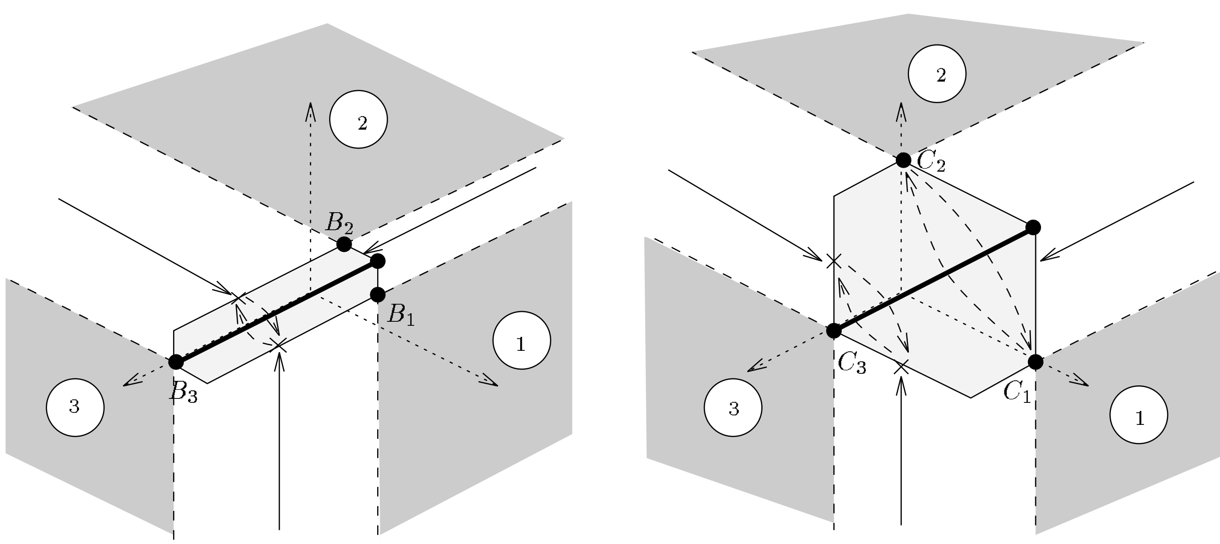

Figure 12: scs2-cyc2, domains of attraction, matrices B and C.

The graphical representations correspond to $a = -0.2, b = -0.5, c = -0.5, d = -0.9$.

3.4 Transient regimes

We will now take a closer look at transient regimes of matrices. The matrices we have been considering so far were chosen in order to be stationary or at least to have a very short transient regime. To emphasize the transient behaviour, we will, on the other hand, consider matrices with long transient regimes.

First of all, one has to remark that a matrix can have an arbitrarily long transient regime. Let us take an example.

$$M = \begin{pmatrix} e & -1 \\ -1 & -\eta \end{pmatrix},\ 0 < \eta \ll 1,\ M^2 = \begin{pmatrix} e & -1 \\ -1 & -2\times\eta \end{pmatrix},$$

$$M^k = \begin{pmatrix} e & -1 \\ -1 & -k\times\eta \end{pmatrix},\ k < [\frac{2}{\eta}]+1,\ \ M^k = \begin{pmatrix} e & -1 \\ -1 & -2 \end{pmatrix},\ k \geq [\frac{2}{\eta}]+1\ .$$

The length of the transient regime is thus $[\frac{2}{\eta}]$. The matrix M is scs1-cyc1, its unique eigenvector is $E = (e,-1)'$. As we have seen previously, it implies that $\forall u \in I\!R^J$, $\lim_k \pi(M^k u) = \pi(e,-1)'$. Let us consider the initial condition $u = (e,3)'$. We have $\pi(Mu) = \pi(e, 1-\eta)'$, $\pi(M^2u) = \pi(e, 1-2\times\eta), \ldots$. We have represented the sequence $\{\pi(M^k u)\}$ in the projective space $I\!PI\!R^2$. We have also represented the same sequence for three other initial conditions.

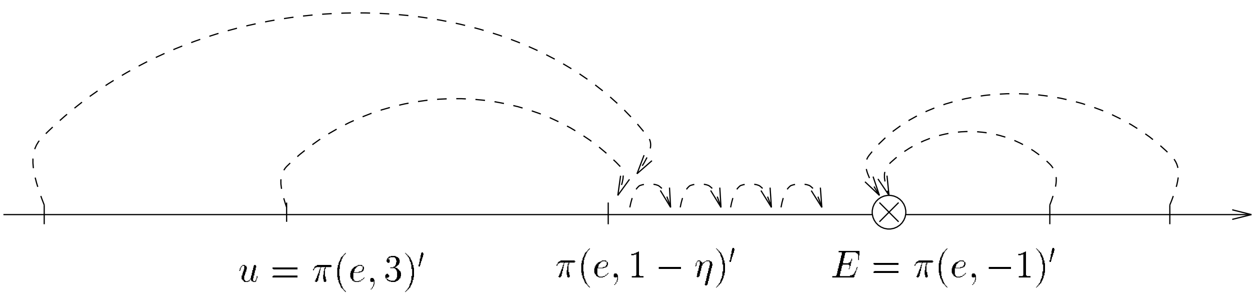

Figure 13: Dimension 2, transient regimes.

We are now going to present analog figures corresponding to matrices of size 3.

First of all we consider the example of scs1-cyc1 matrices.

$$A = \begin{pmatrix} e & . & . \\ . & . & -\eta \\ . & -\eta & . \end{pmatrix},\ \ B = \begin{pmatrix} e & . & . \\ . & -2 & -\eta \\ . & -\eta & -2 \end{pmatrix},\ 0 < \eta \ll 1,\ (.) = -1\ .$$

Both matrices have the same stationary regime, given by the matrix:

$$\lim_k A^k = \lim_k B^k = \begin{pmatrix} e & . & . \\ . & -2 & -2 \\ . & -2 & -2 \end{pmatrix},\ (.) = -1\ .$$

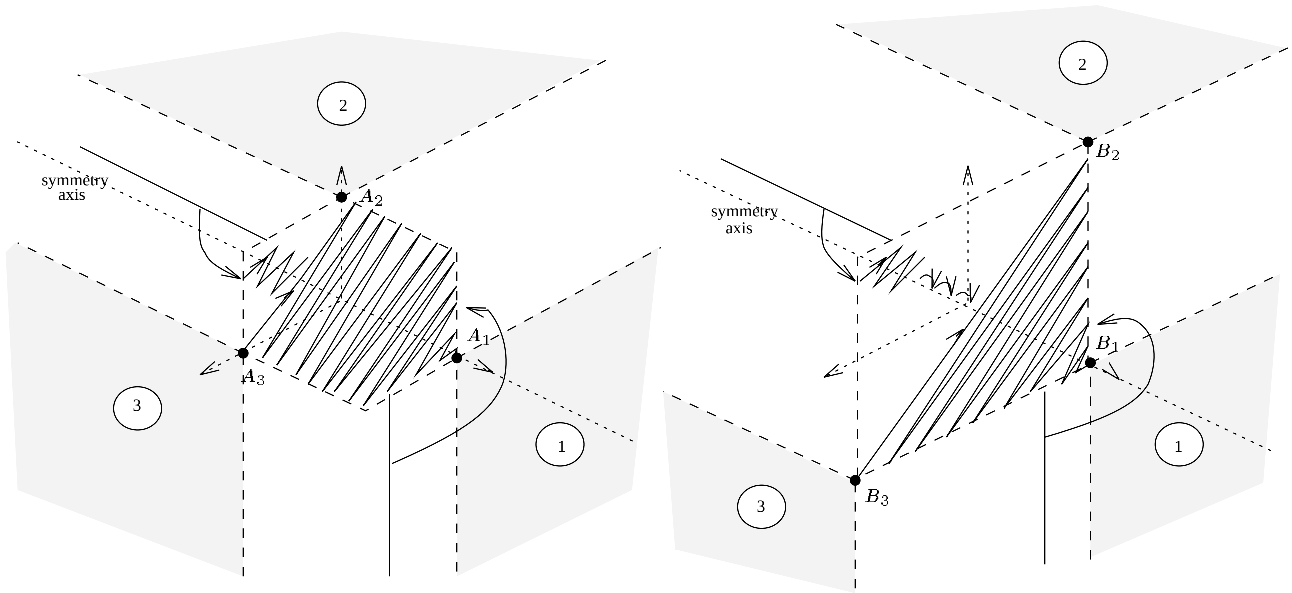

Figure 14: Dimension 3, scs2-cyc2, transient regimes.

Matrix A is obtained by a small perturbation of the matrix presented in paragraph **scs2-cyc2** (Figure 11). The transient behaviour reflects it, as the figure we obtain is very close to Figure 11. As a comparison, we have also represented the matrix B whose behaviour is asymptotically identical.

Let us comment on the figure corresponding to A a little further. There is a symmetry axis for the whole picture. The three points A_1, A_2 and A_3 are the projections of the columns of matrix A. If the initial condition is in zone 1, there is convergence in one step to $A_1 = \pi(e, -1, -1)'$, the unique eigenvector. If the initial condition is in zone 2 (resp. 3), we have $\pi(Ax_0) = A_3$ (resp. $\pi(Ax_0) = A_2$). We have represented the whole sequence $\{\pi(A^k x_0)\}$ for an initial condition $x_0 = A_3$. For an initial condition in one of the three white strips, for example let us consider u_0 (or u_0'), then $\pi(Au_0)$ (or $\pi(u_0')$) is the point pointed by the arrow in the picture (it is the symmetric of the "nearest" point on the set $Im(A)$). For initial condition u_0', we have also drawn the beginning of the sequence $\{\pi(A^n u_0')\}$.

For matrix B, the set of periodic regimes is the same one as A. But the domains of attraction are quite modified. It emphasizes the possible influence of transient regimes, especially in stochastic models. Here we have drawn the sequences $\{\pi(A^n u_0)\}$ (or part of them) for several different initial conditions. One of them is in zone 2, another one in the white strip between zones 2 and 3 and the last one is on the symmetry axis.

We consider now the transient regime of a scs2-cyc1 matrix.

$$C = \begin{pmatrix} e & . & . \\ . & e & . \\ . & . & -\eta \end{pmatrix},\ 0 < \eta \ll 1,\ (.) = -1\ .$$

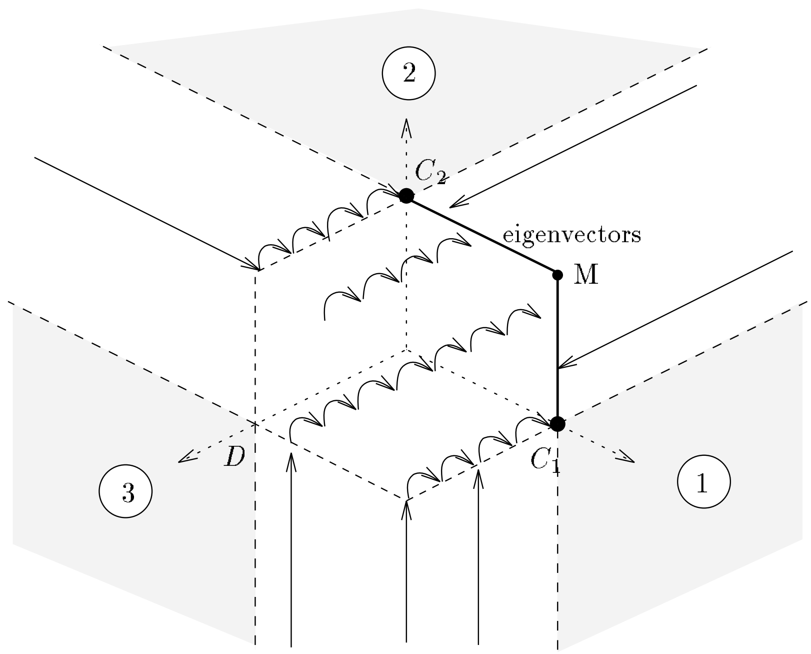

Figure 15: Dimension 3, scs2-cyc1, transient regimes.

The stationary regime of C is the canonical example of the **scs2-cyc1** paragraph (i.e. the same matrix with $-\eta = -2$). We can also view C as a small perturbation of the canonical example of **scs3-cyc1** matrix (i.e the same matrix with $\eta = 0$). The figure reflects these remarks.

The points C_1 and C_2 are the projection of the first two columns of the matrix. The point D is $\pi(e, e, 1-\eta)'$. If the initial condition is in zone 1 (resp. 2) we have convergence to C_1 (resp. C_2) in one step. If it is in the strip between zone 1 and 2, convergence occurs in one step according to the arrows. The hexagon represented in dotted line is $Im(C) = C(\mathbb{R}^3)$. We have represented the whole sequence $\{\pi(C^k x_0)\}$ for several initial conditions.

4 Applications

We are now going to propose examples where the graphical representation in the projective space appears to be useful in understanding some (Max,+) phenomena. The first example is a deterministic one and comes from an article by Braker and Olsder [3].

Example 1 *An alternative to the power algorithm.*

The framework is the following. We consider an irreducible and normalized matrix, A. We suppose that A is scs1-cyck ($k \leq J$). For simplicity of notations, we suppose also that the critical circuit of A is $(1, 2, \ldots, k, 1)$ (we can obtain this just by considering a permutation of the coordinates, see Lemma 3.1). Our goal is to find its unique eigenvector, u. The natural way to do it is to compute $\bigoplus_{k \geq 0} A^k$. This method was presented in the algorithm of Section 3.3. It is the exact equivalent of the power algorithm of the traditional linear algebra. Braker and Olsder propose another method for computing this eigenvector, which we are going to analyze.

Let M be the smallest integer such that A^M is stationary (i.e. $\forall m \geq M, A^{m+k} = A^m$). Here is the algorithm.

1. Take an initial vector $x_0 \neq (\varepsilon, \varepsilon, \ldots, \varepsilon)'$.
2. Define the smallest integer m such that $A^{m+k} x_0 = A^m x_0$ ($m \leq M!$).
3. Consider $v = \frac{1}{k} \times (A^M x_0 + A^{M+1} x_0 + \ldots + A^{M+k-1} x_0)$.

We first give two examples where this algorithm is successful. The figure corresponds to matrices:

$$A = \begin{pmatrix} . & e & . \\ . & . & e \\ e & . & . \end{pmatrix},\ B \begin{pmatrix} . & e & . \\ e & . & . \\ e & . & . \end{pmatrix},\ (.) = -1\ .$$

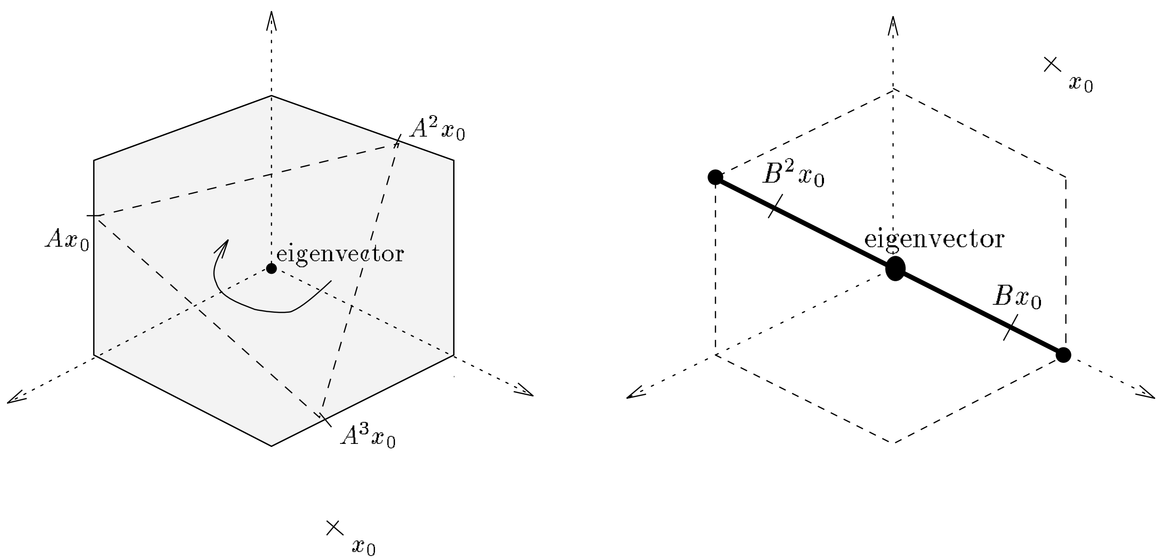

Figure 16: Matrices A and B. The algorithm is successful.

Matrix A is scs1-cyc3, its unique eigenvector is $\mathbf{e} = (e, e, e)'$. It is quite easy to see that for any periodic regime of period 3, $\{u_1, u_2, u_3\}$, we have $\pi(\mathbf{e}) = \pi((u_1 + u_2 + u_3)/3)$. More generally, for all scs1-cyc3 matrices, the algorithm will provide the eigenvector. Matrix B is scs1-cyc2, its unique eigenvector is $\mathbf{e} = (e, e, e)'$. We see on the figure that the algorithm still works.

Braker and Olsder prove the following results:

- The first k components, $v_1, \ldots, v_k$, of the vector v of the algorithm are the correct eigenvector components.
- The vector v is not always the eigenvector of A.

This last statement proves that the algorithm of Braker and Olsder is less efficient than the natural power algorithm (if i_0 is such that $\exists k$, $(A^k)_{i_0 i_0} = e$ then $\tilde{v} = \bigoplus_{k \geq 0} A^k \delta_{i_0}$ is the eigenvector of A, see the algorithm of Section 3.3) which always works.
In order to determine whether v is the eigenvector of A or not, one has only to check if $Av = v$ or not. When the algorithm has failed and v is not the eigenvector of A, Braker and Olsder propose another algorithm which they call extended algorithm.

1. Define the vector $\tilde{v}$ in the following way:

$$\begin{aligned} \hat{v}_i &= v_i \;\; if \;\; (Av)_i = v_i \; , \\ \hat{v}_i &= \varepsilon \;\; if \;\; (Av)_i \neq v_i \; . \end{aligned}$$

2. Let m be the smallest integer such that $A^{m+1}\hat{v} = A^m \hat{v}$.

The integer m is finite (and $m \leq M$). The vector $A^m \hat{v}$ is the unique eigenvector of A.

There exist a technical criteria to determine if the algorithm of Braker and Olsder is going to work or if the extended algorithm is needed (see [3]). But the graphical representation in the projective space gives a very simple illustration of the phenomenon.

Here is now an example where the use of the extended algorithm is necessary.

$$C = \begin{pmatrix} . & e & . \\ e & . & . \\ . & . & -2 \end{pmatrix} , \; (.) = -1 \; .$$

We have represented this example on Figure 17. Matrix C is scs1-cyc2, its unique eigenvector is $(e, e, -1)'$. By the algorithm of Braker and Olsder, we obtain $v = (e, e, e)'$. With the extended algorithm, we get $C \otimes (e, e, \varepsilon)' = (e, e, -1)'$. In most cases, the extented algorithm will be needed for scs1-cyc2 matrices. Of course, the natural power algorithm gives directly the eigenvector. Here for example $\tilde{v} = C(e, \varepsilon, \varepsilon)' \oplus C^2(e, \varepsilon, \varepsilon)' = (e, e, -1)'$.

For matrices of size $J \times J$, we have in the same way that the algorithm is always successful for scs1-cycJ matrices. For scs1-cyck $(1 < k \leq J)$ matrices, the extended algorithm will, in general, be needed.

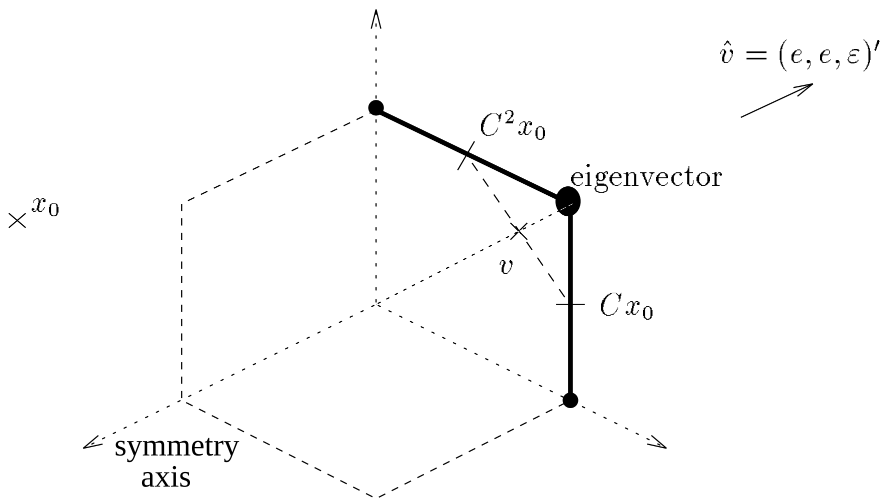

Figure 17: Matrix C. The algorithm of Braker and Olsder fails.

Example 2 *A projectively infinite semigroup of matrices.*

We consider a finite number of matrices $A_1, \ldots, A_k \in \mathbb{R}_{Max}^{J\times J}$. We denote respectively by $< A_1, \ldots, A_k >$ and $\pi < A_1, \ldots, A_k >$ the semigroup generated by $A_1, \ldots, A_k$ and its projection.

$$< A_1, \ldots, A_k > = \{(A_{u_N} \cdots A_{u_2} A_{u_1}),\ u_1, \ldots, u_N \in \{1, \ldots, k\}, N\ finite\},$$

$$\pi < A_1, \ldots, A_k > = \{\pi(A_{u_N} \cdots A_{u_2} A_{u_1}),\ u_1, \ldots, u_N \in \{1, \ldots, k\}, N\ finite\},$$

where π is here the canonical projection of $\mathbb{R}_{Max}^{J\times J}$ into $\mathbb{P}\mathbb{R}_{Max}^{J\times J}$. The problem we are interested in is the finiteness of $\pi < A_1, \ldots, A_k >$. It is in fact a version of the Burnside problem in the special case of the (Max,+) algebra (see [7]).

Let us consider the projective semigroup generated by a single and irreducible matrix $\pi < A > = \{\pi(A), \pi(A^2), \ldots\}$. Theorem 3.3 tells us that $\pi < A >$ is finite.

Remark It is the finiteness of the projective semigroup and not the finiteness of the semigroup which is interesting. Indeed any irreducible matrix A with an eigenvalue different from e is such that $\# < A >$ is infinite.

The next theorem was proved in [7] in a slightly stronger version.

Theorem 4.1 *Let $A_1, \ldots, A_k \in \mathbb{Q}_{Max}^{J\times J}$. We assume that:*

$$\forall u \in \{1, \ldots, k\}, \forall (i,j),\ (A_u)_{ij} > \varepsilon\ .$$

Then the projective semigroup $\pi < A_1, \ldots, A_k >$ is finite.

This theorem does not extend to the case of matrices with non rational entries. Nice counter-examples can be found using the graphical representation in the projective space. We consider the semigroup generated by the matrices:

$$A_1 = \begin{pmatrix} -\eta_1 & . & . \\ . & e & . \\ . & . & e \end{pmatrix}, \; A_2 = \begin{pmatrix} e & . & . \\ . & -\eta_2 & . \\ . & . & e \end{pmatrix}, \; A_3 = \begin{pmatrix} e & . & . \\ . & e & . \\ . & . & -\eta_3 \end{pmatrix},$$

where $(.) = -1$, $0 < \eta_i \ll 1$ and $\eta_i \notin \mathbb{Q}$. We suppose also that $\eta_i/\eta_j \notin \mathbb{Q}$, $i,j \in \{1,2,3\}$, $i \neq j$. An easy way to show that the semigroup $\pi < A_1, A_2, A_3 >$ is infinite is to consider the initial condition $\mathbf{e} = (e,e,e)'$ and to prove that $\Pi = \pi(< A_1, A_2, A_3 > \otimes \mathbf{e}) = \{\pi(M\mathbf{e}),\ M \in < A_1, A_2, A_3 >\}$ is infinite. We obtain a nice illustration of the phenomenon with the help of the graphical representation in the projective space.

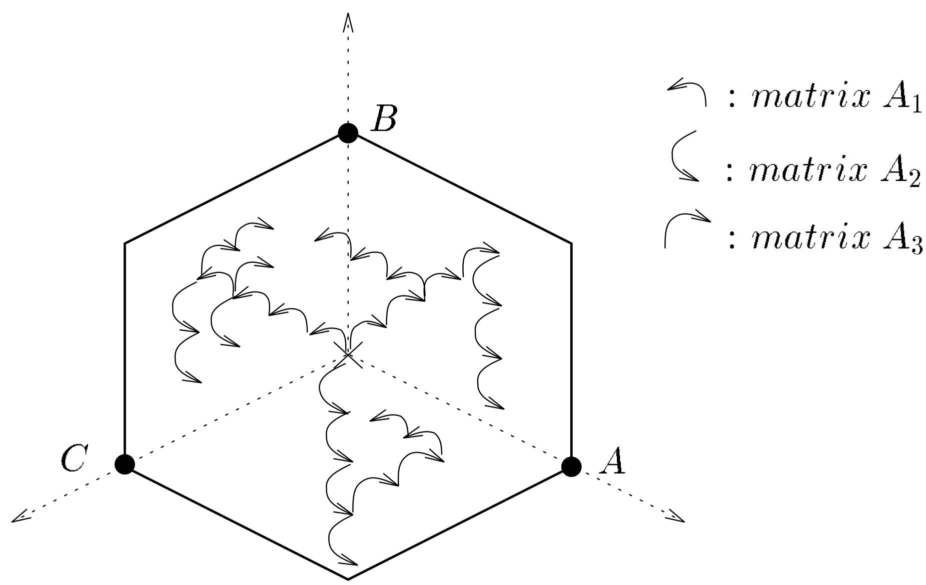

Figure 18: A finitely generated but projectively infinite semigroup of matrices

The extremal eigenvectors of A_1, A_2 and A_3 are $(B,C), (A,C)$ and (A,B) respectively. The picture is analog to the one of Figure 15, with three different transient regimes interacting.

For a point $\mathbf{u} = (u_1, u_2, u_3)'$ such that $d(\mathbf{u}, \mathbf{e}) < 1 - \sup_{i=1,2,3} \eta_i$, where d is the projective distance (def. 2.4), we have:

$$A_1\mathbf{u} = \begin{pmatrix} u_1 - \eta_1 \\ u_2 \\ u_3 \end{pmatrix}, \; A_2\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 - \eta_2 \\ u_3 \end{pmatrix}, \; A_3\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 - \eta_3 \end{pmatrix}.$$

It is very easy to prove that Π is dense in the unit ball of the projective space (the hexagon delimited by A, B and C). In fact let us consider three integers N_1, N_2 and N_3 such that:

$$\sup_{i=1,2,3} (N_i \times \eta_i) - \inf_{i=1,2,3} (N_i \times \eta_i) < 1 \, .$$

Then it is quite obvious that there exists a matrix $M \in < A_1, A_2, A_3 >, M = A_{u_N} \otimes \cdots \otimes A_{u_1}$ with $N = N_1 + N_2 + N_3$ such that:

$$N_i = \#\{k \mid A_{u_k} = A_i\}, \; i = 1, 2, 3 \, ,$$

$$M\mathbf{e} = \begin{pmatrix} -N_1 \times \eta_1 \\ -N_2 \times \eta_2 \\ -N_3 \times \eta_3 \end{pmatrix} .$$

In fact it is easy to understand, watching Figure 18, that we will obtain this formula for $M\mathbf{e}$ iff $\forall n \in \{1, \ldots, N\}$, $\pi(A_{u_n} \otimes \cdots \otimes A_{u_1}\mathbf{e})$ belongs to the interior of the hexagon (A, B, C).

We consider an arbitrary point $\mathbf{v}$ of the interior of the hexagon (A, B, C). As η_1, η_2, η_3 are not co-rational, there exists a sequence of integers $N^{(n)}$ and a sequence of matrices $\{M^{(n)},\ M^{(n)} \in < A_1, \ldots, A_k >\}$ with the following properties.

- The length of $M^{(n)}$ is $N^{(n)}$, i.e $M^{(n)} = A^{(n)}_{u_{N^{(n)}}} \otimes \cdots \otimes A^{(n)}_{u_1}$.
- $N_i^{(n)} = \#\{k \mid A^{(n)}_{u_k} = A_i\}, \; i = 1, 2, 3,$

$$\pi(M^{(n)}\mathbf{e}) = \pi \begin{pmatrix} -N_1^{(n)} \times \eta_1 \\ -N_2^{(n)} \times \eta_2 \\ -N_3^{(n)} \times \eta_3 \end{pmatrix} \stackrel{n \to \infty}{\longrightarrow} \pi(\mathbf{v}) \, .$$

Remark If we consider another initial condition $\mathbf{u} \neq \mathbf{e}$, we will in general obtain a set of reachable points $\pi(< A_1, \ldots, A_k > \mathbf{u})$ dense in the hexagon Π and whose intersection with Π is empty. Let now consider a Markov chain $x(n, x_0)$ whose transition probability $p(.,.)$ verify:

$$\forall \mathbf{v} \in I\!R^3_{Max},\; p(\pi(\mathbf{v}), \pi(A_i\mathbf{v})) = p_i, \; i = 1, 2, 3, \; p_i > 0, \; p_1 + p_2 + p_3 = 1 \, .$$

We take $\mathbf{e}$ as our initial condition. Then Π is a set of transient states for the Markov chain. After the first and before the second hitting time of the border of the hexagon (A, B, C), the Markov chain evolve on a set of transient states dense in the interior of (A, B, C) and whose intersection with Π is empty. It is however possible to show that the chain is positive recurrent. Points A, B or C can be used as regenerative points (for example $\pi(A_3^k A_2^{k'} u) = A$, $\forall u \in I\!R^J$, when k and k' are sufficiently large).

Example 3 *Multiplicity of stationary regimes.*

We consider a stochastic model of product of matrices in the $\mathbb{R}_{Max}$ algebra. The model is the following one:

$$x(n+1) \quad = \quad A(n)x(n)\,, \tag{1}$$

where $x(n+1)$ and $x(n)$ are $\mathbb{R}^J$-valued vectors and $A(n)$ is an irreducible random matrix of size $J \times J$. The exogenous sequence $\{A(n),\ n \in \mathbb{N}\}$ is i.i.d.

The interest for such models arouse from the study of Stochastic Event Graphs, a class of Stochastic Petri Networks. Many networks with synchronization and/or blocking can be modelized this way. Many examples can be found in [1] and [2].
We are interested by the stationary behaviour of $\pi x(n)$. For a network modelized by such a system, we can compute quantities such as queue length, sojourn or idle times from the knowledge of $\pi x(n)$. In [9], the conditions to have a unique stationary regime for $\pi x(n)$ are given.

There is another problem worth considering. What happens for a fixed deterministic initial condition x_0, is it possible to get several stationary regimes? Another way to state the problem is the following one. For a given network with a fixed initial condition and a stochastic dynamic given by equation 1, is there possible to obtain several stationary regimes for queue length or sojourn time. The answer, rather counter-intuitive, is yes.

More precisely, for an i.i.d model $x(n+1) = A(n)x(n)$ and $x_0 \in \mathbb{R}^J$, $\pi x(n, x_0)$ is a Markov chain and this chain can have several classes of recurrence. Let us illustrate this.
We have $A(n) = A$ or B with $P(A(0) = A) = p > 0$ and $P(A(0) = B) = 1 - p > 0$.

$$A = \begin{pmatrix} e & -2 & -2 \\ -2 & e & -2 \\ -2 & -2 & -4 \end{pmatrix}\,,$$

$$B = \begin{pmatrix} e & -1 & 1 \\ -1 & e & 1 \\ -3 & -3 & e \end{pmatrix} = \begin{pmatrix} e & -1 & -1 \\ -1 & e & -1 \\ -1 & -1 & e \end{pmatrix} \circ \begin{pmatrix} e & e & 2 \\ e & e & 2 \\ -2 & -2 & e \end{pmatrix}\,.$$

In the previous line, we have written matrix B in a form which reflects the fact that it is exactly the matrix studied in Figure 4 up to a change of origin (see Lemma 3.2).

On Figure 19, we have materialized the domains of attraction of matrix B. We consider the initial condition $x_0 = (\eta, e, e)'$, where $e < \eta \ll 1$. This initial condition is in the domain of attraction of B_3. As a consequence, we have $\pi(Bx_0) = B_3$ and $\pi(ABx_0) = \pi(AB^k x_0) = \pi(e, e, -2)'$. We check that $\pi(Ax_0) = \pi(e, e-\eta, -2)'$. Both vectors $x_1 = (e, e, -2)'$ and $x_2 = (e, e-\eta, -2)'$ are eigenvectors of both matrices A and B. We conclude that with probability p, we have $\pi x(n, x_0) = \pi(e, e-\eta, -2)'$ and with probability $1-p$, we have

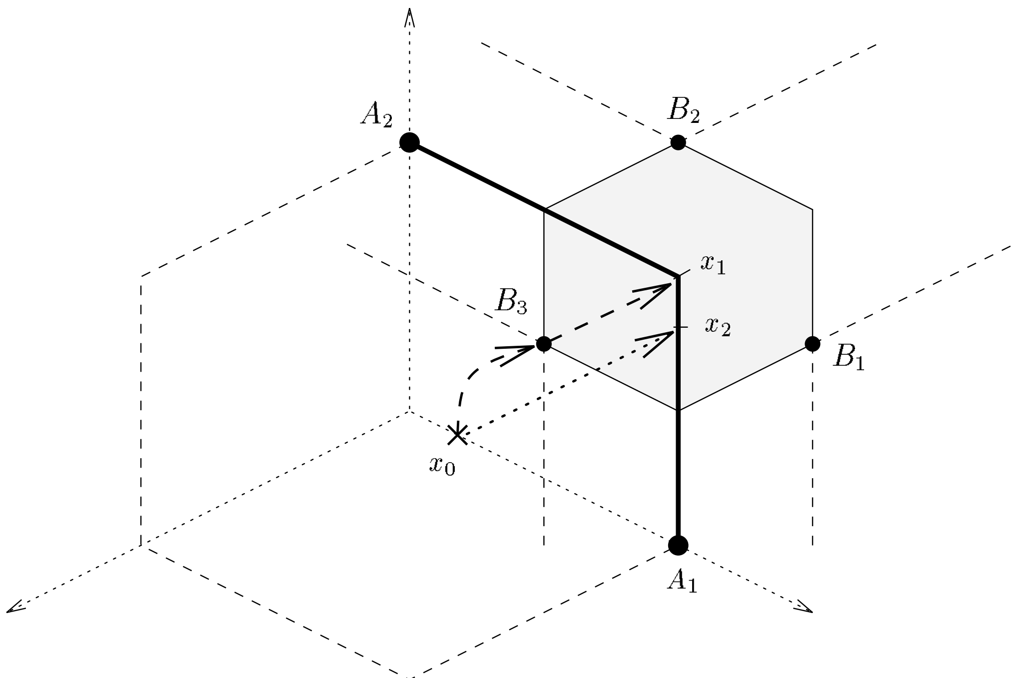

Figure 19: A single initial condition and several stationary regimes.

$\pi x(n, x_0) = \pi(e, e, -2)'$.
There are two absorbing states for the Markov chain $\pi x(n, x_0)$.

This counter-example would probably not have been found without the help of the graphical representation in the projective space. With Figure 19, the multiplicity of stationary regimes becomes rather clear.

5 Software

A C++ program has been written by Bruno Gaujal, which implements the algorithm of section **3**. Given a matrix of dimension 3, this program provides the graphical representation of eigenvectors, periodic regimes and domains of attraction (as in Figures 4 to 12). This program is in free access on Internet. To obtain it, you have to connect via ftp to the site ... You can log with the username 'ftp' and any password using the 'ftp' command. The files is in the subdirectory ... under name .... To use it...

Acknowledgement I would like to thank François Baccelli who introduced me to this subject and whose help has been constant since. I also thank Stéphane Gaubert for many remarks and suggestions to improve this paper.

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