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Abstract: We consider the planar motion of a car-like robot with a bounded derivative of the curvature, with given initial and final configuration (i.e. positions, tangent angles and curvatures). The tangent angle and the curvature of the path are assumed to be continuous. When the distance between the initial and the final point is much greater than the initial and final curvatures and the curvature's derivative, we show how to construct a suboptimal path (the cost is the path length). We admit a finite number of cusps in the path.

Key-words: car-like robot, (sub)optimal path, clothoid, Maximum Principle of Pontryagin

(Résumé : tsvp)

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Chemins sous optimaux du plan dont la dérivée de la courbure est bornée

Résumé : Etant données des configurations initiale et finale (position, orientation et courbure), on cherche un chemin les reliant le long duquel la dérivée de la courbure est bornée. Si les positions initiale et finale sont suffisamment éloignées, on montre comment construire un chemin de longueur sous-optimale, régulier par morceaux et n'ayant qu'un nombre fini de rebroussements. Ce problème intervient dans la planification de trajectoires de robot mobiles.

Mots-clé : robot mobile, chemin (sous)optimal, clothoide, principe du maximum de Pontryagin

1 Introduction.

The problem of constructing an optimal trajectory for the planar motion of a car-like mobile robot between two given positions with given orientations (without obstacles), under more or less realistic constraints, has been the object of several efforts recently.

The basic model of these investigations is an "orientated point" moving in the plane along a trajectory submitted to a bound on the curvature at every point.

In [5] Dubins proves that a unique optimal trajectory exists and is in fact a simple concatenation of at most three pieces, each of which is either a straight line segment or an arc of a circle of fixed radius. Cockayne and Hall in [4] consider the same model but from another point of view: first, they provide the class of trajectories by which a moving "orientated point" can reach a given point in a given direction and they obtain the set of all points reachable at a fixed time.

However these authors solve the simple problem in which the point cannot reverse and cusps are not allowed (the trajectory is C^1 everywhere).

The similar problem admitting cusps on the trajectory (simulating changes from front to rear gear or conversely) is solved by Reeds and Shepp in [9]. These authors obtain a list of all possible (namely – 48) optimal trajectories. These trajectories are again finite concatenations of pieces which are either a straight line segment or an arc of a circle.

However, the very particular methods of their proofs seem very difficult to generalize, and they are replaced in [1] and [10] by simpler arguments based on the Maximum Principle in control theory (this principle is at first stated by Pontryagin in [8]). These arguments look like a promising tool to solve more difficult models of car-like robots.

Finally, a complete synthesis for the Reeds-Shepp model is obtained by Laumond and Souères in [6].

The next step seems to be the treatment of a similar model but now with a bound of the derivative of the curvature, with or without cusps. The case without cusps is considered in [2]. Here we consider the case with cusps.

In §2 we consider the theoretical aspect of the problem, using the Maximum Principle. The piecewise regular trajectories are shown to be concatenations of arcs of clothoids and segments of straight lines. However it is not sure whether the optimal path is piecewise regular or not. In any case it looks rather difficult to compute the optimal path explicitly.

Thus we concentrate our attention on the explicit description of "suboptimal" trajectories.

§3 is devoted to some properties of clothoids. In §4 we construct suboptimal trajectories and in §5 we prove that there exists a constant depending only on the parameter of the clothoid such that the optimal path can be shorter than the suboptimal one by no more than this constant.

In §5 we consider a class of paths to which the suboptimal path belongs and in §6 (Appendix A) we show that the optimal path belongs to the closure of this class (this is the proof of Lemma 7).

Closer to reality is the problem of a planar motion with bounded both the curvature and its derivative. In §4 we explain why the simpler problem which we consider can give abundant information about this more realistic (and more complicated) one.

2 Statement of the problem, existence of an optimal solution and application of the Maximum Principle of Pontryagin to the problem.

Consider the problem to find the shortest C^2 and piecewise C^3 path on the plane joining two given points with given initial and final tangent angles and curvatures along which the derivative of the curvature remains bounded.

The problem has the following origin: consider the planar motion of a car which can equally move forward and backward. One of the couples of wheels of the car is considered mobile, the other is considered fixed. Call "front" the gear of the car which corresponds to the direction of motion at the initial moment; the other gear is called "rear" (the definition of being front or rear doesn't depend on the one to be mobile or fixed). The angle between the mobile wheels and the axis of the car defines the curvature of its trajectory at the given moment, if the direction of motion is fixed; if the direction is reversed, then the curvature changes sign. We assume that during the motion this angle can be changed continuously, with a bounded speed (the speed of the car is assumed constant in absolute value and equal to 1).

This imposes a bound on the derivative of the curvature of the trajectory with respect to the natural parameter (i.e. the path length).

We allow cusps in the trajectory. The cusps are of such a type that at a cusp the tangent angle has a discontinuity (it changes by π) and the curvature is continuous (see Figure 1).

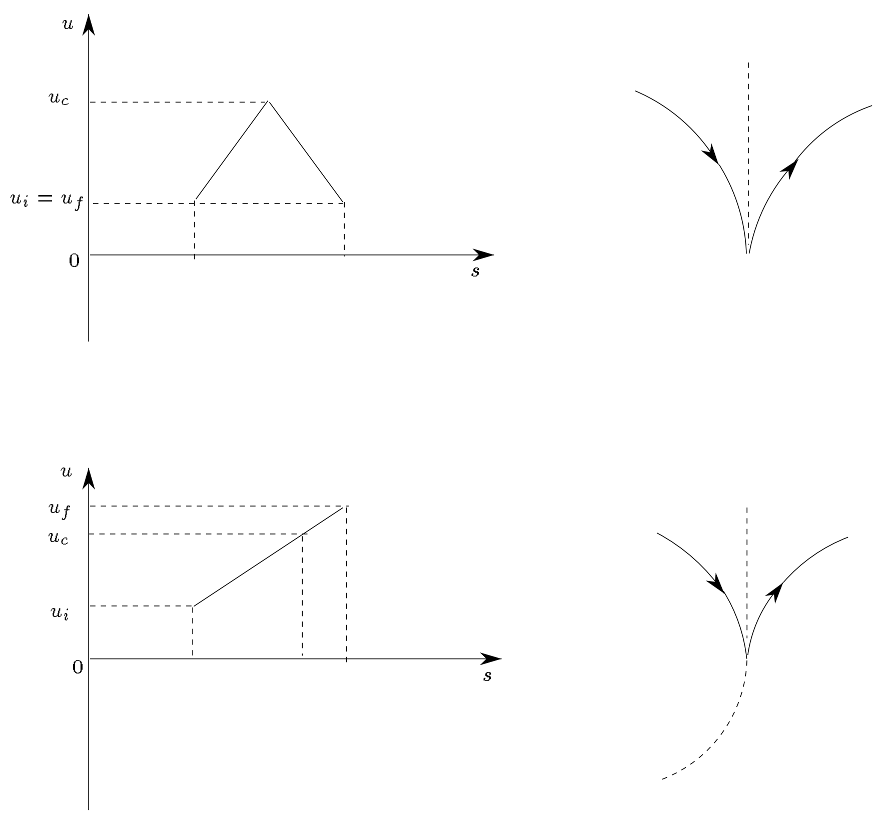

Figure 1

We denote by an arrow the direction of motion of the car. This direction corresponds to the front (to the rear) gear of the car if the car has passed an even (an odd) number of cusps. Introduce a variable which corresponds to the tangent angle and which changes continuously along the path: denote by $\alpha(t)$ the variable which is equal to the tangent angle to the path after an even number of cusps and to the tangent angle $+\pi$ after an odd number of cusps. This variable $\alpha(t)$ is a piecewise C^2 function, the curvature $u(t)$ is a continuous and piecewise C^1 function.

We assume that at a cusp the car can stop for an "infinitely short" instant; at this instant, if δ has been the angle between the mobile wheels and the axis of the car, then it is changed to $-\delta$.

For our car it is not essential whether at the final moment the front wheels are in front or not. Hence, we can correctly define the tangent angle to the trajectory at the initial point, but at the final one we can define only the tangent direction (i.e. the angle modπ, not mod2π). In order to formalise correctly the problem, we consider both possibilities — at the final point the front wheels are in front or not — and then compare the two paths thus obtained.

We have the following system (from now on we use "$\cdot$" for "d/dt"):

$$\dot{X}(t) = \begin{cases} \dot{x}(t) = \cos(\alpha(t) + \varepsilon(t)\pi) & \varepsilon(t) \in \{0,1\} \\ \dot{y}(t) = \sin(\alpha(t) + \varepsilon(t)\pi) & \\ \dot{\alpha}(t) = u(t) & \\ \dot{u}(t) = u'(t) & |u'(t)| \leq B \end{cases} \tag{1}$$

with initial and final conditions:

$$X(0) = (x^0, y^0, \alpha^0, u^0), \qquad X(T) = (x^1, y^1, \alpha^1, u^1) \tag{2}$$

We control the derivative of the curvature (control u') and we allow changes between front and rear gear or conversely at a cusp (control ε, it ensures the continuity of the variable α at cusps). We have $(\varepsilon(t), u'(t)) \in U \subset \mathbf{R}^2$, where $U = \{0,1\} \times [-B, +B]$. The control u' is a measurable, real valued function, the control $\varepsilon \in \{0,1\}$.

We want to find an $X(t)$, satisfying (1) and (2), such that the associated control function $(\varepsilon(t), u'(t))$ should minimize the total length of the path:

$$J(\varepsilon, u') = T = \int_0^T dt \tag{3}$$

Here the variable t is both the arc length and the time because the point moves with speed equal to 1.

At first we prove the controllability of system (1) with initial and final conditions (2). If system (1) with $\varepsilon(t) = 0$ (without cusps) is controllable then system (1) with $\varepsilon(t) \in \{0,1\}$ (i.e. with cusps allowed) is also controllable. So it is sufficient to prove the controllability of the following system:

$$\dot{X}(t) = \begin{cases} \dot{x}(t) = \cos(\alpha(t)) & \\ \dot{y}(t) = \sin(\alpha(t)) & \\ \dot{\alpha}(t) = u(t) & \\ \dot{u}(t) = u'(t) & |u'(t)| \leq B \end{cases} \tag{4}$$

with initial and final conditions (2).

This system can be rewritten in the form:

$$\dot{X}(t) = f(X) + u'g(X)$$

where

$$f(X) = \begin{pmatrix} \cos\alpha \\ \sin\alpha \\ u \\ 0 \end{pmatrix}, \qquad g(X) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The Lie algebra $\mathcal{L}(f,g)$ generated by f and g satisfies the Lie Algebra Rank Condition: for every $X \in \mathbf{R}^4$, $\mathcal{L}(f,g)(X) = \mathbf{R}^4$. Indeed, for every $X \in \mathbf{R}^4$,

$$h(X) = [g,f](X) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

$$i(X) = [h,f](X) = \begin{pmatrix} -\sin\alpha \\ \cos\alpha \\ 0 \\ 0 \end{pmatrix},$$

and the determinant

$$\det \begin{bmatrix} \cos\alpha & 0 & 0 & -\sin\alpha \\ \sin\alpha & 0 & 0 & \cos\alpha \\ u & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = -1, \quad \text{for every } X \in \mathbf{R}^4.$$

The solutions to $\dot{X} = f(X)$ are periodic functions (they are circles). Thus, we can apply Brockett's theorem (see [7], Th.IV-5) and obtain complete controllability of (4) (and, hence, obtain complete controllability of (1)).

We are going to use Filippov's existence theorem for the system (1), (2) (see [3], Th.5.1.ii). In order for the all requirements of the theorem to be satisfied we must consider convex ranges of control. To this end rewrite system (1) in the following form:

$$\dot{X}(t) = \begin{cases} \dot{x}(t) = v(t)\cos\alpha(t) & v(t) \in \{-1,1\} \\ \dot{y}(t) = v(t)\sin\alpha(t) & \\ \dot{\alpha}(t) = u(t) & \\ \dot{u}(t) = u'(t) & |u'(t)| \le B \end{cases} \tag{5}$$

The control functions $(v(t), u'(t)) \in V \subset \mathbf{R}^2$, where $V = \{-1,1\} \times [-B,+B]$. Assume V convex, i.e. $V = [-1,1] \times [-B,+B]$. Modify (5) and consider the system:

$$\dot{X}(t) = \begin{cases} \dot{x}(t) = v(t)\cos\alpha(t) & |v(t)| \le 1 \\ \dot{y}(t) = v(t)\sin\alpha(t) & \\ \dot{\alpha}(t) = |v(t)|u(t) & \\ \dot{u}(t) = u'(t) & |u'(t)| \le B \end{cases} \tag{6}$$

with initial and final conditions (2).

Now t is the time and the arc length s is defined by the equality $ds = |v(t)|dt$. The integral (3) now corresponds to a minimum time problem.

System (6) can be rewritten in the form:

$$\dot{X} = F(X, v, u') \qquad X(t) \in \mathbf{R}^4 \qquad (v, u') \in V$$

For system (6) we can use Filippov's existence theorem, all its requirements being satisfied:

– all scalar functions in the right parts of the equations (6) are continuous together with their partial derivatives;

– the function under the sign of the integral in (3) is continuous;

– the control functions are bounded and the range of control (v, u') is convex;

– $X(t) \in \mathbf{R}^4$ ($\mathbf{R}^4$ is closed);

– the initial and final points $(X(0), X(T))$ are fixed by (2);

– one can verify that there exists a constant $C > 0$ that for every $X(t) \in \mathbf{R}^4$ and $(v, u') \in V$ the following inequality is satisfied: $XF(X) \le C(|X|^2 + 1)$.

Thus we can assume the existence of an optimal solution and an optimal control for system (6) and we are going to apply the Maximum Principle of Pontryagin to this system to obtain necessary conditions for the control functions $(v(t), u'(t))$ and for the trajectory $X(t) = (x(t), y(t), \alpha(t), u(t))$ to be optimal.

Rewrite system (6), (2) and integral (3) as the following system:

$$\begin{cases} \dot{x}(t) = v(t)\cos\alpha(t) & x(0) = x^0 & x(T) = x^1 & |v(t)| \le 1 \\ \dot{y}(t) = v(t)\sin\alpha(t) & y(0) = y^0 & y(T) = y^1 & \\ \dot{\alpha}(t) = |v(t)|u(t) & \alpha(0) = \alpha^0 & \alpha(T) = \alpha^1 & \\ \dot{u}(t) = u'(t) & u(0) = u^0 & u(T) = u^1 & |u'(t)| \le B \\ \dot{x}_0(t) = 1 & x_0(0) = 0 & & \end{cases}$$

Denote by $\Psi(t) = (p, q, \beta, r, e)$ the vector of "dual" variables; the Hamiltonian H is defined by

$$H(t, X, \Psi, v, u') = pv\cos\alpha + qv\sin\alpha + |v|\beta u + ru' + e, \quad \text{for every } t \in [0, T]. \tag{7}$$

We have the following adjoint system for every $t \in [0, T]$:

$$\dot{\Psi}(t) = \begin{cases} \dot{p}(t) = 0 \\ \dot{q}(t) = 0 \\ \dot{\beta}(t) = p(t)v(t)\sin\alpha(t) - q(t)v(t)\cos\alpha(t) \\ \dot{r}(t) = -|v(t)|\beta(t) \\ \dot{e}(t) = 0 \end{cases} \tag{8}$$

So p, q, e are constant on $[0, T]$. Let $p = \lambda\cos\varphi$, $q = \lambda\sin\varphi$; here $\lambda = \sqrt{p^2 + q^2} \ge 0$, $\tan\varphi = q/p$. Then we can rewrite the adjoint system (8) and the Hamiltonian (7):

$$\begin{cases} p(t) \equiv \lambda\cos\varphi \\ q(t) \equiv \lambda\sin\varphi \\ \dot{\beta}(t) = \lambda v(t)\sin(\alpha(t) - \varphi) \\ \dot{r}(t) = -|v(t)|\beta(t) \\ e(t) \equiv e_0 \end{cases} \tag{9}$$

$$H(t, X, \Psi, v, u') = v(t)\lambda\cos(\alpha(t) - \varphi) + |v(t)|\beta(t)u(t) + r(t)u'(t) + e_0$$

Define

$$M(t, X, \Psi) = \min_{\substack{u' \in [-B, +B] \\ v \in [-1, 1]}} H(t, X, \Psi, v, u')$$

where Ψ, X, u' and v are considered as independent variables.

The Maximum Principle of Pontryagin ([8] Chapter I, Th. 1 and [3] Th.5.1i) asserts that if (v_*, u'_*) is an optimal control, then

(a) there exists an absolutely continuous non-zero vector-function $\Psi(t)$ (it is a continuous solution to (9));

(b) for almost every fixed $t \in [0,T]$ the function $H(t,X,\Psi,v,u')$ of the variables $u' \in [-B,+B]$ and $v \in [-1,1]$ only attains its minimum at the point $u' = u'_*, v = v_*$:

$$M(t,X(t),\Psi(t)) = H(t,X(t),\Psi(t),u'_*(t),v_*(t)) \quad , \quad t \in [0,T];$$

(c) the function $M(t) = M(t,X(t),\Psi(t))$ is absolutely continuous in $[0,T]$ and

$$\frac{dM}{dt}(t,X(t),\Psi(t)) = \frac{\partial H}{\partial t}(t,X(t),\Psi(t),u'(t),v(t));$$

(d) at any time $t \in [0,T]$ the relations $e_0 \geq 0$ and $M(t,X(t),\Psi(t)) = 0$ are satisfied.

The function H can reach its minimum with respect to v only at the points $v = 0, \pm 1$. If $v = 0$ during a non-zero time interval then the car doesn't move at all during this interval and doesn't change the value of the variable α, see system (6) (this has nothing to do with the instantaneous stopping at a cusp when the sign of the angle between the mobile wheels and the axis of the car is changed, see the beginning of the section). Hence, to minimise the time, there must be no non-zero time intervals with $v = 0$. So we consider only the cases $v = \pm 1$. Hence, we have $|v(t)| = 1$ and $ds = dt$. Moreover, systems (5) and (6) are identical and we can rewrite H as

$$H(t,X,\Psi,v,u') = v(t)\lambda\cos(\alpha(t)-\varphi) + \beta(t)u(t) + r(t)u'(t) + e_0 \tag{10}$$

To treat condition (b) with respect to $u'(t)$ we consider two cases:

1) $\partial H/\partial u' \equiv 0$ for $t \in [t_*, t_{**}] \subset [0,T]$;

2) $\partial H/\partial u' \not\equiv 0$ for $t \in (t_*, t_{**}) \subset [0,T]$.

In case 1) we have from (10) $r(t) \equiv 0$, then from (9) we obtain that $\beta(t) \equiv 0$, hence $\dot\beta(t) \equiv 0$ and $\alpha(t) = \varphi(\mathrm{mod}\pi)$ for every $t \in [t_*, t_{**}]$ (here the case $\lambda = 0$ is impossible because it contradicts (a)). Then $\dot\alpha(t) \equiv 0$ and thus $u(t) \equiv 0$ and $u'(t) \equiv 0$ for all $t \in [t_*, t_{**}]$.

The corresponding solution is a line segment in the direction φ.

In case 2) we have from (10) $u' = -B\mathrm{sign}(r(t))$.

The corresponding path is a clothoid. Along a clothoid the curvature depends linearly on the arc length and varies continuously from $-\infty$ to $+\infty$. Thus $u'(t) = \pm B$ determines a single clothoid (modulo the group of symmetries of the plane). The clothoid is defined by equation

$$\dot\alpha(t) = \pm Bt, \qquad t \in (-\infty, +\infty)$$

or, equivalently, by equations

$$\begin{cases} x(t) = \sqrt{2/B}\int_0^{t\sqrt{B/2}} \cos(\tau^2)d\tau \\ y(t) = \pm\sqrt{2/B}\int_0^{t\sqrt{B/2}} \sin(\tau^2)d\tau \quad t \in (-\infty,+\infty) \end{cases}$$

A measurable control $(v(t), u'(t))$ and the associated trajectory $X(t)$ of (5) satisfying all conditions of the Maximum Principle of Pontryagin will be called extremal control and extremal trajectory.

A point $X(t_s)$ of an extremal trajectory will be called a switching point if at $t = t_s$ at least one of the control functions $v(t), u'(t)$ has a discontinuity; the time t_s will be called a switching time.

The preceding reasonings lead to:

Lemma 1 *If an extremal path of (1) is regular (i.e. the control functions have finitely many points of discontinuity), then it is the closure of a union of open arcs of clothoids, corresponding to* $u'(t) \equiv \pm B$ *on an open interval of* $[0,T]$, *and line segments in one and the same direction* φ, *corresponding to* $u'(t) \equiv 0$ *on an open interval of* $[0,T]$.

It is not evident how to compute the optimal path explicitly. Moreover, it is not clear whether the optimal path is regular or not. That is why in what follows we concentrate on the explicit construction of a "suboptimal" trajectory. Lemma 1 gives the idea to construct it from arcs of clothoids and a line segment.

In the present paper we consider the case when the distance between the initial and the final point is much greater than the initial and final curvature (u^0, u^1) and B (see the exact definition in §4).

The suboptimal path which we construct is piecewise clothoid or linear, its curvature and tangent angle are continuous and it has a finite number of switching points including cusps, see §4.

Further we show that there exists a constant $C > 0$ depending only on B such that the optimal path is no more than C shorter than the suboptimal one (see §5).

To this end we prove some geometric properties of the clothoid in §3, and we consider in §5 a class $\mathcal{A}$ of piecewise clothoid or linear paths to the closure of which the optimal path belongs (Lemma 7, proved in §6 (Appendix A)).

3 Geometric properties of the clothoid.

Consider a clothoid

$$\begin{cases} \dot{x}(t) = \cos(Bt^2/2) & x(0) = 0 \\ \dot{y}(t) = \pm\sin(Bt^2/2) & y(0) = 0 \end{cases}$$

Call B "the parameter of the clothoid". For simplicity we set $B = 2$ further in the text. So we consider the clothoid

$$\begin{cases} \dot{x}(t) = \cos t^2 & x(0) = 0 \\ \dot{y}(t) = \pm\sin t^2 & y(0) = 0 \end{cases}$$

Here $2t$ is the curvature and at the same time t is the natural parameter, i.e. the length of the clothoid from the inflexion point.

Consider a half clothoid, i.e. $t \geq 0$. Fix a direction α_*(modπ , not mod2π) in $\mathbf{R}^2$ and let $P_1, P_2, \ldots$ denote the consecutive points on the clothoid with a tangent line at them of the chosen direction (with $t_1 < t_2 < \ldots$). Set $P_i = (x_i, y_i)$, $x_i = x(t_i)$, $y_i = y(t_i)$ (Figure 2); on Figure 2 we show a halh-clothoid corresponding to sign "+" before $\sin t^2$ in its equations.

Lemma 2 $\widehat{P_1P_2}$ *is the longest among the arcs* $\widehat{P_iP_{i+1}}$. *Its length depends continuously and monotonously on the choice of the direction* α_*. *We have*

$$|\widehat{P_iP_{i+1}}| = O(1/\sqrt{i})$$

and

$$|\,|\widehat{P_iP_{i+1}}| - |\widehat{P_{i-1}P_i}|\,| = O(1/(i\sqrt{i}))$$

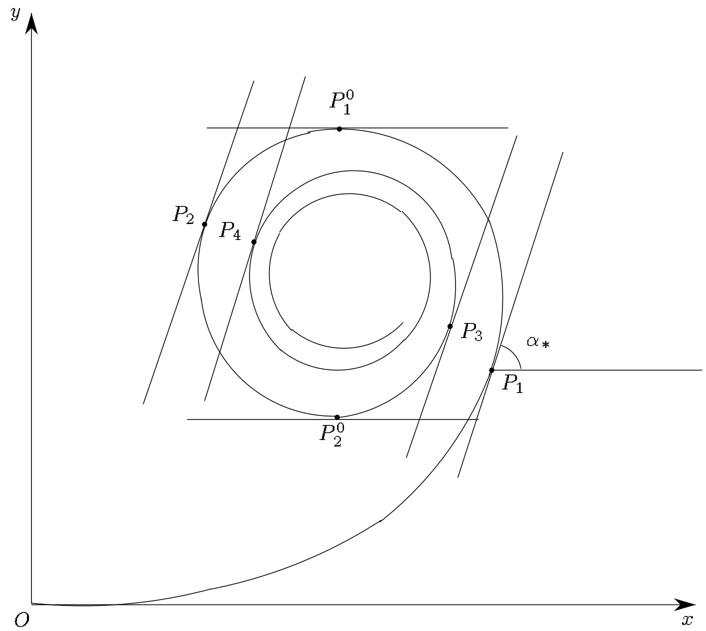

Figure 2

Proof.

$$|\widehat{P_iP}_{i+1}| = \int_{\sqrt{\alpha_*+(i-1)\pi}}^{\sqrt{\alpha_*+i\pi}} \sqrt{\cos^2 t^2 + \sin^2 t^2}dt = \sqrt{\alpha_* + i\pi} - \sqrt{\alpha_* + (i-1)\pi} =$$

$$= \frac{\pi}{\sqrt{\alpha_* + i\pi} + \sqrt{\alpha_* + (i-1)\pi}} = \frac{\pi}{\sqrt{i\pi}} \frac{1}{\sqrt{1+\alpha_*/i\pi} + \sqrt{1+(\alpha_*-\pi)/i\pi}} = O(1/\sqrt{i})$$

$$|\ |\widehat{P_iP}_{i+1}| - |\widehat{P_{i-1}P_i}|\ | = \left| \frac{\pi}{\sqrt{\alpha_* + i\pi} + \sqrt{\alpha_* + (i-1)\pi}} - \frac{\pi}{\sqrt{\alpha_* + (i-1)\pi} + \sqrt{\alpha_* + (i-2)\pi}} \right| =$$

$$= \frac{\pi}{\sqrt{i\pi}} * \left| \frac{1}{\sqrt{1+\alpha_*/i\pi} + \sqrt{1+(\alpha_*-\pi)/i\pi}} - \frac{1}{\sqrt{1+(\alpha_*-\pi)/i\pi} + \sqrt{1+(\alpha_*-2\pi)/i\pi}} \right| =$$

$$= \frac{\sqrt{\pi}}{\sqrt{i}} * \left| \frac{1}{1+\frac{\alpha_*}{2i\pi} + O(\frac{1}{i^2}) + 1 + \frac{(\alpha_*-\pi)}{2i\pi} + O(\frac{1}{i^2})} - \frac{1}{1+\frac{\alpha_*-\pi}{2i\pi} + O(\frac{1}{i^2}) + 1 + \frac{\alpha_*-2\pi}{2i\pi} + O(\frac{1}{i^2})} \right| =$$

$$= \frac{\sqrt{\pi}}{\sqrt{i}} * \left| \frac{1}{2+\frac{2\alpha_*-\pi}{2i\pi} + O(1/i^2)} - \frac{1}{2+\frac{2\alpha_*-3\pi}{2i\pi} + O(1/i^2)} \right| =$$

$$= \frac{\sqrt{\pi}}{2\sqrt{i}} * \left| \frac{1}{1+\frac{2\alpha_*-\pi}{4i\pi} + O(1/i^2)} - \frac{1}{1+\frac{2\alpha_*-3\pi}{4i\pi} + O(1/i^2)} \right| =$$

$$= \frac{\sqrt{\pi}}{2\sqrt{i}} * \left| 1 - \frac{2\alpha_* - \pi}{4i\pi} - 1 + \frac{2\alpha_* - 3\pi}{4i\pi} + O(1/i^2) \right| =$$

$$= \frac{\sqrt{\pi}}{2\sqrt{i}} \left(-1/2i + O(1/i^2) \right) = O(1/i\sqrt{i})$$

The first two statements of the lemma follow directly from the first chain of equalities.

Lemma 3 *The sum*

$$S(\alpha_*) = \sum_{i=1}^{\infty} |\mathrm{proj}_{\alpha_*} \overrightarrow{P_i P_{i+1}}|$$

is finite and there exists a constant $\tilde{C} > 0$ *such that* $S(\alpha_*) < \tilde{C}$ *for every* α_*. *Here* $|\mathrm{proj}_{\alpha_*} \overrightarrow{P_i P_{i+1}}|$ *denotes the absolute value of the projection of the vector* $\overrightarrow{P_i P_{i+1}}$ *on a line parallel to the chosen direction* α_*.

Proof.

$$|\mathrm{proj}_{\alpha_*} \overrightarrow{P_i P_{i+1}}| = \left| \int_{\sqrt{\alpha_* + (i-1)\pi}}^{\sqrt{\alpha_* + i\pi}} (\dot{x} \cos \alpha_* + \dot{y} \sin \alpha_*) dt \right| = I_i(\alpha_*)$$

For convenience, we consider further α_* to be a usual angle, $\alpha_* \geq 0$. Hence,

$$I_i(\alpha_*) = \left| \int_{\sqrt{\alpha_* + (i-1)\pi}}^{\sqrt{\alpha_* + i\pi}} \cos(t^2 - \alpha_*) dt \right| = \left| \int_{\alpha_* + (i-1)\pi}^{\alpha_* + i\pi} \frac{\cos(\tau - \alpha_*)}{2\sqrt{\tau}} d\tau \right| =$$

$$= \left| \int_0^{\pi} \frac{\cos(\tau' + (i-1)\pi)}{2\sqrt{\tau' + \alpha_* + (i-1)\pi}} d\tau' \right| = \left| \int_0^{\pi} \frac{\cos \tau'}{2\sqrt{\alpha_* + (i-1)\pi}} * \frac{d\tau'}{\sqrt{1 + \tau'/(\alpha_* + (i-1)\pi)}} \right| =$$

$$= \left| \int_0^{\pi} \frac{\cos \tau'}{2\sqrt{\alpha_* + (i-1)\pi}} * \left(1 - \frac{\tau'}{2(\alpha_* + (i-1)\pi)} + f\left(\frac{\tau'}{\alpha_* + (i-1)\pi} \right) \right) d\tau' \right|$$

where

$$f(w) = \frac{1}{\sqrt{1+w}} - 1 + \frac{w}{2}.$$

We have

$$\frac{df(w)}{dw} = -\frac{1}{2(\sqrt{1+w})^3} + \frac{1}{2} > 0 \qquad \text{if } w > 0.$$

Hence, $f(w)$ is an increasing function of the variable w, i.e. it takes its maximal value if $\tau' = \pi$ and $\alpha_* = 0$ (because $\tau' \in [0, \pi]$, $\alpha_* \geq 0$). Thus

$$f(w) \leq \frac{1}{\sqrt{1 + 1/(i-1)}} - 1 + \frac{1}{2(i-1)} = O\left(\frac{1}{i^2} \right)$$

and

$$I_i(\alpha_*) \leq \left|\int_0^\pi \frac{\cos\tau' d\tau'}{2\sqrt{\alpha_* + (i-1)\pi}}\right| + \left|\int_0^\pi \frac{\tau'\cos\tau' d\tau'}{4(\alpha_* + (i-1)\pi)\sqrt{\alpha_* + (i-1)\pi}}\right| +$$

$$+ \left|\int_0^\pi \frac{\cos\tau'}{2\sqrt{\alpha_* + (i-1)\pi}} f\left(\frac{\tau'}{\alpha_* + (i-1)\pi}\right) d\tau'\right| \leq$$

$$\leq 0 + \frac{1}{4(\alpha_* + (i-1)\pi)\sqrt{\alpha_* + (i-1)\pi}}\left|\int_0^\pi \tau'\cos\tau' d\tau'\right| + O\left(\frac{1}{i^2}\right) = O\left(\frac{1}{i\sqrt{i}}\right)$$

The series $\sum_{i=1}^\infty 1/(i\sqrt{i})$ is convergent. The lemma is proved.

Lemma 4 $|\mathrm{proj}_{\alpha_*^\perp}\ \overrightarrow{P_iP_{i+1}}\ | = O(1/\sqrt{i})$ *and it is a monotonous function of* α_* *if* α_* *is understood as a usual angle* $\in [0, +\infty)$. *Also*

$$|\ |\mathrm{proj}_{\alpha_*^\perp}\ \overrightarrow{P_{i+1}P_{i+2}}\ | - |\mathrm{proj}_{\alpha_*^\perp}\ \overrightarrow{P_iP_{i+1}}\ |\ | = O(1/(i\sqrt{i}))$$

Both estimations are uniform for $\alpha_* \in [0, +\infty)$. *Here* $\mathrm{proj}_{\alpha_*^\perp}\ \overrightarrow{P_iP_{i+1}}$ *denotes the projection of* $\overrightarrow{P_iP_{i+1}}$ *on a line perpendicular to the chosen direction* α_*.

Proof.

Similarly to the proof of Lemma 3 we obtain the formula

$$|\mathrm{proj}_{\alpha_*^\perp}\ \overrightarrow{P_iP_{i+1}}\ | = \left|\int_{\alpha_*+(i-1)\pi+\pi/2}^{\alpha_*+i\pi+\pi/2} \frac{\cos(\tau - \alpha_*)}{2\sqrt{\tau}} d\tau\right| = \left|\int_{\alpha_*-\pi/2}^{\alpha_*+\pi/2} \frac{\cos(\tau' - \alpha_* + i\pi)}{2\sqrt{\tau' + i\pi}} d\tau'\right| =$$

$$= \frac{1}{2\sqrt{i\pi}}\left|\int_{\alpha_*-\pi/2}^{\alpha_*+\pi/2} \cos(\tau' - \alpha_*)\left(1 - \frac{\tau'}{2i\pi} + O(1/i^2)\right) d\tau'\right| =$$

$$= \frac{1}{\sqrt{i\pi}}\left(1 - \left|\int_{\alpha_*-\pi/2}^{\alpha_*+\pi/2} \cos(\tau' - \alpha_*)\left(\frac{\tau'}{4i\pi} + O(1/i^2)\right) d\tau'\right|\right) = O(1/\sqrt{i})$$

Then we obtain the second assertion of the lemma:

$$|\ |\mathrm{proj}_{\alpha_*^\perp}\ \overrightarrow{P_{i+1}P_{i+2}}\ | - |\mathrm{proj}_{\alpha_*^\perp}\ \overrightarrow{P_iP_{i+1}}\ |\ | =$$

$$= \left|\ \left|\int_{\alpha_*-\pi/2}^{\alpha_*+\pi/2} \frac{\cos(\tau' - \alpha_*)}{2\sqrt{(i+1)\pi}}\left(1 - \frac{\tau'}{2(i+1)\pi} + O(1/i^2)\right) d\tau'\right| - \right.$$

$$\left. - \left|\int_{\alpha_*-\pi/2}^{\alpha_*+\pi/2} \frac{\cos(\tau' - \alpha_*)}{2\sqrt{i\pi}}\left(1 - \frac{\tau'}{2i\pi} + O(1/i^2)\right) d\tau'\right|\ \right| =$$

$$= \left|\ \left|\frac{1}{\sqrt{(i+1)\pi}} + O(1/i\sqrt{i})\right| - \left|\frac{1}{\sqrt{i\pi}} + O(1/i\sqrt{i})\right|\ \right| = \left|\frac{1}{\sqrt{(i+1)\pi}} - \frac{1}{\sqrt{i\pi}} + O(1/i\sqrt{i})\right| =$$

$$= \left|\frac{\pi}{\sqrt{(i+1)\pi}\sqrt{i\pi}(\sqrt{(i+1)\pi} + \sqrt{i\pi})} + O(1/(i\sqrt{i}))\right| = O(1/i\sqrt{i})$$

The lemma is proved.

4 Construction of the suboptimal path.

We construct the suboptimal path in the case when $\mathrm{dist}((x^0,y^0),(x^1,y^1)) \gg \max(B,|u^0|,|u^1|)$ (this means that there exist constants $a>0$, $b>0$, $c\geq 0$ such that $\mathrm{dist}((x^0,y^0),(x^1,y^1)) \geq aB + b\max(|u^0|,|u^1|)+c$). Remember that we set $B=2$ for simplicity.

Without loss of generality we consider the case $y^0=y^1=0$.

Assume that $\cos\alpha^0>0$ (if this is not so, then see below **4 D)**). We are going to show that one can construct a piecewise clothoid or linear path from $(x^0,0)$ to $(x^1,0)$ with a finite number of cusps and switching points where the variable α and the curvature u are continuous (the initial and final values of the variable α and of the curvature u being respectively α^0,α^1 and u^0,u^1). We construct the path from $(x^0,0)$ to $(x^1,0)$ from pieces of equal clothoids and a line segment.

A) The tangent lines at all the cusps of the path will be vertical and the path will be the graph of a continuous function on $[x^0,x^1]$.

The curvature as a function of the arc length s (i.e. the natural parameter) will be piecewise linear and continuous; the absolute value of the angular coefficient is the same, i.e. every piece of the piecewise linear function is of the kind $u=\pm 2s+\tilde{u}_0$.

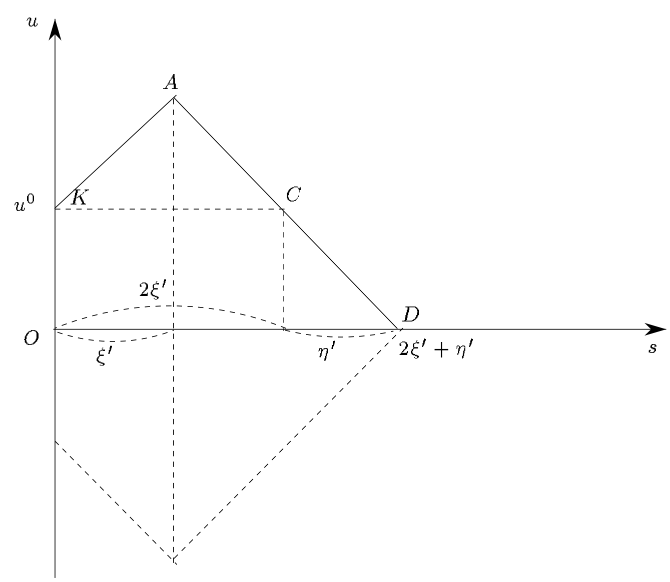

Figure 3

B) Construct the path from $(x^0,0)$ to some point (x',y') such that the curvature as a function of the arc length s is of the form shown on Figure 3, i.e. linear on $[0,\xi']$ and $[\xi',\eta'+2\xi']$, zero at the point $(\eta'+2\xi')$ (here ξ' and η' are path lengths, $x'=x(\eta'+2\xi')$, $y'=y(\eta'+2\xi')$). We impose the condition that all switching points in the interval $(\xi',\eta'+2\xi')$ are cusps (switching points of v).

Let us remark at once that it will be impossible to "read off" the form of the path (which is the part AW of the path shown on Figure 4 from Figure 3 without knowing the positions of the cusps. The arcs $\widehat{P_1P_2}, \widehat{P_2P_3}, \ldots$, see Figure 4, are equal to the corresponding arcs of a clothoid as shown on Figure 2.

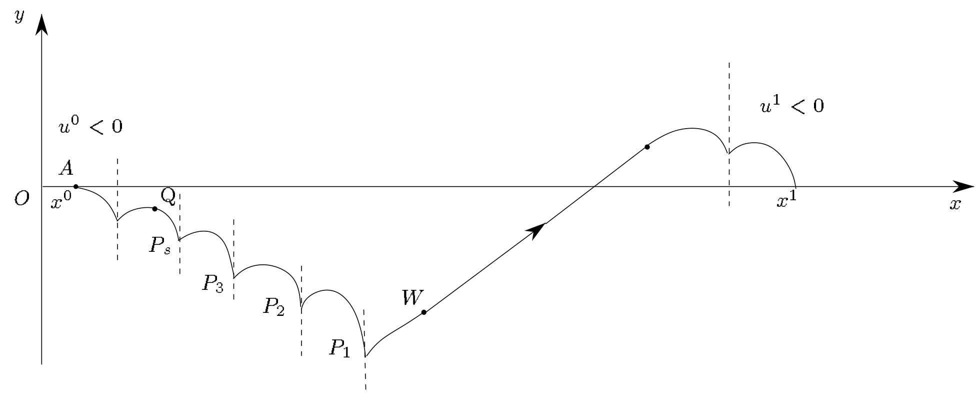

Figure 4

Denote the value of the variable $\alpha(t)$ at the point (x', y') by α' (the corresponding value of the curvature is equal to zero).

The number η' is defined by u^0 ($\eta' = 0.5u^0$). Regard ξ' as a parameter.

The curvature doesn't change sign on $[0, \eta' + 2\xi']$, suppose that it is positive (see Figure 3); the variable $\alpha' - \alpha^0$ is the integral of the curvature on $[0, \eta' + 2\xi']$:

$$\alpha' - \alpha^0 = \int_0^{\eta'+2\xi'} u(t)dt$$

Hence, increasing ξ' monotonously leads to increasing of this integral, i.e. of the absolute value of the variable $\alpha' - \alpha^0$. That is why there exist $d' > 0$ and d^*, $0 \leq d^* < d'$ such that when ξ' varies in $[d^*, d']$ the value of the variable α' at the point (x', y') would assume continuously all values from $[-\pi/2, \pi/2]$ (mod2π), hence, $\cos\alpha' \geq 0$ (if u^0 is negative, then $\alpha' - \alpha^0$ is negative as well and α' varies continuously in $[\pi/2, -\pi/2]$ (mod2π)).

For d' one can take the maximal length of an arc of half clothoid on which the tangent angle makes a full turn (2π). The existence of this maximal length is provided by Lemma 2.

Estimate the area where we can find the point (x', y') if $\xi' \in [d^*, d']$.

Lemma 5 *For $\xi' \in [d^*, d']$ the point (x', y') remains within some disc D' of radius δ'. For x^0 and α^0 fixed the coordinates (x'_c, y'_c) of its centre (which we assume to be the point (x', y') for $\xi' = 0$) depend only on u^0; δ' doesn't depend on any of the constants α^0, u^0, x^0. There exists a constant E' such that $|y'_c| \leq E'$ for all values of u^0, α^0, x^0.*

When B is not fixed then the radius δ', the constant E' and the coordinates (x'_c, y'_c) of the centre depend on B as well.

Proof.

$\mathbf{1^0}$. The absolute value of y'_c is bounded (this follows from Lemma 3 with α_* the direction of the axis Oy).

The contribution of the interval $[0, \eta'+2\xi']$ to the changing of y' is finite and can be estimated by a constant depending only on B; this follows from Lemma 3 with α_* being the direction of the axis Oy (applied to each of the intervals $[0, \xi']$ and $[\xi', \eta' + 2\xi']$). That is why when ξ' varies from d^* to d' then the y-coordinate of the point (x', y') is bounded.

$\mathbf{2^0}$. The contribution of the interval $[0, 2\xi']$ to the changing of x' can be estimated by a constant depending only on B (because $\xi' \leq d'$). When ξ' varies then the contribution of the interval $[2\xi', \eta' + 2\xi']$ to the changing of x' varies as well, but it is bounded from below and above by constants depending only on B. This follows from Lemma 4 and the detailed proof looks like this:

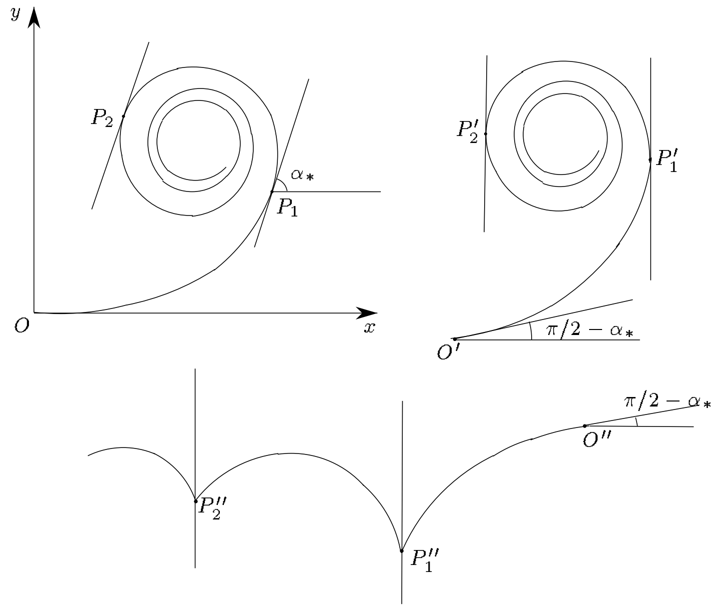

Figure 5

The arcs $\widehat{P_1P_2}, \widehat{P_2P_3}, \ldots$, see Figure 4, are equal to the corresponding arcs of a clothoid as shown on Figure 2. Their projections on the axis Ox are monotonous functions of the tangent angle at the point W (see Figure 4) where the curvature is 0, see Lemma 4. Really, W plays the role of O on Figure 2, i.e. the tangent angle at W is equal to the difference between $\pi/2$ and the tangent angle at P_1 on Figure 2 (see Figure 5). Denote by Q the point of the path corresponding to the point C of Figure 3, i.e. the curvature at Q is equal to u^0, and denote by P_s the point P_i closest to Q, see Figure 4. Denote by $P_1^0, P_2^0, \ldots$ the points of a half-clothoid at which the tangent lines are parallel to Ox (see Figure 2). Then the maximal length L_{max} of the projection on Ox of the path $QP_s \ldots P_1W$ can be estimated from above as follows: preserve the lengths of $\mathrm{proj}_{Ox}\widehat{QP_s}$ and $\mathrm{proj}_{Ox}\widehat{P_1W}$ and replace the projections of $\widehat{P_jP_{j+1}}$ $(j = 1, \ldots, s-1)$ by their maximal possible values. These values are obtained (see Lemma 2) for $\widehat{P_1P_2} = \widehat{OP_1^0}$, $\widehat{P_2P_3} = \widehat{P_1^0P_2^0}, \ldots, \widehat{P_{s-1}P_s} = \widehat{P_{s-2}^0P_{s-1}^0}$. Hence

$$L_{max} < \mathrm{proj}_{Ox}\widehat{OP_1^0} + \mathrm{proj}_{Ox}\widehat{P_1^0P_2^0} + \ldots + \mathrm{proj}_{Ox}\widehat{P_{s-2}^0P_{s-1}^0} + \mathrm{proj}_{Ox}\widehat{QP_s} + \mathrm{proj}_{Ox}\widehat{P_1W}$$

The minimal length L_{min} of $\mathrm{proj}_{Ox}QP_s \ldots P_1W$ can be estimated from below if $\mathrm{proj}_{Ox}\widehat{QP_s}$ and $\mathrm{proj}_{Ox}\widehat{P_1W}$ are neglected and $\mathrm{proj}_{Ox}\widehat{P_1P_2}, \ldots, \mathrm{proj}_{Ox}\widehat{P_{s-1}P_s}$ are replaced by their minimal values (see Lemma 2) that is by $\mathrm{proj}_{Ox}\widehat{P_1^0P_2^0}, \ldots, \mathrm{proj}_{Ox}\widehat{P_{s-1}^0P_s^0}$. Hence

$$L_{min} > \mathrm{proj}_{Ox}\widehat{P_1^0P_2^0} + \ldots + \mathrm{proj}_{Ox}\widehat{P_{s-1}^0P_s^0}$$

Hence

$$L_{max} - L_{min} < \mathrm{proj}_{Ox}\widehat{OP_1^0} + \mathrm{proj}_{Ox}\widehat{QP_s} + \mathrm{proj}_{Ox}\widehat{P_1W} - \mathrm{proj}_{Ox}\widehat{P_{s-1}P_s} < 3\mathrm{proj}_{Ox}\widehat{OP_1^0}$$

because $\mathrm{proj}_{Ox}\widehat{OP_1^0}$ is the maximal possible value for $\mathrm{proj}_{Ox}\widehat{P_iP_{i+1}}$, $(i = 1, 2, \ldots)$, $\mathrm{proj}_{Ox}\widehat{QP_s}$ and $\mathrm{proj}_{Ox}\widehat{P_1W}$.

Construct in the same way the path from $(x^1, 0)$ to some point (x'', y'') (from the right to the left). For this path we have a parameter ξ'' and for this parameter we have the interval $[d^{**}, d'']$ respectively. Similarly to the proof of Lemma 5 we can prove

Lemma 6 *For $\xi'' \in [d^{**}, d'']$ the point (x'', y'') remains within some disc D'' of radius δ''. For x^1 and α^1 fixed the coordinates (x''_c, y''_c) of its centre (which we assume to be the point (x'', y'') for $\xi'' = 0$) depend only on u^1; δ'' doesn't depend on any of the constants α^1, u^1, x^1. There exists a constant E'' such that $|y''_c| \leq E''$ for all values of u^1, α^1, x^1.*

When B is not fixed then the radius δ'', the constant E'' and the coordinates (x''_c, y''_c) of the centre depend on B as well.

We remind the reader that the construction of the suboptimal path is performed in the case when $\mathrm{dist}((x^0, 0), (x^1, 0)) \gg \max(B, |u^0|, |u^1|)$, see the beginning of the section; this condition implies that $D' \cap D'' = \emptyset$ and $x' \ll x''$.

C) Vary ξ' and ξ'' so that the tangent lines at (x', y') and (x'', y'') should be parallel. For $\alpha' = \pi/2$, $\alpha'' = -\pi/2$ and for $\alpha' = -\pi/2$, $\alpha'' = \pi/2$ the angles between the tangent vector to the path at (x', y') and the vector $((x', y'), (x'', y''))$ have different signs. Really, we have always $x' \ll x''$. Hence, for some values of ξ', ξ'' which vary in the intervals $[d^*, d']$, $[d^{**}, d'']$ respectively, (remember that they don't vary independently) this angle equals 0. This gives the suboptimal path from $(x^0, 0)$ to $(x^1, 0)$ (Figure 4). One checks directly that it satisfies all the initial requirements.

Remark 1. It is practically impossible to feel the presence of a switching point without a cusp on the graph of the path (Figure 4), because the first and the second derivatives are continuous there. On Figure 6 we show such a switching point between the clothoids $\mathcal{C}_1$ and $\mathcal{C}_2$ — the path MKL contains an arc MK of the clothoid $\mathcal{C}_1$ and an arc KL of the clothoid $\mathcal{C}_2$.

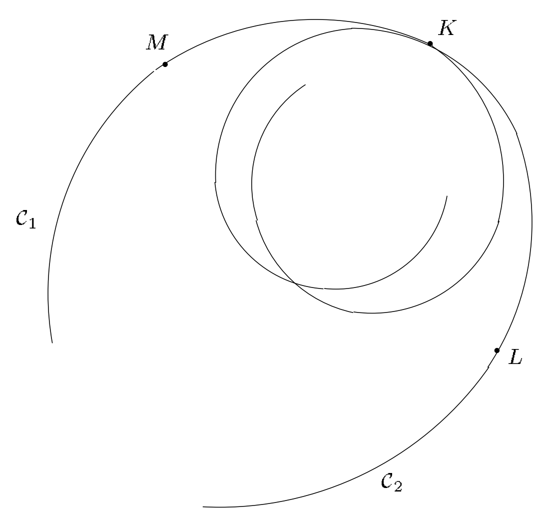

Figure 6

Remark 2. In fact, a "suboptimal" path can belong to one of the four types of path, see Figures 4, 7–9. It depends of the signs of u^0 and u^1. On these figures we denote by an arrow the direction of motion independently on whether this is a motion on front or on rear gear.

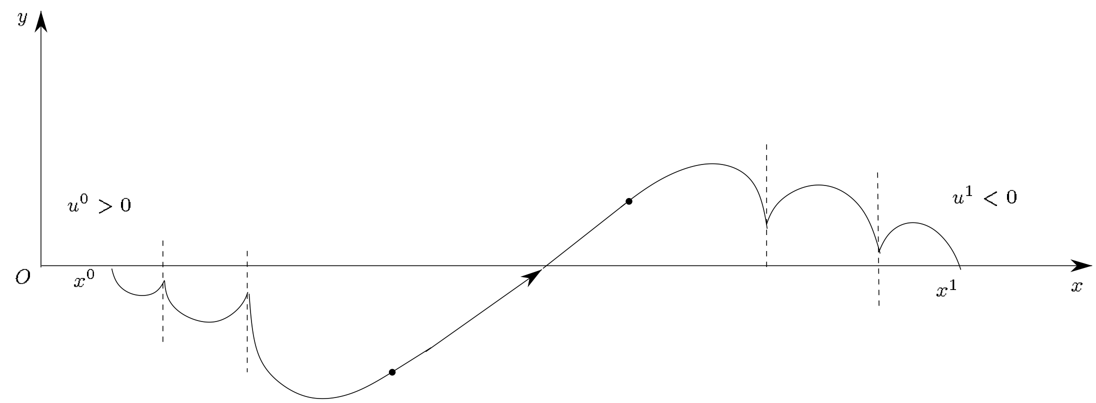

Figure 7

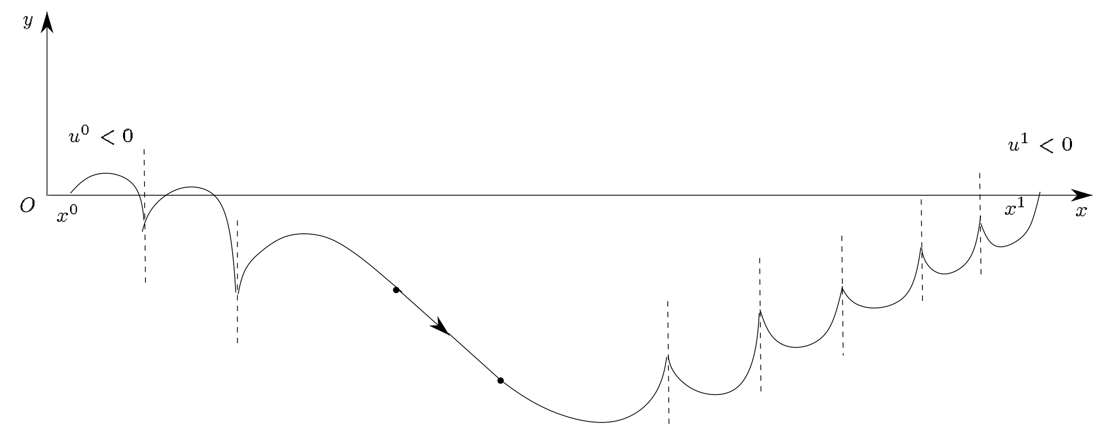

Figure 8

D) If $\cos\alpha^0 < 0$, then we construct a similar curve with an additional cusp, arbitrarily close to $(x^0, 0)$, see Figure 10, and we replace the point $(x^0, 0)$ by a point $(x^{0'}, y^{0'})$ arbitrarily close to it (this is sufficient for the purpose of this paper). At the new point we have $\cos\alpha^{0'} > 0$. The tangent angle and the curvature at $(x^{0'}, y^{0'})$ will be arbitrarily close to the ones at $(x^0, 0)$.

If $\alpha^0 = \pi/2 \pmod{2\pi}$, $u^0 > 0$ or $\alpha^0 = -\pi/2 \pmod{2\pi}$, $u^0 < 0$ we can also construct a similar curve with an additional cusp, as on Figure 10.

The cases $\alpha^0 = \pi/2 \pmod{2\pi}$, $u^0 < 0$ and $\alpha^0 = -\pi/2 \pmod{2\pi}$, $u^0 > 0$ are treated in the same way as the case $\cos\alpha^0 > 0$.

E) Remark 3. The number of cusps of the suboptimal path is $\sim |u^0|^2 + |u^1|^2$ (here $a \sim b$ means that there exist constants c_1, c_2, $0 < c_1 < c_2$ such that $c_1 b \le a \le c_2 b$).

Really, the length of the part of the suboptimal path from the point corresponding to the point C (see Figure 3) to the point (x', y') is equal to $|u^0|/2$. Denote the number of cusps between

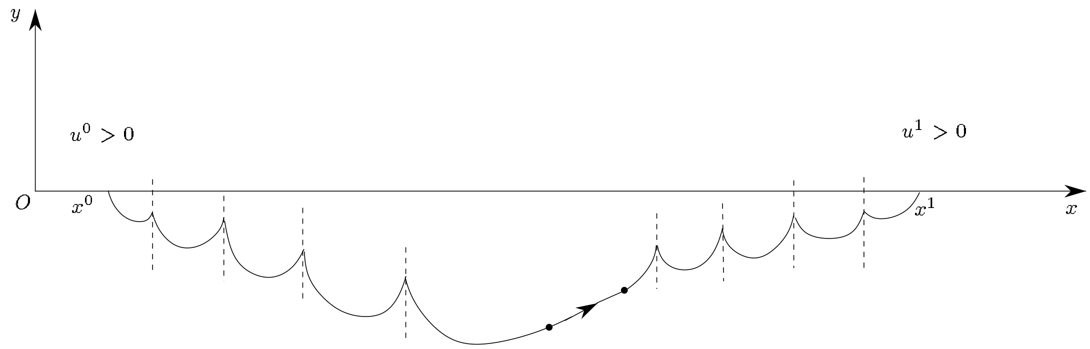

Figure 9

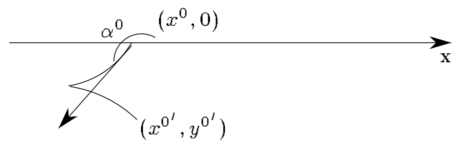

Figure 10

these points by q'. The length of the arc $\widehat{A_\mu A}_{\mu+1}$ is $\sim 1/\sqrt{\mu}$ (see Lemma 2). The sum of the lengths of all arcs $\widehat{A_\mu A}_{\mu+1}$ $(\mu = 1, \ldots, q')$ is equal to $|u^0|/2$ and is $\sim \sum_{\mu=1}^{q'} 1/\sqrt{\mu} \sim \sqrt{q'}$. Hence, $q' \sim |u^0|^2/4$. Between the points $(x^0, 0)$ and the point corresponding to the point C there is no more than one cusp. Hence, the number of cusps of the part of the suboptimal path between the points $(x^0, 0)$ and (x', y') is $\sim |u^0|^2$.

Analogously the number of cusps belonging to the part of the path between the points $(x^1, 0)$ and (x'', y'') is $\sim |u^1|^2$.

Hence, the number of cusps of the suboptimal path is $\sim |u^0|^2 + |u^1|^2$.

F) Remark 4. There exists a constant $F > 0$ depending only on B such that the length of the suboptimal path ψ_{subopt} is $\leq |x^1 - x^0| + F + (|u^0| + |u^1|)/2 \ll (1 + \varepsilon)|x^1 - x^0|$, where $\varepsilon \in (0, \pi/2 - 1)$ is fixed. Its choice defines the choice of the constants a, b, c which define the condition $\mathrm{dist}((x^0, 0), (x^1, 0)) \gg \max(B, |u^0|, |u^1|)$, see the beginning of the section. It becomes clear from the proof of Proposition 12 why $\varepsilon \in (0, \pi/2 - 1)$.

Really, the length of the suboptimal path from the initial point $(x^0, 0)$ to the point corresponding to the point C, see Figure 3, is estimated by some constant G depending only on B. Its length from the point corresponding to the point C to the point (x', y') is equal to $|u^0|/2$. Analogously for the final point we have the same constant G and $|u^1|/2$.

For the length l of the segment $((x', y'), (x'', y''))$ we have

$$l \leq |x'' - x'| + |y'' - y'| \leq |x^1 - x^0| + F'$$

where the constant F' $(|y'' - y'| \leq F')$ depends only on B.

Hence, for the length of the suboptimal path $|\psi_{subopt}|$ we have the following inequality:

$$|\psi_{subopt}| \leq |x^1 - x^0| + F' + 2G + (|u^0| + |u^1|)/2 \ll 1.1|x^1 - x^0|$$

because $\mathrm{dist}((x^0,0),(x^1,0)) \gg (\max(B,|u^0|,|u^1|)$.

Denote by ψ_{opt} the optimal path. For its length we have the evident inequality:

$$|\psi_{opt}| \geq |x^1 - x^0|$$

Hence,

$$|\psi_{subopt}| \ll 1.1|\psi_{opt}|$$

In §5 we prove a better estimation:

$$|\psi_{subopt}| \leq |\psi_{opt}| + C$$

for some constant $C > 0$ depending only on B.

G) Remark 5. Call *Problem 0* (*Problem 1*) the problem of planar motion with bounded derivative of the curvature (with bounded both curvature and its derivative). Evidently, the optimal paths for *Problem 1* are not shorter than the ones for *Problem 0*. One can construct suboptimal paths for *Problem 1* in a similar way as for *Problem 0*, the graph of the curvature as a function of the path length being piecewise linear. The graph might contain one or two horizontal segments with non-zero curvature if the curvature reaches its upper (lower) bound. These segments correspond to arcs of circles which are parts of the suboptimal path. The proof of the suboptimality can be carried out in the same way.

5 The path constructed in §4 is suboptimal.

Denote by $\mathcal{A}$ the class of paths from $(x^0,0)$ to (x^1,ζ) (ζ is arbitrary, the initial and final values of the variable α and of the curvature u are respectively α^0, u^0 or α^1, u^1) which are piecewise clothoid or linear with at most a finite number of cusps and switching points, with continuous curvature and with tangent angle continuous everywhere where the curvature is not zero and there is no cusp.

Denote by $\mathcal{B}$ the class of paths of the class $\mathcal{A}$ which have at most one interval or point of zero curvature. Clearly, $\mathcal{B} \subset \mathcal{A}$ and the path constructed in §4 belongs to the class $\mathcal{B}$.

Denote by $\mathcal{C}$ the class of paths whose coordinates x, y, the variable α and the curvature u satisfy system (5) (for some measurable control functions $v(t), u'(t)$) and satisfy the initial and final conditions $x(0) = x^0$, $y(0) = 0$, $\alpha(0) = \alpha^0$, $u(0) = u^0$, $x(T) = x^1$, $y(T) = \zeta$ ($\zeta \in \mathbf{R}$ is arbitrary), $\alpha(T) = \alpha^1$, $u(T) = u^1$. Obviously, the optimal path for problem (5), (2), (3) belongs to the class $\mathcal{C}$.

Lemma 7 *Every path from the class $\mathcal{C}$ belongs to the closure in the $\mathbf{C}^1$ norm of the class $\mathcal{A}$. Hence, the optimal path for problem (5), (2), (3) belongs to the closure of $\mathcal{A}$.*

This lemma will be proved in §6 (Appendix A).

Lemma 8 *Every path of the class $\mathcal{A} \setminus \mathcal{B}$ is longer than some path of the class $\mathcal{B}$ which contains a point with zero curvature.*

Proof.

Consider any path of the class $\mathcal{A}\backslash\mathcal{B}$. Let (x^i, y^i) and (x^f, y^f) be the initial and final point with zero curvature of this path. Hence, there exists a point (x^a, y^a) between them where the curvature is not zero. Change the path as follows: delete its part from (x^i, y^i) to (x^f, y^f) preserving the rest, shift vertically (i.e. in the direction of Oy) the part of the path from (x^f, y^f) to (x^1, ζ) so that $(x^f, y^f) \mapsto (x^f, y^i)$ (and, hence, $(x^1, \zeta) \mapsto (x^1, \zeta + y^i - y^f)$), connect (x^i, y^i) with (x^f, y^i) by a horizontal line segment.

The new path is shorter than the initial one (due to the absence of points like (x^a, y^a)) and belongs to the class $\mathcal{B}$. The lemma is proved.

Plan of the proof of suboptimality of the path constructed in §4.

We compare the lengths of the optimal path ψ_{opt} and of the one constructed in §4 (the suboptimal one, denote it by ψ_{subopt}) with the help of the path ψ_A constructed in A) (see below). We show (see Corollary 11) that any path $\psi \in \mathcal{A}$ containing a point with zero curvature is either longer than ψ_A or

$$|\psi| \geq |\psi_A| - D_1$$

where the constant $D_1 > 0$ depends only on the parameter B of the clothoid. If a path $\psi \in \mathcal{A}$ contains no point with zero curvature, then (Proposition 12) it is either much longer than the optimal one or it can be replaced by a path of the class $\mathcal{B}$ containing such a point and by no more than $D_1^{'} > 0$ times longer than the path ψ, $D_1^{'}$ depends only on B.

The optimal curve belongs to the closure of the class $\mathcal{A}$ (see Lemma 7). Hence

$$|\psi_{opt}| \geq |\psi_A| - D_1 - D_1^{'}.$$

We have from Corollary 13 that

$$|\psi_A| \geq |\psi_{subopt}| - D_2$$

where $D_2 > 0$ depends only on B. Hence

$$|\psi_{opt}| \geq |\psi_{subopt}| - D_1 - D_1^{'} - D_2$$

and

$$|\psi_{subopt}| \leq |\psi_{opt}| + D_1 + D_1^{'} + D_2$$

where the constant $D_1 + D_1^{'} + D_2$ depends only on B.

A) Construct a path from the class $\mathcal{B}$ as follows: there exist points (x^2, y^2), (x^3, y^3), $x^0 < x^2 < x^3 < x^1$, such that the curvature as a function of the arc length is linear on $I_1 = ((x^0, 0), (x^2, y^2))$ and $I_3 = ((x^3, y^3), (x^1, \zeta))$, zero on $I_2 = ((x^2, y^2), (x^3, y^3))$. The line segment I_2 is parallel to the axis Ox, see Figure 11. The curve is the graph of a function of x, i.e. there is a vertical tangent line at and only at a cusp and/or $(x^0, 0)$ and/or (x^1, ζ).

Denote by $A_1, A_2, \ldots, A_s, B_1, B_2, \ldots, B_p$ the cusp points of this curve. Fix their positions on the graph of the curvature as a function of the path length, see Figure 12; note the order of the points A_i, B_j.

B) Consider any path of the class $\mathcal{B}$ (not necessarily the graph of a function on $[x^0, x^1]$) which contains a point with zero curvature. Then the graph of its curvature as a function of the path length is obtained from the one shown on Figure 12 as follows:

1) stretch the interval of zero curvature (this is not always necessary);

2) cut the segments I_1 and I_3 into several pieces and at every cut insert the graph of a continuous piecewise linear function (with equal values at the ends of the interval) as shown on Figure 13

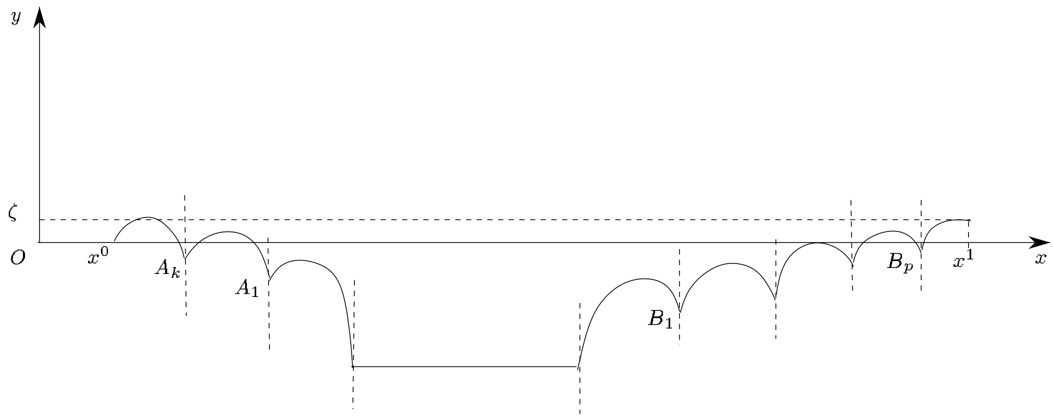

Figure 11

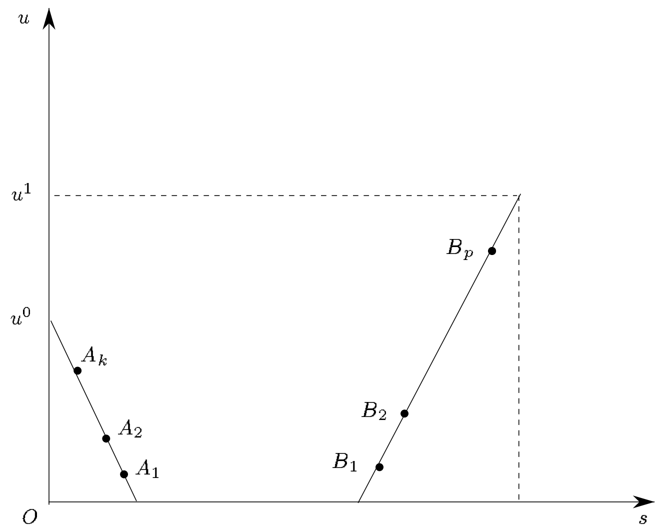

Figure 12

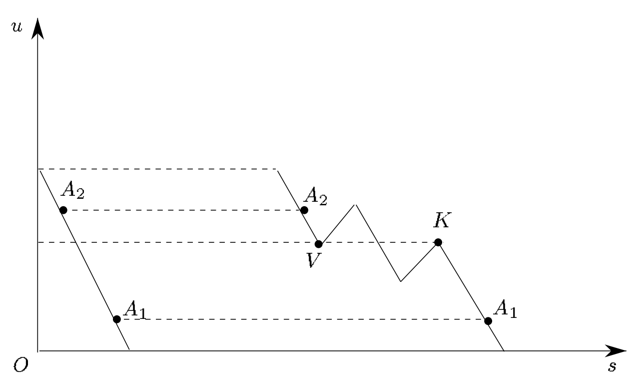

Figure 13

We remind that it is impossible to construct the path whose curvature is shown on Figure 13 without knowing the positions of the cusps on it (if there are such).

C) Let $C_1, \ldots, C_l, D_1, \ldots, D_k$ be the points of the path from B) (it belongs to the class $\mathcal{B}$) where it has a vertical tangent line (the points C_i lie on I_1, the points D_j lie on I_3). We admit that $\widehat{C_iC}_{i+1}$ can be composed from several arcs as well, i.e. that it contains cusps.

Proposition 9 *Every piece $[C_i, C_{i+1})$ (respectively $[D_j, D_{j+1})$) contains at most one of the points A_μ (respectively B_ν). For this new path the point A_μ is, in general, an ordinary point (the tangent line at it isn't necessarily vertical).*

Proof.
Really, between A_μ and $A_{\mu+1}$ on the path constructed in A) the tangent angle makes a half turn (i.e. π); hence, on the one constructed in B) it makes not less than a half turn, because while passing from the old graph of the curvature (Figure 12) to the new one (Figure 13) a piece VK (which contributes to the increasing of the angle) might have been inserted. As the difference between the tangent angles at C_i and C_{i+1} is π, this proves the proposition.

The curvature at A_μ is equal to $2\sqrt{\mu\pi + a^*}$ with $|a^*| < \pi$. As the curvature of a path of the class $\mathcal{B}$ changes as a Lipschitz function, this allows us to compare the length of $\widehat{C_iC}_{i+1}$ with the length of the arc $A_\mu \widehat{A_{\mu+1}}$ of the path constructed in A). We consider only the points A_μ and C_i (for B_ν and D_j the reasoning is the same).

Lemma 10 *Consider an arc $\widehat{C_iC}_{i+1}$ of the path constructed in B) which contains the point A_μ of the path constructed in A). Denote the length of the arc $\widehat{A_\mu A}_{\mu+1}$ of the path constructed in A) by γ_μ and denote the projections of $\widehat{C_iC}_{i+1}$ and $\widehat{A_\mu A}_{\mu+1}$ on Ox by $\mathrm{proj}_{Ox}\widehat{C_iC}_{i+1}$ and $\mathrm{proj}_{Ox}\widehat{A_\mu A}_{\mu+1}$. Then there exist $\mu_0 \in \mathbf{N}$ and $C > 0$ such that for $\mu > \mu_0$*

$$\left| \, |\widehat{C_iC}_{i+1}| - \gamma_\mu \right| \leq C/\mu\sqrt{\mu} \tag{11}$$

$$\left| \, |\mathrm{proj}_{Ox}\widehat{C_iC}_{i+1}| - |\mathrm{proj}_{Ox}\widehat{A_\mu A}_{\mu+1}| \, \right| \leq C/\mu\sqrt{\mu} \tag{12}$$

The arc $\widehat{C_iC}_{i+1}$, in general, isn't a graph of a function. If several points project onto one and the same point of Ox then the projection length is counted with the multiplicity.

The lemma will be proved in §7 (Appendix B).

Corollary 11 *Let a path ψ of the class $\mathcal{B}$ contain a point with zero curvature. Then either ψ is longer than the path constructed in* A) *(denote it by ψ_A) or there exists a constant $D_1 > 0$ such that*

$$|\psi| \geq |\psi_A| - D_1. \tag{13}$$

The constant D_1 depends only on the parameter of the clothoid. Hence, inequality (13) is true for any path ψ of the class $\mathcal{A}$ as well , see Lemma 8, and the optimal path ψ_{opt} isn't more than D_1 shorter than ψ_A (because it belongs to the closure of the class $\mathcal{A}$, see Lemma 7).

Proof.
The path ψ is obtained from ψ_A as shown in B), see Figure 13. To obtain (from ψ_A) a shorter path one must decrease the lengths of the part I_1 and I_3 of the path ψ_A because its line segment I_2 is parallel to Ox and has the minimal possible length. The changes performed to obtain ψ

from ψ_A change the length of the arc containing A_μ (respectively B_ν) by no more than $C/\mu\sqrt{\mu}$ (see Lemma 10). Its projection on Ox can increase by no more than $C/\mu\sqrt{\mu}$ (see Lemma 10), i.e. it can decrease the one of the line segment parallel to Ox by no more than $C/\mu\sqrt{\mu}$. The changes performed on Figure 13 can add new arcs $\widehat{C_iC}_{i+1}$ (respectively $\widehat{D_jD}_{j+1}$), not containing points A_μ (recpectively B_ν), we ignore their effect because they can only increase the length of the path ψ. Hence, we obtain

$$|\psi_A| - |\psi| \leq 2C \sum_{\mu=1}^{\infty} 1/(\mu\sqrt{\mu}) = constant = D_1$$

The corollary is proved.

Proposition 12 *A path ψ of the class $\mathcal{A}$ containing no point with zero curvature (suppose that the curvature is positive) is either longer than $(1+\varepsilon)|x^1 - x^0|$, where $\varepsilon \in (0, \pi/2 - 1)$ is fixed, (i.e.much longer than the optimal one, see Remark 4 in §4) or can be replaced by a path of the class $\mathcal{B}$ containing a point with zero curvature which is no more than a constant (depending only on B) longer than the path ψ.*

Proof.
Let u^* be the lowest curvature of the path ψ. There exists $u^{**} > 0$ such that if $u^* \geq u^{**}$, then

$$|\psi| \geq (1+\varepsilon)|x^1 - x^0|, \qquad \varepsilon \in (0, \pi/2 - 1) \text{ is fixed} \tag{14}$$

Really, we have

$$\lim_{\mu\to\infty} \frac{|\mathrm{proj}_{Ox}\widehat{A_\mu A}_{\mu+1}|}{|\widehat{A_\mu A}_{\mu+1}|} = \lim_{\mu\to\infty} \left|\int_{\sqrt{\mu\pi}}^{\sqrt{\mu\pi+\pi}} \sin t^2 dt\right| \Big/ \left|\int_{\sqrt{\mu\pi-\pi/2}}^{\sqrt{\mu\pi+\pi/2}} dt\right| =$$

$$= \lim_{\mu\to\infty} \left|\int_{\mu\pi}^{\mu\pi+\pi} \frac{\sin\tau}{2\sqrt{\tau}} d\tau\right| \Big/ \left|\sqrt{\mu\pi + \pi/2} - \sqrt{\mu\pi - \pi/2}\right| =$$

$$= \lim_{\mu\to\infty} \left|\int_0^\pi \frac{\sin(\tau+\mu\pi)}{2\sqrt{\tau+\mu\pi}} d\tau\right| \Big/ \left|\frac{\pi}{\sqrt{\mu\pi}(\sqrt{1+\pi/(2\mu\pi)} + \sqrt{1-\pi/(2\mu\pi)})}\right| =$$

$$= \lim_{\mu\to\infty} \frac{1}{\sqrt{\mu\pi}} \left|\int_0^\pi \frac{\sin\tau}{2\sqrt{1+\tau/\mu\pi}} d\tau\right| \Big/ \left(\frac{\pi}{\sqrt{\mu\pi}(\sqrt{1+1/2\mu} + \sqrt{1-1/2\mu})}\right) = 2/\pi < 1$$

Hence, for the large values of μ (i.e. for large values of the curvature) we have

$$|\psi| \geq \tilde{C}|x^1 - x^0|$$

where $\tilde{C}$ can be any number between 1 and $\pi/2$. Hence, such a curve is much longer than the optimal one (see Remark 4 in §4).

Consider $u^* < u^{**}$. Consider on the graph of the curvature as a function of the path length the initial and final point $((x_i, y_i),\ (x_f, y_f)\ x^0 \leq x_i \leq x_f \leq x^1)$ where the curvature is equal to u^{**} (before the initial and after the final point it is $> u^{**}$, between them it can be arbitrary positive). On this graph (see Figure 14) the point N corresponds to the point (x_i, y_i), the point M corresponds to the point (x_f, y_f). Change the graph as follows: from $(x^0, 0)$ to (x_i, y_i) and

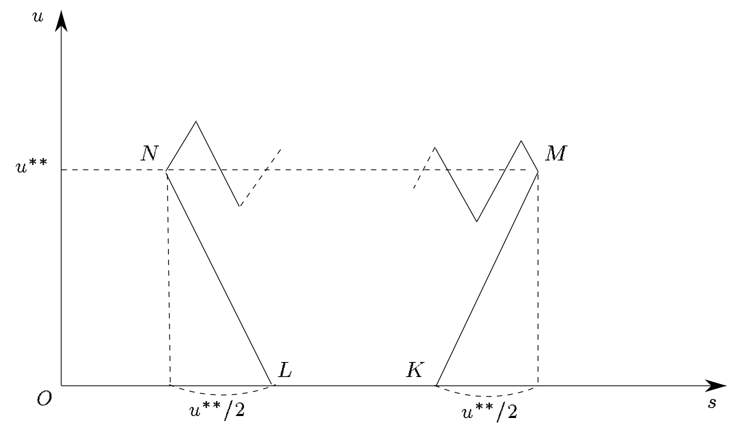

Figure 14

from (x_f, y_f) to $(x^1, 0)$ the graph isn't changed, from (x_i, y_i) the curvature descends linearly to 0, then keeps equal to 0, then ascends linearly to u^{**} at (x_f, y_f); one might have to stretch or contract the interval of zero curvature. The interval of zero curvature (between the points L and K on the graph) corresponds to a horizontal line segment of the path as in A); we consider the path thus changed to be the graph of a function of x between (x_i, y_i) and (x_f, y_f). By doing so we increase the length of the path by no more than a constant depending on u^{**}. Really, the horizontal line segment (corresponding to the interval of zero curvature) does not increase at all the length of the path. The length of the part of the path between the points corresponding to the points N and L on the graph u as a function of s is equal to $u^{**}/2$; analogously for the points K and M. The descend from u^{**} to 0 and the ascend from u^{**} can increase the length by no more than the sum of the lengths of the corresponding parts of the path (i.e. by u^{**}).

If x_i and x_f are so close each to the other that it is impossible for the curvature to descend linearly to 0 and then to ascend linearly to u^{**} again, then the path is again much longer (i.e. a constant (e.g. $1 + \varepsilon$) times longer) than the optimal one. The proposition is proved.

Corollary 13 *Let ψ_{subopt} denote the path constructed in §4 and let ψ_A denote the one constructed in* A). *There exists a constant $D_2 > 0$ such that*

$$|\psi_A| \geq |\psi_{subopt}| - D_2$$

The constant D_2 depends only on the parameter of the clothoid (i.e. on B).

Proof.

Decompose the path ψ_{subopt} into three parts corresponding to the intervals I_1, I_2, I_3 (similarly to the decomposition of the path ψ_A into three parts corresponding to I_1, I_2, I_3), where I_2 is the interval of zero curvature. Then the parts corresponding to I_1, I_3 of ψ_{subopt} are no more than a constant longer than the ones of ψ_A, this is proved in the same way as the similar statement in the proof of Corollary 11. The only difference in the estimation of the part corresponding to I_2 is the fact that the points (x', y') and (x'', y'') might have different $y-$coordinates, i.e. the line segment of ψ_{subopt} is not necessarily parallel to Ox. But $|y'|$, $|y''|$, $|x'|$, $|x''|$ are bounded from below and above by constants depending only on the parameter of the clothoid, see Lemma 3, Lemma 5 and Lemma 6. The length of the line segment is not greater than $|x' - x''| + |y' - y''|$.

Hence, the fact that $y' \neq y''$ adds no more than a constant depending only on B to the length of ψ_{subopt}. The corollary is proved.

Proposition 14 *The path constructed in §4 is no more than a constant longer than the optimal one. This constant depends only on the parameter of the clothoid.*

Proof.
The curve constructed in §4 is no more than a constant longer than the curve constructed in A), see Corollary 13. The infinum of the lengths of the paths of the class $\mathcal{A}$ is no more than a constant smaller than the length of the curve constructed in A), see Corollary 11 and Proposition 12. The optimal curve belongs to the closure of the class $\mathcal{A}$ (see Lemma 7). Hence, the one constructed in §4 is no more than a constant longer than the optimal curve. All constants mentioned above depend only on the parameter of the clothoid. The proposition is proved.

6 Appendix A (Proof of Lemma 7)

$\mathbf{1^0}$. Fix some path $\mathcal{P}$ of the class $\mathcal{C}$ corresponding to some fixed value of ζ. Consider the graph of the curvature u of the path $\mathcal{P}$ as a function $f(s)$ of the path length s. Further we write $s(x,y)$ or $s(A)$ for the length of the path from $(x^0,0)$ to (x,y) (we set $s(x^0,0)=0$) and, respectively, from $(x^0,0)$ to the point which corresponds to the point A from the graph of the curvature. Then $f(s) \in \text{Lip}(2)$. Set $b = \max_{[0,s(x^1,\zeta)]} f(s)$. Suppose that $b > 0$ (if $b < 0$, then we consider $\min_{[0,s(x^1,\zeta)]} f(s) = \max_{[0,s(x^1,\zeta)]}(-f(s))$ instead of b; the case $f \equiv 0$ is trivial).

$\mathbf{2^0}$. Fix $\Delta > 0$. We approximate a path from the class $\mathcal{C}$ by paths from the class $\mathcal{A}$ with precision Δ. This means that we define a diffeomorphism of the axis $Os : s \mapsto \tau(s)$, $d\tau/ds \geq 1$, and a new measurable control (v, u') such that

1) if $f(s)$ and $g(s)$ are the graphs of the curvatures of the old and the new path (f is defined on $[0, s(x^1,\zeta)]$, g is defined on $[0, s']$, $s' > s(x^1,\zeta)$), then $|f(s) - g(s)| < \Delta$ on $[0, s(x^1,\zeta)]$;

2) the distance between every point (x,y) of the old path and its corresponding point $(\tilde{x},\tilde{y})$ of the new one (the correspondence is given by τ) is $< \Delta$;

3) the tangent angles to the paths at (x,y) and $(\tilde{x},\tilde{y})$ are the same (for all points (x,y)).

$\mathbf{3^0}$. We say that the point B $(s_B, f(s_B))$ of the graph of the curvature is *good* if $f(s_B) > 0$ and if the straight lines passing through any couple of points A, C sufficiently close to B (A is to the left, C is to the right of B) with angular coefficients (-2) and 2 respectively (see Figure 15) intersect at a point D (s_D, u_D) which is strictly below the graph of the curvature and $u_D > 0$.

Set $S^+ = \{s \in [0, s(x^1,\zeta)] | f(s) > 0\}$. According to 1^0, we assume that $S^+ \neq \emptyset$. Then either S^+ possesses a good point or the set S^+ possesses the following characteristics:

1) S^+ has no point of local (not necessarily strict) maximum (because a point of local maximum in S^+ is a good point). Hence, there is no interval $(s_*, s_{**}) \subset S^+$ such that $u(s_*) = 0$ and $u(s_{**}) = 0$. Hence, the set S^+ consists of at most two intervals such that the initial point of the first interval is the point $s(x^0,0) = 0$ and the final point of the second interval is the point $s(x^1,\zeta)$ (remember that we consider the interval $[0, s(x^1,\zeta)]$).

2) Denote by P an arbitrary point of the graph $f(s)$ of the set S^+. Then in the ε-neighbourhood of the point P the function $f(s)$ can belong to one of the following types:

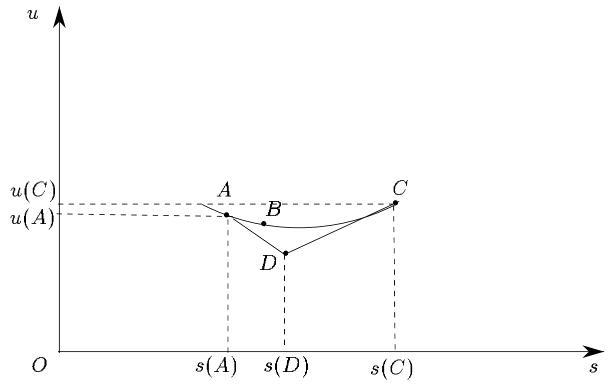

Figure 15

a) $f(s) = \begin{cases} -2(s - s_P) + u(s_P) & s \in (s_P - \varepsilon, s_P] \\ 2(s - s_P) + u(s_P) & s \in [s_P, s_P - \varepsilon) \end{cases}$

b) $f(s) = -2(s - s_P) + u(s_P), \quad s \in (s_P - \varepsilon, s_P + \varepsilon)$

c) $f(s) = 2(s - s_P) + u(s_P), \quad s \in (s_P - \varepsilon, s_P + \varepsilon)$

We leave the proof of 2) to the reader; use the fact that if the point B isn't good, then B lies on the union of the two segments $AD \cup DC$ for all choices of A and C sufficiently close to B.

3) In fact, $f(s)$ cannot belong to the type a) because in this case there exists a point of local maximum in the set S^+ to the left or to the right of P. Hence, in the neighbourhood of P the function $f(s)$ can belong only to the types b) or c).

$\mathbf{4^0}$. Consider the case when S^+ possesses a good point. In the case when there is no good point in S^+ the changing of the graph $f(s)$ of the curvature is described in 9^0.

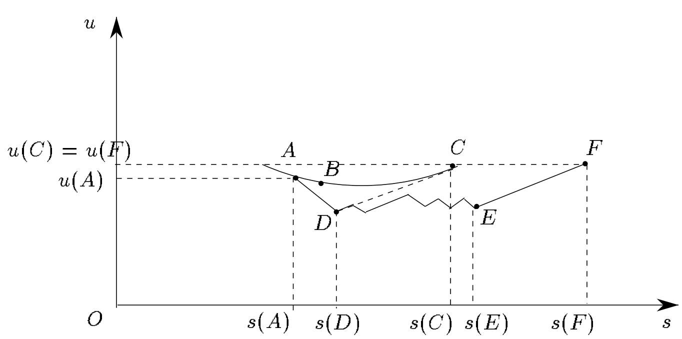

Figure 16

Let $B \in S^+$ be a good point. Replace the part ABC of the graph f of the curvature by the piecewise-linear curve $ADEF$ (see Figure 16) so that the points C and F have the same u-coordinate, the lowest u-coordinate on DE is the same as the one of D and E and

$$\int_{s(A)}^{s(C)} f(s)ds = \int_{s(A)}^{s(F)} g(s)ds$$

where g denotes the new graph of the curvature. Hence, the tangent angles at the points corresponding to C and to F (from the old and from the new path) are the same. To the right of F the graph of the curvature of the new path is obtained from the one of the old path (considered to the right of C) by a translation. The control function v is unchanged to the left of A. To the right of F it is the same as the old one to the right of C. Hence, the new path (considered to the right of F) is obtained from the old one (considered to the right of C) by a translation. On $ADEF$ the control function v is defined in $5^0 - 6^0$.

5^0. Suppose that the tangent angle to any point of the path corresponding to a part ABC is not vertical. Then we define v to be constant on $[s(A), s(F)]$ so that the x-coordinate of the point of the path should increase. Prove that then the x-coordinate of the endpoint of the new path should be $> x^1$ (this is equivalent, see 4^0, to say that the point corresponding to F has a greater x-coordinate than the one corresponding to C).

Let α_1, α_2 be the tangent angles to the old and the new path and let the x-coordinate of the old path be increasing with s. Compare the coordinates x_F and x_C. We find them from the following formulas:

$$x_C = \int_{s(A)}^{s(C)} \cos\alpha_1(s)ds$$

$$x_F = \int_{s(A)}^{s(F)} \cos\alpha_2(s)ds \qquad s(F) > s(C)$$

Here the functions $\cos\alpha_1(s)$ and $\cos\alpha_2(s)$ are some continuous functions of s and $\alpha_1(s) > \alpha_2(s)$ on $(s(A), s(C)]$ $(\alpha_1(s(A)) = \alpha_2(s(A)))$. There exists an absolutely continuous and increasing function $\tau^* : [s(A), s(C)] \to \mathbf{R}$ such that for $s \in [s(A), s(C)]$ we have

$$\cos\alpha_2(\tau^*(s)) \equiv \cos\alpha_1(s), \qquad \tau^*(s(A)) = s(A), \qquad \tau^*(s(C)) = s(F)$$

Really, α_1 and α_2 are monotonously increasing with s. Hence $\tau^*(s) > s$, $d\tau^*/ds > 1$ (this follows from $\alpha_1(s) > \alpha_2(s)$). Compare x_F with x_C.

$$x_C = \int_{s(A)}^{s(C)} \cos\alpha_1(s)ds = \int_{s(A)}^{s(C)} \cos\alpha_2(\tau^*(s))ds =$$

$$= \int_{s(A)}^{s(F)} \frac{\cos\alpha_2(\tau^*)d\tau^*}{d\tau^*/ds} < \int_{s(A)}^{s(F)} \cos\alpha_2(\tau^*)d\tau^* = x_F$$

The function τ^* will define the change of coordinate τ on $[s(A), s(C)]$ (see 2^0).

6^0. If on $[s(A), s(C)]$ the tangent angle to the old path becomes equal to $\pm\pi/2$ (this can happen at isolated points only, due to the constant sign of the curvature), then we perform the same change of the graph of the curvature but the new control function v will change from 1 to -1 or from -1 to 1 at points corresponding to points of the path with vertical tangent lines. Hence, the new path (on $[s(A), s(F)]$) will have a finite number of cusps corresponding to the discontinuities of v. Denote thus obtained path by $\mathcal{W}$.

7^0. Denote $|s(C) - s(A)|$ by δ and $f(B)$ by d; $d > 0$. We choose C and A so close to B, that $u(D) > d/2$, see Figure 16. Fix some small number δ' and divide the segments $[0, s(A)]$

and $[s(C), s(x^1, \zeta)]$ into intervals of length $\leq \delta'$. We change the graph of the curvature on each such interval, denote the new graph by $h(s)$. This change is shown on Figure 17 (consider, for example, the part of the graph $f(s)$ between two points Q, R): we replace the part QR of the graph $f(s)$ by the piecewise-linear graph (with angular coefficients ± 2) $QQ'R'R$ chosen such that

$$\int_{s(Q)}^{s(R)} f(s) = \int_{s(Q)}^{s(R)} h(s)$$

(i.e. the values of the variable α at the point R for the functions $f(s)$ and $h(s)$ are equal). Such a choice of Q', R' is possible because $\int_{QQ''R} u \leq \int_{QR} u \leq_{QR''R} u$. After such a change we don't change the curvature and the value of the variable α at (x^1, ζ). Denote this path by $\mathcal{V}_{\delta'}$.

For δ fixed and $\delta' \to 0$ we obtain an approximation of the path $\mathcal{W}$ by paths $\mathcal{V}_{\delta'}$ belonging to the class A with the same $x^0, u^0, \alpha^0, u^1, \alpha^1$. The x and y coordinates will be uniformly approximated, too. Thus for δ' sufficiently small the x coordinate of the final point of the path $\mathcal{V}_{\delta'}$ will be $> x^1$.

To obtain the approximation claimed by the lemma set $\delta \to 0$, $\delta' \to 0$, $\delta' \ll \delta$, approximate the control functions (v, u') as shown below and add two cusps (see 8^0) in order the x-coordinate of the final point of the thus obtained path from the class $\mathcal{A}$ to be equal to x^1.

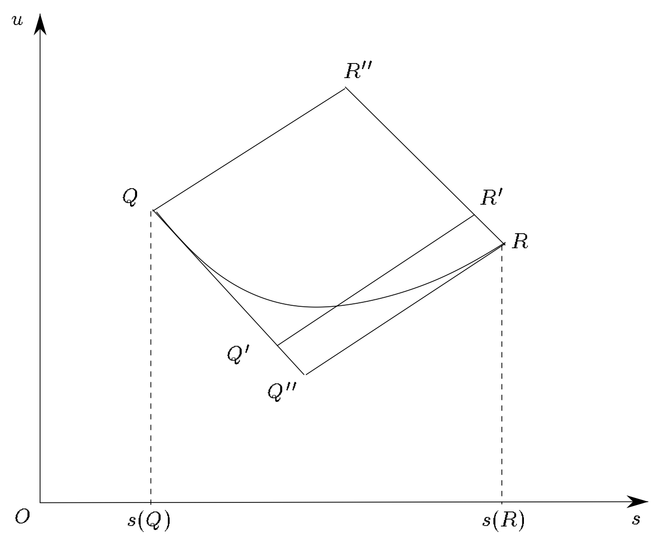

Figure 17

We approximate v by a piecewise-constant function (i.e. assuming only the values ± 1) at the same time as u' and this will give an approximation of (v, u') and of the corresponding paths by paths with a finite numbers of cusps and with a piecewise linear and continuous curvature (the curvature's derivative in absolute value being bounded by 2). We don't change v on the interval $[s(A), s(F)]$ after 6^0.

8^0. Then we add two cusps as shown on Figure 18 (we replace the piece KNL first by $KNMP$ and then by $KNMS$, N and M being cusp points, the arcs NL and PN, MP and MS being symmetric to one another with respect to N and M) and obtain an approximation of the class $\mathcal{A}$.

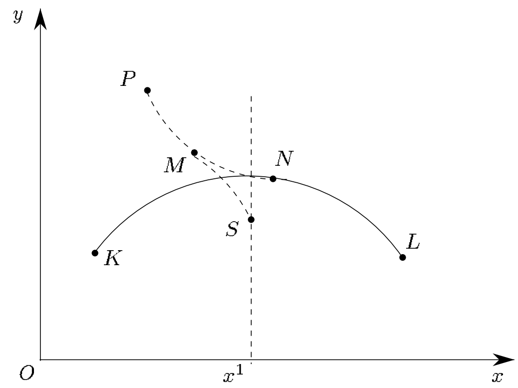

Figure 18

9^0. If the graph of $f(s)$ where $f > 0$ is like the one shown on Figure 12 (or like its left or right part), then we try to perform the transformations from $1^0 - 7^0$ with the set where $f < 0$. If we fail there as well, then the graph of f consists of three linear parts, the one in the middle being $f = 0$. Then we add a new piece to the graph, as shown on Figure 19, which doesn't change $\int_0^{s(x_1,\zeta)} f(s)ds$; we change the control function v on the inserted piece so that the x-coordinate of a point of the path should be increasing there (hence, v will be piecewise linear there). Thus, the endpoint of the changed path will have a greater x-coordinate than the endpoint of the initial one. After this we approximate (u', v) by piecewise constant functions as in 7^0 and add two cusps like in 8^0 to obtain an approximation of the class $\mathcal{A}$. The lemma is proved.

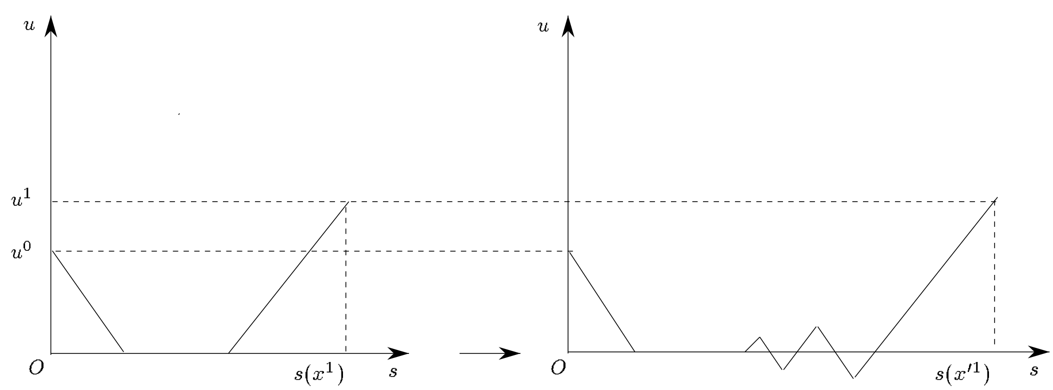

Figure 19

7 Appendix B (Proof of Lemma 10)

1^0. Consider an arc of a clothoid φ. Let the curvatures at the initial and final point of this arc be equal to $2\sqrt{a}$ and $2\sqrt{a+b}$ and the tangent angles α — to $a+c$ and $a+b+c$. Then the length of φ is equal to

$$\int_{\sqrt{a}}^{\sqrt{a+b}} \sqrt{\cos^2(t^2+c)+\sin^2(t^2+c)}dt = \sqrt{a+b}-\sqrt{a} \tag{15}$$

The length of the projection of φ on Ox is equal to

$$\int_{\sqrt{a}}^{\sqrt{a+b}} \cos(t^2+c)dt = \int_a^{a+b} \frac{\cos(\tau+c)}{2\sqrt{\tau}}d\tau$$

$\mathbf{2^0}$. It follows from 1^0 that $|\widehat{C_iC}_{i+1}|$ is a finite sum of the kind

$$\sum_{k=1}^{q}(\sqrt{\mu\pi+a^*+b'_k}-\sqrt{\mu\pi+a^*+b''_k})$$

where $(\mu\pi+a^*+b'_k+c)$ and $(\mu\pi+a^*+b''_k+c)$ are the tangent angles at the ends of the arcs of a half-clothoid belonging to $\widehat{C_iC}_{i+1}$;

$$\sum_{k=1}^{q}|b'_k-b''_k|=\pi,\quad |b'_k|\leq\pi,\quad |b''_k|\leq\pi \tag{16}$$

We assume that $\mu\geq 3$ (i.e. we set $\mu_0=3$), that is why the expressions under the square roots will be positive.

$\mathbf{3^0}$. We have from 1^0 that

$$\gamma_\mu = |\widehat{A_\mu A}_{\mu+1}| = \sqrt{\mu\pi+a^*+\pi}-\sqrt{\mu\pi+a^*} = \frac{\pi}{\sqrt{\mu\pi+a^*+\pi}+\sqrt{\mu\pi+a^*}}$$

It follows from 2^0 that

$$|\widehat{C_iC}_{i+1}| = \sum_{k=1}^{q}(\sqrt{\mu\pi+a^*+b'_k}-\sqrt{\mu\pi+a^*+b''_k}) = \sum_{k=1}^{q}\frac{b'_k-b''_k}{\sqrt{\mu\pi+a^*+b'_k}+\sqrt{\mu\pi+a^*+b''_k}}$$

Hence, using (16) we obtain

$$\frac{\pi}{2\sqrt{\mu\pi+a^*+\pi}} \leq |\widehat{C_iC}_{i+1}| \leq \frac{\pi}{2\sqrt{\mu\pi+a^*-\pi}}$$

Thus, we can estimate

$$M = |\ |\widehat{C_iC}_{i+1}| - |\widehat{A_\mu A}_{\mu+1}|\ | \leq \max(a_+,a_-)$$

where

$$a_\pm = \left|\frac{\pi}{2\sqrt{\mu\pi+a^*\pm\pi}} - \frac{\pi}{\sqrt{\mu\pi+a^*+\pi}+\sqrt{\mu\pi+a^*}}\right| =$$

$$= \frac{\pi}{\sqrt{\mu\pi}}\left|\frac{1}{2\sqrt{1+\frac{a^*\pm\pi}{\mu\pi}}} - \frac{1}{\sqrt{1+\frac{a^*+\pi}{\mu\pi}}+\sqrt{1+\frac{a^*}{\mu\pi}}}\right| = \frac{\sqrt{\pi}}{\sqrt{\mu}}O(1/\mu)$$

Hence, $M = O(1/\mu\sqrt{\mu})$ and formula (11) is proved.

$\mathbf{4^0}$. The projection on Ox of the arc $\widehat{C_iC}_{i+1}\ni A_\mu$ (the tangent angle and the curvature at A_μ being fixed) is minimal if the graph of the curvature as a function of the arc length is like the

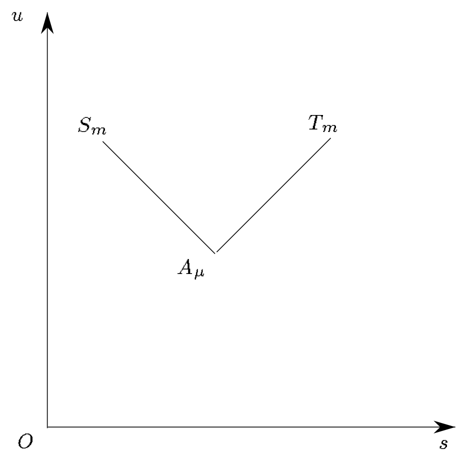

Figure 20

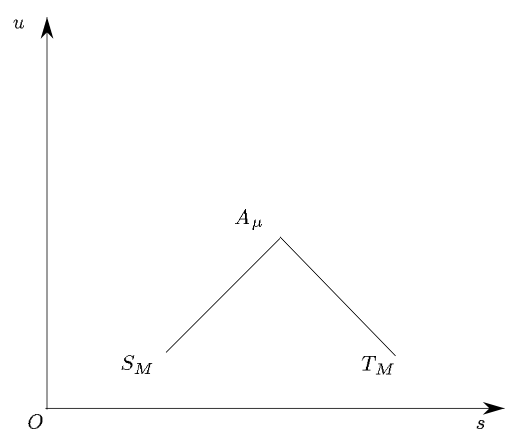

Figure 21

one on Figure 20 and maximal when it is like the one on Figure 21. The points $S_m, S_M, (T_m, T_M)$ correspond to the curvature at C_i (at C_{i+1}). Really, the curvature at A_μ is known. To obtain the shortest (longest) projection, one must integrate the highest (lowest) possible curvature.

5^0. We have

$$|\text{proj}_{Ox}\widehat{C_iC}_{i+1}| = \int_{\alpha=-\pi/2}^{\alpha=\pi/2} \cos\alpha(t)dt$$

Let the indices M, m mean "maximal" and "minimal" in the sense of Figures 20–21. Note that at A_μ we have $\alpha_M = \alpha_m$ and $\dot\alpha_M = \dot\alpha_m$. Hence, the functions $\cos\alpha_M(t)$ and $\cos\alpha_m(t)$ have a contact of second order at A_μ.

The lengths of the intervals of integration (for M and m) are $O(1/\sqrt{\mu})$, see Lemma 2, and, hence, if I is the intersection of the intervals of integration (for M and m), we have

$$\int_I (\cos\alpha_M(t) - \cos\alpha_m(t))dt = O(1/\mu\sqrt{\mu})$$

because $|I| = O(1/\sqrt{\mu})$ and $\sup_I |\cos\alpha_M(t) - \cos\alpha_m(t)| = O(1/\mu)$.

6^0. Let $\dot\alpha_m(A_\mu) = \dot\alpha_M(A_\mu) = \sqrt{\mu\pi + a^*}$ and $\alpha_M(A_\mu) = \alpha_m(A_\mu) = \mu\pi + a^* + c$. Let $\alpha_m(S_m) = \mu\pi + a^* + c + b$ and $\alpha_M(S_M) = \mu\pi + a^* + c - b$. Thus, using formula (15) from 1^0 we obtain

$$\widehat{S_m A_\mu} = \sqrt{\mu\pi + a^* + b} - \sqrt{\mu\pi + a^*}$$

and

$$\widehat{S_M A_\mu} = \sqrt{\mu\pi + a^*} - \sqrt{\mu\pi + a^* - b}$$

Hence

$$\sigma' \equiv ||\widehat{S_M A_\mu}| - |\widehat{S_m A_\mu}|| = O(1/\mu\sqrt{\mu})$$

(see the proof of Lemma 2). Similarly we obtain that

$$\sigma'' \equiv ||\widehat{T_M A_\mu}| - |\widehat{T_m A_\mu}|| = O(1/\mu\sqrt{\mu})$$

Here σ' and σ'' are the differences of the lengths of the corresponding intervals of integration because t is the natural parameter, i.e. the arc length.

Hence the difference of the lengths of the intervals of integration is $O(1/\mu\sqrt{\mu})$.

7^0. Denote

$$\chi_m = |\text{proj}_{Ox} S_m\widehat{A_\mu}T_m|, \quad \chi_M = |\text{proj}_{Ox} S_M\widehat{A_\mu}T_M|$$

We have the obvious inequalities:

$$\chi_m \le \left[\begin{array}{l} |\text{proj}_{Ox}\widehat{C_iC}_{i+1}| \\ |\text{proj}_{Ox} S_M\widehat{A_\mu}T_m| \end{array} \right. \le \chi_M \tag{17}$$

We proved in $5^0 - 6^0$ that

$$|\chi_m - \chi_M| = O(1/\mu\sqrt{\mu}) \tag{18}$$

Denote

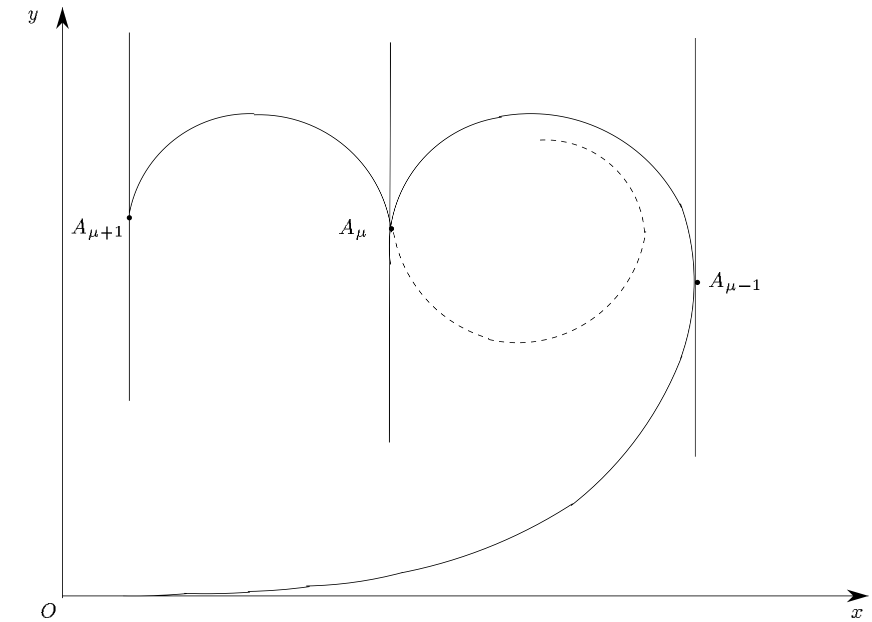

Figure 22

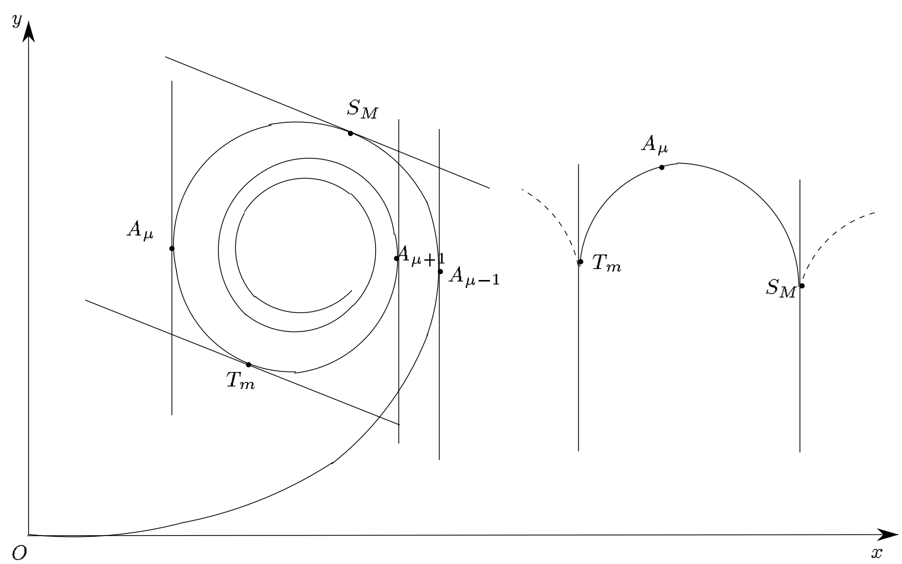

Figure 23

$$\chi' = |\mathrm{proj}_{Ox} A_\mu \widehat{A_{\mu+1}}|, \quad \chi'' = |\mathrm{proj}_{Ox} A_{\widehat{\mu-1}} A_\mu|$$

We choose as a direction α_* the direction of the axis Oy. Then for the points $A_{\mu-1}, A_\mu, A_{\mu+1}$ (see Figure 22) we can use Lemma 4 and obtain

$$|\chi' - \chi''| = O(1/\mu\sqrt{\mu}) \tag{19}$$

On Figure 23 we show the relative position of the points of a clothoid at which it has the same curvature as at the points $A_{\mu-1}, S_M, A_\mu, T_m, A_{\mu+1}$. On the right hand-side of Figure 23 we show the arc $S_M \widehat{A_\mu} T_m$ as it looks on the path constructed in B) provided that there are no cusps between S_M and T_m. For the points $A_{\mu-1}, S_M, A_\mu, T_m, A_{\mu+1}$ we can use Lemma 2 and obtain

$$\chi' \leq |\mathrm{proj}_{Ox} S_M \widehat{A_\mu} T_m| \leq \chi'' \tag{20}$$

Comparing (19) and (20) we get

$$|\mathrm{proj}_{Ox} S_M \widehat{A_\mu} T_m| - \chi' = O(1/\mu\sqrt{\mu}) \tag{21}$$

and

$$|\mathrm{proj}_{Ox} S_M \widehat{A_\mu} T_m| - \chi'' = O(1/\mu\sqrt{\mu})$$

From (17) we have

$$|\ |\mathrm{proj}_{Ox} \widehat{C_i C}_{i+1}| - |\mathrm{proj}_{Ox} S_M \widehat{A_\mu} T_m|\ | \leq |\chi_M - \chi_m| \tag{22}$$

Thus we get (using (22))

$$|\ |\mathrm{proj}_{Ox} \widehat{C_i C}_{i+1}| - \chi'| \leq |\ |\mathrm{proj}_{Ox} \widehat{C_i C}_{i+1}| - |\mathrm{proj}_{Ox} S_M \widehat{A_\mu} T_m|\ |+$$

$$+|\ |\mathrm{proj}_{Ox} S_M \widehat{A_\mu} T_m| - \chi'| \leq |\chi_M - \chi_m| + |\ |\mathrm{proj}_{Ox} S_M \widehat{A_\mu} T_m| - \chi'|$$

Hence, from (18) and (21) we obtain

$$|\ |\mathrm{proj}_{Ox} \widehat{C_i C}_{i+1}| - |\mathrm{proj}_{Ox} \widehat{A_\mu A}_{\mu+1}|\ | = O(1/\mu\sqrt{\mu})$$

The lemma is proved.

8 Acknowledgement

J.D.Boissonnat stated the problem and together with A.Cérézo and J.Leblond participated in numerous and helpful discussions. A.Cérézo made many useful remarks concerning the text (both the logical structure and the better presentation).To all of them we express our most sincere gratitude.

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