

# A Min-Max certainty equivalence principle for non-linear discrete-time control problems

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INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

*A Min-Max Certainty  
Equivalence Principle  
for Nonlinear Discrete-Time  
Control Problems*

Pierre BERNHARD

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# A min-max certainty equivalence principle for nonlinear discrete-time control problems

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September, 1993

**Abstract** We extend to a setup comparable to that provided in a recent paper coauthored with Alain Rapaport, [INRIA report 2019] for the continuous time case, the main result of our paper [INRIA report 2020] in the discrete time case, thus obtaining the most comprehensive certainty equivalence principle for the discrete time problem to date.

## Un principe d'équivalence à la certitude min-max pour des problèmes de commande non linéaires à temps discret

**Résumé** Nous étendons à un contexte comparable à celui proposé dans le rapport de recherche INRIA 2019 (en coauteur avec Alain rapaport) pour les problèmes à temps continu, le principal résultat de notre rapport de recherche INRIA 2020 en temps discret. Nous obtenons ainsi le principe d'équivalence à la certitude le plus général disponible à ce jour pour les problèmes à temps discret.

# A min-max certainty equivalence principle for nonlinear discrete-time control problems

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September, 1993

## Abstract

We extend to a setup comparable to that provided in [4] for the continuous time case, the main result of [3] in the discrete time case, thus obtaining the most comprehensive certainty equivalence principle for the discrete time problem to date.

## 1 Introduction

The first min-max certainty equivalence principle ever seems to be due to Whittle [6], and concerns the standard discrete time linear quadratic problem. (Although it appeared in the investigation of a stochastic exponential quadratic or “risk sensitive” problem.) Since then, the author introduced systematically this concept, for both discrete time and continuous time problems in [2], and it was used in [1]. The continuous time case was further studied in [5] and [4]. In [3], we proposed a more general theory, with a certainty equivalence principle as one of its consequences, but in a less general setup. Here, we extend the result of [3] to a set up comparable to that of [4], thus providing the most general such result to date for discrete time systems, with a very simple proof.

## 2 The framework

### 2.1 The system

We are given a disturbed discrete time control system in  $\mathbb{R}^n$

$$x_{t+1} = f_t(x_t, u_t, w_t). \quad (1)$$

The time  $t$  is a non negative integer, the control input  $u$ , with value  $u_t$  at time  $t$ , ranges over  $U$ , and the disturbance input  $w$  over  $W$ . Moreover, initial state  $x_0$  is also considered a disturbance ranging over a set  $X_0 \subset \mathbb{R}^n$ .

The control sequence  $\{u_t\}_{t \in \mathbb{N}}$ , or equivalently the time function  $t \mapsto u_t \in U$  is in  $\mathcal{U}$  and similarly the disturbance sequence  $\{w_t\}_{t \in \mathbb{N}}$  belongs to  $\mathcal{W}$ . They will often be written  $u_{(\cdot)}$  and  $w_{(\cdot)}$ .

The complete disturbance is

$$\omega = (x_0, w_{(\cdot)}) \in \Omega = X_0 \times \mathcal{W}.$$

Let

$$w^\tau = \{w_t, 0 \leq t \leq \tau\} \in \mathcal{W}^\tau$$

be the restriction to  $[0, \tau]$  of the sequence  $w_{(\cdot)}$ , and similarly for other time sequences. Let also

$$\omega^\tau = (x_0, w^\tau) \in \Omega^\tau.$$

### 2.2 The observation and admissible strategies

The controller does not have a complete knowledge of the past disturbance nor of the state. An *observation process* has been defined that to each pair  $(u_{(\cdot)}, \omega)$  makes correspond a sequence  $\{\Upsilon_t\}_{t \in \mathbb{N}}$  of subsets of  $\Omega$ . The observation process is assumed to enjoy the following three properties :

#### Hypothesis A

**A1** Consistency :

$$\forall (u_{(\cdot)}, \omega), \forall t, \quad \omega \in \Upsilon_t,$$

**A2** Strict non anticipativeness :

$$\omega \in \Upsilon_t \Leftrightarrow \omega^{t-1} \in \Upsilon_t^{t-1},$$

**A3** Perfect recall :

$$\Upsilon_{t+1} \subset \Upsilon_t.$$

The set  $\mathcal{M}$  of admissible strategies for the controller will be that of functions  $\mu$  of the form

$$u_{(\cdot)} = \mu(\Upsilon_{(\cdot)}) : u_t = \mu_t(\Upsilon_t). \quad (2)$$

A typical instance is when an output

$$y_t = h_t(x_t, w_t)$$

is measured by the controller, who is allowed to use strictly causal controls of the form  $u_t = \mu_t(y^{t-1})$ . From the output sequence  $y_{t-1}$  observed up to time  $t-1$  the controller can infer the equivalence class  $\Upsilon_t$  of disturbances which, together with the past controls  $u_{t-1}$  used, generate that same output.

### 2.3 The performance index

A terminal set  $\mathbb{T} \subset \mathbb{N} \times \mathbb{R}^n$  is given. The problem terminates at the first time instant such that the state reaches  $\mathbb{T}$ ,

$$t_f = \min\{t \mid (t, x_t) \in \mathbb{T}\}. \quad (3)$$

In the sequel,  $x_f$  always stands for  $x_{t_f}$ .

It turns out to be convenient to introduce the section  $\mathbb{T}_t \in \mathbb{R}^n$  of  $\mathbb{T}$  at time  $t$  as

$$\mathbb{T}_t = \{x \mid (t, x) \in \mathbb{T}\}$$

so that the termination condition  $(t, x_t) \in \mathbb{T}$  can equivalently be written  $x_t \in \mathbb{T}_t$ .

A performance index is given as <sup>1</sup>

$$J(u_{(\cdot)}, \omega) = M_{t_f}(x_f) + \sum_{t=0}^{t_f-1} L_t(x_t, u_t, w_t) + N(x_0), \quad (4)$$

The controller's objective is to minimize the worst possible case, thus to chose a control strategy  $\mu^*$  such that

$$\max_{\omega \in \Omega} J(\mu^*, \omega) = \min_{\mu \in \mathcal{M}} \max_{\omega \in \Omega} J(\mu, \omega). \quad (5)$$

---

<sup>1</sup>the term  $N(x_0)$  is unnecessary. It can be absorbed in  $L_0$ . It turns out to be convenient to keep it there in applications. See the linear quadratic case in [1] for instance.

Instead of restricting  $x_0$  to  $X_0$ , we may equivalently let  $N(x) = -\infty$  for all  $x$  not in  $X_0$ . This causes the maximizing  $\omega$  to always have  $x_0 \in X_0$ .

## 2.4 An alternate performance index

All the sequel extends to a problem with no target set and performance index

$$J(u_{(\cdot)}, \omega) = \min_{t_f} [M_{t_f}(x_f) + \sum_{t=0}^{t_f-1} L_t(x_t, u_t, w_t) + N(x_0)],$$

Isaacs' equation (6) is then replaced by its obvious generalization

$$V_t(x) = \min \left\{ M_t(x), \min_u \max_w [V_{t+1}(f_t(x, u, w)) + L_t(x, u, w)] \right\}$$

and  $T$  by the set  $\bar{T} = \{(t, x) \mid V_t(x) = M_t(x)\}$ .

Such a formulation lets one solve the qualitative problem of whether one can insure that the state reach a given target set defined by  $T(x) \leq 0$ . Let  $M = T$ ,  $L = 0$ ,  $\forall x \in X_0$ ,  $N(x) = 0$ . The sign of the minimax value of the game yields the answer.

We do not develop this in the sequel, the adaptation is straightforward.

## 2.5 Minima and maxima

We assume that a proper set of hypotheses hold to insure the existence of the minima and maxima we use hereafter. One possibility is to assume that  $f_t$  and  $L_t$  are of class  $C^1$  for all  $t$ , as well as  $M$  and  $N$ , and that  $U$ ,  $W$ , and  $X_0$  are compact. But this does not account for the classical linear quadratic case, where  $U$ ,  $W$ , and  $X_0$  are whole vector spaces, existence of the extrema being insured by the behavior at infinity of  $M$ ,  $L$ , and  $N$ .

# 3 The auxiliary problem

## 3.1 Basic formulation

We consider the full information dynamic game defined by (1), (3), and (4), but without the  $N(x_0)$  term which has no meaning in a full information

game, and state feedbacks as admissible strategies. Let  $V_t(x)$  be its upper value. It satisfies Isaacs' equation

$$V_t(x) = \min_{u \in U} \max_{w \in W} [V_{t+1}(f_t(x, u, w)) + L_t(x, u, w)], \quad (6)$$

with the boundary condition

$$\forall (t, x) \in \mathbb{T}, \quad V_t(x) = M_t(x). \quad (7)$$

We notice that the game with the original cost  $J$  as in (4) and  $\omega$  as maximizing control has an upper value

$$A_0 = \max_{x \in X_0} [V_0(x) + N(x)]. \quad (8)$$

We now introduce an assumption concerning the full information game (we shall say what to do if it is not satisfied)

**Hypothesis B** We assume that the minimum in  $u$  in (6) is, for each  $(t, x)$ , reached at a unique point  $u = \phi_t^*(x)$ .

We introduce a fictitious observation process : let

$$\Gamma_t = \{\omega \in \Omega \mid \forall s \leq t, x_s \notin \mathbb{T}_s\}$$

It is straightforward to check that for  $t < t_f$ , this process satisfies hypothesis **A**. And also, because both  $\Upsilon_{(\cdot)}$  and  $\Gamma_{(\cdot)}$  satisfy hypothesis **A**, so does their intersection. Introduce therefore the modified observation process

$$\Omega_t = \Upsilon_t \cap \Gamma_t,$$

it satisfies hypothesis **A** for  $t \in [0, t_f - 1]$ .

The *auxiliary problem* is defined for each  $t$  as the following maximization problem: let

$$G_t(u^{t-1}, \omega) = G_t(u^{t-1}, \omega^{t-1}) = \left[ V_t(x_t) + \sum_{s=0}^{t-1} L_s(x_s, u_s, w_s) + N(x_0) \right] \quad (9)$$

$$A_t = \max_{\omega \in \Omega_t} G_t(u_{t-1}, \omega) \quad (10)$$

For each  $t$ , the minimizer knows the past controls  $u^{t-1}$  it has used, and the above problem can therefore be solved, yielding a solution only depending, beyond  $u^{t-1}$ , on the available information  $\Omega_t$ . We call  $\hat{\omega}_t$  a maximizing  $\omega$  above, and  $\hat{x}_t$  the current "worst state"  $x_t$  it leads to.

Notice that the notation  $A_t$  is consistent with the notation  $A_0$  introduced in (8).



### 3.2 Alternate formulation

We need a further notation : for  $\xi \notin \mathbb{T}_t$ , let

$$\Omega_t(\xi) = \{\omega \in \Omega_t \mid x_t = \xi\}$$

As all the other subsets of  $\Omega$  we have introduced, it is a function of the past controls  $u^{t-1}$ . Contrary to  $\Omega_t$ , it may be empty even before termination.

We define the *conditional cost to come* function  $W$  from  $\mathbb{N} \times \mathbb{R}^n$  into  $\mathbb{R} \cup \{-\infty\}$  as

$$W_t(x) = \max_{\omega \in \Omega_t(x)} \left[ \sum_{s=0}^{t-1} L_s(x_s, u_s, w_s) + N(x_0) \right]$$

where it is understood that as usual, the max over an empty set is  $-\infty$ . Notice that we have

$$\forall x \in \mathbb{R}^n, \quad W_0(x) = N(x). \quad (11)$$

The problem (12) may be written as

$$A_t = \max_x \max_{\omega \in \Omega_t(x)} \left[ V_t(x_t) + \sum_{s=0}^{t-1} L_s(x_s, u_s, w_s) + N(x_0) \right],$$

and using the definition of the conditional cost to come, as

$$A_t = \max_x [V_t(x) + W_t(x)]. \quad (12)$$

And  $\hat{x}_t$ , as defined above, is any maximizing  $x$  in (12).

## 4 Main result

Introduce the notation

$$S_t(x, u) = \max_{w \in \mathbb{W}} [V_{t+1}(f_t(x, u, w)) + L_t(x, u, w) + W_t(x)]. \quad (13)$$

We introduce the main hypothesis :

**Hypothesis C** For  $u_{(\cdot)}$  generated by the strategy  $\hat{\mu}$  of the theorem, the function  $S_t$  has, for all  $\omega \in \Omega$  and all  $t$  a saddle-point  $\hat{x}_t, \hat{u}_t$ , so that

$$\max_x S_t(x, \hat{u}_t) = \max_x \min_u S_t(x, u). \quad (14)$$

(We shall see that  $\hat{x}_t$  as defined here necessarily coincides with the definition in the auxiliary problem, so that this notation is consistent.)

We may now state the theorem :

**Theorem 1 (Certainty Equivalence Principle)** *Under hypotheses A, B, and C, let  $\hat{x}_t$  be a worst case state as defined by the auxiliary control problem, then  $u_t = \hat{\mu}_t(\Omega_t) := \phi_t^*(\hat{x}_t)$  is uniquely defined for all  $t$  and is an optimal controller in the sense of (5), leading to a value  $A_0$  as in (8).*

**Proof** Unicity of  $\phi^*(\hat{x}_t)$  easily follows from assumption B.

Let us investigate the behavior of  $A_t$  under the effect of the strategy  $\hat{\mu}$  of the theorem. Take the expression of  $A_t$  in (12) and use Isaacs' equation (6) to replace  $V_t$  in terms of  $V_{t+1}$ . It comes

$$A_t = \max_x \min_{u \in U} \max_{w \in W} [V_{t+1}(f_t(x, u, w)) + L_t(x, u, w) + W(x)],$$

The maximum in  $x$  is by definition reached at  $x = \hat{x}_t$  and the minimum in  $u$  at  $u = \phi_t^*(\hat{x}_t) = \hat{\mu}_t(\Omega_t)$ . Using (13),

$$A_t = \max_x \min_{u \in U} S_t(x, u) = S_t(\hat{x}_t, \hat{\mu}_t(\Omega_t)). \quad (15)$$

Now, we also have, using (10)

$$A_{t+1} = \max_{\omega \in \Omega_{t+1}} \left[ V_{t+1}(x_{t+1}) + L_t(x_t, u_t, w_t) + \sum_{s=0}^{t-1} L_s(x_s, u_s, w_s) + N(x_0) \right].$$

According to hypothesis A.3, we get

$$A_{t+1} \leq \max_{\omega \in \Omega_t} \left[ V_{t+1}(x_{t+1}) + L_t(x_t, u_t, w_t) + \sum_{s=0}^{t-1} L_s(x_s, u_s, w_s) + N(x_0) \right].$$

But according to hypothesis A.2,  $\omega \in \Omega_t$  is equivalent to  $\omega^{t-1} \in \Omega_t^{t-1}$ , and thus  $w_t$  free. We therefore get

$$A_{t+1} \leq \max_{\omega^{t-1} \in \Omega_t^{t-1}} \max_{w \in W} \left[ V_{t+1}(x_{t+1}) + L_t(x_t, u_t, w_t) + \sum_{s=0}^{t-1} L_s(x_s, u_s, w_s) + N(x_0) \right]$$

Use (1) to substitute for  $x_{t+1}$ , and again write

$$\max_{\omega^{t-1} \in \Omega_t^{t-1}} [\dots] = \max_x \max_{\omega^{t-1} \in \Omega_t^{t-1}(x)} [\dots]$$

to obtain

$$A_{t+1} \leq \max_x \max_{w \in W} [V_{t+1}(f_t(x, u_t, w)) + L_t(x, u_t, w) + W(x)].$$

We are assessing the effect of the strategy  $u_t = \hat{\mu}_t(\Omega_t)$ . We therefore obtain

$$A_{t+1} \leq \max_x S_t(x, \hat{\mu}_t(\Omega_t)). \quad (16)$$

Compare (15) with (16) using hypothesis **C**, and specifically (14). It results that  $A_{t+1} \leq A_t$ , and therefore, that

$$A_{t_f-1} \leq A_0. \quad (17)$$

Let  $\omega$  be fixed in  $\Omega$ , but arbitrary. Together with the strategy  $\hat{\mu}$  they generate an observation process  $\Omega_{(\cdot)}$ . By definition, hypothesis **A.1** holds up to time  $t_f - 1$ , and according to the definition (10),

$$\forall \omega \in \Omega, \quad G_{t_f-1}(u^{t_f-1}, \omega) \leq A_{t_f-1}.$$

and also

$$\forall \omega \in \Omega, \quad G_{t_f}(\hat{\mu}, \omega) \leq \max_{\omega | \omega^{t_f-1} \in \Omega_{t_f-1}^{t_f-1}} G_{t_f}(\hat{\mu}, \omega).$$

But observe that the proof above concerning  $A_{t+1}$  begins by dropping any constraint on  $w_t$ . Thus it is also true that

$$\max_{\omega | \omega^{t_f-1} \in \Omega_{t_f-1}^{t_f-1}} G_{t_f}(\hat{\mu}, \omega) \leq A_{t_f-1}.$$

Now, comparing (9) and (4), and using (7), it comes  $G_{t_f} = J$ . This together with the above inequality and (17) yield

$$\forall \omega \in \Omega, \quad J(\hat{\mu}, \omega) \leq A_0.$$

We have pointed out that  $A_0$  is the value of the *full information* game. Hence no partial information strategy can do better, and  $\hat{\mu}$  is optimal.

## 4.1 Sufficient conditions

We investigate some sufficient conditions that insure hypothesis **C**. The first corollary is obvious:

**Corollary 1** *If for all  $\omega \in \Omega$ , the function  $S_t$  is, for all  $t$  convex s.c.i. in  $u$ , concave s.c.s. in  $x$ , and diverges to  $\infty$  for  $\|u\| \rightarrow \infty$  in  $\mathbf{U}$ , and to  $-\infty$  for  $\|x\| \rightarrow \infty$ , then the certainty equivalence principle holds.*

**Proof** Simply apply the Von Neumann-Sion Theorem.

**Corollary 2** *If the set of first order necessary conditions concerning the extrema in Isaacs' equation and the maximization problem in (12) has a unique solution for all  $\omega$  and all  $t$ , then the certainty equivalence principle holds.*

**Proof** Under the hypothesis of the corollary, the maximum in  $w$  in Isaacs' equation (6) is unique. Applying Danskin's theorem, the partial derivative of  $S_t$  in  $u$  is the same as that of the right hand side of (6), and using the unicity of the minimum in  $u$ , that in  $x$  is the same as that of the right hand side of (12). Thus the hypothesis of the theorem says that the first order necessary conditions for the minimum in  $u$  and the maximum in  $x$  of  $S_t$  have a unique solution. Using Danskin's theorem again, it classically follows that it has a saddle point. (These conditions are those needed to extend to non linear problems the proof of [1])

**Corollary 3** *If the minimization problem in Isaacs equation is always convex, and the auxiliary problem concave for all  $t$ , (both, of course, having solutions) then the certainty equivalence principle holds.*

**Proof** Apply the theorem of Von Neumann-Sion to the function

$$(u, \omega) \mapsto G_{t+1}(u^{t-1} \cdot u, \omega)$$

and notice that

$$\max_{\omega} G_{t+1}(u^{t-1} \cdot u, \omega) = \max_x S_t(x, u).$$

(This condition is that given in [2])

## 5 Conclusions

We have extended to a more general case, specifically variable end time, whether defined by a capture condition or taken as a minimization parameter, the discrete time certainty equivalence principle of [3]. Taking advantage of the fact that we focus on the case where the principle holds, we were able to give a simpler proof, very much in the spirit of that of [4]. We conjecture that condition **C** above is the most general that can be given. Notice that if the unicity condition **B** is not met, one should chose among the certainty equivalent controls the one that provides the saddle point in **C**. The rest of the proof holds unchanged.

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**pax:bolot**

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Tue Sep 7 18:27:08 1993

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Received: from sophia.inria.fr by pax.inria.fr with SMTP  
 (5.65c8/IDA-1.2.8) id AA24525; Tue, 7 Sep 1993 18:22:58 +0200  
 Received: from thumper.bellcore.com by sophia.inria.fr with SMTP  
 (5.65c8/IDA-1.2.8) id AAL7672; Tue, 7 Sep 1993 18:22:52 +0200  
 Received: from tesla.bellcore.com by thumper.bellcore.com (4.1/4.7)  
 id <AA08292> for Jean.Bolot@sophia.inria.fr; Tue, 7 Sep 93 12:20:57 EDT  
 Received: by tesla.bellcore.com (4.1/SMI-4.1)  
 id AA00474; Tue, 7 Sep 93 12:20:51 EDT  
 Date: Tue, 7 Sep 1993 12:12:02 -0400 (EDT)  
 From: Dipak Ghosal <ghosal@tesla.bellcore.com>  
 Subject: Re: Visit  
 To: Jean-Chrysostome Bolot <Jean.Bolot@sophia.inria.fr>  
 In-Reply-To: <199309071601.AA24387@pax.inria.fr>  
 Message-Id: <Pine.3.05.9309071259.F331-b100000@tesla>  
 Mime-Version: 1.0  
 Content-Type: TEXT/PLAIN; charset=US-ASCII

On Tue, 7 Sep 1993, Jean-Chrysostome Bolot wrote:  
 > I think I'll take the NJ transit like last time. But you  
 > said you moved to a new place. Do you still live near Red Bank?  
 NJ transit sounds fine. You can call me form the Penn Station and  
 I will pick you up from the Red Bank station.  
 I am in the same apartment complex but in a different apartment. Here is the  
 address and telephone number.

118 Victoria Drive  
 Eatontown, NJ 07724  
 (908) 544 8772 (home)  
 (908) 758 5060 (office)

> Also could you please send me your home phone just in case? And  
 > finally is Sunday late afternoon/night fine with you, or will you  
 > be away for the weekend (in which case I'll see you in Bellcore  
 > on Monday)

We just came back from a long trip this past weekend and so we will stay put at  
 home the coming weekend. So Sunday late afternoon/night is fine with us.

Dipak  
 -----  
 Bellcore (908) 758-5060 (office)  
 331 Newman Springs Road (908) 758-4369 (fax)  
 Red Bank, NJ 07724



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