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Classification of Markov chains describing the evolution of random strings

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Abstract

We consider LIFO queue with several customer types, batch arrivals and batch services. We get necessary and sufficient conditions for this queue to be ergodic, null reccurent and transient. These are given in terms of positive solutions of finite system of equations $p_i = F_i(p_1, ..., p_k), i = 1, ..., k$, where F_i are polynomials with positive coefficients.

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Classification de chaînes de Markov décrivant l' évolution de suites aléatoires de charactères

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Résumé

On considère des files d'attente avec plusieurs types de clients et avec arrivées et services simultanées. On obtient les conditions nécessaires et suffisantes d'ergodicité et de récurrence.

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1 Introduction.

We recall main definitions from [1]. Consider a discrete time homogeneous countable Markov chain $\mathcal{L}$ with the set of states

$$\mathcal{A} = \bigcup_{n=0}^{\infty} \{1, \ldots, r\}^n.$$

In other words, the states of this Markov chain are ordered r-symbol sequences (strings) $\alpha = x_n \ldots x_1, x_i \in \{1, \ldots, r\}$, of arbitrary length $n = n(\alpha) = |\alpha|$. We denote $\emptyset$ an empty string (of length 0). For two arbitrary strings $\alpha = x_n \ldots x_1, \beta = y_m \ldots y_1$ we define their concatenation (of length $m + n$)

$$\alpha\beta = x_n \ldots x_1 y_m \ldots y_1$$

For the one-step transition probabilities $p_{\alpha,\beta}, \alpha \to \beta$,we assume the following conditions.

Condition B (boundedness of jumps): for some $d < \infty$
$p_{\alpha,\beta} \neq 0$ only if $||\alpha| - |\beta|| \leq d$;

Condition H (space homogeneity):
Let $|\alpha| \geq d$ and $\alpha = \gamma\rho, \beta = \theta\rho$ for some γ, θ, ρ with $|\gamma| = d, |\theta| \leq 2d$. Then $p_{\alpha,\beta}$ does not depend on ρ but only on γ and θ. So we can denote this transition probability by $q(\gamma, \theta)$: we delete γ from the left and append θ instead.

Condition ND (nondegeneracy): for any strings $\gamma, \theta : |\gamma| = d, |\theta| \leq d, \quad q(\gamma, \theta) > 0$. We assume also positivity of all transition probabilities $p_{\alpha,\beta}$ for $|\alpha| < d, \quad |\beta| \leq |\alpha| + d$.

Then all states are essential and the Markov chain is irreducible, aperiodic. Note that Condition ND is assumed only to simplify the formulations.

The goal of the paper is to give necessary and sufficient conditions for this Markov chain to be ergodic, null recurrent or transient.These conditions are given in terms of maximal eigenvalue of some finite matrix A. To give the main ideas we consider first in section 2 the case $d = 1$. In this case the matrix has matrix elements which are linear combinations of some $q(\gamma, \theta)$. In case $d > 1$ matrix elements of A are given in terms of some positive solutions of a *fundamental* nonlinear algebraic equation. We study this equation in section 3. Classification for the case $d > 1$ is considered in sections 4,5. In section 5 we get also convergent (exponentially fast) algorithms for the desired solutions of the fundamental equation.

There was a lot of activity concerning close problems in the algebraic situation. Consider the free group G with l generators $a_1, ..., a_l$ and put $a_{-i} = a_i^{-1}$.The papers [7] , [8] , [6] , [9] are concerned with the product of independent random variables with values in G. Ergodic case cannot occur here (and mostly transient cases were considered in those papers), so for our purposes this situation is not interesting but some ideas are relevant (see Remark 6 at the end of section 3).

2 Linear recurrent equations: case $d = 1$

In this section it is always assumed that $d = 1$. Let $\alpha(t)$, $t = 0, 1, 2, \ldots$ be the state of the process at time t. Put

$$\tau_i = min\{t : |\alpha(t)| < |i\rho|, \alpha(0) = i\rho\}, i \in \{1, \ldots, r\}$$

The distribution of τ_i does not depend on ρ. If the process is ergodic, then $E_i = E\tau_i < \infty$ and $\{E_i\}_{i=1,\ldots,r}$ satisfy the following system of equations

$$E_i = q(i, \emptyset) + \sum_j q(i,j)(1 + E_j) + \sum_{j,k} q(i, jk)(1 + E_j + E_k)$$

$$= 1 + \sum_j q(i,j)E_j + \sum_{j,k} q(i, jk)(E_j + E_k) \quad (1)$$

We can rewrite (1) in the matrix form as

$$\overrightarrow{E} = \overrightarrow{1} + A\overrightarrow{E} \quad (2)$$

where $\overrightarrow{E}$ is the vector $(E_i : i = 1, \ldots, r)$, A is a positive matrix.Then A has the unique maximal positive eigenvalue λ. The equation (2) has a positive finite solution iff $\lambda < 1$. So, if the process is ergodic, then $\lambda < 1$. However, far more than this is true.

Theorem 2.1 *The process $\alpha(t)$ is*
(i) ergodic iff $\lambda < 1$,
(ii) null recurrent iff $\lambda = 1$,
(iii) transient iff $\lambda > 1$.
Proof:

We will give the proof of the theorem by constructing appropriate Lyapounov functions.

Let

$$\tau(\alpha) = min\{t : |\alpha(t)| = 0, \alpha(0) = \alpha\}$$

be random time to reach the empty string $\emptyset$, starting from $\alpha \neq \emptyset$. Then

$$T(\alpha) = E\tau(\alpha) = \sum_{k=1}^{n} E_{x_k}$$

if

$$\alpha = x_n \dots x_1,$$

Put $f(\alpha) = \sum_{k=1}^{n} e_{x_k}$. where $\{e_i\}$ are the components of the eigenvector $\overrightarrow{e} = (e_1, \dots, e_r)$ corresponding to the maximal eigenvalue λ, i.e.

$$\lambda e_i = \sum_j q(i,j)e_j + \sum_{j,k} q(i,jk)(e_j + e_k) \tag{3}$$

By Perron-Frobenius theorem $e_i > 0$ for all i. We need the following lemma

Lemma 2.1
(i) if $\lambda > 1$, then there exists $\epsilon > 0$ such that

$$E[f(\alpha(t+1)) - f(\alpha(t))|\alpha(t) = \alpha] > \epsilon$$

for all $\alpha : |\alpha| \geq 1$
(ii) if $\lambda = 1$, then

$$E[f(\alpha(t+1)) - f(\alpha(t))|\alpha(t) = \alpha] = 0$$

for all $\alpha : |\alpha| \geq 1$
(iii) if $\lambda < 1$, then there exists $\epsilon > 0$ such that

$$E[f(\alpha(t+1)) - f(\alpha(t))|\alpha(t) = \alpha] < -\epsilon$$

for all $\alpha : |\alpha| \geq 1$

Proof:
It is just a computation for $|\alpha| > 1$. Let $\alpha = i\rho$, then

$$E[f(\alpha(t+1)) - f(\alpha(t))|\alpha(t) = \alpha] =$$

$$= -q(i,\emptyset)e_i + \sum_j q(i,j)(e_j - e_i) + \sum_{j,k} q(i,jk)(e_j + e_k - e_i) =$$

$$= -e_i + \sum_j q(i,j)e_j + \sum_{j,k} q(i,jk)(e_j + e_k) =$$

$$= (\lambda - 1)e_i$$

The lemma is proved. ■

To prove the theorem note that

a) for all α

$$f(\alpha) \geq 0, f(\alpha) \to +\infty \textit{ when } |\alpha| \to \infty,$$

b) there exists $L > 0$ such that

$$|f(\alpha) - f(\beta)| > L \textit{ implies } p_{\alpha,\beta} = 0.$$

Now Theorem 1 follows from the well known criteria (see Appendix).■

The associated branching process.
Let us introduce the following set of probabilities $\{p_i\}$

$$p_i = P(\tau_i < \infty),$$

they satisfy the following equations

$$p_i = q(i, \emptyset) + \sum_j q(i,j)p_j + \sum_{j,k} q(i,jk)p_j p_k.$$

Looking at this equations we get an idea of a branching process which is associated with our string. Consider the branching process with r particle types where a particle of type i dies with the probability $q(i, \emptyset)$, gives one offspring j with the probability $q(i,j)$ and gives two offsprings j, k with the probability $q(i, jk)$. Then the previous equation can be interpreted as the equation for the probability p_i that ,starting from a particle i the system will ever die. Then as it is known from the branching processes theory the matrix A governs the behaviour of this process: if $\lambda > 1$ then the process dies with a probability less than 1 (this corresponds exactly to the transience of the string), if $\lambda = 1$ then the process is critical (this corresponds to the null recurrence of the string), if $\lambda < 1$ then we have subcritical region and the string is ergodic.

Simplest queue.

Let us consider the simplest and well-known example of LIFO queue, when the customers of type i arrive with intensity λ_i and are served with intensity μ_i.The well known ergodicity condition here is

$$\sum \frac{\lambda_i}{\mu_i} < 1$$

This condition can be easily proved with the following heuristic arguments. During time T approximately $\lambda_i T$ customers of type i arrive. The mean time necessary to serve them is approximately $\sum_i \lambda_i T(\mu_i)^{-1}$. But it should be less than T. We get from this $\sum \frac{\lambda_i}{\mu_i} < 1$.

Here we shall show how this condition follows from theorem 2.1. Instead of the continuous time LIFO network consider the discrete time Markov chain with parameters

$$q(i,i) = 1 - \epsilon(\mu_i + \lambda_i)$$

$$q(i,\emptyset) = \epsilon\mu_i$$

$$q(i,ij) = \epsilon\lambda_i$$

$$q(i,kj) = 0 \quad if \quad k \neq i$$

where ϵ is sufficiently small so that each $q(i,i) > 0$.
Then our matrix

$$A = (a_{ij}), \quad a_{ij} = \delta_{ij} + \epsilon(-\mu_i\delta_{ij} + \lambda_j).$$

Note that theorem 2.1 can be reformulated as follows.
Theorem 2.1'
Our Markove chain is ergodic iff there exists a positive vector $e = (e_1, ..., e_n)$ such that

$$Ae < e, \quad i.e. \quad for\ all\ i\ (Ae)_i < e_i;$$

null recurrent iff there exists a positive vector $e = (e_1, ..., e_n)$ such that

$$Ae = e;$$

transient iff there exists a positive vector $e = (e_1, ..., e_n)$ such that

$$Ae > e, \quad i.e. \quad for\ all\ i\ (Ae)_i > e_i.$$

Let e be a vector with components $e_i = 1/\mu_i$ then

$$Ae = e + \epsilon(\sum \frac{\lambda_k}{\mu_k} - 1).$$

so from the Theorem 2' we have the above result.

Now we want to discuss the above heuristic arguments for more general case.
Let us consider the case when arrival of type j depends on the type of the left end of the string, i.e. if $\alpha = i\beta$ then j arrives with intensity λ_{ij}, $\mu_i$is again the service time of type i . Let the Markov chain be transient, then there exists the limiting stationary disribution π_i for the state of the left end of the string. So during time T approximately $\sum_i \pi_i \lambda_{ij} T$ customers of type j will arrive. The mean time necessary to serve them is approximately $\sum_j \frac{\sum_i \pi_i \lambda_{ij} T}{mu_j}$ and it should be greater than T. We get from this $\sum_j \frac{\sum_i \pi_i \lambda_{ij}}{mu_j} > 1$. The condition of this form can be obtained if we take the left hand eigenvector $v = (v_1, ..., v_r)$ of the matrix A correspondings to the maximal eigenvalue λ of A . If $\sum_i v_i = 1$ then it is easy to show that $\lambda <,=,> 1$ iff $\sum_j \frac{\sum_i v_i \lambda_{ij} T}{mu_j} <,=,> 1$ correspondingly.

3 The fundamental nonlinear equation: case $d > 1$.

For our Markov chain $\alpha(t)$ put

$$\alpha(0) = \gamma\rho, \quad |\gamma| = d, \quad |\gamma\rho| = n.$$

Let $\tau_{\gamma,n}$ be the first time when the length $\alpha(t)$ becomes less than n, i.e.

$$\tau_{\gamma,n} = min\{t :| \alpha(t) |< n\}$$

Note that the distribution of $\tau_{\gamma,n}$ and $E_\gamma = E\tau_{\gamma,n}$ does not depend on ρ and on n. Introduce the following notation

$$\overrightarrow{p} = \{p(\gamma, \delta) : |\gamma| = d, |\delta| < d\} \tag{4}$$

$$p(\gamma, \delta) = P(\tau_{\gamma,n} < \infty, \alpha(\tau_{\gamma,n}) = \delta\rho | \alpha(0) = \gamma\rho) \tag{5}$$

If the Markov chain is recurrent then

$$\sum_\delta p(\gamma, \delta) = 1 \tag{6}$$

To get equations for the vector $\overrightarrow{p}$ we need the following definition.

Definition 3.1 *Ordered pair of strings* (α, β) *is a matching pair if* $\alpha = \gamma\rho, |\gamma| = d, \beta = \delta\rho, |\delta| < d$, *for some* ρ, γ, δ. *For such* α *and* β *we put* $p(\alpha, \beta) = p(\gamma, \delta)$.

Definition 3.2 *A sequence of strings* $\Gamma = \alpha_1, \dots, \alpha_n$ *is called a path, if each* (α_i, α_{i+1}) *is a matching pair.*

Let S be a set of strings and $\Gamma(\alpha, S)$ be the set of paths Γ such that:
i) the first element α_1 of Γ is α;
ii) the last element$\alpha_n(\Gamma) = \alpha_n$ of Γ belongs to S and is the unique element of Γ,belonging to S. Let S_n be the set of strings with length less than n

$$S_n = \{\alpha : |\alpha| < n\}$$

and put

$$P_\Gamma = \prod_{i=1}^{n-1} p(\alpha_i, \alpha_{i+1}) \tag{7}$$

for $\Gamma = \alpha_1, \dots, \alpha_n$. It is easy to see that the following system of equations

$$p(\gamma, \delta) = q(\gamma, \delta) + \sum_{|\theta| \geq d} q(\gamma, \theta) \sum_{\Gamma \in \Gamma(\theta, S_d) : \alpha_n(\Gamma) = \delta} P_\Gamma \tag{8}$$

holds. We can write this system (8) as

$$\overrightarrow{p} = F(\overrightarrow{p}) \tag{9}$$

In the following sections we shall use some positive solutions of the fundamental equation (9). At least one such solution of (9) always exists. It is the vector of probabilities $p(\gamma, \delta)$ as defined above. The following lemma shows that other solutions can exist as well.

Lemma 3.1 *For any Markov chain satisfying conditions B, H, ND there exists a positive solution* $\overrightarrow{p^*}$ *of system (8) such that for any string* γ

$$\sum_\delta p^*(\gamma, \delta) = 1 \tag{10}$$

Remark 1. It follows that in transient cases there exist at least two positive solutions of the fundamental equation.

Remark 2. For the case $d = 1$ Lemma 3.1 is trivial, because (8) takes the form (we put $p_i = p(i, \emptyset)$)

$$p_i = q(i, \emptyset) + \sum_j q(i, j)p_j + \sum_{j,k} q(i, jk)p_j p_k \tag{11}$$

and it is easy to see, that $p_i^* \equiv 1$ is a solution.

Proof of Lemma 3.1:
Let

$$\Delta = \{\overrightarrow{p^*} = \{p^*(\gamma, \delta)\} : |\gamma| = d, |\delta| < d, p^*(\gamma, \delta) \geq 0, \forall \gamma \sum_\delta p^*(\gamma, \delta) = 1\}$$

Let us prove that $F : \Delta \to \Delta$, then the statement of Lemma 3.1 follows by the fixed point theorem . If $\overrightarrow{g} = F(\overrightarrow{p})$, $\overrightarrow{p} \in \Delta$, then

$$\sum_{|\delta|<d} g(\gamma, \delta) = \sum_{|\delta|<d} [q(\gamma, \delta) + \sum_{|\theta|\geq d} q(\gamma, \theta) \sum_{\Gamma\in\Gamma(\theta, S_d):\alpha_n(\Gamma)=\delta} P_\Gamma] =$$

$$= \sum_{|\delta|<d} q(\gamma, \delta) + \sum_{|\theta|\geq d} q(\gamma, \theta) \sum_{\Gamma\in\Gamma(\theta, S_d)} P_\Gamma \tag{12}$$

As $\overrightarrow{p} \in \Delta$ we can interprete $p(\gamma, \delta)$ as transition probabilities of some Markov chain to jump from the state $\gamma\rho$ to the state $\delta\rho$. In this case $\sum_{\Gamma(\theta, S_d)} P_\Gamma$ is the probability to reach the set S_d, starting from θ . It is equal to 1. From (10) we get

$$\sum_{|\delta|<d} g(\gamma, \delta) = \sum_{|\delta|<d} q(\gamma, \delta) + \sum_{|\theta|\geq d} q(\gamma, \theta) = 1\blacksquare$$

Minimal solution of the equation $F(p) = p$.

In many places in this section, it will be convenient to be able to compare vectors of R^n element by element. This we may do by means of the natural (or component-wise) partial ordering of R^n: for $p, v \in R^n$ $p < v$ iff $p_i \leq v_i$, $i = 1, \ldots, n$ and $p \neq v$.

Definition 3.3 *Let Λ be a subset of R^n. Then a mapping $G : \Lambda \subset R^n \to R^n$ is isotone on Λ if $G(p) \leq G(v)$ whenever $p \leq v$, $p, v \in \Lambda$.*

Clear that F (see (9)) is isotone on $\{p \geq 0\} = R^n_+$.

In the remainder of this section, we shall use the notation $p^k \uparrow p,\ k \to \infty$, to mean that $p^0 \leq p^1 \leq \ldots \leq p,\ \lim_{k\to\infty} p^k = p$.

The following lemma is a variant of the Kantorovich lemma (see [5])

Lemma 3.2 *Let $G : R^n_+ \to R^n_+$ be such that*

$$G \text{ is isotone and continuous on } R^n_+ \tag{13}$$

$$\text{and for some } u \geq 0, u \in R^n,\ G(u) \leq u. \tag{14}$$

Then the sequence $G^{(k)}(0) = G(G^{(k-1)}(0)),\ k = 0, 1, \ldots, G^{(k)}(0) \uparrow \tilde{p}$ so that $G(\tilde{p}) = \tilde{p}$ and for any other solution $p \geq 0$ of $G(p) = p$

$$\tilde{p} \leq p$$

Remark 3. Note that $F(p)$ satisfies condition (14) if we take u equal to the probabilistic solution.
Proof:
By isotonicity we have

$$0 \leq u,$$

$$0 \leq G(0) \leq G(u) \leq u,$$

$$0 \leq G(0) \leq G^{(2)}(0) \leq u, \quad etc$$

$$0 \leq G(0) \leq \ldots \leq G^{(k)}(0) \leq u.$$

Hence, the monotone sequence $\{G^{(k)}(0)\}$ has the limit $\tilde{p} \leq u$.
Clearly, as G is continuous, then

$$\tilde{p} = G(\tilde{p})$$

If $p \geq 0$ is another solution, we can put $u = p$ and then $\tilde{p} \leq p$. ■

We shall call $\tilde{p}$ the minimal solution of $G(p) = p$.

Now we shall prove that the minimal solution coincides with the probabilistic solution.

Lemma 3.3 *The vector of the probabilities $p = \{p(\gamma, \delta\}$ (defined in (5)) is the minimal solution of $F(p) = p$.*

Proof:

Let $p^k = F^{(k)}(0)$, $p^k = \{p^k(\gamma, \delta)\}$. We can consider p^k as the sum over some trajectories, i.e.

$$p^k(\gamma, \delta) = \sum_{(\alpha_0, \ldots, \alpha_n) \in S_k(\gamma, S_d) : \alpha_n(\Gamma) = \delta} p_{\alpha_0, \alpha_1} \cdot p_{\alpha_1, \alpha_2} \cdot \ldots \cdot p_{\alpha_{N-1}, \alpha_N}$$

It follows that

$$p^k(\gamma, \delta) \uparrow p(\gamma, \delta) \text{ as } k \to \infty$$

■

Remark 4. We can rephrase all this in the following way. Write our system as

$$y_0 = q_0 + F_0(y_0), y_0 = p$$

where q_0 is a constant and $F_0(0) = 0$. The positive solutions of (9) correspond to the solutions $y_0 \geq q_0$. Putting $y_0 = y_1 + q_0$ we get

$$y_1 = q_1 + F_1(y_1), y_1 \geq 0.$$

By induction we get y_n, q_n and F_n. It is clear that $\sum_{n=0}^{\infty} q_n$ is the minimal solution and any solution can be represented as the sum of the minimal solution and the solution of the equation

$$z = F_\infty(z), F_\infty = \lim_{n \to \infty} F_n.$$

Again F_∞ is the polynomial of the same degree and $F_\infty(0) = 0$.

Remark 5. For $d = r = 1$ we have the simple random walk on Z_+ (denote q the probability of (-1)-jump) and the fundamental equation is

$$p = F(p) = q + (1 - q)p^2.$$

In this case we have exactly two positive solutions for ergodic and transient cases, but only one positive solution for the null-recurrent case.

Remark 6. There was a lot of activity concerning close problem in the algebraic situation. Consider the free group G with l generators $a_1, ..., a_l$ and put $a_{-i} = a_i^{-i}$.The papers [7] , [8] , [6] , [9] are concerned with the product of independent random variables with values in G .It means that probabilities $u_i, i = 1, ..., l, -1, ..., -l$ are given for the transition $\alpha \to \alpha i, \alpha \in G$. Note that words with adjacent a_i and a_{-i} are not allowed and it is a particular case of our problem. To see this put

$$q(i, \emptyset) = u_{-i}, q(i, ij) = u_j, q(i, kj) = 0 \quad if \ k \neq i.$$

Ergodic case cannot occur here, mostly transient cases were considered. This activity was started in ([7]), in this paper the similar fundamental equation was derived in this case with

$$p_i = u_{-i} + p_i \sum_{k \neq -i} u_k p_k$$

It appeared that this equation could be explicitely solved and many subsequent papers used this fact. In our case there is no explicit solution.

Remark 7. From lemma 3, chapter 5 in [10] it follows that for the case $d = 1$ there exists only one solution of the fundamental equation satisfying the condition $0 \leq p_i \leq 1$. Unfortunately the proof cannot be generalised to prove the following important statement:the solution $p^* \in \Delta$ is unique. This is an important statement and we hope to come back to this problem in the next paper.

Remark 8. Let $Q = \{q(\gamma, \delta)\}$ be the parameter space (disregarding the jumps near the boundary).We want to prove that our Markov chain cannot be null-recurrent only on a subset of Lebesgue measure zero on the parameter space.

Assume that for the parameters $\{q(\gamma, \delta)\}$ $\mathcal{L}$ is null-recurrent and p^* is a solution of the fundamental equation. Then we can consider the following small perturbation of the parameters q.

Let us fix some sufficiently small $\epsilon(\gamma, \theta) > 0$,for all γ, δ such that $|\gamma| = d, |\theta| \geq d$. We want to find the other $\epsilon(\gamma, \delta)$ such that p^* remains a solution of the fundamental equation for the parameters

$$\tilde{q}(\gamma, \theta) = q(\gamma, \theta) + \epsilon(\gamma, \theta)$$

$$\tilde{q}(\gamma,\delta)+\epsilon(\gamma,\delta)>0$$

$\epsilon(\gamma,\delta)$ can be found from the following equation (define for brevity $\Gamma(\theta,\delta)=\Gamma(\theta,S_d):\alpha_n(\Gamma)=\delta$)

$$p^*(\gamma,\delta)=q(\gamma,\delta)+\sum_{|\theta|\geq d}q(\gamma,\theta)\sum_{\Gamma\in\Gamma(\theta,\delta)}P_\Gamma^*=$$

$$=q(\gamma,\delta)+\sum_{|\theta|\geq d}(q(\gamma,\theta)+\epsilon(\gamma,\theta)-\epsilon(\gamma,\theta))\sum_{\Gamma\in\Gamma(\theta,\delta)}P_\Gamma^*=$$

$$=q(\gamma,\delta)-\sum_{|\theta|\geq d}\epsilon(\gamma,\theta)\sum_{\Gamma\in\Gamma(\theta,\delta)}P_\Gamma^*+\sum_{|\theta|\geq d}(q(\gamma,\theta)+\epsilon(\gamma,\theta))\sum_{\Gamma\in\Gamma(\theta,\delta)}P_\Gamma^*=$$

so we have

$$\epsilon(\gamma,\delta)=-\sum_{|\theta|\geq d}\epsilon(\gamma,\theta)\sum_{\Gamma\in\Gamma(\theta,\delta)}P_\Gamma^*$$

It is clear that for the matrix $\tilde{A}(p^*)$ (see in the next section equation (22)) defined by $\tilde{q}$ we have $\tilde{A}(p^*)>A(p^*)$. Then it follows that the Markov chain, defined by $\tilde{q}$, is transient.

Similarly we could fix $\epsilon(\gamma,\theta)>0$, for all γ,δ such that $|\gamma|=d,|\theta|\leq d$ and get ergodicity.

4 Criteria via Lyapounov functions: case $d>1$.

Similar to the case $d=1$, we get criteria of ergodicity, null recurrence and transience, using the maximal eigenvalue of a positive matrix A, depending howevever (this is the essential difference with the case $d=1$) on the solution $p^*(\gamma,\delta)$,defined in Lemma 3.1.

Assuming the Markov chain to be ergodic, we get now the equations similar to the equation (2) of the section 2. Introduce

$$E_\gamma=E[\tau_{\gamma,n}|\alpha(0)=\gamma\rho],\quad |\gamma|=d,\ |\gamma\rho|=n, \tag{15}$$

It was mentioned in the section 3 that E_γ does not depend on ρ.

For $\alpha=\gamma\rho,|\gamma|=d$ we set $E_\alpha=E_\gamma$ and

$$T(\alpha) \stackrel{def}{=} E\tau(\alpha, d) \tag{16}$$

where $\tau(\alpha, d)$ is the time to reach S_d, starting from α. We put $T(\alpha) = 0$ for $\alpha : |\alpha| < d$. It is easy to see, that

$$T(\gamma\rho) = E_\gamma + \sum_\delta p(\gamma, \delta)T(\delta\rho) \tag{17}$$

or

$$T(\gamma\rho) = \sum_\delta p(\gamma, \delta)(E_\gamma + T(\delta\rho)) \tag{18}$$

Then from (18)

$$T(\alpha) = \sum_{\Gamma\in\Gamma(\alpha, S_d)} P_\Gamma E_\Gamma \tag{19}$$

where

$$E_\Gamma = \sum_{i=1}^{N} E_{\alpha_i} \tag{20}$$

for $\Gamma = \alpha_1 \ldots \alpha_N$. Now we can write for E_γ

$$E_\gamma = 1 + \sum_{|\theta|\geq d} q(\gamma, \theta)T(\theta) = 1 + \sum_{|\theta|\geq d} q(\gamma, \theta) \sum_{\Gamma\in\Gamma(\theta, S_d)} P_\Gamma E_\Gamma \tag{21}$$

or in matrix terms

$$\overrightarrow{E} = \overrightarrow{1} + A(\overrightarrow{p})\overrightarrow{E} \tag{22}$$

In case $d > 1$ the matrix A depends on the vector of probabilities $\overrightarrow{p}$ from (5).

Now, let us go over from ergodic to the general case. Now we can formulate the main criteria.

Theorem 4.1 *Let $\overrightarrow{p}^*$ be any positive solution of (9), existing by Lemma 3.1, and satisfying (10). λ be the maximal eigenvalue of the matrix $A(\overrightarrow{p}^*)$. The process $\alpha(t)$ is ergodic, null recurrent, transient iff $\lambda < 1, \lambda = 1, \lambda > 1$ accordingly.*

Proof.
We use Lyapounov functions. Similar to the case $d = 1$, we try to find a Lyapounov function f as the mean time to reach the set S_d (see (18)). Define the function f by the following recurrent equations

$$f(\gamma\rho) = \sum_{\delta} p^*(\gamma,\delta)(f(\delta\rho) + e_\gamma) \tag{23}$$

$$f(\delta) = 0 \quad for\ \delta : |\delta| < d,$$

where $\{e_\gamma\}_{|\gamma|=d}$ are the components of the eigenvector corresponding to the maximal eigenvalue λ , i.e.

$$\lambda e_\gamma = \sum_{|\theta|\geq d} q(\gamma,\theta) \sum_{\Gamma\in\Gamma(\theta,S_d)} P^*_\Gamma \tag{24}$$

and by the Perron-Frobenious theorem for all γ

$$e_\gamma > 0 \tag{25}$$

Note, that from recurrent equations (23) it follows, that
for all $\theta, \rho : |\theta| \geq d$

$$f(\theta\rho) = \sum_{\Gamma\in\Gamma(\theta,S_d)} P^*_\Gamma(e_\Gamma + f(\delta\rho)) \tag{26}$$

Here e_Γ is defined by $\overrightarrow{e}$ with the following formulas (similar to the definition of E_Γ, see (20):

$$e_\Gamma = \sum_{i=1}^{N} e_{\alpha_i}\ for\ \Gamma = \alpha_1 \ldots \alpha_N$$

Lemma 4.1 *Let f is defined by (23), then the folowing statements hold*
(i) if $\lambda > 1$, then there exists $\epsilon > 0$ such that

$$E[f(\alpha(t+1)) - f(\alpha(t))|\alpha(t) = \alpha] > \epsilon$$

for all $\alpha, |\alpha| \geq d$ and for all t;
(ii) if $\lambda = 1$, then

$$E[f(\alpha(t+1)) - f(\alpha(t))|\alpha(t) = \alpha] = 0$$

for all $\alpha, |\alpha| \geq d$ *for and all* t*:*

(iii) if $\lambda < 1$, *then there exists* $\epsilon > 0$ *such that*

$$E[f(\alpha(t+1)) - f(\alpha(t))|\alpha(t) = \alpha] < -\epsilon$$

for all $\alpha, |\alpha| \geq d$ *and for all* t.

Proof:
It is a matter of computation

$$E[f(\alpha(t+1)) - f(\alpha(t))|\alpha(t) = \gamma\rho] =$$

$$= \sum_{|\delta|<d} q(\gamma,\delta)f(\delta\rho) + \sum_{|\theta|\geq d} q(\gamma,\theta)f(\theta\rho) - f(\gamma\rho) \overset{(26)}{=}$$

$$= \sum_{|\delta|<d} q(\gamma,\delta)f(\delta\rho) + \sum_{|\theta|\geq d} q(\gamma,\theta) \sum_{|\delta|<d} \sum_{\Gamma\in\Gamma(\theta\rho,\{\delta\rho\})} P^*_\Gamma[e_\Gamma + f(\delta\rho)] - f(\gamma\rho) =$$

$$= \sum_{|\delta|<d} f(\delta\rho)[q(\gamma,\delta) + \sum_{|\theta|\geq d} \sum_{\Gamma\in\Gamma(\theta_\rho,\{\delta\rho\})} P^*_\Gamma] + \sum_{|\theta|\geq d} q(\gamma,\theta) \sum_{\Gamma\in\Gamma(\theta,S_d)} P^*_\Gamma e_\Gamma - f(\gamma\rho) =$$

using the fact, that $\overrightarrow{p^*}$ is a solution of (9), we can continue in the following way

$$= \sum_{|\delta|<d} f(\delta\rho)p^*(\gamma,\delta) + \sum_{|\theta|} q(\gamma,\theta) \sum_{\Gamma\in\Gamma(\theta,S_d)} P^*_\Gamma e_\Gamma - f(\gamma\rho) \overset{(24)}{=}$$

$$= \sum_{|\delta|<d} f(\delta\rho)p^*(\gamma,\delta) + \lambda e_\gamma - f(\gamma\rho) \overset{(23)}{=}$$

$$= \lambda e_\gamma - e_\gamma = (\lambda - 1)e_\gamma \qquad \blacksquare$$

To use the criteria of the appendix it is necessary that the function f satisfy the following properties.

Lemma 4.2

(i) $f(\alpha) \to \infty$ *if* $|\alpha| \to \infty$

(ii) There exist $L > 0$ *such that the inequality*

$$|f(\alpha) - f(\beta)| > L \text{ implies } p_{\alpha,\beta} = 0 \tag{27}$$

Proof:

The first property easily follows from (23) and (10).The second is more difficult. We will prove that there exists $K > 0$ such that for all $\alpha, \delta_1, \delta_2 :$ $|\delta_1| < d, |\delta_2| < d$

$$|f(\delta_1\alpha) - f(\delta_2\alpha)| < K. \tag{28}$$

Then (27) follows from Condition B (boundedness of jumps), (23) and (28).

Let

$$M_\alpha = \max_{|\delta|<d} f(\delta\alpha), \quad m_\alpha = \min_{|\delta|<d} f(\delta\alpha), \quad \Delta_\alpha = M_\alpha - m_\alpha.$$

Our goal is to prove the following inequality.

There exist $a > 0, 1 > b > 0$ such that for all $\alpha = \gamma\rho, |\gamma| = d$

$$\Delta_\alpha \leq a + b\Delta_\rho. \tag{29}$$

So we will get

$$\Delta_\alpha \leq a + b(a + b(\ldots)) = a(1 + b + b^2 + \ldots) = K < \infty$$

and we have (28) . Let us prove (29). From (26) we have

$$f(\delta\alpha) = f(\delta\gamma\rho) = \sum_{\Gamma \in \Gamma(\delta\alpha, S_{|\alpha|})} P^*_\Gamma(e_\Gamma + f(\ \kappa_\Gamma)) \tag{30}$$

where κ_Γ is the last element of Γ.

From the condition $\Gamma \in \Gamma(\delta\alpha, S_{|\alpha|})$ it follows that $\kappa_\Gamma = \tilde{\delta}\rho$ for some $\tilde{\delta} : |\tilde{\delta}| < d$. So we can rewrite (30) in the form

$$f(\delta\alpha) = a(\delta\gamma) + \sum_{|\tilde{\delta}|<d} p_\gamma(\delta, \tilde{\delta}) f(\tilde{\delta}\rho)$$

where

$$a(\delta\gamma) = \sum_{\Gamma \in \Gamma(\delta\alpha, S_{|\alpha|})} P^*_\Gamma e_\Gamma$$

$$p_\gamma(\delta, \tilde{\delta}) = \sum_{\Gamma \in \Gamma(\delta\alpha, \{\tilde{\delta}\rho\})} P^*_\Gamma.$$

Let us remark that

$$2 \max_{\delta\gamma} a(\delta\gamma) = a < \infty$$

$$\sum_{|\tilde{\delta}|<d} p_\gamma(\delta, \tilde{\delta}) = 1$$

This fact can be proved similar to Lemma 3.1. We have

$$\min_{\delta,\gamma,\tilde{\delta}} p_\gamma(\delta, \tilde{\delta}) = \epsilon > 0$$

It follows from Condition ND (nondegeneracy).
So we have the following

$$|f(\delta_1\gamma\rho) - f(\delta_2\gamma\rho)| \leq a + |\sum_{|\tilde{\delta}|<d} p_\gamma(\delta_1, \tilde{\delta}) f(\tilde{\delta}\rho) - \sum_{|\tilde{\delta}|<d} p_\gamma(\delta_2, \tilde{\delta}) f(\tilde{\delta}\rho)| =$$

$$= a + |\sum_{|\tilde{\delta}|<d} [p_\gamma(\delta_1, \tilde{\delta}) - \epsilon] f(\tilde{\delta}\rho) - \sum_{|\tilde{\delta}|<d} [p_\gamma(\delta_2, \tilde{\delta}) - \epsilon] f(\tilde{\delta}\rho)| =$$

$$= a + |\sum_{|\tilde{\delta}|<d} [p_\gamma(\delta_1, \tilde{\delta}) - \epsilon]\{f(\tilde{\delta}\rho) - m_\rho\} - \sum_{|\tilde{\delta}|<d} [p_\gamma(\delta_2, \tilde{\delta}) - \epsilon]\{f(\tilde{\delta}\rho) - m_\rho\}| \leq$$

$$\leq a + \max_{\delta_1,\delta_2} \sum_{|\tilde{\delta}|<d} [p_\gamma(\delta_i, \tilde{\delta}) - \epsilon]\{f(\tilde{\delta}\rho) - m_\rho\} \leq$$

$$\leq a + [1 - \sum_{|\tilde{\delta}|<d} \epsilon]\{M_\rho - m_\rho\}.$$

Put $b = 1 - \sum_{|\tilde{\delta}|<d} \epsilon$ we get (29); ■
Now the statement of Theorem 4.1 follows from Lemmas 4.1,4.2 and the theorems of the Appendix.

5 Classification via the minimal solution

If we can find the minimal solution of the fundamental equation then we have the following classification criteria.

Theorem 5.1 *(i) The Markov chain is recurrent iff the minimal solution* $\tilde{p}(\gamma, \delta)$ *satisfies the property*

$$\sum_{\delta} \tilde{p}(\gamma, \delta) = 1$$

for all γ.

(ii) Assume that our Markov chain is recurrent. Then it is null recurrent iff the spectral radius (maximal positive eigenvalue) of the matrix $F' = \left(\frac{\partial F_i}{\partial p_j}(\tilde{p})\right)$ *is equal to* 1.

Note that (i) is trivial.The proof of (ii) is contained in the following lemmas. To make these criteria practically handable we also provide exponentially fast iterative algorithms to find the minimal solution.

Lemma 5.1 *Let* $G : R^n_+ \to R^n_+$*, satisfy (13), (14),*

$$G(p) > 0 \ , \ p \geq 0, \tag{31}$$

and G *has the derivative* $D = \left(\frac{\partial G_i}{\partial p_j}(\tilde{p})\right)$ *such that*

$$D > 0 \ , i.e. \ for \ all \ i,j \ \ \frac{\partial G_i}{\partial p_j}(\tilde{p}) > 0. \tag{32}$$

Then the spectral radius of D*,* $R(D) \leq 1$.

Proof:

Suppose that $R(D) > 1$, then for some $v \in R^n, v > 0$, we have $Dv > v$. It follows that there exists $\epsilon > 0$ such that $\tilde{p} > u = \tilde{p} - \epsilon v > 0$ and

$$G(u) = \tilde{p} - \epsilon Dv + o(\epsilon) = u + \epsilon(v - Dv) + o(\epsilon) < u$$

we get

$$0 < G(u) < u < \tilde{p}.$$

On the other hand $\tilde{p} \leq u$ (see the proof of Lemma 3.2) represents a contradiction. Thus $R(D) \leq 1$. ■

Lemma 5.2 *Let* G *satisfy (13), (14), (31), (32) and*

$$for \ all \ i,j,s \ \ \frac{\partial^2 G_i}{\partial p_j \partial p_s}(p) > 0 \ \ if \ p \geq 0, \ p \neq 0. \tag{33}$$

Then

a) $R(G'(p)) < 1$ *if* $p \leq \tilde{p}, p \neq \tilde{p}$,
b) $R(D) = 1$ *if there exists the unique positive solution of* $G(p) = p$ *(then this solution is* $\tilde{p}$ *).*

Proof:

a) (33) and (13) imply that

$$0 \leq G'(p) < G'(\tilde{p}) = D.$$

Hence,

$$R(G'(p)) < R(D) \leq 1$$

b)$\Rightarrow$

Suppose $p \geq 0, G(p) = p$ is another solution, then $p \geq \tilde{p}, p \neq \tilde{p}$ (see Lemma 3.2). (33) ensures that G is strictly order-convex (see theorem ?? in [5]) i.e.

$$G(z) \geq G(y) + G'(y)(z-y) \quad if \ z - y \geq 0, \ or \ z - y \leq 0$$

and the inequality is strict if $z \neq y$.

Therefore

$$p = G(p) > G(\tilde{p}) + D(p - \tilde{p}) \ or$$

$$p - \tilde{p} > D(p - \tilde{p})$$

Hence $R(D) < 1$.
■

Lemma 5.3 *(a) If the Markov chain is ergodic or transient then* $R(F') < 1$.
(b) If it is null recurrent then $R(F') = 1$.
(c) In the recurrent case the positive solution of the fundamental equation, satisfying $\sum_\delta p(\gamma, \delta) = 1$ *is unique.*

Proof.

Cases (a) and (b) and null-recurrent case (c) were proved above. In ergodic case (c) the minimal solution $\tilde{p}$ satisfies this condition. For any other solution $p, p > \tilde{p}$.So $\sum_\delta p(\gamma, \delta) > 1$.

Lemma 5.4 *Let G satisfy (13), (14), (31), (32), (33) then*
a) if $R(D) < 1$ then there exist $c_1, c_2 > 0$ such that

$$\|\tilde{p} - G^{(k)}(0)\| < c_1 e^{-c_2 k}$$

b) if $R(D) = 1$ then there exist $c_1, c_2 > 0$ such that

$$\frac{c_1}{k} < \|\tilde{p} - G^{(k)}(0)\| < \frac{c_2}{k} \tag{34}$$

Proof:

a) It follows from Theorem 5.2.

b) We first prove the case $n = 1$. From the conditions (13)- (33) we have

$$G(p) = \tilde{p} + (p - \tilde{p}) + b(p - \tilde{p})^2 + o(|p - \tilde{p}|^2),\ b > 0$$

Denote $G^{(k)}(0)$ by p^k we obtain

$$p^k = p^{k-1} + b(p^{k-1} - \tilde{p})^2 + o(|p^{k-1} - \tilde{p}|^2)$$

$$\frac{1}{\tilde{p} - p^k} - \frac{1}{\tilde{p} - p^{k-1}} = \frac{b(p^{k-1} - \tilde{p})^2 + o(|p^{k-1} - \tilde{p}|^2)}{(p^{k-1} - \tilde{p})^2 + o(|p^{k-1} - \tilde{p}|^2)} = b + O(|p^{k-1} - \tilde{p}|).$$

Consequently

$$\frac{1}{\tilde{p} - p^k} = b \cdot k + o(k)$$

and (34) follows directly.

Now, let $n > 1$. As in the case $n = 1$

$$G(p) = \tilde{p} + D(p-\tilde{p}) + \frac{1}{2}B(p-\tilde{p})\cdot(p-\tilde{p}) + o(\|p-\tilde{p}\|^2) \qquad (35)$$

where
$B(y) = (b_{i,j}(y)),\ b_{i,j}(y) = \sum_s \frac{\partial^2 G_i}{\partial p_j \partial p_s}(\tilde{p}) y_s$.

Let π be left eigenvector of D with eigenvalue 1, then

$$\pi > 0 \qquad (36)$$

and

$$\pi p^k = \pi p^{k-1} + \pi B(p^{k-1}-\tilde{p})\cdot(p^{k-1}-\tilde{p}) + o(\|p-\tilde{p}\|^2).$$

From conditions (33) and (36) it follows that there exist $b_1, b_2 > 0$ such that

$$b_1\left(\pi(p^{k-1}-\tilde{p})\right)^2 \leq \pi B(p^{k-1}-\tilde{p})\cdot(p^{k-1}-\tilde{p}) \leq b_2\left(\pi(p^{k-1}-\tilde{p})\right)^2$$

therefore (34) follows in a manner analogous to the case $n = 1$. ■

Theorem 5.2 *The chain $\mathcal{L}$ is*

a) ergodic or transient iff there exist $c_1, c_2 > 0$ such that

$$\|\tilde{p} - F^{(k)}(0)\| < c_1 e^{-c_2 k} \qquad (37)$$

b) null-recurrent iff there exist $c_1, c_2 > 0$ such that

$$\frac{c_1}{k} < \|\tilde{p} - F^{(k)}(0)\| < \frac{c_2}{k} \qquad (38)$$

Proof:

First of all we will prove that

$$R(F'(\tilde{p})) = R(A(\tilde{p}))\ if\ p > 0\ and\ for\ all\ \gamma, |\gamma| = d\ \sum_{\delta} p(\gamma,\delta) = 1 \qquad (39)$$

Let $v = \{v(\gamma,\delta)\}$ be an eigenvector of $F'(p)$ corresponding maximal eigenvalue $0 < \lambda = R(F'(p))$. In coordinate form we have

$$(F'(p)v)(\gamma,\delta) = \sum_{\theta} q(\gamma,\theta) \sum_{\Gamma\in\Gamma(\theta,\delta)} P_\Gamma v^*_\Gamma = \lambda v(\gamma,\delta)$$

where

$$v^*_\Gamma = \sum_{i=0}^{N} \frac{v(\gamma_i,\delta_i)}{p(\gamma_i,\delta_i)} \quad if\ \Gamma = ((\gamma_0,\delta_0),\ldots,(\gamma_N,\delta_N)).$$

If $\tilde{v}(\gamma) = \sum_\delta v(\gamma,\delta)$ from the property $\sum_\delta p(\gamma,\delta) = 1$ we obtain

$$\sum_{\delta} (F'(p)v)(\gamma,\delta) = \sum_{\theta} q(\gamma,\theta) \sum_{\Gamma\in\Gamma(\theta,S_d} P_\Gamma \tilde{v}_\Gamma = \lambda\tilde{v}(\gamma) \tag{40}$$

where

$$\tilde{v}_\Gamma = \sum_{i=0}^{N} \tilde{v}(\gamma_i) \quad if\ \Gamma = ((\gamma_0,\delta_0),\ldots,(\gamma_N,\delta_N)).$$

In matrix form (40) is

$$A(p)\tilde{v} = \lambda\tilde{v},$$

and we have (39).

Let us assume $p = \tilde{p}$. It follows from the Theorem 2 that if $\mathcal{L}$ is null-recurent $R(F'(\tilde{p})) = R(A(\tilde{p})) = 1$ and if $\mathcal{L}$ is ergodic $R(F'(\tilde{p})) = R(A(\tilde{p})) < 1$.

Let us consider the transient case.
Lemma 3.1 and Lemma3.2 give two different solutions of equation $F(p) = p$. Those are p^* and $\tilde{p}$. If (33) holds for $F(p)$ then from (5.2) we would have $R(F'(\tilde{p})) < 1$. In fact (33) does not hold for F.But it holds for

$$G(p) = F^{(2)}(p).$$

p^* and $\tilde{p}$ are solutions of $G(p) = p$ too, therefore

$$R(G'\tilde{p})) < 1 \tag{41}$$

but $G'(\tilde{p}) = F'(\tilde{p}) \cdot F'(\tilde{p})$. Hence

$$R(F'(\tilde{p})) < 1.$$

Therefore in the ergodic and transient cases

$$R(F'p)) \leq R(F'(\tilde{p})) < 1 \quad for\ 0 \leq p \leq \tilde{p}.$$

a) follows directly.
Let us prove b).
Since because of $F^{(k)}(0)$ is monotone sequence, it is enough to prove (38) for $F^{(2k)}(0) = G^{(k)}(0)$. For the least (38) follows from (5.4). ■

Theorem 5.3 *Let $G : R^n \to R^n$ and (13), (14), (31), (32), (33) hold, then the sequence $p^k, k = 0, 1, \dots$ is well defined by*

$$p^0 = 0, \ p^{k+1} = H(p^k)$$

where for $0 \leq p < \tilde{p}$ $H(p) = (E - G'(p))^{-1}(G(p) - G'(p)p)$

$$p^k \uparrow \tilde{p}, \ k \to \infty, p^k < \tilde{p}$$

and the rate of convergence is exponential i.e. there exist $c_1, c_2 > 0$ such that

$$\|\tilde{p} - p^k\| < c_1 e^{-kc_2}$$

Proof:

If $R(D) < 1$ (see Lemma 5.2)then Theorem 5.3 follows from standart results (see [5]). Therefore we assume $R(D) = 1$.

Lemma 5.5 *Let $0 \leq p < \tilde{p}$, $G(p) > p$, then*

$$y = H(p) < \tilde{p}$$

$$y > G(p) \tag{42}$$

$$y < G(y) \tag{43}$$

Proof:

First of all $H(p)$ is defined, since because of $R(G'(p)) < R(D) = 1$ (see Lemma 5.2) $E - G'(p)$ is invertible.

It is clear that $y > p$ because

$$y = (E - G'(p))^{-1}(G(p) - G'(p)p) = p + (E - G'(p))^{-1}(G(p) - G'(p)).$$

y is also the solution of the equation $S(y) = y$ where

$$S(y) = G(p) + G'(p)(y - p) > G(p)$$

and we have (42).

G is strictly order-convex then

$$G(z) > S(z) \; if \; z \geq p, \; z \neq p.$$

Hence $\tilde{p} \geq y$.

For G is isotone

$$\tilde{p} \geq G(y) \tag{44}$$

and again from order-convexity it follows

$$G(y) > S(y) = y, \tag{45}$$

we obtain from (44) and (45)

$$y < G(y) \leq \tilde{p} \; \blacksquare$$

From Lemma 5.5 it follows that sequence $\{p^k\}$ is well defined and $p^k \uparrow \tilde{p}$.

Lemma 5.6 *There are an open neighbourhood $U \subset R^n$ of 0 and $0 < c < 1$ such*

that for all $p < 0$, $p \in U$

$$H(\tilde{p} + p) = \tilde{p} + C(p) \cdot p$$

where $C(p)$ is a positive matrix continuous on $U \cap \{p \leq 0\}$ and $R(C(p)) <$ c.

Proof:

From (35) we have

$$G(\tilde{p} + p) = \tilde{p} + (D + \frac{1}{2}B(p) + o(\|p\|)) \cdot p \tag{46}$$

(33) yields that

$$\|p\| = O(\|B(p)\|).$$

So that (46) becomes

$$G(\tilde{p} + p) = \tilde{p} + (D + \frac{1}{2}B(p) + o(\|B(p)\|)) \cdot p.$$

For $G'(p)$ we get

$$G'(\tilde{p} + p) = D + B(p) + o(\|B(p)\|),$$

and for H

$$H(\tilde{p} + p) - \tilde{p} = (E - G'(\tilde{p} + p))^{-1} (G(\tilde{p} + p) - G'(\tilde{p} + p)(\tilde{p} + p)) - \tilde{p} =$$

$$= (E - G'(\tilde{p} + p))^{-1} (G(\tilde{p} + p) - \tilde{p} - G'(\tilde{p} + p)p) =$$

$$= (E - D - B(p) + o(\|B(p)\|))^{-1} \left(-\frac{1}{2}B(p) + o(\|B(p)\|\right) \cdot p = C(p)p$$

$C(p)$ is continuous on $p < 0$ and positive on some open neighbourhood of 0 for $-B(p) > 0$, $if\ p < 0$.

Let $\tilde{U}$ be an open neighbourhood of 0 such that for $p < 0,\ p \in \tilde{U}$

$$D + B(p) > 0,$$

then

$$\tilde{C}(p) = -\left(E - D - B(p)\right)^{-1} B(p) > 0 \; if \; p < 0, \; p \in \tilde{U}.$$

We will prove that $R(C(p)) = 1$ for all $p < 0, \; p \in \tilde{U}$.

Really, let v be eighenvector of D

$$Dv = v, \; v > 0$$

then

$$\tilde{C}(p)v = \left(E - D - B(p)\right)^{-1} \left(D - D - B(p)\right) v =$$

$$= \left(E - D - B(p)\right)^{-1} \left(E - D - B(p)\right) v = v.$$

So that v is the eigenvector for $\tilde{C}(p)$.

We get also the estimate

$$0 \leq \tilde{c}_{ij} \leq \frac{\max_s v_s}{\min_s v_s}$$

Therefore $\tilde{C}(p)$ is uniformly bounded on $p < 0, \; p \in \tilde{U}$.

For $C(p)$ we have

$$C(p) = \frac{1}{2}\tilde{C}(p) + O(\|B(p)\|).$$

Hence $C(p)$ is uniformly bounded, continuous, positive and $R(C(p)) < \frac{1}{3}$ on some open neighbourhood $0 \in U$ for $p < 0$.It follows we can continue $C(p)$ on $U \cap \{p \leq 0\}$. ■

The end of the proof:

Let $C = C(0)$. From Lemma 5.6 we have $R(C) < 1$. Let $\epsilon > 0$ be such that $R(C) + \epsilon = c < 1$. There exists a norm (see [5])$\|\cdot\|$ such that

$$\|C\| < R(C) + \frac{\epsilon}{2}$$

Let $0 \in U$ be an open neighborhood such that

$$\|C - C(p)\| < \frac{\epsilon}{2} \quad p \in U \cap \{p \leq 0\},$$

then

$$\|C(p)\| \leq \|C\| + \|C - C(p)\| \leq c < 1$$

As we know $p^k \uparrow \tilde{p}$ then there exists k_0 such that for all $k > k_0$ $p^- \tilde{p} \in U$. It follows that

$$\|p^{k+1} - \tilde{p}\| = \|H(p^k) - \tilde{p}\| = \|C(p^k - \tilde{p}) \cdot (p^k - \tilde{p})\| \leq c\|p^k - \tilde{p}\|$$

Theorem is proved. ■

Corollary 5.1 *Let $G = F^{(2)}$ then the sequence H^{2k} converges to vector of probabilities $p = \{p(\gamma, \delta)\}$ (see 5) exponentially fast.*

6 Appendix

Here we remind the theorems which were used in sections 2 and 3. We give them in a simpler form than in [3] , [4]. Let $\mathcal{L}$ be a discrete time homogeneous Markov chain with the countable state space $\mathcal{A} = \{\alpha_i, i = 1, 2, \ldots\}$. The one-step transition probabilities will be denoted by $p_{\alpha,\beta}$. $\mathcal{L}$ is supposed to be irreducible and aperiodic. The state of the chain at time t is denoted by ξ_t.

Theorem 6.1 *The Markov chain $\mathcal{L}$ is recurrent iff, there exist a positive function $f(\alpha), \alpha \in \mathcal{A}$ and a finite set A, such that*

(i) $E[f(\xi_{t+1}) - f(\xi_t)|\xi_t = \alpha_i] \leq 0, \forall \alpha_i \notin A$

(ii) $f(\alpha_i) \to \infty$ *if* $i \to \infty$

Theorem 6.2 *The Markov chain $\mathcal{L}$ is ergodic iff there exist a positive function $f(\alpha), \alpha \in \mathcal{A}$, a number $\epsilon > 0$ and a finite set A, such that*

(i) $E[f(\xi_{t+1}) - f(\xi_t)|\xi_t = \alpha_i] \leq -\epsilon, \forall \alpha_i \notin A$

(ii) $E[f(\xi_{t+1})|\xi_t = \alpha_i] < \infty, \forall \alpha_i \in A$

Theorem 6.3 *For the Markov chain $\mathcal{L}$ to be non ergodic, it is sufficient that there exist a function $f(\alpha), \alpha \in \mathcal{A}$ and constants C, L, such that*

(i) $E[f(\xi_{t+1}) - f(\xi_t)|\xi_t = \alpha_i] \geq 0, \forall \alpha_i \in \{f(\alpha) > C\}$, *where the sets* $\{f(\alpha) > C\}$ *and* $\{f(\alpha) \leq C\}$ *are not empty*

(ii) $E[f(\xi_{t+1}) - f(\xi_t)|\xi_t = \alpha_i] \leq L, \forall \alpha_i \in A$

Theorem 6.4 *For the Markov chain $\mathcal{L}$ to be transient, it is sufficient that there exist a positive function $f(\alpha), \alpha \in \mathcal{A}$ and constants $\epsilon, C, L > 0$, , such that*

(i) $E[f(\xi_{t+1}) - f(\xi_t)|\xi_t = \alpha_i] \geq \epsilon, \forall \alpha_i \in \{f(\alpha) > C\}$, *where the set* $\{f(\alpha) > C\}$ *is not empty;*

(ii) the inequality $|f(\alpha) - f(\beta)| > L$ *implies* $p_{\alpha,\beta} = 0$

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