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ON THE CONVERGENCE OF A D.K.T. METHOD VALID FOR SHELLS OF ARBITRARY SHAPE[*]

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Abstract

In a recent paper by the same Authors, we have thoroughly described how to extend to the case of general shells the well known D.K.T. methods (i.e. Discrete Kirchhoff Triangle) which are now classically used to solve plate problems. In this paper we have also detailed how to realize the implementation and we have reported some numerical results obtained over classical benchmarks.

The aim of this paper is to prove the convergence of a closely related method and to obtain corresponding error estimates.

SUR LA CONVERGENCE D'UNE METHODE D.K.T. VALABLE POUR DES COQUES DE FORME ARBITRAIRE

Résumé

Dans un article récent des mêmes auteurs, nous avons soigneusement décrit comment étendre au cas de coques générales les méthodes D.K.T. (i.e. Discrete Kirchhoff Triangle) qui sont maintenant très utilisées pour approcher les problèmes de plaques. Dans cet article, nous avons également détaillé comment réaliser l'implémentation et nous avons donné des résultats numériques obtenus sur des exemples tests classiques.

L'objet de ce travail est de démontrer la convergence de cette méthode et d'obtenir les estimations d'erreur correspondantes.

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1 INTRODUCTION

There exists a large number of approximation methods for plate and shell problems. This great diversity can be explained :

- by the number of continuous models which have to be adapted to the problem into consideration : geometrical characteristics (thickness, curvature), small or large displacements, small or large strains... ;
- for a given model, by the number of possible variational formulations which lead to conforming or nonconforming displacement, mixed, hybrid, equilibrium... finite element methods ;
- for a given model and for a given variational formulation, by the variety of finite elements : two or three dimensional, triangular, quadrilateral, use of low or high degrees of polynomials, exact or approximate integrations...

Thus, there are many different parameters to take into account in order to choose a finite element method and this is not so easy ! In order to make easier such approximations, the computational mechanicians have tried to propose a general and effective method valid for the most part of the problems.

In this way, the methods based upon the so-called Mindlin/Reissner plate or shell theories [1,2] seemed very attractive since they only require C^0-continuity for shape functions and since they can take into account the effect of shear deformations. These models are well adapted to the study of moderately thick plates or shells. Unfortunately a direct discretization of such Mindlin/Reissner models by conforming finite element methods leads to numerical instabilities as soon as the thickness becomes too small (locking phenomenon...).

The introduction of D.K.T. (Discrete Kirchhoff Triangles) methods has improved such approximations ; they have been satisfactorily applied to a large class of thin to moderately thick, linear or nonlinear plate problems. The basic ideas of these D.K.T. methods are mainly :

i) to neglect the shear strain energy in the computation of the strain energy ; this part is generally small when compared to the membrane or bending strain energy, specially for sufficiently thin plates or shells ;

ii) to introduce on the discrete model some constraints like :

- ⋆ Kirchhoff-Love conditions at mesh nodes ;
- ⋆ "tangential" Kirchhoff-Love conditions at some nodes located along the sides of the finite elements ;
- ⋆ and, if necessary, some other constraints in order to insure a good definition of the discrete problem.

The basic ideas of these D.K.T. methods were introduced in [3,4,5]. But these methods became really popular more recently, particularly after the works of [6,7,8].

Parallely, [9,10] have performed the numerical analysis of these D.K.T. methods for plate problems. They have proved the convergence of these methods to the solution of the Kirchhoff plate model and they have obtained the corresponding error estimates.

Thus, after these successes obtained in the approximation studies of plate problems, it seemed to be natural to extend these methods to the case of thin to moderately thick shell problems with arbitrary shape. In [11,12], we have thoroughly described how to make such an extension and we have illustrated the efficiency of this method by giving its numerical results on a set of classical benchmarks. In this way, like in [13,14,15,16,17], we have assumed that the middle surface of the shell is defined as the image of a plane reference domain by a regular mapping. Such a construction is interesting from practical but also theoretical point of view, since we have just to analyze the D.K.T. approximation without any additional geometrical approximation. Then, we have detailed how to realize the implementation and we have reported some numerical results obtained over classical benchmarks. Finally, the second Author [18] has considered a light modification of Kirchhoff-Love restrictions in order to simplify the mathematical study of the convergence.

The aim of this paper is now to prove the *convergence* of this last method [18] and to obtain corresponding *error estimates*. For clarity, we recall in paragraph 2 the equations of thin shells into consideration which include, or not, the effect of shear deformation. Next, in paragraph 3, we define a D.K.T. finite element method valid for general thin shells. Finally, in paragraph 4, we prove that such a D.K.T. finite element method is nothing but a nonconforming finite element approximation of the Koiter's modelization and then, we prove the convergence and we obtain the corresponding error estimates.

2 THE EQUATIONS OF THIN SHELLS INTO CONSIDERATION

The geometrical description of the shell before deformation is given for example in [17]. In particular, the middle surface of the shell is defined as the image of a domain Ω by a mapping $\vec{\phi}$ which will be assumed sufficiently regular. Subsequently, we will use the notations of this referred book. For convenience, we briefly record hereunder the two different models into consideration.

2.1 Naghdi's model including shear effects

A general presentation of this model is given for instance in [16] or [19]. Here we take a more simple statement from [8] valid for plane stress assumptions, for shear strains which are constant through the thickness and for sufficiently thin shells.

The displacement field $\vec{U}$ of any point M of the 3D shell can be approximated as follows :

$$\vec{U} = \vec{u} + \zeta \beta_\alpha \vec{a}^\alpha,$$

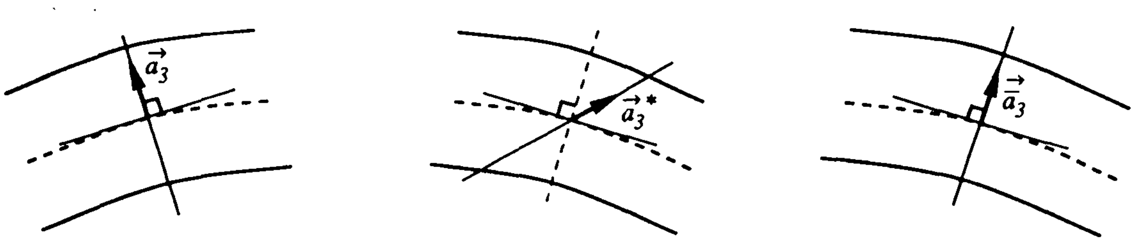
where $\vec{u}$ is the displacement field of the orthogonal projection P of the point M upon the middle surface of the shell before deformation, i.e.,

$$\vec{u} = u_i \vec{a}^i,$$

where ζ denotes the coordinate through the thickness of the shell and where the parameters β_α are the components of the rotation of the normal $\vec{a}^3$. Let us explain this last point : the assumption of shear deformation involves that the particles located upon the normal $\vec{a}_3$ before deformation remain aligned after deformation and define a possibly non unit vector $\vec{a}_3^*$, i.e.,

$$\vec{a}_3^* = \vec{a}_3 + \beta_\alpha \vec{a}^\alpha.$$

This assumption is illustrated in Fig. 1.



before deformation after deformation (Naghdi) after deformation (Koiter)

Fig. 1. Geometrical aspects of the deformation of the shell (transverse sections)

Loading of the shell :

Subsequently we assume that the shell is :

i) *clamped* along $\Gamma_0 \subset \Gamma = \partial\Omega$ with measure $(\Gamma_0) > 0$;

ii) *loaded by a distribution of volume forces* whose resultant is $\vec{p}$ and whose resultant moment is $\vec{0}$ along the middle surface S of the shell ;

iii) *loaded by a distribution of surface forces* applied to the part $(\Gamma_1 = \Gamma - \Gamma_0)\times]-e/2, e/2[$ of its lateral surface. We denote by $\vec{N}$ the corresponding resultant along Γ_1 and by $\underset{\sim}{M} = \varepsilon_{\alpha\beta} M^\beta \vec{a}^\alpha$ the corresponding resultant moment along Γ_1.

The corresponding work of these loads associated to a displacement $(\vec{v}, \underset{\sim}{\delta})$ of the particles of the shell is approximated by :

$$\ell(\vec{v}, \underset{\sim}{\delta}) = \int_\Omega \vec{p}\vec{v}\,\sqrt{a}\,d\xi^1 d\xi^2 + \int_{\Gamma_1} (\vec{N}\vec{v} - M^\alpha \delta_\alpha) d\gamma. \tag{2.1}$$

Then, the corresponding variational formulation of this model can be stated as follows :

PROBLEM 2.1. For any $\vec{p} \in (L^2(\Omega))^3$, $\vec{N} \in (L^2(\Gamma_1))^3$, $\underset{\sim}{M} \in (L^2(\Gamma_1))^2$, find $(\vec{u}, \underset{\sim}{\beta}) \in \vec{V}$, $\underset{\sim}{\beta} = \beta_\alpha \vec{a}^\alpha$, such that :

$$a[(\vec{u}, \underset{\sim}{\beta}); (\vec{v}, \underset{\sim}{\delta})] + b[(\vec{u}, \underset{\sim}{\beta}); (\vec{v}, \underset{\sim}{\delta})] = \ell(\vec{v}, \underset{\sim}{\delta}), \quad \forall (\vec{v}, \underset{\sim}{\delta}) \in \vec{V}, \tag{2.2}$$

where :

$$a[(\vec{u}, \underset{\sim}{\beta}); (\vec{v}, \underset{\sim}{\delta})] = \int_\Omega e E^{\alpha\beta\lambda\mu} [\gamma_{\alpha\beta}(\vec{u}) \gamma_{\lambda\mu}(\vec{v}) + \frac{e^2}{12}\, \chi_{\alpha\beta}(\vec{u}, \underset{\sim}{\beta}) \chi_{\lambda\mu}(\vec{v}, \underset{\sim}{\delta})]\, \sqrt{a}\, d\xi^1 d\xi^2, \tag{2.3}$$

$$b[(\vec{u}, \underset{\sim}{\beta}); (\vec{v}, \underset{\sim}{\delta})] = \int_\Omega \frac{eEa^{\alpha\beta}}{2(1+\nu)}\, (\phi_\alpha(\vec{u}) + \beta_\alpha)(\phi_\beta(\vec{v}) + \delta_\beta)\, \sqrt{a}\, d\xi^1 d\xi^2, \tag{2.4}$$

$$E^{\alpha\beta\lambda\mu} = \frac{E}{2(1+\nu)}\, (a^{\alpha\lambda} a^{\beta\mu} + a^{\alpha\mu} a^{\beta\lambda} + \frac{2\nu}{1-\nu}\, a^{\alpha\beta} a^{\lambda\mu}),$$

$$\vec{V} = (\{v \in H^1(\Omega)\ ;\ v|_{\Gamma_0} = 0\})^5.$$

Moreover, the components $\gamma_{\alpha\beta}(\vec{v})$, $\chi_{\alpha\beta}(\vec{v}, \underset{\sim}{\delta})$ of the tensors of deformation and of change of curvature of the middle surface of the shell are given by :

$$\gamma_{\alpha\beta}(\vec{v}) = \frac{1}{2}\, (v_{\alpha|\beta} + v_{\beta|\alpha}) - b_{\alpha\beta} v_3, \tag{2.5}$$

$$\chi_{\alpha\beta}(\vec{v},\underline{\delta}) = \frac{1}{2}\left(\delta_{\alpha|\beta}+\delta_{\beta|\alpha}\right) - \frac{1}{2}\left(b_\alpha^\lambda v_{\lambda|\beta}+b_\beta^\lambda v_{\lambda|\alpha}\right) + c_{\alpha\beta}v_3, \tag{2.6}$$

while :

$$\phi_\alpha(\vec{v}) = v_{3,\alpha} + b_\alpha^\lambda v_\lambda. \tag{2.7}$$

It is interesting to record that :

THEOREM 2.1. ([20,21]) : Problem 2.1 has one and only one solution.

PROOF. The main step is the obtention of the $\vec{V}$-ellipticity property of the bilinear form, i.e. : there exists a constant $\alpha > 0$ such that :

$$a[(\vec{v},\underline{\delta});(\vec{v},\underline{\delta})] + b[(\vec{v},\underline{\delta});(\vec{v},\underline{\delta})] \geq \alpha\left\{\|\vec{v}\|_{1,\Omega}^2 + \|\underline{\delta}\|_{1,\Omega}^2\right\}, \quad \forall(\vec{v},\underline{\delta})\in\vec{V}. \tag{2.8}$$

■

2.2 *Koiter's model*

This model is a refinement of the classical theories [22,23] : now the normal to the middle surface remains normal to the middle surface during the deformation (see Fig. 1c) so that :

$$\bar{\vec{a}}_3 = \vec{a}_3 - \phi_\mu(\vec{u})\vec{a}^\mu, \quad \phi_\mu(\vec{u}) = u_{3,\mu} + b_\mu^\lambda u_\lambda.$$

In particular, let us note that the Naghdi's model gives back the Koiter's model as soon as $\vec{a}_3^* = \bar{\vec{a}}_3$, i.e.,

$$\beta_\alpha + \phi_\alpha(\vec{u}) = 0. \tag{2.9}$$

By substituting this relation, and more generally relation $\delta_\alpha + \phi_\alpha(\vec{v}) = 0$, into relations (2.1) to (2.6), we get the variational formulation of Koiter's model :

PROBLEM 2.2. For any $\vec{p} \in (L^2(\Omega))^3$, $\vec{N} \in (L^2(\Gamma_1))^3$, $\underline{M} \in (L^2(\Gamma_1))^2$, find $\vec{u}^* \in \vec{V}^*$ such that :

$$a^*(\vec{u}^*,\vec{v}) = f^*(\vec{v}), \quad \forall\vec{v}\in\vec{V}^*, \tag{2.10}$$

with :

$$\begin{aligned}
a^*(\vec{u},\vec{v}) &= \int_\Omega eE^{\alpha\beta\lambda\mu}[\gamma_{\alpha\beta}(\vec{u})\gamma_{\lambda\mu}(\vec{v}) + \frac{e^2}{12}\,\bar{\rho}_{\alpha\beta}(\vec{u})\bar{\rho}_{\lambda\mu}(\vec{v})]\,\sqrt{a}\,d\xi^1 d\xi^2,\\
\vec{V}^* &= \{\vec{v}\in(H^1(\Omega))^2\times H^2(\Omega),\ \ \vec{v}|_{\Gamma_0} = \vec{0},\ \ \partial_n v_3|_{\Gamma_0} = 0\},\\
\bar{\rho}_{\alpha\beta}(\vec{v}) &= v_{3|\alpha\beta} - c_{\alpha\beta}v_3 + b_\alpha^\lambda v_{\lambda|\beta} + b_\beta^\lambda v_{\lambda|\alpha} + b_{\alpha|\beta}^\lambda v_\lambda,\\
f^*(\vec{v}) &= \int_\Omega \vec{p}\vec{v}\,\sqrt{a}\,d\xi^1 d\xi^2 + \int_{\Gamma_1}[\vec{N}\vec{v} + M^\alpha(v_{3,\alpha} + b_\alpha^\lambda v_\lambda)]d\gamma.
\end{aligned} \tag{2.11}$$

It is worth to note that :

THEOREM 2.2. ([20] or [24]) : Problem 2.2 has one and only one solution.

PROOF. Here again the main step is to prove the $\vec{V}^*$-ellipticity of bilinear form, i.e., there exists a constant $\alpha^* > 0$ such that :

$$a^*(\vec{v},\vec{v}) \geq \alpha^*\|\vec{v}\|_{\vec{V}^*}^2, \quad \forall\vec{v}\in\vec{V}^*.$$

■

3 THE D.K.T. FINITE ELEMENT METHOD

The direct approximation of the equation (2.2) by conforming finite elements is easy to implement since we have just to use $\mathcal{C}^0$-elements. However, for small thickness, we observe numerical instabilities (locking phenomenon) as quoted for example by [25]. Several methods have been proposed to fight such instabilities but none of them meets a systematic success until recent works by [26,27].

On the other side, the direct approach of the equations (2.10) by conforming finite element methods leads to the use of finite elements of class $\mathcal{C}^1$ whose implementation is always delicate. Moreover, these equations give a good modelization of thin shells but they are less performant for semi-thick or thick shells for which it is not really reasonable to neglect shear strain effects.

The object of *D.K.T. finite element methods* is to propose an "intermediate" discrete model between the two discrete models which can be obtained from equations (2.2) and (2.10). In this way :

i) one defines a mesh of the plane domain Ω ;

ii) one approaches the components of the unknowns $\vec{u}, \underline{\beta}$ in suitable finite element spaces ;

iii) one introduces some constraints over the discrete space which are some types of Kirchhoff-Love conditions imposed in a finite number of nodes ;

iv) one neglects the shear strain energy, i.e., $b[.;.]$.

In this paragraph, we detail this D.K.T. approximation and then, we give two equivalent variational formulations.

3.1 The mesh in use

From now on, we assume for simplicity that the plane domain Ω is polygonal. Then, we consider partitions of this domain by regular family of triangulations which satisfy an inverse assumption [28].

Subsequently, we detail the most representative of these D.K.T. methods in case of triangles ; the other methods based upon triangular or rectangular elements are briefly described in [7].

3.2 The finite element spaces used to approximate the unknowns $\vec{u}$ and $\underline{\beta}$

Every components u_i and β_α will be approximated in a finite dimensional space associated to the finite elements displayed in Fig. 2.

3.3 The discrete space associated to the D.K.T. method

Let $\vec{Z}_h$ be the finite element space defined by :

$$\left.\begin{array}{c}\vec{Z}_h = \{(\vec{v}_h, \underline{\delta}_h) \in \vec{V} \;;\; v_{h\alpha} \in V_{h1},\ \delta_{h\alpha} \in V_{h1},\ \alpha = 1,2 \;;\; v_{h3} \in V_{h2} \;; \\ (\vec{v}_h, \underline{\delta}_h)|_T \text{ satisfies the twelve "discrete Kirchhoff} \\ \text{constraints" for any } T \in \mathcal{T}_h\},\end{array}\right\} \qquad (3.1)$$

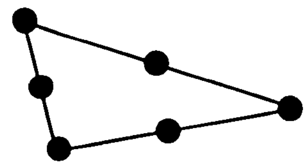
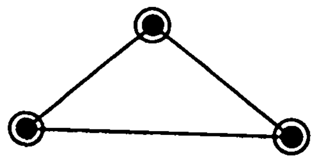
Unknown	Finite element type	Representation of the finite element	Associated finite element spaces including boundary conditions $v\|_{\Gamma_0} = 0$
$u_1, u_2, \beta_1, \beta_2$	P_2-Lagrange		V_{h1}
u_3	P_3'-Hermite		V_{h2}

Fig. 2. Finite elements in use

where the twelve constraints are :

(i) *Kirchhoff-Love assumption is satisfied at the vertices*, i.e., with (2.7) and (2.9) :

$$\delta_{h\alpha}(a_i) = -v_{h3,\alpha}(a_i) - b_\alpha^\lambda(a_i)v_{h\lambda}(a_i), \quad \alpha = 1,2 \; ; \; i = 1,2,3, \tag{3.2}$$

that means six constraints.

To express the other constraints, it is worth to introduce in the plane reference domain Ω the unit external normal $\vec{n}_i$ vector to the side $a_{i+1}a_{i-1}$ of the triangle into consideration (see Fig. 3) at midpoint b_i, so that $\vec{n}_i = n_i^\alpha \vec{e}_\alpha$. Likewise, we will use the unit tangential vector $\vec{t}_i = \vec{n}_i \times \vec{e}_3$ with $\vec{t}_i = t_i^\alpha \vec{e}_\alpha$. For convenience, we will also use $t_{i\alpha} = t_i^\alpha$ and $n_{i\alpha} = n_i^\alpha$.

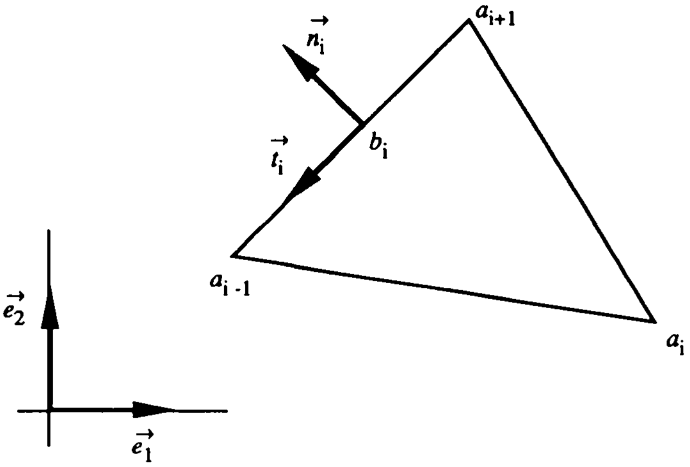


Fig. 3. Local basis attached to a triangle

Then, the other six constraints can be stated as follows :

(ii) *Kirchhoff-Love assumption is "tangentially" satisfied at the midsides b_i of the triangle :*

$$t_i^\alpha \delta_{h\alpha}(b_i) = -t_i^\alpha [v_{h3,\alpha}(b_i) + b_\alpha^\lambda(b_i) v_{h\lambda}(b_i)], \quad i = 1,2,3\ ; \tag{3.3}$$

(iii) *Kirchhoff-Love assumption is "normally" satisfied in mean at the midsides b_i of the triangle, i.e. :*

$$n_i^\alpha \delta_{h\alpha}(b_i) = -n_i^\alpha \left[\tfrac{1}{2} \left(v_{h3,\alpha}(a_{i-1}) + v_{h3,\alpha}(a_{i+1})\right) + b_\alpha^\lambda(b_i) v_{h\lambda}(b_i)\right], \quad i = 1,2,3. \tag{3.4}$$

Thus, by imposing these twelve independent constraints, we have reduced to twenty one elements the set of effective degrees of freedom. For proving the convergence, it will be useful to keep the set :

$$\{v_{h\alpha}(a_i),\ v_{h\alpha}(b_i),\ v_{h3}(a_i),\ v_{h3,\alpha}(a_i),\ \alpha = 1,2\ ;\ i = 1,2,3\}, \tag{3.5}$$

while, for implementation purpose, we have seen in [12] that it is most convenient to replace $v_{h3,\alpha}(a_i)$ by $\delta_{h\alpha}(a_i)$.

3.4 Variational formulation of the D.K.T. approximation problem

Since we specify discrete Kirchhoff conditions, this D.K.T. approximation is mainly addressed to *thin* shell approximations. So, we define the associate discrete problem by
- looking for approximate solutions lying in discrete space $\vec{Z}_h$;
- neglecting in (2.2) the term coming from shear energy, that is $b[.;.]$.

Then the D.K.T. approximation problem can be defined as follows :

PROBLEM 3.1. For any $\vec{p} \in (L^2(\Omega))^3$, $\vec{N} \in (L^2(\Gamma_1))^3$, $\underline{M} \in (L^2(\Gamma_1))^2$, find $(\vec{u}_h, \underline{\beta}_h) \in \vec{Z}_h$ such that

$$a[(\vec{u}_h, \underline{\beta}_h); (\vec{v}_h, \underline{\delta}_h)] = \ell(\vec{v}_h, \underline{\delta}_h), \quad \forall (\vec{v}_h, \underline{\delta}_h) \in \vec{Z}_h. \tag{3.6}$$

■

Subsequently, we will introduce another equivalent formulation of Problem 3.1 which will be more interesting for the study of convergence and for the obtention of error estimates.

First, let us give some remarks concerning the definition of Problem 3.1.

REMARK 3.1. If we introduce the subspace $\vec{Z}$ of the space $\vec{V}$ defined by :

$$\vec{Z} = \{(\vec{v}, \underline{\delta}) \in \vec{V} \text{ such that } \delta_\alpha + v_{3,\alpha} + b_\alpha^\lambda v_\lambda = 0 \text{ in } L^2(\Omega)\}, \tag{3.7}$$

then, we can check that :

i) the problem : *For any $\vec{p} \in (L^2(\Omega))^3$, $\vec{N} \in (L^2(\Gamma_1))^3$, $\underline{M} \in (L^2(\Gamma_1))^2$, find $(\vec{u}, \underline{\beta}) \in \vec{Z}$ such that :*

$$a[(\vec{u}, \underline{\beta}), (\vec{v}, \underline{\delta})] = \ell(\vec{v}, \underline{\delta}), \quad \forall (\vec{v}, \underline{\delta}) \in \vec{Z}, \tag{3.8}$$

is equivalent to Problem 2.2 since we have :

$$\ell(\vec{v}, \underline{\delta}) = f^*(\vec{v}), \quad \chi_{\alpha\beta}(\vec{v}, \underline{\delta}) = -\bar{\rho}_{\alpha\beta}(\vec{v}), \quad \forall (\vec{v}, \underline{\delta}) \in \vec{Z}\ ;$$

ii) Problem 3.1 is nothing but a nonconforming approximation of problem (3.8), and then of Koiter's model given by Problem 2.2, since $\vec{Z}_h \not\subset \vec{Z}$. It is easy to check that $(\vec{v}_h, \underline{\delta}_h) \in \vec{Z}_h$

implies $\vec{v}_h \in (\mathcal{C}^0(\bar{\Omega}))^3$ but since $\partial_n v_{h3}$ is not continuous along the interface, we do not have $v_{h3} \in H^2(\Omega)$. In fact the constraints (3.4) introduce a control for the jump of these normal derivatives along the common sides between adjacent triangles ; these contraints are essential to ensure the convergence of this nonconforming method. For nonconforming finite element approximations of thin shell problems, we refer to [29,30]. ■

As we have mentionned in the comments of relation (3.5), it is convenient to eliminate the unknowns $\underset{\sim}{\delta}_h$ and to give another equivalent statement of Problem 3.1. The elimination of the unknowns $\underset{\sim}{\delta}_h$ leads to an approximation space $\vec{V}_h = V_{h1} \times V_{h1} \times V_{h2}$ which is not included in $\vec{V}^*$.

It will be convenient to define the discrete derivative :

$$\partial_{h\alpha}(\vec{v}_h) = -\delta_{h\alpha} - b^\lambda_\alpha v_{h\lambda}, \quad \forall (\vec{v}_h, \underset{\sim}{\delta}_h) \in \vec{Z}_h, \tag{3.9}$$

whose interesting properties are :

i) $\partial_{h\alpha}(\vec{v}_h)$ is supposed to be a good approximation of $v_{h3,\alpha}$ (see the constraints introduced in the definition (3.1) of $\vec{Z}_h$). This result will be stated precisely in Lemma 3.1 ;

ii) $\partial_{h\alpha}(\vec{v}_h) \in H^1(\Omega)$ while $v_{h3,\alpha}$ only belongs to $L^2(\Omega)$.

For any $\vec{v}_h \in \vec{V}_h$, we define :

$$\left.\begin{array}{l} \bar{\rho}_{h\alpha\beta}(\vec{v}_h) \stackrel{\text{def}}{=} \partial_{h\alpha\beta}(\vec{v}_h) - \Gamma^\lambda_{\alpha\beta}\partial_{h\lambda}(\vec{v}_h) - c_{\alpha\beta}v_{h3} + b^\lambda_\alpha v_{h\lambda|\beta} + b^\lambda_\beta v_{h\lambda|\alpha} + b^\lambda_\alpha|_\beta v_{h\lambda}, \\ \text{where} \\ \partial_{h\alpha\beta}(\vec{v}_h) = \frac{1}{2}\left(\partial_\beta\partial_{h\alpha}(\vec{v}_h) + \partial_\alpha\partial_{h\beta}(\vec{v}_h)\right), \end{array}\right\} \tag{3.10}$$

so that :

$$\bar{\rho}_{h\alpha\beta}(\vec{v}_h) = -\chi_{\alpha\beta}(\vec{v}_h, \underset{\sim}{\delta}_h), \quad \forall (\vec{v}_h, \underset{\sim}{\delta}_h) \in \vec{Z}_h. \tag{3.11}$$

For following purposes, it is worth to note that relations (3.9) to (3.11) can be extended to functions of space $\vec{Z}$ defined by relation (3.7), by setting :

$$\partial_{h\alpha}(\vec{v}) \stackrel{\text{def}}{=} v_{3,\alpha} \; ; \; \bar{\rho}_{h\alpha\beta}(\vec{v}) \stackrel{\text{def}}{=} -\chi_{\alpha\beta}(\vec{v}, \underset{\sim}{\delta}) = \bar{\rho}_{\alpha\beta}(\vec{v}), \text{ for any } (\vec{v}, \underset{\sim}{\delta}) \in \vec{Z}. \tag{3.12}$$

With these definitions it is easy to prove that :

$$\left.\begin{array}{l} a[(\vec{u}_h, \underset{\sim}{\beta}_h), (\vec{v}_h, \underset{\sim}{\delta}_h)] = a_h(\vec{u}_h, \vec{v}_h), \quad \forall (\vec{u}_h, \underset{\sim}{\beta}_h), (\vec{v}_h, \underset{\sim}{\delta}_h) \in \vec{Z}_h, \\ \ell(\vec{v}_h, \underset{\sim}{\delta}_h) = f_h(\vec{v}_h), \quad \forall ((\vec{v}_h, \underset{\sim}{\delta}_h) \in \vec{Z}_h, \end{array}\right\} \tag{3.13}$$

where :

$$\left.\begin{array}{l} a_h(\vec{u}_h, \vec{v}_h) = \int_\Omega eE^{\alpha\beta\lambda\mu}[\gamma_{\alpha\beta}(\vec{u}_h)\gamma_{\lambda\mu}(\vec{v}_h) + \frac{e^2}{12}\,\bar{\rho}_{h\alpha\beta}(\vec{u}_h)\bar{\rho}_{h\lambda\mu}(\vec{v}_h)]\,\sqrt{a}\,d\xi^1 d\xi^2, \\ f_h(\vec{v}_h) = \int_\Omega \vec{p}\vec{v}_h\,\sqrt{a}\,d\xi^1 d\xi^2 + \int_{\Gamma_1}[\vec{N}\vec{v}_h + M^\lambda(\partial_{h\lambda}(\vec{v}_h) + b^\mu_\lambda v_{h\mu})]d\gamma. \end{array}\right\} \tag{3.14}$$

From the identities (3.13), we can give the following equivalent formulation of the D.K.T. approximation (3.6) :

PROBLEM 3.2. For any $\vec{p} \in (L^2(\Omega))^3$, $\vec{N} \in (L^2(\Gamma_1))^3$, $\underline{M} \in (L^2(\Gamma_1))^2$, find $\vec{u}_h \in \vec{V}_h$ such that :

$$a_h(\vec{u}_h, \vec{v}_h) = f_h(\vec{v}_h), \quad \forall \vec{v}_h \in \vec{V}_h.$$

■

Therefore, we have defined a nonconforming approximation of Koiter's model given by Problem 2.2, with three kinds of nonconformity :

i) $\vec{V}_h \not\subset \vec{V}^*$;

ii) $a^*(.,.)$ is approximated by $a_h(.,.)$;

iii) $f^*(.)$ is approximated by $f_h(.)$.

To prepare the convergence studies, it is convenient to equip the space $\vec{V}_h$ with the norm

$$\|\vec{v}_h\|_h = \left\{ \|\vec{v}_h\|_{1,\Omega}^2 + \sum_{T \in \mathcal{T}_h} |v_{h3}|_{2,T}^2 \right\}^{1/2}, \quad \forall \vec{v}_h \in \vec{V}_h, \tag{3.15}$$

and to prove the following lemmas :

LEMMA 3.1. Let us assume that the mapping $\vec{\phi}$ which defines the middle surface of the shell belongs to the space $[W^{4,\infty}(\Omega)]^3$. Then, there exists a constant $C > 0$, independent of h_T, such that for $m = 0, 1$ and for $\alpha = 1, 2$, we have :

$$|v_{h3,\alpha} - \partial_{h\alpha}(\vec{v}_h)|_{m,T} \le C h_T^{1-m} \{ \|v_{h1}\|_{1,T} + \|v_{h2}\|_{1,T} + |v_{h3}|_{2,T} \}, \quad \forall (\vec{v}_h, \underline{\delta}_h) \in \vec{Z}_h.$$

PROOF. By using the definitions (3.1) and (3.9), we obtain for every $(\vec{v}_h, \underline{\delta}_h) \in \vec{Z}_h$ over each triangle $T \in \mathcal{T}_h$:

$$\left.\begin{aligned} \partial_{h\alpha}(\vec{v}_h)(\xi) &= -\delta_{h\alpha}(\xi) - b_\alpha^\lambda(\xi) v_{h\lambda}(\xi) \\ &= -\sum_{i=1}^{3} [\lambda_i(2\lambda_i - 1)\delta_{h\alpha}(a_i) + 4\lambda_{i+1}\lambda_{i-1}\delta_{h\alpha}(b_i)] \\ &\quad - b_\alpha^\lambda(\xi) \sum_{i=1}^{3} [\lambda_i(2\lambda_i - 1) v_{h\lambda}(a_i) + 4\lambda_{i+1}\lambda_{i-1}\, v_{h\lambda}(b_i)]. \end{aligned}\right\} \tag{3.16}$$

Next the combination of relations (3.2) (3.3) and (3.4) gives

$$\delta_{h\alpha}(a_i) = -v_{h3,\alpha}(a_i) - b_\alpha^\lambda(a_i) v_{h\lambda}(a_i) \ ; \tag{3.17}$$

$$\begin{aligned} \delta_{h\alpha}(b_i) &= -t_{\alpha i} t_i^\nu [v_{h3,\nu}(b_i) + b_\nu^\lambda(b_i) v_{h\lambda}(b_i)] \\ &\quad - n_{\alpha i} n_i^\nu \left[\tfrac{1}{2} (v_{h3,\nu}(a_{i-1}) + v_{h3,\nu}(a_{i+1})) + b_\nu^\lambda(b_i) v_{h\lambda}(b_i) \right] \\ &= -[t_{\alpha i} t_i^\nu v_{h3,\nu}(b_i) + \tfrac{1}{2}\, n_{\alpha i} n_i^\nu (v_{h3,\nu}(a_{i-1}) + v_{h3,\nu}(a_{i+1}))] - b_\alpha^\lambda(b_i) v_{h\lambda}(b_i), \end{aligned}$$

where we have used notations $t_{\alpha i} = t_i^\alpha$, $n_{\alpha i} = n_i^\alpha$ (since these components are taken with respect to an orthonormal basis) and the identity $t_{\alpha i} t_i^\nu + n_{\alpha i} n_i^\nu = \delta_\alpha^\nu$. Moreover, since $v_{h3|T} \in$ Hermite triangle of type (3'), we can write :

$$\tfrac{1}{2}\, n_i^\nu [v_{h3,\nu}(a_{i-1}) + v_{h3,\nu}(a_{i+1})] = n_i^\nu v_{h3,\nu}(b_i) + \tfrac{1}{8}\, D^3 v_{h3}(b_i)((a_{i+1} - a_{i-1})^2, \vec{n}_i),$$

so that :

$$\delta_{h\alpha}(b_i) = -v_{h3,\alpha}(b_i) - b_\alpha^\lambda(b_i)v_{h\lambda}(b_i) - \frac{1}{8}\, n_{\alpha i} D^3 v_{h3}(b_i)((a_{i+1} - a_{i-1})^2, \vec{n}_i). \tag{3.18}$$

Then, by substituting (3.17) and (3.18) into (3.16), we obtain for any $\xi \in T$:

$$\begin{aligned}\partial_{h\alpha}(\vec{v}_h)(\xi) = &\sum_{i=1}^{3}[\lambda_i(2\lambda_i - 1)(b_\alpha^\lambda(a_i) - b_\alpha^\lambda(\xi))v_{h\lambda}(a_i) + 4\lambda_{i+1}\lambda_{i-1}(b_\alpha^\lambda(b_i) - b_\alpha^\lambda(\xi))v_{h\lambda}(b_i)]\\ &+ \sum_{i=1}^{3}[\lambda_i(2\lambda_i - 1)v_{h3,\alpha}(a_i) + 4\lambda_{i+1}\lambda_{i-1}v_{h3,\alpha}(b_i)]\\ &+ \frac{1}{2}\sum_{i=1}^{3}[\lambda_{i+1}\lambda_{i-1}n_{\alpha i}D^3 v_{h3}(b_i)((a_{i+1} - a_{i-1})^2, \vec{n}_i)],\end{aligned}$$

or again,

$$\begin{aligned}\partial_{h\alpha}(\vec{v}_h)(\xi) &= \pi_T^2(b_\alpha^\lambda v_{h\lambda})(\xi) - b_\alpha^\lambda(\xi)\pi_T^2(v_{h\lambda})(\xi) + \pi_T^2(v_{h3,\alpha})(\xi)\\ &+ \frac{1}{2}\sum_{i=1}^{3}[\lambda_{i+1}\lambda_{i-1}n_{\alpha i}D^3 v_{h3}(b_i)((a_{i+1} - a_{i-1})^2, \vec{n}_i)],\end{aligned}$$

where π_T^2 denotes the $P_2(T)$-interpolation operator. Since by definition $v_{h\lambda} \in P_2(T)$ and $v_{h3,\alpha} \in P_2(T)$, we can write :

$$\left.\begin{aligned}\partial_{h\alpha}(\vec{v}_h)(\xi) = v_{h3,\alpha}(\xi) + \pi_T^2(b_\alpha^\lambda v_{h\lambda})(\xi) - b_\alpha^\lambda(\xi)v_{h\lambda}(\xi)\\ + \frac{1}{2}\sum_{i=1}^{3}[\lambda_{i+1}\lambda_{i-1}n_{\alpha i}D^3 v_{h3}(b_i)((a_{i+1} - a_{i-1})^2, \vec{n}_i)], \quad \forall \xi \in T,\end{aligned}\right\} \tag{3.19}$$

so that we get for $m = 0$ or 1 and for any $T \in \mathcal{T}_h$:

$$|v_{h3,\alpha} - \partial_{h\alpha}(\vec{v}_h)|_{m,T} \leq \sqrt{2}\{|b_\alpha^\lambda v_{h\lambda} - \pi_T^2(b_\alpha^\lambda v_{h\lambda})|_{m,T} + |r_T(v_{h3})|_{m,T}\}, \tag{3.20}$$

where :

$$r_T(v_{h3}) = \sum_{i=1}^{3}\left[\frac{1}{2}\,\lambda_{i+1}\lambda_{i-1}n_{\alpha i}D^3 v_{h3}(b_i)((a_{i+1} - a_{i-1})^2, \vec{n}_i)\right].$$

Now, if we assume that $b_\alpha^\lambda \in W^{2,\infty}(\Omega)$, we obtain for $m = 0$ or 1 and for any $T \in \mathcal{T}_h$ (see [28, Theorem 3.1.5]) :

$$|b_\alpha^\lambda v_{h\lambda} - \pi_T^2(b_\alpha^\lambda v_{h\lambda})|_{m,T} \leq C h_T^{2-m}|b_\alpha^\lambda v_{h\lambda}|_{2,T},$$

so that (see [28, p. 192]) :

$$|b_\alpha^\lambda v_{h\lambda} - \pi_T^2(b_\alpha^\lambda v_{h\lambda})|_{m,T} \leq C h_T^{2-m}\sum_{\lambda=1}^{2}\{|v_{h\lambda}|_{0,T} + |v_{h\lambda}|_{1,T} + |v_{h\lambda}|_{2,T}\},$$

and by using $|v_{h\lambda}|_{2,T} \leq C h_T^{-1}|v_{h\lambda}|_{1,T}$ (see [28, Theorem 3.1.2]), we have :

$$|b_\alpha^\lambda v_{h\lambda} - \pi_T^2(b_\alpha^\lambda v_{h\lambda})|_{m,T} \leq C h_T^{1-m}\{\|v_{h1}\|_{1,T} + \|v_{h2}\|_{1,T}\}. \tag{3.21}$$

Finally, if we observe that $|v_{h3}|_{3,\infty,T} \leq C h_T^{-2}|v_{h3}|_{2,T}$ and $|\lambda_{i+1}\lambda_{i-1}|_{m,T} \leq C h_T^{1-m}$ (see [28, Theorems 3.1.2 and 3.1.3]) we obtain :

$$|r_T(v_{h3})|_{m,T} \leq C h_T^{1-m}|v_{h3}|_{2,T}. \tag{3.22}$$

Then conclusion arises from inequalities (3.20) to (3.22). ■

LEMMA 3.2. Let us assume that the mapping $\vec{\phi}$ which defines the middle surface of the shell belongs to the space $[W^{4,\infty}(\Omega)]^3$. Then, there exists a constant $C > 0$, independent of h_T, such that for $\alpha = 1, 2$ we have :

$$|v_{h3,\alpha} - \partial_{h\alpha}(\vec{v}_h)|_{-\frac{1}{2},\partial T} \leq C h_T\{\|v_{h1}\|_{1,T} + \|v_{h2}\|_{1,T} + |v_{h3}|_{2,T}\}, \quad \forall(\vec{v}_h, \underline{\delta}_h) \in \vec{Z}_h.$$

PROOF. The boundary ∂T of triangle T is the union of the three sides S_i, $i = 1, 2, 3$. Let us examine the contribution of one given side $S_i = [a_{i+1}, a_{i-1}]$. The specification of development (3.19) to this side gives :

$$\left.\begin{aligned} [\partial_{h\alpha}(\vec{v}_h) - v_{h3,\alpha}]|_{S_i} &= [\pi_T^2(b_\alpha^\lambda v_{h\lambda}) - b_\alpha^\lambda v_{h\lambda} \\ &+ \tfrac{1}{2}\,\lambda_{i+1}\lambda_{i-1} n_{\alpha i} D^3 v_{h3}(b_i)((a_{i+1} - a_{i-1})^2, \vec{n}_i)]|_{S_i}. \end{aligned}\right\} \tag{3.23}$$

Next by using an extension of [28, Theorem 3.1.5] (in this direction, see also [31]) done by [32, Lemma 6] :

$$|b_\alpha^\lambda v_{h\lambda} - \pi_T^2(b_\alpha^\lambda v_{h\lambda})|_{-\frac{1}{2},S_i} \leq C h_T^2 |b_\alpha^\lambda v_{h\lambda}|_{2,T},$$

and, like in the proof of Lemma 3.1, we deduce :

$$|b_\alpha^\lambda v_{h\lambda} - \pi_T^2(b_\alpha^\lambda v_{h\lambda})|_{-\frac{1}{2},S_i} \leq C h_T\{\|v_{h1}\|_{1,T} + \|v_{h2}\|_{1,T}\}. \tag{3.24}$$

Again, by using [28, Theorems 3.1.2 and 3.1.3] we get $|v_{h3}|_{3,\infty,T} \leq C h_T^{-2} |v_{h3}|_{2,T}$ and, in accordance with [32, Lemma 3], we obtain $|\lambda_{i+1}\lambda_{i-1}|_{-\frac{1}{2},S_i} \leq C h_T$, so that :

$$\left| \tfrac{1}{2}\,\lambda_{i+1}\lambda_{i-1} n_{\alpha i} D^3 v_{h3}(b_i)((a_{i+1} - a_{i-1})^2, \vec{n}_i)\right|_{-\frac{1}{2},S_i} \leq C h_T |v_{h3}|_{2,T}. \tag{3.25}$$

Finally, by combining relation (3.23) with the estimates (3.24) and (3.25), we obtain the expected result. ■

4 CONVERGENCE AND ERROR ESTIMATES

As usual, let us start by giving an abstract error estimate which will give the way to prove the convergence and to get error estimates.

LEMMA 4.1. (Abstract error estimate). Assume that the bilinear form $a_h(.,.)$ is uniformly continuous on $\vec{V}_h \cup \vec{Z}$ and elliptic on $\vec{V}_h$, i.e., there exists constants $\gamma > 0$ and $\delta > 0$, independent of h, such that :

$$\left.\begin{aligned} &\gamma\|\vec{v}_h\|_h^2 \leq a_h(\vec{v}_h, \vec{v}_h), \quad \forall \vec{v}_h \in \vec{V}_h, \\ &a_h(\vec{v}_h, \vec{w}_h) \leq \delta\|\vec{v}_h\|_h\,\|\vec{w}_h\|_h, \quad \forall \vec{v}_h, \vec{w}_h \in \vec{V}_h \cup \vec{Z}. \end{aligned}\right\} \tag{4.1}$$

Then, there exists a constant $C > 0$, independent of h, such that

$$\left.\begin{aligned} \|\vec{u}^* - \vec{u}_h\|_h \leq C\Bigg\{ &\inf_{\vec{v}_h \in \vec{V}_h} \|\vec{u}^* - \vec{v}_h\|_h + \sup_{\vec{w}_h \in \vec{V}_h} \frac{|f^*(\vec{w}_h) - f_h(\vec{w}_h)|}{\|\vec{w}_h\|_h} \\ &+ \sup_{\vec{w}_h \in \vec{V}_h} \frac{|f^*(\vec{w}_h) - a_h(\vec{u}^*, \vec{w}_h)|}{\|\vec{w}_h\|_h}\Bigg\}, \end{aligned}\right\} \tag{4.2}$$

where $\vec{u}^*$ and $\vec{u}_h$ are respectively the solutions of Problems 2.2 and 3.2.

PROOF. From (3.12), (3.14) and from the assumption (4.1), we obtain for any $\vec{v}_h \in \vec{V}_h$:

$$\gamma\|\vec{u}_h - \vec{v}_h\|_h^2 \leq a_h(\vec{u}_h - \vec{v}_h, \vec{u}_h - \vec{v}_h) = a_h(\vec{u}^* - \vec{v}_h, \vec{u}_h - \vec{v}_h)$$

$$+ [f_h(\vec{u}_h - \vec{v}_h) - f^*(\vec{u}_h - \vec{v}_h)] + [f^*(\vec{u}_h - \vec{v}_h) - a_h(\vec{u}^*, \vec{u}_h - \vec{v}_h)],$$

and, by using the uniform continuity of $a_h(.,.)$ on $\vec{V}_h \cup \vec{Z}$:

$$\gamma\|\vec{u}_h - \vec{v}_h\|_h \leq \delta\|\vec{u}^* - \vec{v}_h\|_h + \sup_{\vec{w}_h \in \vec{V}_h} \frac{|f_h(\vec{w}_h) - f^*(\vec{w}_h)|}{\|\vec{w}_h\|_h} + \sup_{\vec{w}_h \in \vec{V}_h} \frac{|f^*(\vec{w}_h) - a_h(\vec{u}^*, \vec{w}_h)|}{\|\vec{w}_h\|_h},$$

where δ is a constant independent of h. By combining this inequaliy with the triangular one, i.e.,

$$\|\vec{u}^* - \vec{u}_h\|_h \leq \|\vec{u}^* - \vec{v}_h\|_h + \|\vec{v}_h - \vec{u}_h\|_h,$$

and by taking the infinimum with respect to $\vec{v}_h \in \vec{V}_h$, we get (4.2). ■

To use this lemma, we have to check the uniform continuity on $\vec{V}_h \cup \vec{Z}$ (see Lemma 4.2) and the uniform $\vec{V}_h$-ellipticity (see Lemma 4.6) of the bilinear form, on the one hand, and we need to estimate the interpolation error term (see Lemma 4.3) and the two consistency terms (see Lemmas 4.4 and 4.5) which appear in the right hand side of estimate (4.2).

LEMMA 4.2. (Uniform continuity of $a_h(.,.)$ on $\vec{V}_h \cup \vec{Z}$). Let us assume that the mapping $\vec{\phi}$ which defines the middle surface of the shell belongs to the space $[W^{4,\infty}(\Omega)]^3$. Then, there exists a constant $\delta > 0$, independent of h, such that

$$|a_h(\vec{v}_h, \vec{w}_h)| \leq \delta\|\vec{v}_h\|_h \, \|\vec{w}_h\|_h, \quad \forall \vec{v}_h, \vec{w}_h \in \vec{V}_h \cup \vec{Z}. \tag{4.3}$$

PROOF. From Lemma 3.1, there exists a constant $C > 0$, independent of h, such that :

$$\|\partial_{h\alpha}(\vec{v}_h)\|_{1,T} \leq \|\partial_{h\alpha}(\vec{v}_h) - v_{h3,\alpha}\|_{1,T} + \|v_{h3,\alpha}\|_{1,T} \leq C[\|v_{h1}\|_{1,T} + \|v_{h2}\|_{1,T} + \|v_{h3}\|_{2,T}].$$

Then, by using [23, Théorème 6.2.1] and the expressions (2.5) (3.10) and (3.14), we obtain the inequality (4.3) for any $\vec{v}_h, \vec{w}_h \in \vec{V}_h$. This result can be immediately extended to $\vec{v}_h$ or $\vec{w}_h \in \vec{Z}$ by using (3.12). ■

LEMMA 4.3. (Interpolation error estimate) Let us assume that the solution $\vec{u}^*$ of Koiter's Problem 2.2 belongs to the space $\vec{V} \cap [(H^2(\Omega))^2 \times H^3(\Omega)]$. Then, there exists a constant $C > 0$, independent of h, such that :

$$\inf_{\vec{v}_h \in \vec{V}_h} \|\vec{u}^* - \vec{v}_h\|_h \leq Ch\{|u_1^*|_{2,\Omega}^2 + |u_2^*|_{2,\Omega}^2 + |u_3^*|_{3,\Omega}^2\}^{1/2}. \tag{4.4}$$

PROOF. Let $\vec{\pi}_h\vec{u}^*$ be the $\vec{V}_h$-interpolation of the solution $\vec{u}^*$. From [28, Theorem 3.1.5], we obtain :

$$|u_\alpha^* - \pi_T^2 u_\alpha^*|_{m,T} \leq Ch_T^{2-m}|u_\alpha^*|_{2,T}, \; m = 0,1 \; ; \quad |u_3^* - \pi_T^3 u_3^*|_{m,T} \leq Ch_T^{3-m}|u_3^*|_{3,T}, \; m = 0,1,2.$$

Then, with definition (3.15), we have :

$$\left.\begin{aligned}\inf_{\vec{v}_h\in\vec{V}_h}\|\vec{u}^*-\vec{v}_h\|_h \le \|\vec{u}^*-\vec{\pi}_h\vec{u}^*\|_h &= \left(\sum_{m=0}^{1}|\vec{u}^*-\vec{\pi}_h\vec{u}^*|^2_{m,\Omega}+\sum_{T\in\mathcal{T}_h}|u_3^*-\pi_T^3u_3^*|^2_{2,T}\right)^{1/2}\\ &\le Ch\left\{|u_1^*|^2_{2,\Omega}+|u_2^*|^2_{2,\Omega}+|u_2^*|^2_{3,\Omega}\right\}^{1/2},\end{aligned}\right\}$$

which is nothing but the expected result. ∎

LEMMA 4.4. (estimate of the first term of consistency). Let us assume that the mapping $\vec{\phi}$ which defines the middle surface of the shell belongs to the space $[W^{4,\infty}(\Omega)]^3$. Then, for any $\vec{p}\in(L^2(\Omega))^3$, $\vec{N}\in(L^2(\Gamma_1))^3$, $\underline{M}\in(H^{1/2}(\Gamma_1))^2$, there exists a constant $C>0$, independent of h, such that :

$$\sup_{\vec{w}_h\in\vec{V}_h}\frac{|f_h(\vec{w}_h)-f^*(\vec{w}_h)|}{\|\vec{w}_h\|_h}\le Ch|\underline{M}|_{\frac{1}{2},\Gamma_1}. \tag{4.5}$$

PROOF. By using the definitions (2.11) and (3.14), we get for any $\vec{w}_h\in\vec{V}_h$:

$$f_h(\vec{w}_h)-f^*(\vec{w}_h)=\int_{\Gamma_1}M^\lambda(\partial_{h\lambda}(\vec{w}_h)-w_{h3,\lambda})d\gamma. \tag{4.6}$$

But Γ_1 is an union of triangle sides S_j, $j=1,\ldots,J$, i.e.,$\Gamma_1=\bigcup_{j=1}^{J}S_j$.

Let us examine the contribution of such a given side S_j to the integral over Γ_1. With Lemma 3.2, we obtain :

$$\left|\int_{S_j}M^\lambda(\partial_{h\lambda}(\vec{w}_h)-w_{h3,\lambda})d\gamma\right|\le C|M^\lambda|_{\frac{1}{2},S_j}|\partial_{h\lambda}(\vec{w}_h)-w_{h3,\lambda}|_{-\frac{1}{2},S_j}$$

$$\le C_\lambda h_T|M^\lambda|_{\frac{1}{2},S_j}\{\|w_{h1}\|_{1,T}+\|w_{h2}\|_{1,T}+|w_{h3}|_{2,T}\},$$

and then :

$$|f_h(\vec{w}_h)-f^*(\vec{w}_h)|\le Ch|\underline{M}|_{\frac{1}{2},\Gamma_1}\|\vec{w}_h\|_h,$$

so that (4.5) is proved. ∎

REMARK 4.1. Let us precise that for implementation, one can replace $f_h(\vec{w}_h)$ directly by

$$f^*(\vec{w}_h)=\int_\Omega\vec{p}\vec{v}_h\sqrt{a}\,d\xi^1d\xi^2+\int_{\Gamma_1}[\vec{N}\vec{v}_h+M^\lambda(v_{h3,\lambda}+b^\mu_\lambda v_{h\mu})]d\gamma,$$

i.e., one can compute $v_{h3,\lambda}$ instead of $\partial_{h\lambda}(\vec{v}_h)$ in the integral over Γ_1, which reduces the previous consistency error estimate to zero whithout modifying the following analysis. One can check that this slight modification keeps a sense as long as $\underline{M}\in(L^2(\Gamma_1))^2$ and v_{h3} is piecewise polynomial so that $Dv_{h3|\Gamma_1}\in(L^2(\Gamma_1))^2$. ∎

In order to estimate the consistency term $\sup_{\vec{w}_h\in\vec{V}_h}\dfrac{|f^*(\vec{w}_h)-a_h(\vec{u}^*,\vec{w}_h)|}{\|\vec{w}_h\|_h}$, it is convenient to record the following notations :

$$\bar{n}^{\alpha\beta}(\vec{u}^*)=eE^{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}(\vec{u}^*),\quad \bar{m}^{\alpha\beta}(\vec{u}^*)=\frac{e^3}{12}\,E^{\alpha\beta\lambda\mu}\bar{\rho}_{\lambda\mu}(\vec{u}^*),$$

which denotes respectively the (membrane) stress resultant and the (bending) stress couple over the thickness of the shell. Let us also notice that $\vec{u}^*$ is the solution of the Koiter problem so that with (3.12),

$$\bar{\rho}_{h\alpha\beta}(\vec{u}^*) = -\chi_{\alpha\beta}(\vec{u}^*, \underline{\beta}) = \bar{\rho}_{\alpha\beta}(\vec{u}^*), \tag{4.7}$$

where $\beta_\alpha = -u^*_{3,\alpha} - b^\lambda_\alpha u^*_\lambda$. Then, we prove :

LEMMA 4.5. (estimate of the second term of consistency). Under the assumptions of Lemma 4.4 and by assuming that $\vec{u}^* \in (H^2(\Omega))^2 \times H^3(\Omega)$, one has :

$$\sup_{\vec{w}_h \in \vec{V}_h} \frac{|f^*(\vec{w}_h) - a_h(\vec{u}^*, \vec{w}_h)|}{\|\vec{w}_h\|_h} \leq Ch\{\|u^*_1\|^2_{2,\Omega} + \|u^*_2\|^2_{2,\Omega} + \|u^*_3\|^2_{3,\Omega}\}^{1/2}. \tag{4.8}$$

PROOF. First, we decompose the consistency error by using relations (2.5) (2.6) (2.11) (3.11) (3.14) (4.7) and the Green formula (see [33, (1.13.61)], i.e., $\int_\Omega v^\alpha|_\alpha \sqrt{a}\, d\xi^1 d\xi^2 = \int_\Gamma \nu_\alpha v^\alpha d\gamma$, where $\vec{\nu} = \nu_\alpha \vec{a}^\alpha$ denotes the unit outer normal vector to $\vec{\phi}(\Gamma_1)$) :

$$f^*(\vec{w}_h) - a_h(\vec{u}^*, \vec{w}_h) = \int_\Omega [\vec{p}\vec{w}_h - \bar{n}^{\alpha\beta}(\vec{u}^*)\gamma_{\alpha\beta}(\vec{w}_h) - \bar{m}^{\alpha\beta}(\vec{u}^*)\bar{\rho}_{h\alpha\beta}(\vec{w}_h)]\, \sqrt{a}\, d\xi^1 d\xi^2$$

$$+ \int_{\Gamma_1} [\vec{N}\vec{w}_h + M^\lambda(w_{h3,\lambda} + b^\mu_\lambda w_{h\mu})] d\gamma$$

$$= \int_\Omega \{[p^\alpha + (\bar{n}^{\alpha\beta}(\vec{u}^*) + b^\alpha_\lambda \bar{m}^{\lambda\beta}(\vec{u}^*))|_\beta] w_{h\alpha} - \bar{m}^{\alpha\beta}(\vec{u}^*)|_\beta\, \omega_{h\alpha}$$

$$+ [p^3 + b_{\alpha\beta}\bar{n}^{\alpha\beta}(\vec{u}^*) + c_{\alpha\beta}\bar{m}^{\alpha\beta}(\vec{u}^*)] w_{h3}\} \sqrt{a}\, d\xi^1 d\xi^2$$

$$+ \int_{\Gamma_1} \{[N^\alpha - (\bar{n}^{\alpha\beta}(\vec{u}^*) - b^\alpha_\lambda \bar{m}^{\lambda\beta}(\vec{u}^*))\nu_\beta + M^\lambda b^\alpha_\lambda] w_{h\alpha}$$

$$+ \bar{m}^{\alpha\beta}(\vec{u}^*)\nu_\beta\, \omega_{h\alpha} + N^3 w_{h3} + M^\lambda w_{h3,\lambda}\} d\gamma, \quad \forall (\vec{w}_h, \underline{\omega}_h) \in \vec{Z}_h.$$

Since $\vec{u}^*$ is solution of the variational problem (2.10), we have for any $\vec{v} \in \vec{V}^*$:

$$\int_\Omega [\bar{n}^{\alpha\beta}(\vec{u}^*)\gamma_{\alpha\beta}(\vec{v}) + \bar{m}^{\alpha\beta}(\vec{u}^*)\bar{\rho}_{\alpha\beta}(\vec{v})]\, \sqrt{a}\, d\xi^1 d\xi^2$$

$$= \int_\Omega \vec{p}\vec{v}\, \sqrt{a}\, d\xi^1 d\xi^2 + \int_{\Gamma_1} [\vec{N}\vec{v} + M^\lambda(v_{3,\lambda} + b^\mu_\lambda v_\mu)] d\gamma,$$

and since we have assumed that $\vec{u}^* \in (H^2(\Omega))^2 \times H^3(\Omega)$, Green formula gives :

$$\int_\Omega \{[p^\alpha + (\bar{n}^{\alpha\beta}(\vec{u}^*) + 2b^\alpha_\lambda \bar{m}^{\lambda\beta}(\vec{u}^*))|_\beta - b^\alpha_{\lambda|\beta} m^{\lambda\beta}(\vec{u}^*)] v_\alpha$$

$$+ [p^3 + b_{\alpha\beta}\bar{n}^{\alpha\beta}(\vec{u}^*) + c_{\alpha\beta}\bar{m}^{\alpha\beta}(\vec{u}^*)] v_3 + \bar{m}^{\alpha\beta}(\vec{u}^*)|_\beta v_{3,\alpha}\} \sqrt{a}\, d\xi^1 d\xi^2$$

$$+ \int_{\Gamma_1} \{[N^\alpha - M^\lambda b^\alpha_\lambda - (\bar{n}^{\alpha\beta}(\vec{u}^*) - 2\bar{m}^{\lambda\beta}(\vec{u}^*) b^\alpha_\lambda)\nu_\beta] v_\alpha$$

$$+ N^3 v_3 + [M^\alpha - \bar{m}^{\alpha\beta}(\vec{u}^*)\nu_\beta] v_{3,\alpha}\} d\gamma = 0, \quad \forall \vec{v} \in \vec{V}^*.$$

Now, consider the decomposition :

$$v_{3,\alpha} = t_\alpha v_{3,t} + n_\alpha v_{3,n},$$

where t_α and n_α denote the components of the unit tangential and normal vectors as specified in paragraph 3.3. From [14, p. 175], we obtain :

$$\int_{\Gamma_1} [M^\alpha - \bar{m}^{\alpha\beta}(\vec{u}^*)\nu_\beta] n_\alpha v_{3,n}\, d\gamma = 0,$$

so that :

$$\int_\Omega \{[p^\alpha + (\bar{n}^{\alpha\beta}(\vec{u}^*) + 2b^\alpha_\lambda \bar{m}^{\lambda\beta}(\vec{u}^*))|_\beta - b^\alpha_{\lambda|\beta} m^{\lambda\beta}(\vec{u}^*)]v_\alpha$$

$$+ [p^3 + b_{\alpha\beta}\bar{n}^{\alpha\beta}(\vec{u}^*) + c_{\alpha\beta}\bar{m}^{\alpha\beta}(\vec{u}^*)]v_3 + \bar{m}^{\alpha\beta}(\vec{u}^*)|_\beta v_{3,\alpha}\} \sqrt{a}\, d\xi^1 d\xi^2$$

$$+ \int_{\Gamma_1} \{[N^\alpha - M^\lambda b^\alpha_\lambda - (\bar{n}^{\alpha\beta}(\vec{u}^*) - 2\bar{m}^{\lambda\beta}(\vec{u}^*) b^\alpha_\lambda)\nu_\beta] v_\alpha$$

$$+ N^3 v_3 + [M^\alpha - \bar{m}^{\alpha\beta}(\vec{u}^*)\nu_\beta] t_\alpha v_{3,t}\} d\gamma = 0, \quad \forall \vec{v} \in \vec{V}^*.$$

Then, by density and by using [34, Theorem 2.11], we infer that this relation is still true for any $\vec{w}_h \in \vec{W}_h$. Thus, we obtain :

$$f^*(\vec{w}_h) - a_h(\vec{u}^*, \vec{w}_h) = \int_\Omega \{[-b^\alpha_\lambda \bar{m}^{\lambda\beta}(\vec{u}^*)|_\beta w_{h\alpha} - \bar{m}^{\alpha\beta}(\vec{u}^*)|_\beta \omega_{h\alpha} - \bar{m}^{\alpha\beta}(\vec{u}^*)|_\beta w_{h3,\alpha}\} \sqrt{a}\, d\xi^1 d\xi^2$$
$$+ \int_{\Gamma_1} \{b^\alpha_\lambda \bar{m}^{\lambda\beta}(\vec{u}^*)\nu_\beta w_{h\alpha} + \bar{m}^{\alpha\beta}(\vec{u}^*)\nu_\beta \omega_{h\alpha} + \bar{m}^{\alpha\beta}(\vec{u}^*)\nu_\beta w_{h3,\alpha}\} d\gamma,$$

which can also be written with (3.9) :

$$\left.\begin{aligned} f^*(\vec{w}_h) - a_h(\vec{u}^*, \vec{w}_h) = &- \int_\Omega \bar{m}^{\alpha\beta}(\vec{u}^*)|_\beta [w_{h3,\alpha} - \partial_{h\alpha}(\vec{w}_h)] \sqrt{a}\, d\xi^1 d\xi^2 \\ &+ \int_{\Gamma_1} \bar{m}^{\alpha\beta}(\vec{u}^*)\nu_\beta [w_{h3,\alpha} - \partial_{h\alpha}(\vec{w}_h)] d\gamma. \end{aligned}\right\} \tag{4.9}$$

From Lemma 3.1 and by considering the terms which appear in $\bar{m}^{\alpha\beta}(\vec{u}^*)|_\beta$, we obtain :

$$\left.\begin{aligned} &\left|\int_\Omega \bar{m}^{\alpha\beta}(\vec{u}^*)|_\beta (w_{h3,\alpha} - \partial_{h\alpha}(\vec{w}_h)) \sqrt{a}\, d\xi^1 d\xi^2\right| \\ &\le Ch[\|u_1^*\|^2_{2,\Omega} + \|u_2^*\|^2_{2,\Omega} + \|u_3^*\|^2_{3,\Omega}]^{1/2} \|\vec{w}_h\|_h. \end{aligned}\right\} \tag{4.10}$$

On the other hand, the integral along Γ_1 is of the same type than the second hand side of relation (4.6). Since $\vec{u}^* \in (H^2(\Omega))^2 \times H^3(\Omega)$ and since $H^1(\Omega) \hookrightarrow H^{1/2}(\Gamma)$, we have

$$\|m^{\alpha\beta}(\vec{u}^*)\nu_\beta\|_{\frac{1}{2},\Gamma_1} \le C\|m^{\alpha\beta}(\vec{u}^*)\|_{1,\Omega} \le C[\|u_1^*\|^2_{2,\Omega} + \|u_2^*\|^2_{2,\Omega} + \|u_3^*\|^2_{3,\Omega}]^{1/2}.$$

Then, by using Lemma 3.2, we obtain :

$$\left|\int_{\Gamma_1} \bar{m}^{\alpha\beta}(\vec{u}^*)\nu_\beta (w_{h3,\alpha} - \partial_{h\alpha}(\vec{w}_h))\, d\gamma\right| \le Ch[\|u_1^*\|^2_{2,\Omega} + \|u_2^*\|^2_{2,\Omega} + \|u_3^*\|^2_{3,\Omega}]^{1/2} \|\vec{w}_h\|_h. \tag{4.11}$$

The conclusion arises from relations (4.9) (4.10) and (4.11). ■

In order to apply the abstract error estimate (4.2), it just remains to prove that the assumption $(4.1)_1$ is satisfied, that means :

LEMMA 4.6. For any uniformly regular triangulation $\mathcal{T}_h$ in the sense of [28, (3.2.28)] and for h sufficiently small, there exists a constant $\gamma > 0$, independent of h, such that :

$$\gamma \|\vec{v}_h\|_h^2 \leq a_h(\vec{v}_h, \vec{v}_h), \quad \forall \vec{v}_h \in \vec{V}_h. \tag{4.12}$$

PROOF. This lemma is a consequence of the previous interpolation and consistency error estimates and of discrete compactness properties for nonconforming finite element spaces obtained in [35]. The proof takes two steps :

STEP 1 : For h sufficiently small, the bilinear form $a_h(.,.)$ is uniformly $\vec{V}_h$-coercive, e.g., there exists two positive constants C_1 and C_2, independent of h, such that, for any $\vec{v}_h \in \vec{V}_h$, we have :

$$a_h(\vec{v}_h, \vec{v}_h) \geq C_1 \|\vec{v}_h\|_h^2 - C_2[\|v_{h1}\|_{0,\Omega}^2 + \|v_{h2}\|_{0,\Omega}^2 + \|v_{h3}\|_{1,\Omega}^2]. \tag{4.13}$$

According to [24, Theorem 6.1.1] and from (2.5) (3.10) and (3.14), the contribution $a_{hT}(.,.)$ of every triangle $T \in \mathcal{T}_h$ to the bilinear form $a_h(.,.)$ satisfies :

$$\left.\begin{aligned} a_{hT}(\vec{v}_h, \vec{v}_h) \geq \frac{1}{L} \{&\|v_{h1}\|_{1,T}^2 + \|v_{h2}\|_{1,T}^2 + \|v_{h3}\|_{1,T}^2 \\ &+ \|\partial_{h11}(\vec{v}_h) - \Gamma_{11}^\lambda \partial_{h\lambda}(\vec{v}_h)\|_{0,T}^2 + \|\partial_{h12}(\vec{v}_h) - \Gamma_{12}^\lambda \partial_{h\lambda}(\vec{v}_h)\|_{0,T}^2 \\ &+ \|\partial_{h22}(\vec{v}_h) - \Gamma_{22}^\lambda \partial_{h\lambda}(\vec{v}_h)\|_{0,T}^2\} - \frac{M}{L}\{\|v_{h1}\|_{0,T}^2 + \|v_{h2}\|_{0,T}^2 + \|v_{h3}\|_{1,T}^2\}, \ \forall \vec{v}_h \in \vec{V}_h, \end{aligned}\right\} \tag{4.14}$$

where the constants $L > 0$ and $M \geq 0$ are independent of h.

From relations (3.10) and (3.19), i.e.,

$$\partial_{h\alpha\beta}(\vec{v}_h) = \frac{1}{2}(\partial_\alpha \partial_{h\beta}(\vec{v}_h) + \partial_\beta \partial_{h\alpha}(\vec{v}_h)),$$

$$\partial_{h\alpha}(\vec{v}_h)|_T = v_{h3,\alpha} + \pi_T^2(b_\alpha^\lambda v_{h\lambda}) - b_\alpha^\lambda v_{h\lambda} + \frac{1}{2}\sum_{i=1}^3 [\lambda_{i+1}\lambda_{i-1} n_{\alpha i} D^3 v_{h3}(b_i)((a_{i+1} - a_{i-1})^2, \vec{n}_i)],$$

and from [24, Theorem 6.1.1, inequality (6.1.7)], we obtain :

$$\left.\begin{aligned} \|\partial_{h11}(\vec{v}_h) - \Gamma_{11}^\lambda \partial_{h\lambda}(\vec{v}_h)\|_{0,T}^2 &= \|\partial_1 \partial_{h1}(\vec{v}_h) - \Gamma_{11}^\lambda[\partial_{h\lambda}(\vec{v}_h) - v_{h3,\lambda}] - \Gamma_{11}^\lambda v_{h3,\lambda}\|_{0,T}^2 \\ &\geq \frac{\beta}{1+\beta}\|v_{h3,11} + \frac{1}{2}\sum_{i=1}^3 (\lambda_{i+1}\lambda_{i-1})_{,1} n_{1i} D^3 v_{h3}(b_i)((a_{i+1} - a_{i-1})^2, \vec{n}_i)\|_{0,T}^2 \\ &\quad - 5\beta\{\|[\pi_T^2(b_1^\lambda v_{h\lambda}) - b_1^\lambda v_{h\lambda}]_{,1}\|_{0,T}^2 + \rho_1^2[\|\partial_{h1}(\vec{v}_h) - v_{h3,1}\|_{0,T}^2 \\ &\quad + \|\partial_{h2}(\vec{v}_h) - v_{h3,2}\|_{0,T}^2 + \|v_{h3,1}\|_{0,T}^2 + \|v_{h3,2}\|_{0,T}^2]\}, \end{aligned}\right\} \tag{4.15}$$

where $\beta > 0$ is any constant independent of h and where $\rho_1 > 0$ is an upperbound on geometrical characteristics of the middle surface of the shell (see [24, (3.1.1)]). Of course, we can get similar estimates for $\|\partial_{h12}(\vec{v}_h) - \Gamma_{12}^\lambda \partial_{h\lambda}(\vec{v}_h)\|_{0,T}^2$ and for $\|\partial_{h22}(\vec{v}_h) - \Gamma_{22}^\lambda \partial_{h\lambda}(\vec{v}_h)\|_{0,T}^2$.

But

$$\|[\pi_T^2(b_\alpha^\lambda v_{h\lambda}) - b_\alpha^\lambda v_{h\lambda}]_{,\beta}\|_{0,T}^2 \leq C\{\|v_{h1}\|_{1,T}^2 + \|v_{h2}\|_{1,T}^2\},$$

and, from Lemma 3.1, we have :

$$\|\partial_{h\alpha}(\vec{v}_h) - v_{h3,\alpha}\|_{0,T}^2 \leq C[h_T^2\{\|v_{h1}\|_{1,T}^2 + \|v_{h2}\|_{1,T}^2\} + |v_{h3}|_{1,T}^2].$$

Then, by taking β sufficiently small in estimate (4.15) and in their analogues, we can write (4.14) as follows :

$$\left.\begin{array}{c} a_{hT}(\vec{v}_h, \vec{v}_h) \geq C_3\{\|v_{h1}\|_{1,T}^2 + \|v_{h2}\|_{1,T}^2 + \|v_{h3}\|_{1,T}^2 \\ + \|v_{h3,11} + \frac{1}{2}\sum_{i=1}^{3}(\lambda_{i+1}\lambda_{i-1})_{,1}n_{1i}D^3v_{h3}(b_i)((a_{i+1}-a_{i-1})^2, \vec{n}_i)\|_{0,T}^2 \\ + \|v_{h3,12} + \frac{1}{4}\sum_{i=1}^{3}(\lambda_{i+1}\lambda_{i-1})_{,1}n_{2i} + (\lambda_{i+1}\lambda_{i-1})_{,2}n_{1i}]D^3v_{h3}(b_i)((a_{i+1}-a_{i-1})^2, \vec{n}_i)\|_{0,T}^2 \\ + \|v_{h3,22} + \frac{1}{2}\sum_{i=1}^{3}(\lambda_{i+1}\lambda_{i-1})_{,2}n_{2i}D^3v_{h3}(b_i)((a_{i+1}-a_{i-1})^2, \vec{n}_i)\|_{0,T}^2\} \\ - C_2\{\|v_{h1}\|_{0,T}^2 + \|v_{h2}\|_{0,T}^2 + \|v_{h3}\|_{1,T}^2\}, \end{array}\right\} \quad (4.16)$$

where C_2 and C_3 are constants > 0 and independent of h. We obtain (4.13) by proving the existence of a constant $C > 0$, independent of h, such that relation (4.17) below is true and then by adding it upon all $T \in \mathcal{T}_h$.

$$\left.\begin{array}{c} \|v_{h3,11} + \frac{1}{2}\sum_{i=1}^{3}(\lambda_{i+1}\lambda_{i-1})_{,1}n_{1i}D^3v_{h3}(b_i)((a_{i+1}-a_{i-1})^2, \vec{n}_i)\|_{0,T}^2 \\ + \|v_{h3,12} + \frac{1}{4}\sum_{i=1}^{3}(\lambda_{i+1}\lambda_{i-1})_{,1}n_{2i} + (\lambda_{i+1}\lambda_{i-1})_{,2}n_{1i}]D^3v_{h3}(b_i)((a_{i+1}-a_{i-1})^2, \vec{n}_i)\|_{0,T}^2 \\ + \|v_{h3,22} + \frac{1}{2}\sum_{i=1}^{3}(\lambda_{i+1}\lambda_{i-1})_{,2}n_{2i}D^3v_{h3}(b_i)((a_{i+1}-a_{i-1})^2, \vec{n}_i)\|_{0,T}^2 \\ \geq C|v_{h3}|_{2,T}^2, \quad \forall v_{h3} \in V_{h2}. \end{array}\right\} \quad (4.17)$$

This proof follows the lines of [10, Lemma 1] and takes three points :

i) *Proof of the relation (4.18) below :*

$$\left.\begin{array}{c} \sum_{\substack{\alpha,\beta=1 \\ \alpha\leq\beta}}^{2} \|v_{h3,\alpha\beta} + \frac{1}{4}\sum_{i=1}^{3}[(\lambda_{i+1}\lambda_{i-1})_{,\alpha}n_{\beta i} \\ + (\lambda_{i+1}\lambda_{i-1})_{,\beta}n_{\alpha i}]D^3v_{h3}(b_i)((a_{i+1}-a_{i-1})^2, \vec{n}_i)\|_{0,T} = 0 \\ \text{implies } |v_{h3}|_{2,T} = 0. \end{array}\right\} \quad (4.18)$$

Firstly, let us observe from (3.19) that upon any $T \in \mathcal{T}_h$:

$$v_{h3,\alpha} + \frac{1}{2}\sum_{i=1}^{3}\lambda_{i+1}\lambda_{i-1}n_{\alpha i}D^3v_{h3}(b_i)((a_{i+1}-a_{i-1})^2, \vec{n}_i) = \partial_{h\alpha}(\vec{v}_h)|_T + b_\alpha^\lambda v_{h\lambda} - \pi_T^2(b_\alpha^\lambda v_{h\lambda}),$$

so that

$$\partial_{h\alpha}(\vec{v}_h)|_T + b_\alpha^\lambda v_{h\lambda} - \pi_T^2(b_\alpha^\lambda v_{h\lambda}) \in P_2(T), \quad \forall \vec{v}_h \in \vec{V}_h, \ \forall T \in \mathcal{T}_h.$$

Then, the assumption (4.18) involves :

$$\left.\begin{array}{l}\partial_{h1}(\vec{v}_h)|_T + b_1^\lambda v_{h\lambda} - \pi_T^2(b_1^\lambda v_{h\lambda}) = c_1 + c_3\xi^2 \\ \\ \partial_{h2}(\vec{v}_h)|_T + b_2^\lambda v_{h\lambda} - \pi_T^2(b_2^\lambda v_{h\lambda}) = c_2 - c_3\xi^1\end{array}\right\} \tag{4.19}$$

where c_i, $i = 1,2,3$, are appropriate constants attached to the triangle T.

Now, from the discrete Kirchhoff constraints (3.2) and from (3.9), we have at the vertices a_i of the triangle T :

$$[\partial_{h\alpha}(\vec{v}_h) + b_\alpha^\lambda v_{h\lambda} - \pi_T^2(b_\alpha^\lambda v_{h\lambda})](a_i) = \partial_{h\alpha}(\vec{v}_h)(a_i) = v_{h3,\alpha}(a_i),$$

so that, if we define θ_{ht}^i along every side $[a_{i+1}, a_{i-1}]$ of ∂T as follows :

$$\theta_{ht}^i = t_i^\alpha[\partial_{h\alpha}(\vec{v}_h) + b_\alpha^\lambda v_{h\lambda} - \pi_T^2(b_\alpha^\lambda v_{h\lambda})]|_{\partial T_i}, \tag{4.20}$$

we obtain :

$$\theta_{ht}^i(a_{i-1}) = v_{h3,t}(a_{i-1}), \quad \theta_{ht}^i(a_{i+1}) = v_{h3,t}(a_{i+1}).$$

Likewise, from the discrete Kirchhoff constraints (3.3) and from relation (3.9), we obtain :

$$\theta_{ht}^i(b_i) = v_{h3,t}(b_i),$$

and then, since θ_{ht}^i and $v_{h3,t}$ belong to $P_2([a_{i+1}, a_{i-1}])$ we have :

$$\theta_{ht}^i \equiv v_{h3,t} \text{ along } [a_{i+1}, a_{i-1}]. \tag{4.21}$$

Hence, by using relations (4.20) (4.21) and Green's formula in connection with $(t_i^1, t_i^2) = (n_i^2, -n_i^1)$, we obtain (see Fig. 3) :

$$0 = \int_{\partial T} v_{h3,t}\,dt = \sum_{i=1}^{3} \int_{\partial T_i} \theta_{ht}^i\,dt$$
$$= \int_T [\partial_2(\partial_{h1}(\vec{v}_h) + b_1^\lambda v_{h\lambda} - \pi_T^2(b_1^\lambda v_{h\lambda})) - \partial_1(\partial_{h2}(\vec{v}_h) + b_2^\lambda v_{h\lambda} - \pi_T^2(b_2^\lambda v_{h\lambda}))]d\xi^1 d\xi^2,$$

so that with relations (4.19), we get $0 = 2c_3$ area(T). It follows $c_3 = 0$ and then, with (4.19), $[\partial_{h\alpha}(\vec{v}_h) + b_\alpha^\lambda v_{h\lambda} - \pi_T^2(b_\alpha^\lambda v_{h\lambda})]|_T = c_\alpha$, $\alpha = 1,2$, so that :

$$v_{h3,\alpha}(a_i) = c_\alpha, \quad \alpha = 1,2, \;;\; i = 1,2,3.$$

Relation (4.20) involves that θ_{ht}^i is constant along each side ∂T_i of triangle T ; then, by using relation (4.21) we get that $v_{h3,t}$ is constant upon each side ∂T_i of triangle T. In other words, we have :

$$Dv_{h3}(a_{i+1})(a_{i-1} - a_{i+1}) = Dv_{h3}(a_{i-1})(a_{i-1} - a_{i+1}) = v_{h3}(a_{i-1}) - v_{h3}(a_{i+1}), \tag{4.22}$$

and two similar relations when one considers sides ∂T_{i+1} and ∂T_{i-1}. Since $v_{h3|T} \in$ Hermite triangle of type (3'), we obtain :

$$v_{h3} = \sum_i(-2\lambda_i^3 + 3\lambda_i^2 + 2\lambda_1\lambda_2\lambda_3)v_{h3}(a_i) + \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{3} \lambda_i\lambda_j(\lambda_i - \lambda_j + 1)Dv_{h3}(a_i)(a_j - a_i),$$

and then the substitution of relation (4.22) and of their analogs along ∂T_{i+1} and ∂T_{i-1} gives finally :

$$v_{h3|T} = \sum_{i=1}^{3} \lambda_i v_{h3}(a_i).$$

Thus, we get $|v_{h3}|_{2,T} = 0$ and (4.18) is proved.

ii) *Proof of relation (4.17) :*

Over the quotient space P_3'/P_1, the semi-norm $|v_{h3}|_{2,T}$ is a norm. Likewise, the step 1 proves that the left hand member of the relation (4.17) is a norm over the same quotient space. Since these norms are defined over the same finite dimensional space, they are equivalent. Then, it remains to prove that the equivalence constants are independent of the triangle T into consideration. In other words, we have to prove that the constant C in (4.17) is independent of T.

In order to clear the dependence on h of both quantities, we express both hand sides of relation (4.17) over any triangle $T \in \mathcal{T}_h$ as functions of the barycentric coordinates (λ_i, $i = 1, 2, 3$). Firstly, if $v_{h3}|_T \in$ Hermite triangle of type (3'), then (see [12]) :

$$v_{h3}(\xi^1, \xi^2) = [DLLC3(v_{h3})]_{1\times 9} \ [A_3]_{9\times 10} \ [\lambda 3]_{10\times 1},$$

where :

$$[DLLC3(v_{h3})]_{1\times 9} = [v_{h3}(a_1) \ v_{h3}(a_2) \ v_{h3}(a_3) \ Dv_{h3}(a_1)(a_3 - a_1) \ Dv_{h3}(a_1)(a_2 - a_1)$$

$$Dv_{h3}(a_2)(a_1 - a_2) \ Dv_{h3}(a_2)(a_3 - a_2) \ Dv_{h3}(a_3)(a_2 - a_3) \ Dv_{h3}(a_3)(a_1 - a_3)],$$

$$[A_3]_{9\times 10} = \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 & 6 & 6 & 0 & 0 & 0 & 0 & 4 \\ 0 & 2 & 0 & 0 & 0 & 6 & 6 & 0 & 0 & 4 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 6 & 6 & 4 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \end{bmatrix},$$

and :

$${}^t[\lambda 3] = [\lambda_1^3 \ \lambda_2^3 \ \lambda_3^3 \ \lambda_1^2\lambda_3 \ \lambda_1^2\lambda_2 \ \lambda_2^2\lambda_1 \ \lambda_2^2\lambda_3 \ \lambda_3^2\lambda_2 \ \lambda_3^2\lambda_1 \ \lambda_1\lambda_2\lambda_3].$$

Consequently :

$$v_{h3,\alpha\beta}(\xi^1, \xi^2) = [DLLC3(v_{h3})]_{1\times 9} \ [A_3]_{9\times 10} \ [\partial_{\alpha\beta}\lambda 3]_{10\times 1}.$$

Then, we get :

$$|v_{h3}|_{2,T}^2 = [DLLC3(v_{h3})]_{1\times 9} \ [A_3]_{9\times 10} \ [Y]_{10\times 10} \ {}^t[A_3] \ {}^t[DLLC3(v_{h3})],$$

where :

$$[Y]_{10\times 10} = \int_T ([\partial_{11}\lambda 3] \ {}^t[\partial_{11}\lambda 3] + 2[\partial_{12}\lambda 3] \ {}^t[\partial_{12}\lambda 3] + [\partial_{22}\lambda 3] \ {}^t[\partial_{22}\lambda 3]) d\xi^1 d\xi^2,$$

$$[\partial_{11}\lambda 3] = \frac{1}{\Delta^2}\left\{(\xi_2^2-\xi_3^2)^2\left[\frac{\partial^2\lambda 3}{(\partial\lambda_1)^2}\right] + (\xi_3^2-\xi_1^2)^2\left[\frac{\partial^2\lambda 3}{(\partial\lambda_2)^2}\right] + (\xi_1^2-\xi_2^2)^2\left[\frac{\partial^2\lambda 3}{(\partial\lambda_3)^2}\right]\right.$$
$$+ 2(\xi_2^2-\xi_3^2)(\xi_3^2-\xi_1^2)\left[\frac{\partial^2\lambda 3}{\partial\lambda_1\partial\lambda_2}\right] + 2(\xi_3^2-\xi_1^2)(\xi_1^2-\xi_2^2)\left[\frac{\partial^2\lambda 3}{\partial\lambda_2\partial\lambda_3}\right]$$
$$\left. + 2(\xi_1^2-\xi_2^2)(\xi_2^2-\xi_3^2)\left[\frac{\partial^2\lambda 3}{\partial\lambda_3\partial\lambda_1}\right]\right\},$$

and similar expressions for $[\partial_{12}\lambda 3]$ and $[\partial_{22}\lambda 3]$. Then, $\Delta = 2\text{area}(T) = 0(h_T^2)$ so that $[\partial_{11}\lambda 3] = 0(h_T^{-2})$, $[Y] = 0(h_T^{-2})$ and $|v_{h3}|_{2,T}^2 = 0(h_T^{-2})$.

Concerning the first hand side of (4.17), let us observe that $v_{h3,11} = 0(h_T^{-2})$, $(\lambda_{i+1}\lambda_{i-1})_{,1} = 0(h_T^{-1})$, $D^3 v_{h3}(b_i)((a_{i+1}-a_{i-1})^2, \vec{n}_i) = 0(h_T^{-1})$ so that :

$$v_{h3,11} + \frac{1}{2}\sum_{i=1}^{3}(\lambda_{i+1}\lambda_{i-1})_{,1} n_{1i} D^3 v_{h3}(b_i)((a_{i+1}-a_{i-1})^2, \vec{n}_i) = 0(h_T^{-2}),$$

and then the first hand side is in $0(h_T^{-2})$.

Thus, both members of the inequality (4.17) are in $0(h_T^{-2})$ so that the constant C is necessarily independent of h, for h sufficiently small.

iii) *Proof of inequality (4.13) :*

We obtain the inequality (4.13) by substitution of estimate (4.17) into (4.16) and by adding it upon all $T \in \mathcal{T}_h$.

STEP 2 : For h sufficiently small, the bilinear form $a_h(.,.)$ is uniformly $\vec{V}_h$-elliptic, i.e., inequality (4.12) is true.

Let us assume that inequality (4.12) is false. Then, there exists a sequence $\{\vec{v}_{h_k}\}$ of functions of $\vec{V}_{h_k}$, $\lim_{k\to\infty} h_k = 0$, such that :

$$\|\vec{v}_{h_k}\|_{h_k} = 1, \quad [a_{h_k}(\vec{v}_{h_k}, \vec{v}_{h_k})]^{1/2} < \frac{1}{k}.$$

Therefore, in the same way than in [30, Lemma 2.2.2], we deduce that the sequence $\{\vec{v}_{h_k}\}$ is weakly convergent to $\vec{0}$. Now, from the weak discrete compactness of V_{h2} in $\pi_{T\in\mathcal{T}_h} H^1(T)$ (see [35, p. 98]) and from the compactness of the inclusion of V_{h1} in $L^2(\Omega)$, we deduce :

$$\lim_{k\to\infty}(\|\vec{v}_{h_k1}\|_{0,\Omega}^2 + \|\vec{v}_{h_k2}\|_{0,\Omega}^2 + \|\vec{v}_{h_k3}\|_{1,\Omega}^2) = 0,$$

so that

$$\lim_{k\to\infty}[a_{h_k}(\vec{v}_{h_k}, \vec{v}_{h_k}) + C_2(\|\vec{v}_{h_k1}\|_{0,\Omega}^2 + \|\vec{v}_{h_k2}\|_{0,\Omega}^2 + \|\vec{v}_{h_k3}\|_{1,\Omega}^2)] = 0.$$

Finally, the inequality (4.13) involves :

$$0 < C_1 = C_1\|\vec{v}_{h_k}\|_{h_k}^2 \le [a_{h_k}(\vec{v}_{h_k}, \vec{v}_{h_k}) + C_2(\|\vec{v}_{h_k1}\|_{0,\Omega}^2 + \|\vec{v}_{h_k2}\|_{0,\Omega}^2 + \|\vec{v}_{h_k3}\|_{1,\Omega}^2)],$$

which leads to a contradiction. Hence, the inequality (4.12) is satisfied with γ independent of h.

Let us recall that to apply Stummel's result [33], one has to check a "strong continuity property" at each common edge between two adjacent triangles of $\mathcal{T}_h$: this is satisfied because the nodal scheme allows the full continuity of v_{h3} and $v_{h3,\alpha}$ at, at least, one point of such a side (for example at the vertices). ∎

THEOREM 4.1. (Convergence and error estimate theorem). Let us assume that i) the middle surface of the shell is the image of a plane polygonal domain Ω by a mapping $\vec{\phi} \in (W^{4,\infty}(\Omega))^3$; ii) the triangulation $\mathcal{T}_h$ of the domain Ω is uniformly regular in the sense of [28, (3.2.28)] ; iii) the loads $\vec{p} \in (L^2(\Omega))^3$, $\vec{N} \in (L^2(\Gamma_1))^3$ and $\underline{M} \in (H^{1/2}(\Gamma_1))^2$; iv) the solution $\vec{u}^*$ of the Koiter problem (2.10) belongs to the space $\vec{V} \cap [(H^2(\Omega))^2 \times H^3(\Omega)]$.

Then, for h sufficiently small, the Problem 3.2 has one and only one solution $\vec{u}_h \in \vec{V}_h$ which satisfies :

$$\|\vec{u}^* - \vec{u}_h\|_h \leq Ch\{(\|u_1^*\|_{2,\Omega}^2 + \|u_2^*\|_{2,\Omega}^2 + \|u_3^*\|_{3,\Omega}^2)^{1/2} + |\underline{M}|_{\frac{1}{2},\Gamma_1}\}. \tag{4.23}$$

PROOF. From Lemmas 4.2 and 4.6, we can use the abstract error estimate (4.2). With (4.4) (4.5) and (4.8) we obtain (4.23). ∎

REMARK 4.2. Let us add that if one has only $\underline{M} \in (L^2(\Gamma_1))^2$, the discretization error $\|\vec{u} - \vec{u}_h\|_h$ would be $0(h^{1/2})$. Of course when $\underline{M} = \underline{0}$ or measure$(\Gamma_1) = 0$, we recover the full rate of convergence. ∎

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