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Eitan Altman, Serguei Foss

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POLLING ON A GRAPH WITH GENERAL ARRIVAL AND SERVICE TIME DISTRIBUTION

Eitan ALTMAN, Serguei FOSS

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**Institut National
de Recherche
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et en Automatique**

**2004 route des Lucioles
B.P. 93
06902 Sophia-Antipolis
France**

Polling sur un graphe avec des temps d'inter-arrivées et de services avec distributions générales *

Eitan ALTMAN, Serguei FOSS

INRIA Centre Sophia Antipolis
2004 Route des Lucioles
06560 Valbonne, France

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Abstract

Nous considérons un système de polling sur un graphe avec des temps d'inter-arrivées, de services et de déplacements du serveur qui forment des suites i.i.d. Nous présentons des conditions suffisantes et nécessaires de stabilité, en utilisant une analyse de la dérive de la charge dans le système et des temps résiduels des inter-arrivées, pendant les cycles.

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Eitan ALTMAN, Serguei FOSS
INRIA Centre Sophia Antipolis
2004 Route des Lucioles
06560 Valbonne, France

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abstract

We consider a polling system on a graph with general independent arrival, service and walking times. We present necessary and sufficient conditions for stability, based on drift analysis of both the workload and the residual interarrival times within some embedded times called cycles.

Keywords: Polling systems, stability condition, general arrival, service and walking times.

1 Introduction

We consider a polling system on a graph $\mathcal{G}$; a single server follows some fixed path L that starts at some point 0 and returns to that point after having scanned the whole graph (not necessarily in a cyclic way). We consider some general independent arrival process of customers. The location of the arrivals are i.i.d. If the probability distribution of the location of arrivals have some mass at some points then queueing may occur there. The server can serve one customer at a time. In addition to the service time, the server also requires some random time (that may depend on its trajectory) to move from one customer to another.

Such systems have been widely used as models in wide areas of applications in telecommunication systems in general and for Local Area Networks in particular, computer systems, reader-head's movement in a computer's hard disk, flexible manufacturing systems, road traffic control, repair and maintenance. For overviews on polling systems, see Takagi [20] and [24]. Growing attention was given in recent years to establish rigorously sufficient and necessary stability conditions for such systems.

Several different approaches were used to obtain ergodicity as a measure of stability, see [3], [5], [6], [12], [13], [19] and [25] for the stability of "discrete" polling systems (consisting of N queues). Kroese and Schmidt [14], [15], studied the stability of polling on a graph (i.e. "continuous" polling models), and Altman and Levy [4] studied the stability of non cyclic polling in two and higher dimensional plains. In [3], and [6] Altman et al. further present sufficient and necessary conditions for stronger notions of stability, namely the geometric ergodicity and the geometric rate of convergence of the first moment of the process of queues' length, (embedded at polling instants). Sufficient conditions for Central Limit Theorems and the Law of Iterated Logarithm were given in [6]. All the above references assumed Poisson arrivals and general independent walking times and service times. The contribution of this paper is to consider polling with general independent arrival process.

We study several notions of stability in this paper, and obtain in particular necessary and sufficient conditions for the expected workload in the system to be uniformly bounded in time. We show that under some second moment conditions, if the rate ρ at which work arrives to the system (defined later) is smaller than one then the system is stable.

An alternative sample-path approach for studying the stability of polling systems is given by Altman et al. [1]. Different type of conditions are imposed on a single sample path, which yields several kinds of notions of stability under suitable assumptions.

The structure of the paper is as follows. In Section 2 we present the model. We define a "cycle", and present a drift analysis of the behavior of the workload (Section 3) and the residual interarrival times (Section 4) within a single cycle. The stability results are finally presented in Section 5.

2 model

Consider a polling system on a graph $\mathcal{G}$. We assume that a single server follows some fixed path L that starts at some point 0 and returns to that point after having scanned the whole graph. Define such a scan as a cycle. Note that in a cycle, some points (including 0) may be scanned several times. Moreover, n consecutive cycles can be considered as one large cycle.

Customers arrive to the system to arbitrary locations on the graph. Arriving customers are routed to some location according to a routing sequence, assumed to be i.i.d. If the distribution of this location has mass at some points, then queueing phenomena may occur there. The processes of inter-arrival times and service times of customers are assumed to be independent, and each of them forms i.i.d. sequence.

It takes some time for the server to move from one point to another on the graph. The total walking times at different cycles are i.i.d. and independent of arrival and service processes.

The number of customers that are served at point $l \in L$ on the graph at each visit is determined by some polling discipline (such as the gated, exhaustive etc) which may be random and may depend on the history of the processes.

3 Drift analysis in a single cycle

We first focus on drift analysis within a single cycle. We are interested to upper bound in some sense the difference between the work present at the beginning of the cycle and the end of the cycle.

We introduce the following random variables, defined on some probability space $(\Omega, \mathcal{F}, \mathcal{P})$.
T is a random variable representing the time at which the cycle begins.
$\vec{V}$ is a random function $\vec{V} : \mathcal{G} \to \mathbb{R}$, representing the distribution of workload in the graph at time T.
$V = \int_{x \in \mathcal{G}} V(x)dx$ is the total amount of work in the system at time T.
χ is the residual interarrival time (i.e. the time between T and the next arrival).
$\{\tau_i\}_{i=1}^{\infty}$ are i.i.d. random variables representing the interarrival times; τ_1 is the time between the arrival that occurs at time $T + \chi$ and the following arrival. We shall assume that $0 < E\tau_i < \infty$.
$\{\sigma_i\}_{i=1}^{\infty}$ are random variables representing the service times; σ_i is the service time required by the ith arriving customer (after T). We shall assume that $0 \leq E\sigma_i < \infty$.
$\rho := E\sigma_1 / E\tau_1$.

$\{\alpha_i\}_{i=1}^{\infty}$ are random variables representing the location of the arrivals.
D is the total walking time in the cycle.

We assume that $0 < E[D] < \infty$; $\{\sigma_i, \alpha_i\}_{i=1}^{\infty}$ are assumed to be i.i.d, although σ_i and α_i may be dependent.

Define
η:= a nonnegative integer-valued random variable representing the number of arrivals that occur in the cycle.
Define $\mathcal{F}^l$ to be the σ-algebra generated by $\{\{\tau_i\}_{i>l}, \{\sigma_i\}_{i>l}, \{\alpha_i\}_{i>l}\}$. For all $l = 0, 1, 2, ...,$

$$\{\eta \leq l\} \text{ is independent of } \mathcal{F}^l \tag{1}$$

and $\{\eta = 0\} = \{V + D \leq \chi\}$. We further assume that

$$(T, \vec{V}, \chi, D) \text{ are independent of } \mathcal{F}^0 \tag{2}$$

G:= a nonnegative random variable
(1) $\forall l = 0, 1, ...$ and for all sets $A \in \mathcal{F}$, the event $\{G \in A\} \cap \{\eta = l\}$ is independent of $\mathcal{F}^l$.
(2) $G \leq \tau_\eta$ a.s. on the event $\eta \geq 1$;
(3) $G = V + D$ on the event $\{\eta = 0\}$.
(4) G is the past interarrival time of the last arrival that occurs in the cycle, on the event $\{\eta > 0\}$.

Denote $\tau_0 = \chi$, (so that $G \leq \tau_\eta$ a.s.).

Introduce
$T_1 = T + G + \sum_{i=1}^{\eta} \tau_i$ which represents the random time at which the cycle ends (where, by convention, $\sum_{i=1}^{0} \tau_i = 0$).
$\chi_1 = \tau_\eta - G$ which represents the residual interarrival time at time T_1 (i.e. the time between T_1 and the next arrival).
$\Gamma = \sum_{j=1}^{\eta} \sigma_j$ is the total work arriving during the cycle.
V^{out}:= the amount of work that left the system during the cycle. It satisfies $V^{out} + D = T_1 - T$.
V_1:= the amount of work at time T_1. Thus $V_1 = V + \Gamma - V^{out}$.

Lemma 3.1 *Assume $E[\eta] < \infty$. The following hold:*

$$E(\Gamma|T, \vec{V}, \chi, D) = E(\eta|T, \vec{V}, \chi, D)E\sigma_1 \qquad a.s. \tag{3}$$

$$E(\sum_{i=1}^{\eta} \tau_i|T, \vec{V}, \chi, D) = E(\eta|T, \vec{V}, \chi, D)E\tau_1 \qquad a.s. \tag{4}$$

$$E(\Gamma|T, \vec{V}, \chi, D) = \rho E(T_1 + \chi_1 - T - \chi|T, \vec{V}, \chi, D) \qquad a.s. \tag{5}$$

Proof. Follows from Wald's identity, (1), (2) and the fact that

$$T_1 + \chi_1 - T - \chi = \sum_{i=1}^{\eta} \tau_i.$$

■

Denote

$$\tilde{\Delta} = E(\Gamma|T, \vec{V}, \chi, D) - \rho E(T_1 - T|T, \vec{V}, \chi, D)$$

Then it follows that

$$\tilde{\Delta} = E(\chi_1 - \chi|T, \vec{V}, \chi, D) \qquad a.s.$$

We are now ready to present the main result of the section.

Lemma 3.2 *Assume $E[\eta] < \infty$. Then*

$$E(V_1|T, \vec{V}) - V = \rho E[D] + (\rho - 1)E(V^{out}|T, \vec{V}) + \rho E(\chi_1 - \chi|T, \vec{V}) \qquad a.s. \tag{6}$$

Proof. Follows from (5). ■

4 Residual interarrival times

4.1 The Poisson case, and Borovkov's conditions

Below are some cases where the term $E(\chi_1 - \chi|T, \vec{V})$ can be evaluated. A general treatment of this issue is delayed to the next section.

(i) In the case Poisson arrivals, χ and χ_1 are both independent of T and $\vec{V}$, and have the same distribution with finite mean. In that case (6) simplifies to

$$E(V_1|T,\vec{V}) - V = \rho E[D] + (\rho-1)E(V^{out}|T,\vec{V}) \qquad a.s. \tag{7}$$

(ii) In our case the following holds. The conditional distribution of χ_1 given $\eta \geq 1, G, T, \vec{V}, \chi, D$ depends on G only, i.e.

$$P(\chi_1 > t|\eta, G, T, V, \chi, D) = P(\tau_1 > t + x|\tau_1 > x) \qquad a.s.$$

on the event $\{\eta \geq 1\} \cap \{G = x\}$. Assume now that for some $c < \infty$ and for all $t, x > 0$,

$$P(\tau_1 > t + x|\tau_1 > x) \leq cP(\tau_1 > t) \tag{8}$$

(This condition is known as Borovkov's condition, see [9]). Then

$$\rho E(\chi_1 - \chi|T,\vec{V}) \leq \rho E(\chi_1|T,\vec{V}) = E\left[E(\chi_1|\eta, G, T, \vec{V}, \chi, D)\Big|T,\vec{V}\right] \leq cE[\tau_1] < \infty \qquad a.s.$$

Hence we get

$$E(V_1|T,\vec{V}) - V \leq \rho E[D] + (\rho-1)E(V^{out}|T,\vec{V}) + cE\tau_1 \qquad a.s. \tag{9}$$

4.2 2nd moment conditions: Exhaustive and Gated type policies

In this Section we present a drift analysis for the residual interarrival times. For simplicity, we restrict to a cyclic polling system composed of N queues, and begin by analysing the case where each of the queues is surved by either a gated or an exhaustive policy.

In order to have a "stable" behavior of the residual interarrival times, it is not necessary to require that $\rho < 1$. However, if queue n is served according to the exhaustive service discipline then we need to assume that the rate at which work arrives to that queue is less than 1, or equivalently,

$$E[\tau_1] > E[\sigma_1 I\{\alpha_1 = n\}]. \tag{10}$$

We shall consider $2N$ phases in the cycle: at phase $2n-1$, $n = 1, ..., N$, queue n are visited and served. At phase $2n$, $n = 1, ..., N$, the server moves (walks) from queue n to the next queue, during a random time D^n.

Let χ_n be the residual interarrival time at the beginning of phase n, and $\hat{\chi}(t)$ be the the residual interarrival time at time t. Let $\vec{V}_n$ denote the vector of workloads in the different queues at the beginning of the nth phase. In particular, we have $\chi_1 = \chi$, and we shall understand $\chi_{2N+1} = \chi'$ (where χ and χ' were defined in the previous Section, as the residual interarrival time in the beginning and in the end of the cycle, respectively).

Since we consider gated and exhaustive policies only, it is easily seen that $(\vec{V}_n, \chi_n)$ constitute an embedded Markov chain.

We assume throughout the Section that the interarrival times have finite second moments. It then follows from standard renewal arguments that there exists some finite constant C such that

$$\sup_{t \geq 0} E[\hat{\chi}(t)] = C \tag{11}$$

Lemma 4.1 *(i) For each $n = 1, 2, ..., 2N$, there exist some finite constants M_n, C_n and $\epsilon_n \geq 0$ such that*

$$E[\chi_{n+1} | \vec{V}_n, \chi_n] \leq (\chi_n - \epsilon) I\{\chi_n > M_n\} + C_n I\{\chi_n \leq M_n\} \tag{12}$$

(ii) There exist some finite constants $\overline{M}$ and $\overline{C}$ such that

$$E[\chi_{2N+1} | \vec{V}_1, \chi_1] \leq (\chi_1 - \epsilon) I\{\chi_1 > \overline{M}\} + \overline{C} I\{\chi_1 \leq \overline{M}\}$$

where $\epsilon = \sum_{n=1}^{2N} \epsilon_n > 0$.

Proof. If phase n corresponds to a visit to a station where the service discipline is gated, then it follows from (11) that

$$E[\chi_{n+1} | \vec{V}_n, \chi_n] \leq (\chi_n - V_n) I\{\chi_n > V_n\} + C I\{\chi_n \leq V_n\} \leq \max(\chi_n, C).$$

Hence (12) is satisfied with $\epsilon = 0$, $M_n = C$ and $C_n = C$.

Assume that phase n corresponds to a walking time D^i for some i. Choose M such that

$$E[(D^i + C) I\{D^i > M\}] < E[D^i]/2.$$

Then it follows from (11) that

$$E[\chi_{n+1} | \vec{V}_n, \chi_n]$$

$$
\begin{aligned}
&\leq E[(\chi_n - D^i)I\{\chi_n > D^i\} + CI\{\chi_n \leq D^i\}|\chi_n] \\
&\leq I\{\chi_n > M\}\Big[E(\max(\chi_n - D^i, 0)|\chi_n) + CP(M < D^i)\Big] + I\{\chi_n \leq M\}\max(C, M) \\
&= I\{\chi_n > M\}\Big(\chi_n - E[D^i] + E[(D^i + C)I\{D^i > M\}]\Big) + I\{\chi_n \leq M\}\max(C, M) \\
&\leq I\{\chi_n > M\}\Big(\chi_n - E[D^i]/2\Big) + I\{\chi_n \leq M\}\max(C, M).
\end{aligned}
$$

Hence (12) is satisfied with

$$
\epsilon_n = E[D^i]/2, \qquad M_n = M \text{ and } C_n = \max(C, M). \tag{13}
$$

It remains to establish the case that phase n corresponds to a visit to a station, say k, where the service discipline is exhaustive. We have:

$$
\text{If } \chi_n \geq V_n \text{ then } \chi_{n+1} = \chi_n - V_n. \tag{14}
$$

Assume now that $\chi_n < V_n$. Denote
t_n: the moment when phase n begins (and thus t_{n+1} is the moment when it ends).
$\chi_n + \tau_{1,n} + ... + \tau_{l,n}$: the $(l+1)st$ arrival epoch after time t_n,
$\sigma_{l,n}$: the amount of work brought by the customer that is the lth to arrive after t_n,
$\alpha_{n,l}$: the queue to which is routed the customer that is the lth to arrive after t_n,
$\beta_l = \tau_{n,l} - \sigma_{n,l} I\{\alpha_{n,l} = k\}$,
$S_l = \sum_{i=1}^{l} \beta_l$.
$\phi(t) = \min\{l \geq 1 : S_l > t\}$, for all $t \geq 0$.

Note that $\{\tau_{i,n}, \sigma_{i,n}, \alpha_{i,n}\} =_d \{\tau_i, \sigma_i, \alpha_i\}$. Then

$$
\text{For } \chi_n < V_n,\ \chi_{n+1} = S_{\phi(V_n)} - V_n. \tag{15}
$$

It follows from Gut [11] p. 58, that there exists some finite constant $\hat{C}$ such that

$$
\sup_{t \geq 0} E[S_{\phi(t)}] - t \leq \hat{C} \tag{16}
$$

By combining (15) and (16) we conclude that $\chi_{n+1} \leq \hat{C}$. Hence (12) is satisfied with $\epsilon = 0$, $M_n = \hat{C}$ and $C_n = \hat{C}$.

We establish (ii) for $N = 2$. The proof then follows a simple inductive argument for arbitrary N. Let $M = \max(M_1, C_2) + \epsilon_1 + \epsilon_2$, $C = \max(M_1, c_1, c_2)$.

$$
E[\chi_3|\vec{V}_1, \chi_1] \quad \leq \quad E[(\chi_2 - \epsilon_2)I\{\chi_2 \geq M_2\} + C_2 I\{\chi_2 < M_2\}|\vec{V}_1, \chi_1]
$$

$$
\begin{aligned}
&\le I\{\chi_1 > M_1\}(\chi_1 - \epsilon_1 - \epsilon_2)P(\chi_2 > M_2\}|\vec{V}_1, \chi_1) + C_2 P(\chi_2 \le M_2\}|\vec{V}_1, \chi_1) \\
&\quad + I\{\chi_1 \le M_1\}\left(C_1 - \epsilon_1)P(\chi_2 > M_2|\vec{V}_1, \chi_1) + C_2 P(\chi_2 \le M_2\}|\vec{V}_1, \chi_1)\right) \\
&\le I\{\chi_1 > M\}(\chi_1 - \epsilon_1 - \epsilon_2) \\
&\quad + I\{M_1 < \chi_1 \le M\} \max(M_1, C_2) + I\{\chi_1 \le M_1\} \max(C_1, C_2) \\
&\le I\{\chi_1 > M\}(\chi_1 - \epsilon_1 - \epsilon_2) + I\{\chi_1 \le M\} C
\end{aligned}
$$

The fact that $\epsilon > 0$ follows from (i) and the fact that $E[D] > 0$ and from (13). ∎

Remark 4.1 *The results of the previous Lemma easily generalize to any mixture of gated-type and exhaustive-type policies, which are defined in a way similar to [3] and [16]. A gated-type policy is characterized by a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that if upon arrival to a station the amount of work in that station is V, then the amount of service time given by the server in that station is $f(V)$. (f may even be a random variable, independent of any other quantity). A pure gated policy is then characterized by $f(V) = V$. An exhaustive-type policy is characterized by a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that if upon arrival to a station the amount of work in that station is V, then the amount of service time given by the server in that station is such that when leaving the station, the amount of work remaining there is $V - f(V)$. (f again may be a random variable, independent of any other quantity). A pure exhaustive policy is characterized by $f(V) = V$. In the proof of the previous Lemma, we simply replace V_n by $f(V_n)$, in (14) and (15).*

4.3 General stopping-type policies

Next we generalize the last results to a much larger class of policies. We consider again a cyclic polling system composed of N queues, where in each station the policy to be defined as follows.

Assume that the phase n corresponds to a visit to some station, say k, with an initial workload of V_n and residual interarrival time χ_n. We shall use again the notation introduced in the previous Subsection. For each phase n, we define the customers that arrive after the beginning of the phase (i.e. after t_n) to be "new customers". Consider phase n. Denote by $\varphi_n(u)$ the number of the last new customer, being served during the phase n, if policy u is used.

Definition: We say that a policy u is of the stopping type (ST policy) if it satisfies

(i) Upon arriving to the station, some of the initial work in the system is served. The amount of work served is a function of the amount of work found at t_n in the station, say $f(V_n)$ (with $f(V_n) \leq V_n$).

(ii) After that, some of the new customers are served according to the FCFS policy.

(iii) for each $i \geq 0$, the event $\varphi_n(u) \leq i$ is independent of the sequences

$$(\{\tau_{l,n}\}_{l>i}, \{\sigma_{i,n}\}_{l>i}, \{\alpha_{i,n}\}_{l>i}).$$

Note that the exhaustive and gated type policies are special cases of ST-policies.

Lemma 4.2 *Assume that ST-policies are used in all stations. Then the statements of Lemma 4.1 still hold.*

Proof. Consider a ST-service u that uses some function f in step (i) above. Consider the exhaustive type policy $\hat{u}$ with the same f (i.e. service will be given until the amount of work in the system decreases by $f(V_n)$). Note that $\{\chi_n > V_n\} \subset \{\varphi_n(u) = 0\}$ and $\varphi_n(u) \leq \varphi_n(\hat{u})$. Then

$$t_{n+1} - t_n = f(V_n) + \sum_{i=1}^{\varphi_n(u)} s_i(k)$$

where $s_i(k) = \sigma_{i,n} I\{\alpha_{i,n} = k\}$; and

$$\chi_{n+1} = (\chi_n - f(V_n)) I(\chi \geq f(V_n)) + (S_{\mu,n} + \chi_n - (t_{n+1} - t_n)) I\{\chi_n < f(V_n)\}$$

where $S_{l,n} = \tau_{1,n} + ... + \tau_{l,n}$ and $\mu = \min\{l \geq 1 : S_{l,n} + \chi_n \geq (t_{n+1} - t_n)\}$. The second term in the right hand side of the last equality can be rewritten in the form:

$$\begin{aligned}
&[(S_{\mu,n} + \chi_n - (t_{n+1} - t_n)) I\{\varphi_n(u) = \varphi_n(\hat{u})\} \\
&+ \quad (S_{\mu,n} + \chi_n - (t_{n+1} - t_n)) I\{\varphi_n(u) < \varphi_n(\hat{u})\}] I\{\chi_n < f(V_n)\}
\end{aligned}$$

But on the event $\{\varphi_n(u) = \varphi_n(\hat{u})\}$, we have

$$\mu = \varphi_n(u) = \varphi_n(\hat{u}) \qquad a.s.$$

and on the event $\{\varphi_n(u) < \varphi_n(\hat{u})\}$, we have

$$\mu > \varphi_n(u) \qquad a.s.$$

So

$$S_{\mu,n} = S_{\varphi_n(u)} + \sum_{\varphi_n(u)+1}^{\mu} \tau_{n,i}.$$

Therefore

$$\begin{aligned}
& E\left\{(S_{\mu,n} + \chi_n - (t_{n+1} - t_n))I\{\varphi_n(u) = \varphi_n(\hat{u})\} \Big| \chi_n, \vec{V}_n \right\} \\
\leq \quad & E\left\{(S_{\mu,n} + \chi_n - \sum_{i=1}^{\varphi_n(\hat{u})} s_i(k)) \Big| \chi_n, \vec{V}_n \right\} \leq \hat{C}
\end{aligned}$$

where $\hat{C}$ is given in (16) (and the last inequality follows from Gut [11] p. 58 exactly as in (16)). On the other hand,

$$\begin{aligned}
& E\left\{(S_{\mu,n} + \chi_n - (t_{n+1} - t_n))I\{\varphi_n(u) < \varphi_n(\hat{u})\} \Big| \chi_n, \vec{V}_n \right\} \\
= \quad & E\left\{(S_{\mu,n} + \chi_n - (t_{n+1} - t_n))I\{S_{\varphi_n(u),n} + \chi_n - (t_{n+1} - t_n) < 0\} \Big| \chi_n, \vec{V}_n \right\} \\
= \quad & E\left\{\left(E\left\{ \sum_{i=\varphi_n(u)+1}^{\mu} \tau_{i,n} \middle| \mathcal{F}_{\varphi_n(u)} \right\} + \chi_n + S_{\varphi_n(u)} + (t_{n+1} - t_n)\right)\right. \\
& \qquad \left. \times I\left\{S_{\varphi_n(u)} + \chi_n - (t_{n+1} - t_n < 0\right\} \middle| \chi_n, \vec{V}_n \right\} \\
\leq \quad & CE\left\{I\left\{S_{\varphi_n(u)} + \chi_n - (t_{n+1} - t_n < 0\right]\Big| \chi_n, \vec{V}_n\right\} \\
\leq \quad & C \qquad a.s.
\end{aligned}$$

Here $\mathcal{F}_{\phi_n(u)}$ is the σ field generated by

$$\vec{V}_n, \chi_n, \{\tau_{i,n}, \sigma_{i,n}, \alpha_{i,n}; 1 \leq i \leq \varphi_n(u)\}.$$

Finally we get

$$E\left(\chi_{n+1} \Big| \chi_n, \vec{V}_n\right) \leq (\chi_n - f(V_n))I\{\chi_m \geq f(V_n)\} + 2CI\{\chi_m < f(V_n)\}.$$

We can thus choose M_n and C_n in the same way we did in the proof of Lemma 4.1 for the gated or exhaustive policies. ∎

5 Positive recurrence.

In this section we consider quantities that correspond to the sequence of consecutive cycles. We add the $i = 1, 2, ...$ to the quantities defined in Section 3 in order to denote a quantity refering to the ith cycle. Thus $T(i)$ is the beginning of the ith cycle etc. Let $X(i) = (T(i), \vec{V}(i), \chi(i))$.

Define

$$U(1) = \{\tau_j, \sigma_j, \alpha_j\}_{j=1}^{\infty}. \tag{17}$$

Let θ^n be the shift operator, such that

$$(\tau_{n+j}, \sigma_{n+j}, \alpha_{n+j}) = (\tau_j, \sigma_j, \alpha_j) \circ \theta^n \tag{18}$$

a.s. Define recursively

$$U(i+1) = U(i) \circ \theta^{\eta(i)} \equiv \{\tau_j(i+1), \sigma_j(i+1), \alpha_j(i+1)\}_{j=1}^{\infty}. \tag{19}$$

As before, we assume $\eta(i)$ to be a stopping time (for each i), i.e. we make the following assumption:

(A) $(X(i), \{\tau_j(i), \sigma_j(i), \alpha_j(i); 1 \leq j \leq \eta(i)\}, \eta(i), G(i)\}$ are independent of $U(i+1)$ for each $i \geq 1$.

As a corollary of this assumption, we have:

(i) for each i

$$U(i) =_d U(1); \tag{20}$$

(ii) each of the sequences $\{X(i)\}$ and $\{Y(i) \equiv (\vec{V}(i), \chi(i)\}$ forms a time-homogeneous Markov chain.

Theorem 5.1 *Assume that there exist some finite constants M, C and $\epsilon > 0$, such that*

$$\mathbf{E}\{\chi(i+1)/X(i)\} \leq (\chi(i) - \epsilon) \cdot I(\chi(i) > M) + C \cdot I(\chi(i) \leq M) \tag{21}$$

a.s. for all i.

Then

(i) for each initial condition $X(1)$

$$\lambda \equiv \min\{i \geq 1 : \chi(i) \leq M\} < \infty \tag{22}$$

a.s.;

(ii) $\sup \mathbf{E}\{\lambda / X(1)\} < \infty$,

where we take a supremum on all non-random initial conditions $X(1)$*, such that* $\chi(1) \leq M$.

Proof. Follows directly from the Tweedie's criterion ([22] Section 6). ■

Corollary 5.2 *Assume, in addition to the conditions of Theorem 5.1, that*

(i) $\mathbf{E}\eta(i)$ *is finite for all* i*;*

(ii) there exists a non-negative function h*, such that* $h(V) \to \infty$ *as* $V \to \infty$*, and*

$$\mathbf{E}\{V^{out}(i)/X(i)\} \geq h(V(i)) \tag{23}$$

a.s. for all i*;*

(iii) $\rho < 1$.

Then one can choose constants $\tilde{L}$ *and* $\tilde{M}$*, such that*

(i) for each initial condition $X(1)$

$$\tilde{\lambda} \equiv \min\{i \geq 1 : V(i) \leq \tilde{L}; \chi(i) \leq \tilde{M}\} < \infty \tag{24}$$

a.s.;

(ii) $\sup \mathbf{E}\{\tilde{\lambda} / X(1)\} < \infty$,

where we take a supremum on all non-random initial conditions $X(1)$*, such that* $V(1) \leq \tilde{L}$ *and* $\chi(1) \leq \tilde{M}$.

(iii) For arbitrary initial condition $X(1)$*, introduce a sequence* $\{\psi_n\}$, $\psi_{n+1} > \psi_n$ *a.s., where*

$$\psi_1 = \min\{i \geq 1 : V(i) \leq \tilde{L}; \chi(i) \leq \tilde{M}\} \tag{25}$$

and for $n \geq 1$

$$\psi_{n+1} = \min\{i \geq \psi_n + 1 : V(i) \leq \tilde{L}; \chi(i) \leq \tilde{M}\}. \tag{26}$$

Then there exists a constant $U < \infty$*, such that*

$$\mathbf{E}\{\psi_{n+1} - \psi_n | \mathcal{F}_{\psi_n}\} \leq U \tag{27}$$

a.s. for all n*.*

Remark 5.1 *The statement of corollary 5.2 means, that the set*

$$B \equiv [0, \tilde{L}] \times [0, \tilde{M}] \tag{28}$$

is uniformly positive recurrent for the Markov chain $\{Y(i)\}$*.*

Remark 5.2 *The condition (23) is satisfied for ST policies with* $f_n(V) \geq K \cdot V$ *for each phase* n *(where* $K > 0$ *is some fixed constant).*

Proof of Corollary 5.2: Set

$$X = (T, \vec{V}, \chi). \tag{29}$$

Lemma 3.2 implies the inequality:

$$\mathbf{E}(V_1/X) \leq V + \rho \cdot \mathbf{E}(D) + (\rho - 1) \cdot KV + \rho \cdot C. \tag{30}$$

Therefore for each $R > 0$

$$\begin{aligned} &\mathbf{E}\{V_1 + R \cdot \chi_1/X\} \leq \\ &\quad V + R(\chi - \epsilon) \cdot I(\chi > M) + RC \cdot I(\chi \leq M) + \rho C + \rho \mathbf{E}(D) + (\rho - 1)h(V). \end{aligned} \tag{31}$$

Choose $R >> 1$ such that

$$R \cdot \epsilon - \rho \cdot (C + \mathbf{E}(D)) \equiv \delta_1 > 0, \tag{32}$$

and $L >> 1$ such that

$$(1 - \rho) \cdot h(V) - (R + \rho) \cdot C - \rho \cdot \mathbf{E}(D) \equiv \delta_2 > 0 \tag{33}$$

for all $V \geq L$. Set

$$\delta = \min\{\delta_1, \delta_2\}. \tag{34}$$

Therefore, if either $\chi > M$ or $V > L$, then

$$\mathbf{E}(V_1 + R \cdot \chi_1/X) \le V + R \cdot \chi - \delta \tag{35}$$

a.s., and if $\chi \le M$ and $V \le L$, then

$$\mathbf{E}(V_1 + R \cdot \chi_1/X) \le L + R \cdot \max(C, M) + \rho \cdot C + \rho \cdot \mathbf{E}(D). \tag{36}$$

So we can apply Tweedie's criterion [22] Section 6, and this completes the proof. ∎

Remark 5.3 *For arbitrary initial condition $X(1)$, introduce a sequence $\{\psi_n\}$, $\psi_{n+1} > \psi_n$ a.s., where*

$$\psi_1 = \min\{i \ge 1 : V(i) \le \tilde{L}; \chi(i) \le \tilde{M}\} \tag{37}$$

and for $n \ge 1$

$$\psi_{n+1} = \min\{i \ge \psi_n + 1 : V(i) \le \tilde{L}; \chi(i) \le \tilde{M}\}. \tag{38}$$

It follows from Corollary 5.2 [22], that there exists a constant $U < \infty$, such that

$$\mathbf{E}\{\psi_{n+1} - \psi_n | \mathcal{F}_{\psi_n}\} \le U \tag{39}$$

a.s. for all n.

Next we restrict to the setting and assumptions of Subsection 4.2. In particular, we restrict to a cyclic polling system composed of N queues.

Proposition 5.3 *Assume that ST policies are used in all station. Assume moreover that $\rho < 1$ and conditions (ii) of Corollary 5.2 holds.*
Then (21) is satisfied, $\mathbf{E}\eta(i) < \infty$; and hence the statements of Theorem 5.1 and Corollary 5.2 hold. Moreover,

$$\sup \mathbf{E}\{T(\tilde{\lambda}) - T(1)/X(1)\} < \infty, \tag{40}$$

where we take a supremum on all non-random initial conditions $X(1)$, such that $V(1) \le \tilde{L}$ and $\chi(1) \le \tilde{M}$.

Proof. The result follows from the comparison with the exhaustive policy. Indeed, for each i the r.v. $\eta(i)$ can be represented as a sum

$$\eta(i) = \eta_1(i) + \ldots + \eta_{2N}(i), \tag{41}$$

where $\eta_{2k+1}(i)$ is a number of arrivals during the service phase on the station k, $k = 1, \dots, N$. Then we can use the induction arguments. Assume, that for fixed k a r.v. $V_{2k+1}(i)$ has a finite expectation. Consider an exhaustive policy on this phase (with the same initial conditions), and denote for it by $\hat{\eta}_{2k+1}(i)$ a number of arrivals during the service phase. Then $\eta_{2k+1}(i) \leq \hat{\eta}_{2k+1}(i)$ a.s. and

$$\hat{\eta}_{2k+1}(i) = \min\{n : \chi_{2k+1}(i) + \sum_{j=1}^{n} \tau_{2k+1,j} > V_{2k+1}(i) + \sum_{j=1}^{n} \sigma_{2k+1,j}\} \equiv$$

$$\min\{n : \sum_{j=1}^{n} (\tau_{2k+1,j} - \sigma_{2k+1,j}) > V_{2k+1}(i) - \chi_{2k+1}(i)\}.$$

Since $\mathbf{E}(\tau_{2k+1,1} - \sigma_{2k+1,1}) > 0$ and $\mathbf{E}V_{2k+1}(i) < \infty$, then $\mathbf{E}\hat{\eta}_{2k+1}(i) < \infty$ (see, e.g., [11]) and, therefore, $\mathbf{E}\eta_{2k+1}(i) < \infty$ and $\mathbf{E}V_{2k+2}(i) < \infty$. The similar arguments can be used to prove $\mathbf{E}\eta_{2k+2} < \infty$ and (37).

$\blacksquare$

Remark 5.4 *Note, that if $\rho > 1$, then $V(i) \to \infty$ a.s. as $i \to \infty$. Indeed, for each t denote*

$\lambda(t) = \min\{n : \chi(1) + \tau_1 + \ldots + \tau_n \geq t\}$;

$A(t) = \sum_{n=1}^{\lambda(t)} \sigma_n$*: the amount of work, the customers bring to the system during time interval* $[0, t)$;

$S(t)$*: the amount of work, is served during the time interval* $[0, t)$.

By definition, $S(t) \leq t$ for all t, and renewal theorem implies

$$\lambda(t)/t \to (\mathbf{E}\tau_1)^{-1} \tag{42}$$

a.s. Since $T(i+1) - T(i) \geq D(i)$ a.s. and $\mathbf{E}D(1) > 0$, then

$$\lambda(T(i))/T(i) \to (\mathbf{E}\tau_1)^{-1} \tag{43}$$

and

$$A(T(i))/T(i) = \{A(T(i))/\lambda(T(i))\} \cdot \{\lambda(T(i))/T(i)\} \to \rho \tag{44}$$

a.s. But $V(i) = V(1) + A(T(i)) - S(T(i))$. Therefore

$$\liminf V(i)/T(i) = \lim A(T(i))/T(i) - \limsup S(T(i))/T(i) \geq \rho - 1 > 0 \tag{45}$$

and

$$V(i) \to \infty \tag{46}$$

a.s. as $i \to \infty$.

Remark 5.5 *If we assume the Borovkov's condition on the inter-arrival times, then the condition (21) is always true (even for an arbitrary graph* $\mathcal{G}$ *).*

6 Ergodicity

There is a family of well-known additional sufficient conditions, when positive recurrence implies the ergodicity (and, moreover, coupling- ergodicity) of the sequence $\{X(i)\}$ and of the continuous-time processes (see, e.g., Asmussen [7], Lindvall [8], Meyn and Tweedie [17], and others.). We present two examples for sets of such sufficient conditions. We restrict to the setting and assumptions of Subection 4.2.

6.1 Example 1

Assume that

- (i) A r.v. τ_1 has a non-lattice distribution.
- (ii) The ST-policy on each station is fixed (i.e. it is the same for all cycles) and such, that $f(V) = V$ for all V.
- (iii) There exist some constants c_1 and c_2, such that

  $$q_1 \equiv \mathbf{P}(\sigma_1 \leq c_1) > 0, \quad q_2 \equiv \mathbf{E}(D(1) \leq c_2) > 0 \tag{47}$$

  and

  $$q_3 \equiv \mathbf{P}(\tau_1 > \tilde{L} + \max(C, c_1) + c_2) > 0. \tag{48}$$

- (iv) There exist some constants $0 \leq a < b$ and $p > 0$, such that

  $$\mathbf{P}(D(1) \in G) \geq p \cdot \lambda(G) \tag{49}$$

  for each Borel set $G \subseteq [a, b]$ (here λ is a Lebesgue measure).

Proposition 6.1 *Under the above conditions one can construct a stationary ergodic sequence $\{Y^i\}$ on the same probability space with $\{Y(i)\}$, such that*

$$\mathbf{P}(Y(j) = Y^j \quad \forall j \geq i) \to 1 \tag{50}$$

as $i \to \infty$.

Proof. For each integer $m \geq 1$ we can represent a set $B \equiv [0, \tilde{L}] \times [0, \tilde{M}]$ as a union of the sets:

$$B = \bigcup_{l=1}^{m} \bigcup_{r=1}^{m} [\tilde{L} \cdot (l-1)/m, \tilde{L} \cdot l/m] \times [\tilde{M} \cdot (r-1)/m, \tilde{M} \cdot r/m]. \tag{51}$$

Since the set B is uniformly positive recurrent for the Markov chain $\{Y(i)\}$, and a "non-latticity" of τ_1 implies irreducibility and aperiodicity of $\{Y(i)\}$, then for each m one can choose at least one pair (l_m, r_m), such that a set

$$B(m) \equiv [\tilde{L} \cdot (l_m - 1)/m, \tilde{L} \cdot l_m/m] \times [\tilde{M} \cdot (r_m - 1)/m, \tilde{M} \cdot r_m/m] \tag{52}$$

is uniformly positive recurrent.

Choose $m >> 1$ such that $1/m < (b-a)/2$. Then

$$\mathbf{P}(Y(i+3) \in (\cdot) | Y(i)) \geq \tilde{p} \cdot \gamma(\cdot) \tag{53}$$

a.s. on the event $\{Y(i) \in B(m)\}$, where $\tilde{p} = const > 0$ and γ is a probability measure on $[0, \infty) \times [0, \infty)$, and $\tilde{p}$ and γ can be represented in terms of q_1, q_2, q_3, p. So the Markov chain $\{Y(i)\}$ is Harris ergodic, and this completes the proof. ∎

Corollary 6.2 *Let $\vec{V}_t$ be a vector of workloads on the stations at time t. Under the above conditions, there exists a probability distribution $\mathbf{P}(\cdot)$, such that the distributions $\mathbf{P}(\vec{V}_t \in (\cdot))$ converge weakly to $\mathbf{P}(\cdot)$, as $t \to \infty$.*

6.2 Example 2

We present below conditions that imply not only ergodicity, but also the finiteness of the first moment of the workload in steady state, and the convergence of the first moments of the workload to the steady state moment.

ASSUMPTIONS:

- (Ai) In all stations, a ST policy is used.
- (Aii) $\rho < 1$ and the following stronger version of conditions (ii) of Corollary 5.2 holds. There exists some $K > 0$ such that

$$\mathbf{E}\{V^{out}(i)/Y(i)\} \geq KV(i) \tag{54}$$

  a.s. for all i;
- (Aiii)

$$P(\tau > D(1) + D(2) + \sigma + M \text{ and } D(1) > M) > 0$$

  where M is given in (21) (which is satisfied according to Proposition (5.3).
- (Aiv) Let B_0 be a Borel set in $\mathbb{R}_+$.

$$P(\{-2L+\tau_1-\sigma_1-D(1)-D(2) \in B_0\} \cap \{\tau_1 > D(1)+D(2)+\sigma+2L\} \cap \{D(1) > M\}) > 0$$

  implies

$$\inf_{-2L \leq l \leq M} P(\{l + \tau_1 - \sigma_1 - D(1) - D(2) \in B_0\} \\ \cap \{\tau > D(1) + D(2) + \sigma + 2L\} \cap \{D(1) > M\}) > 0$$

  where L is given in (32).

Conditions (Ai) and (Aii) will ensure the positive recurrence. Condition (Aii) will imply moreover the existence of first moments of the workloads. Conditions (Aiii) and (Aiv) are required to construct a "small set", as will be required in Theorem 6.4 below. A simple sufficient condition for both (Aiii) and (Aiv) to hold is both $D(1)$ and τ_1 have both positive probabilities to belong to Borel sets having finite measures.

Proposition 6.3 *Under the above assumptions, $\{Y(i)\}$ is ergodic and $V(i)$ has finite expectation for all i. Let π be the steady state distribution of the workload. Then*

$$E[\hat{V}] < \infty, \qquad \lim_{i \to \infty} E[V(i)|X(1) = x] \to E[\hat{V}] \tag{55}$$

for all $x \in \mathbf{X}$.

To prove the proposition we shall use the following Theorem ([23, 24, 17]):

Theorem 6.4 *Consider a strongly aperiodic Markov chain X_n on a state space $\mathbf{X}$ with transition probabilities $P : \mathbf{X} \times \mathcal{B}(\mathbf{X}) \to [0,1]$. Assume that there exists a set $\mathcal{K} \in \mathbf{X}$ and a functions $g, f : \mathbf{X} \to \mathbb{R}$, $g(\cdot), f(\cdot) \geq 1$, such that*
(i) there exists some $\epsilon > 0$ such that $E[g(X_{n+1}) - g(X_n)|X_n] \leq -\epsilon f(X_n)$ for $X_n \in \mathcal{K}^c$;
(ii) $E[g(X_{n+1})|X_n] < \infty$ for $X_n \in \mathcal{K}$;
(iii) $\mathcal{K}$ is a small set.
Then
(a) X_n is ergodic.
(b.1) The steady state expectation exists: $E[f(\hat{X})] < \infty$;
(b.2)

$$E[f(X_n)|X_1 = x] \to E[f(\hat{X})]$$

for all $x \in \mathbf{X}$.

Remark:
In the above Lemma, a set $\mathcal{K}$ is said to be small if there exists some positive measure ϕ on $\mathbf{X}$, such that for any $B \subset \mathbf{X}$ with $\phi(B) > 0$ there exists j such that

$$\inf_{x \in \mathcal{K}} \sum_{n=1}^{j} P^n(x, B) > 0.$$

where $P(x, B)$ is the transition probabilities of the Markov chain.

Proof of Proposition 6.3: Let $Y = (V, \chi)$. We apply Theorem 6.4 by identifying $g(Y)$ with $g(Y) = 1 + V + R\chi$ where R is given in (32), and $Y = (V, \chi)$, $f(Y) = 1 + (\rho - 1)KV/2$, $\mathcal{K} = \{(v, \chi) : v \leq 2L, \chi \leq M\}$; The measure ϕ related to the small set $\mathcal{K}$ is defined as follows. Let $B_0 = \{\chi | (0, \chi) \in B\}$. Then

$$\phi(B) := P(\{-2L + \tau_1 - \sigma_1 - D(1) - D(2) \in B_0\} \cap \{\tau_1 > D(1) + D(2) + \sigma_1 + 2L\} \cap \{D(1) > M\})$$

With the above choice of g and f, (31) implies conditions (Ai) and (Aii) of Theorem 6.4. Next we show that $\mathcal{K}$ is small.

Assume that we are at the beginning of a cycle, say cycle 1, in some state $(V(1), \chi(1)) \in \mathcal{K}$. Then

- (C1) $D(1) > M$ implies that an arrival will occur during cycle 1, since $\chi(1) \in \mathcal{K}$ and hence smaller or equal to M.

- (C2) $\tau_1 > D(1) + D(2) + \sigma_1 + 2L$ implies that the next arrival will not occur in the current cycle nor in the next one. Indeed, in that case, the total length of cycles 1 and 2 is

$$V(1) + D(1) + D(2) + \sigma_1 \leq D(1) + D(2) + \sigma_1 + 2L$$

since $V(1) \in \mathcal{K}$.

Hence (C1) and (C2) imply that $V(3) = 0$. $\chi(3)$ is then given by

$$\chi(3) = \chi(1) + \tau_1 - [V(1) + D(1) + D(2) + \sigma_1]$$

Assume now that B is such that $\phi(B) > 0$, i.e.

$P(\{-2L + \tau_1 - \sigma_1 - D(1) - D(2) \in B_0\} \cap \{\tau_1 > D(1) + D(2) + \sigma_1 + 2L\} \cap \{D(1) > M\})$.

Then for any $Y(1) \in \mathcal{K}$

$$\begin{aligned}
& P(Y(3) \in B | Y(1)) \\
\geq \; & P(Y(3) \in B \cap \{\tau_1 > D(1) + D(2) + \sigma_1 + 2L\} \cap \{D(1) > M\} | Y(1)) \\
= \; & P(\chi(3) \in B_0 \cap \{\tau_1 > D(1) + D(2) + \sigma_1 + 2L\} \cap \{D(1) > M\} | Y(1)) \\
= \; & P(\chi(1) + \tau_1 - [V(1) + D(1) + D(2) + \sigma_1] \in B_0 \\
& \qquad \cap \{\tau_1 > D(1) + D(2) + \sigma_1 + 2L\} \cap \{D(1) > M\} | Y(1)) \\
\geq \; & \inf_{-2L \leq l \leq M} P(l + \tau_1 - [+D(1) + D(2) + \sigma_1] \in B_0 \\
& \qquad \cap \{\tau_1 > D(1) + D(2) + \sigma_1 + 2L\} \cap \{D(1) > M\} | Y(1)) \\
> \; & 0
\end{aligned}$$

uniformly for all $Y(1) \in \mathcal{K}$. This follows from the definition of the measure ϕ and from condition (Aiv). ■

▪

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