



## Contiguity orders

Vincent Bouchitté, Abdelmajid Hilali, Roland Jégou, Jean-Xavier Rampon

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Vincent Bouchitté, Abdelmajid Hilali, Roland Jégou, Jean-Xavier Rampon. Contiguity orders. [Research Report] RR-1970, INRIA. 1993. inria-00074703

**HAL Id: inria-00074703**

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Vincent Bouchitté, Abdelmajid Hilali, Roland Jégou, Jean-Xavier Rampon

**N° 1970**

Mars 1993

PROGRAMME 1

Architectures parallèles,  
bases de données,  
réseaux et systèmes distribués ***Rapport  
de recherche*****1993**





## Contiguity Orders

Vincent Bouchitté\*, Abdelmajid Hilali\*\*, Roland Jégou\*\*\*,  
Jean-Xavier Rampon\*\*\*\*

Programme 1 — Architectures parallèles, bases de données, réseaux  
et systèmes distribués  
Projet Pampa

Rapport de recherche n° 1970 — Mars 1993 — 25 pages

**Abstract:** This paper is devoted to the study of contiguity orders i.e. orders having a linear extension  $L$  such that all upper (or lower) cover sets are intervals of  $L$ . This new class is a strict generalization of both interval orders and  $N$ -free orders, and is linearly recognizable. It is proved that computing the number of contiguity extensions is  $\#P$ -complete, and that the dimension of height one contiguity orders is polynomially tractable. Moreover the membership is a comparability invariant on bi-contiguity orders. Finally for strong-contiguity orders the calculation of the dimension is  $NP$ -complete.

(Résumé : *tsvp*)

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## Ordres de contiguïté

**Résumé :** Nous définissons les ordres de contiguïté comme possédant une extension linéaire pour laquelle chaque ensemble de successeurs directs apparaît comme un intervalle. Nous étudions cette nouvelle classe d'ordres dont la reconnaissance est linéaire, ainsi que certaines restrictions. Cette classe contient en particulier les ordres d'intervalles et les ordres sans "N", ce qui entraîne la  $NP$ -complétude de la dimension et du nombre de sauts. Nous montrons que le calcul du nombre d'extensions de contiguïté est  $\#P$ -complet. Concernant les ordres de contiguïté de hauteur un, nous prouvons que la dimension est un problème polynomial. Nous établissons également l'invariance de comparabilité de l'appartenance à la classe des ordres de bi-contiguïté ainsi que la  $NP$ -complétude de la dimension pour les ordres de forte contiguïté.

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# Contiguity Orders

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## 1 Introduction

In [4] we introduced the class of lower-contiguity orders and we showed that such orders can be represented by translating line-segments in the plane. This latter problem has been defined and investigated first by Rival and Urrutia in [19]. These notions have also been studied by Bouchitté et al. in [3]. The representation of a contiguity order  $P = (X, \leq_P)$  by translating line-segments leads to an efficient encoding of its Hasse diagram in  $\mathcal{O}(|X|)$  space. This encoding allows to answer in constant time to the question: given  $x$  and  $y$ , is  $x$  covered by  $y$ ? It can be obtained in time proportional to the size of the Hasse diagram by using [3].

Roughly speaking a lower-contiguity order (resp. an upper-contiguity order) admits a linear extension  $L$  such that each lower-cover set (resp. upper-cover set) appears as an interval of  $L$ . This notion is closely related to the notion of interval hypergraph. On one hand, every interval hypergraph  $\mathcal{H} = (X, \mathcal{E})$  can be seen as an height one upper-contiguity order  $P = (X \cup \mathcal{E}, \leq_P)$  where  $e <_P x$  whenever  $x \in e$ ,  $x \in X$  and  $e \in \mathcal{E}$ . Then upper-contiguity orders give a natural generalization of interval hypergraphs to arbitrary height orders. On the other hand, the interval hypergraph theory provides an interesting tool to study contiguity orders.

The class of contiguity orders is quite large since it contains among others the well-known classes: interval orders,  $N$ -free orders, planar lattices. Therefore it is  $NP$ -complete to compute the dimension and the jump number of contiguity orders because of the  $NP$ -completeness of the dimension of  $N$ -free orders (see Kierstead and Penrice [16]) and of the jump number of interval orders (see Mitas [17]). However, the height one contiguity orders which are also called convex graphs by Brandstädt in [5] have a polynomial time algorithm for the jump number see Dahlhaus [7]. We prove in this paper the existence of a polynomial time algorithm for computing their dimension.

In Section 2, we recall the main definitions and fix some notation. In Section 3, we give a characterization of upper-contiguity orders in terms of interval hypergraphs which leads to a linear time recognition algorithm by using the  $PQ$ -tree data structure of Booth and Lueker [1]. Then, we show that computing the number of upper-contiguity extensions is  $\#P$ -complete. While computing the dimension is difficult for contiguity orders it becomes polynomially tractable on height one contiguity orders for which the dimen-

sion is proved to be less or equal than 3. This section is ended by the characterization of total–upper–contiguity orders. In Section 4, we deal with bi–contiguity orders which are both upper–contiguity and lower–contiguity orders. This class contains interval orders,  $N$ –free orders and planar lattices. An interesting feature of this class is the membership comparability invariance. In the last section, we study a restriction of bi–contiguity orders, namely strong–contiguity orders. For such orders there exists a linear extension which is both an upper–contiguity and a lower–contiguity extension. The recognition problem remains linear for this class and the dimension problem remains  $NP$ –complete.

## 2 Definitions and notation

Throughout this paper, all orders are assumed to be finite.

In order to present our results, we have to recall some basic facts and to fix some notation related to an order  $P = (X, \leq_P)$ .

Let  $x, y \in X$  with  $x \neq y$ . We say that  $x$  and  $y$  are *comparable* in  $P$  (denoted by  $x \sim_P y$ ), when either  $x \leq_P y$  or  $y \leq_P x$  holds. On the other hand,  $x$  and  $y$  are said to be *incomparable* in  $P$  (denoted by  $x \parallel_P y$ ), if neither  $x \leq_P y$  nor  $y \leq_P x$  hold. The undirected graph  $G = (X, E)$  where two vertices are joined by an edge iff the corresponding pair of elements of  $P$  is comparable, is called the *comparability graph* of  $P$ . A parameter or a function is said to be a *comparability invariant* if it has the same value for all orders with the same comparability graph (see Kelly [13] for a survey on comparability graphs and comparability invariants, and see Dreesen et al. [8] for an efficient tool for proving comparability invariance in the finite case).

We say that  $x$  is *covered* by  $y$  (or  $y$  *covers*  $x$ ) and we denoted by  $x \prec_P y$  if  $x <_P y$  and  $\forall z \in X$  with  $x \leq_P z \leq_P y$  we have  $z = x$  or  $z = y$ . This binary relation defines the *Hasse diagram* of  $P$  while the associated undirected graph is called the *covering graph*.

A *chain* (resp. an *antichain*) in  $P$  is a subset  $A$  of  $X$  such that every pair of distinct elements of  $A$  are comparable (resp. incomparable). The *length* of a chain in  $P$  is one less than its cardinality and the *size* of an antichain in  $P$  is its cardinality. The *height* of  $P$ , denoted  $h(P)$ , is the maximum length of a chain in  $P$ .



An element  $x \in X$  is called a *minimal* element (respectively, *maximal* element) if there is no element  $y \in X$  with  $y <_P x$  (respectively,  $x <_P y$ ). We denote by  $Min(P)$  (resp.  $Max(P)$ ) the set of minimal (resp. maximal) elements in  $P$ .

We denote by  $P^- = (X, \leq_{P^-})$  the *dual order* of  $P$ , that is  $\forall x, y \in X$ ,  $(x \leq_{P^-} y) \iff (y \leq_P x)$ .

Let  $A \subseteq X$  we denote by  $P|_A$  the *order restriction* of  $P$  on  $A$ . That is,  $P|_A = (A, \leq_{P|_A})$  and  $\forall x, y \in A$ ,  $(x \leq_{P|_A} y) \iff (x \leq_P y)$ . Let  $A \subseteq X$ , we denote by  $P - A$  the order  $P|_{(X-A)}$ .

An order  $Q = (X, \leq_Q)$  is said to be an *order extension* of  $P$  if  $\forall x, y \in X$ ,  $x \leq_P y \implies x \leq_Q y$ . A *linear extension*  $L$  of  $P$  is an order extension of  $P$  which is a total order, then  $L$  can be written  $x_1 < x_2 < \dots < x_n$  if  $|X| = n$ . We denote by  $\mathcal{L}(P)$  the set of all linear extensions of  $P$ . Let  $G = (X, U)$  an acyclic directed graph, a permutation  $\sigma$  of  $X$  such that  $(x, y) \in U$  implies  $\sigma(x) < \sigma(y)$ , is also called a linear extension of  $G$ .

A *realizer* of  $P$  is a subset of  $\mathcal{L}(P)$  such that the intersection of all its elements is  $P$ . The minimum cardinality of a realizer is called the *dimension* of  $P$  and denoted by  $dim(P)$  (see Kelly and Trotter [15] for a survey on the dimension theory).

A *jump* in a linear extension  $L$  of  $P$  is a pair of elements  $x, y$  such that  $x <_L y$  and  $x \parallel_P y$ . The *jump number* of  $P$ , denoted  $s(P)$ , is the minimum number of jumps yielded by one of its linear extensions (see Bouchitté and Habib [2] for a survey on the jump number tractability).

We denote by  $U_P(x)$  (resp.  $D_P(x)$ ) the upset (resp. downset) of  $x$  in  $P$ ,  $U_P(x) = \{y \in X, x \leq_P y\}$  (resp.  $D_P(x) = \{y \in X, y \leq_P x\}$ ) and by  $UC_P(x)$  (resp.  $LC_P(x)$ ) the upper cover (resp. lower cover) set of  $x$  in  $P$ ,  $UC_P(x) = \{y \in X, x <_P y\}$  (resp.  $LC_P(x) = \{y \in X, y <_P x\}$ ). These notation can be extended to subsets  $Y$  of  $X$ .  $UC_P(Y)$  (resp.  $LC_P(Y)$ ) is the upper cover set (resp. lower cover set) belonging to  $X - Y$  of all elements of  $Y \subseteq X$  in  $P$ ,  $UC_P(Y) = \{z \in X - Y, \exists y \in Y, y <_P z\}$  (resp.  $LC_P(Y) = \{z \in X - Y, \exists y \in Y, z <_P y\}$ ).

Let  $\{Q_x = (Y_x, \leq_{Q_x}), x \in X\}$  be an order family. We denote by  $Q = \sum_{x \in P} Q_x$

the order  $Q = (\bigcup_{x \in X} Y_x, \leq_Q)$ , where  $z \leq_Q t$  iff  $(z \in Y_x, t \in Y_y)$  and  $((x <_P y)$  or  $(x = y$  and  $z \leq_{Q_x} t)$ ).

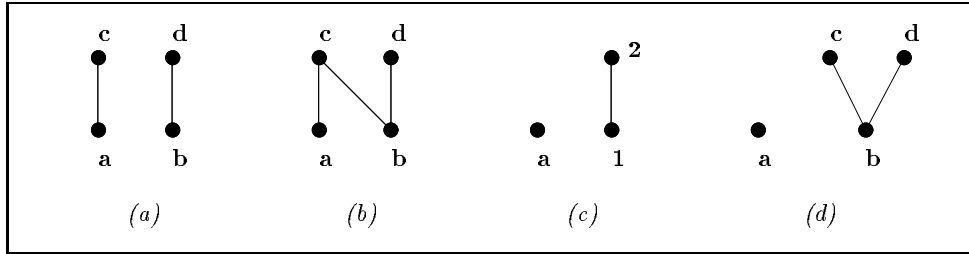


Figure 1: (a) the  $2 \oplus 2$ ; (b) the "N"; (c) the  $1 \oplus 2$  and (d) the  $1 \oplus V$ .

### 3 Contiguity orders

**Definition 1** An order  $P = (X, \leq_P)$  is an upper-contiguity (resp. a lower-contiguity) order, if there exists  $L \in \mathcal{L}(P)$  such that for any  $x \in X$ ,  $UC_P(x)$  (resp.  $LC_P(x)$ ) is an interval of  $L$ , that is if  $x <_P a$  and  $x <_P b$  and if  $a \leq_l b$  then  $\forall c$  such that  $a \leq_l c \leq_l b$  we have  $x <_P c$ . Such an extension is called an upper-contiguity (resp. a lower-contiguity) extension of  $P$ .

Directly from the above definition we can deduce the following property on upper-contiguity extensions.

**Proposition 1** Any upper-contiguity order has an upper-contiguity extension starting with all its minimal elements.

Since the dual of an upper-contiguity order is a lower-contiguity order, all properties and definitions given on upper-contiguity orders are dually available on lower-contiguity orders. The property of upper-contiguity is not stable by duality (see Figure 2). When an order is either an upper-contiguity order or a lower-contiguity order, it will be called a contiguity order.

#### 3.1 Characterization of upper-contiguity orders

In this section, we present a characterization of upper-contiguity orders in terms of interval hypergraphs. This characterization leads to an efficient recognition algorithm of such orders.

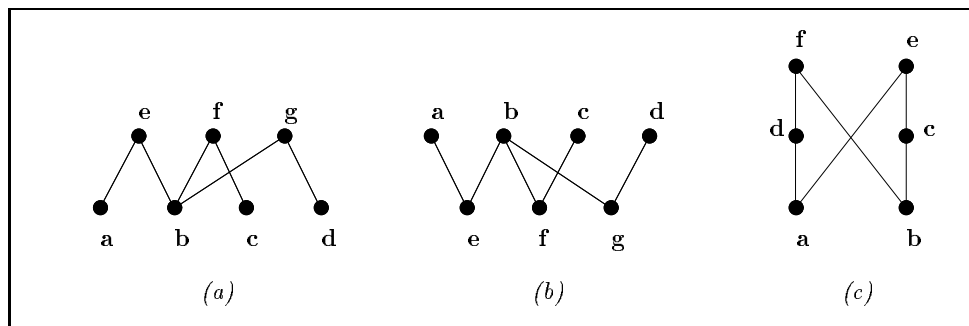


Figure 2: (a) an upper-contiguity order; (b) a non upper-contiguity order; (c) nor upper-contiguity neither lower-contiguity order.

An *hypergraph*  $\mathcal{H} = (X, \mathcal{E})$  on a set  $X$  of *vertices*, is a family  $\mathcal{E}$  of subsets of  $X$ , called the *edges*. Every hypergraph  $\mathcal{H} = (X, \mathcal{E})$  can be represented by its incidence bipartite graph  $B(\mathcal{H}) = (X, \mathcal{E}, E)$  where  $x \in E$  iff  $x \in e$  with  $x \in X$  and  $e \in \mathcal{E}$ . Then the connected components of  $B(\mathcal{H})$  are exactly those of  $\mathcal{H}$ . Our interest will focus on the particular class of interval hypergraphs which has been widely studied last years (see for example Duchet [9] or Trotter and Moore [22]).

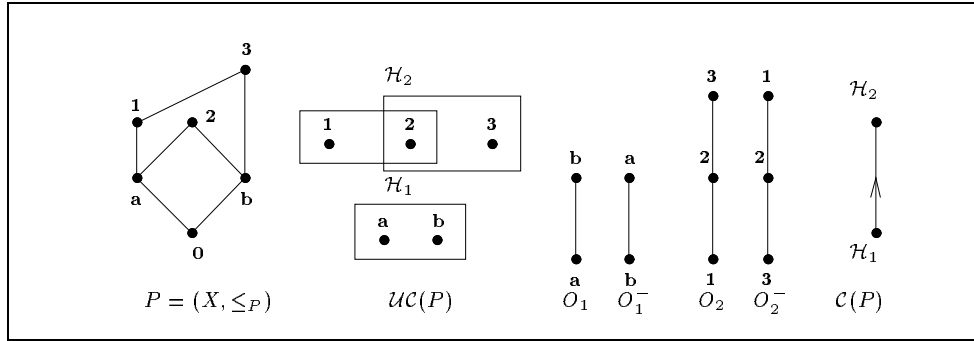
**Definition 2** Let  $\mathcal{H} = (X, \mathcal{E})$  be an hypergraph,  $\mathcal{H}$  is an interval hypergraph if there exists a permutation  $\sigma = (x_1, \dots, x_n)$  of  $X$  such that each edge of  $\mathcal{H}$  is an interval of  $\sigma$ .

**Definition 3** Let  $\mathcal{H} = (X, \mathcal{E})$  be an interval hypergraph, a total order " $<$ " on  $X$  is said to be induced by  $\mathcal{H}$  if  $\mathcal{E}$  is a family of intervals of  $(X, <)$ .

We now, introduce some definitions related to an order  $P = (X, \leq_P)$ .

**Definition 4** The hypergraph  $\mathcal{UC}(P) = (X - \text{Min}(P), \mathcal{E})$  is the upper cover hypergraph associated to  $P$  where  $\mathcal{E} = \{UC_P(x), x \in X\}$ .

**Definition 5** The connexion graph of  $\mathcal{UC}(P)$  is the directed graph  $\mathcal{C}(P)$  whose vertices are the connected components of  $\mathcal{UC}(P)$  (i.e.  $\{\mathcal{H}_i, i \in I$  where  $\mathcal{H}_i = (X_i, \mathcal{E}_i)\}$ ) and such that  $(\mathcal{H}_i, \mathcal{H}_j)$  is an arc iff  $i \neq j$  and there exists  $x \in X_i, y \in X_j$  such that  $x <_P y$ .

Figure 3: An order  $P$  and its associated hypergraphs and graphs.

**Theorem 1** Let  $P = (X, \leq_P)$  be an order, the two following conditions are equivalent:

- (i)  $P$  is an upper-contiguity order
- (ii)  $P$  satisfies the properties:
  - (a) Each connected component  $\mathcal{H}_i = (X_i, \mathcal{E}_i)$  of  $\mathcal{UC}(P)$ , is an interval hypergraph, and there exists a total order on  $X_i$  induced by  $\mathcal{H}_i$  which is a linear extension of  $P|_{X_i}$ .
  - (b)  $\mathcal{C}(P)$  is acyclic.

**Proof:** Let  $\{\mathcal{H}_i, i \in \{1, \dots, k\}\}$  with  $\mathcal{H}_i = (X_i, \mathcal{E}_i)$  be the set of connected components of  $\mathcal{UC}(P)$ . Assume that  $P = (X, \leq_P)$  is an upper-contiguity order and that  $L$  is one of its upper-contiguity extension. By definition of  $L$ , each  $\mathcal{H}_i$  is clearly an interval hypergraph and  $L|_{X_i}$  is a linear extension of  $P|_{X_i}$ . Since the  $X_i$ 's are pairwise disjoint intervals of  $L$ , the sequence  $\sigma(1) < \dots < \sigma(k)$ , defined by  $\sigma(i) < \sigma(j)$  if  $X_{\sigma(i)}$  is on the left of  $X_{\sigma(j)}$  in  $L$ , satisfies:  $(X_{\sigma(i)}, X_{\sigma(j)})$  is in  $\mathcal{C}(P)$  implies  $\sigma(i) < \sigma(j)$ . Thus  $\mathcal{C}(P)$  is acyclic. Let  $P = (X, \leq_P)$  be an order satisfying the two conditions of part (ii). Wlog, let us denote by  $O_i$  a total order on  $X_i$  induced by  $\mathcal{H}_i$  which is a linear extension of  $P|_{X_i}$ . Since  $\forall i, j \in \{1, \dots, k\}$  with  $i \neq j$  we have  $X_i \cap X_j = \emptyset$ , it is then clear that for each linear extension  $(\sigma(1), \dots, \sigma(k))$  of  $\mathcal{C}(P)$ , the total order  $O_{\sigma(1)} < \dots < O_{\sigma(k)}$  is a linear extension of  $P - \text{Min}(P)$ . Thus, for any total order  $O(\text{Min}(P))$  on  $\text{Min}(P)$ , the total order  $O(\text{Min}(P)) < O_{\sigma(1)} < \dots < O_{\sigma(k)}$  is a linear extension of  $P$ . By construction such a total order is an upper-contiguity extension of  $P$ .

□

### 3.2 Recognition of upper–contiguity orders

In the above characterization of upper–contiguity orders, the major difficulty, from an algorithmical point of view, is the recognition of interval hypergraphs. Since this recognition can be achieved in linear time using the  $PQ$ –tree data structure introduced in [1], we are able to state.

**Theorem 2** *Upper–contiguity orders are linearly recognizable in the size of their Hasse diagrams.*

**Proof:** Let  $P = (X, \leq_P)$  be an order given by its upper covering relations. The set of elements is stored in an array where each element is pointing on its upper cover list. In order to compute the connected components of  $\mathcal{UC}(P)$  we have to build the incidence bipartite graph of  $\mathcal{UC}(P)$ . This graph  $B(\mathcal{UC}) = (U, V, E)$  is defined by  $U = \mathcal{E}$ ,  $V = X - \text{Min}(P)$  and  $\forall (u, v) \in U \times V$ ,  $(u, v) \in E$  iff  $v \in u$ . The data structure representing  $B(\mathcal{UC})$  is in fact the same as the one representing  $P$ . Thus one can compute the connected components of  $\mathcal{UC}(P)$  in linear time. Let  $\mathcal{H}_i$  be a connected component of  $\mathcal{UC}(P)$  taken as input for the  $PQ$ –tree construction algorithm. If the  $PQ$ –tree cannot be produced then  $P$  is not an upper–contiguity order. Otherwise we obtain a total order, say  $O(X_i)$  on  $X_i$ , it is now sufficient to check that the covering relations of  $P$  between elements of  $X_i$  are all compatible either with  $O(X_i)$  or with  $O^-(X_i)$ , that is  $O(X_i)$  or  $O^-(X_i)$  is a linear extension of  $P|_{X_i}$ . These operations can be done in linear time on all the connected components of  $\mathcal{UC}(P)$ . It remains to compute  $\mathcal{C}(P)$ , which can be done in linear time, and to check that this graph is acyclic.

□

### 3.3 Counting upper–contiguity extensions

The characterization theorem of upper–contiguity orders given in the previous section states links between the upper–contiguity extensions of an order and the linear extensions of the connexion graph of its upper cover hypergraph. We will take advantage of these relations in order to show that it is  $\#P$ –complete to compute the number of upper–contiguity extensions.

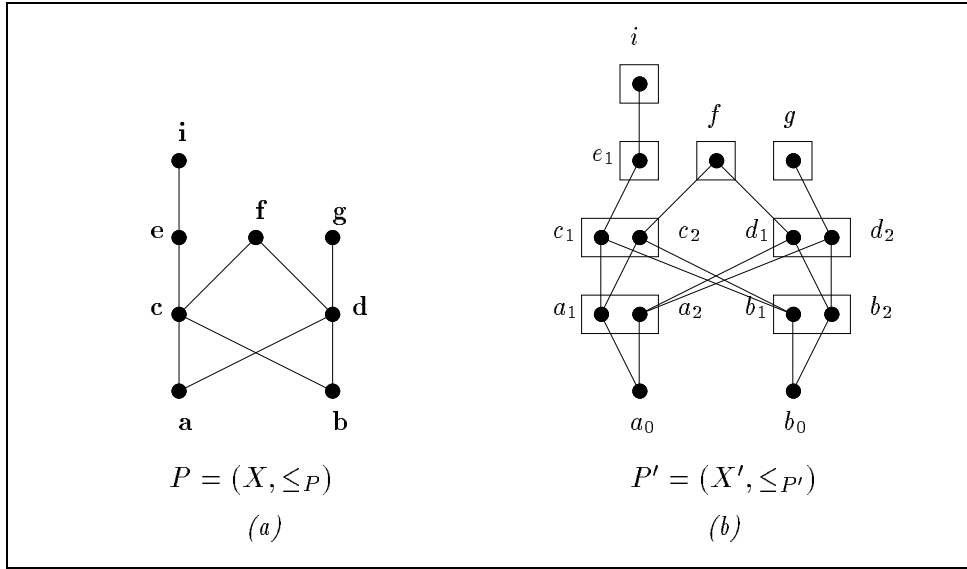


Figure 4: An order  $P$  and its associated order  $P'$ .

Roughly speaking an enumeration problem is  $\#P$ -complete if it is as hard as the problem of counting the number of true assignments for an instance of the satisfiability problem (for a formal definition see Garey and Johnson [10]).

To an order  $P = (X, \leq_P)$  we associate the order  $P' = (X', \leq_{P'})$  defined as follows. For  $UC_P(x) = \{y_1, \dots, y_k\}$  we set  $A(x) = \{x_1, \dots, x_k\}$  if  $k \geq 1$  and  $A(x) = \{x\}$  if  $k = 0$ . The underlying set of  $P'$  is then  $X' = X_0 \cup (\bigcup_{x \in X} A(x))$  where  $X_0 = \{x_0, x \in \text{Min}(P)\}$ . Now, for all  $x \in \text{Min}(P)$  we set  $UC_{P'}(x_0) = A(x)$  and for every  $x \prec_P y_j$  we set  $UC_{P'}(x_j) = A(y_j)$ .

With the above construction, we are now able to show that any order is isomorphic to the transitive closure of the connexion graph of an upper cover hypergraph associated to a given order.

**Lemma 1** *The transitive closure of  $\mathcal{C}(P')$  is isomorphic to  $P$ .*

**Proof:** By construction the connected components of  $\mathcal{UC}(P')$  are exactly the sets  $A(x)$ . Note that this implies that  $P'$  is an  $N$ -free order (see Section 4.2 for a formal definition). We also denote by  $A(x)$  the element of  $\mathcal{C}(P')$  corresponding to the set  $A(x)$ . If  $x \prec_P y$  then there exists  $x' \in A(x)$  such that  $UC_{P'}(x') = A(y)$  so  $(A(x), A(y))$  is an arc of  $\mathcal{C}(P')$ . Conversely suppose that  $(A(x), A(y))$  is an arc of  $\mathcal{C}(P')$  then there exists  $x' \in A(x)$  such that  $x' \leq_{P'} y'$  for every  $y' \in A(y)$ . For any  $y' \in A(y)$  there exists a chain  $(x' = z_1, \dots, z_k = y')$  where  $z_i \prec_{P'} z_{i+1}$ ,  $1 \leq i < k$ . If we denote by  $t_i$  the element of  $P$  such that  $z_i \in A(t_i)$  then  $(x = t_1, \dots, t_k = y)$  is a chain of  $P$ , so  $x \leq_P y$ .

□

In order to establish our next theorem, we will use a result of Brightwell and Winkler given in [6] and stating that the computation of the number of linear extensions of an order is  $\#P$ -complete.

**Theorem 3** *Computing the number of upper-contiguity extensions of an order is  $\#P$ -complete.*

**Proof:** Let  $P = (X, \leq_P)$  be an order with a minimal element, say  $x^1$ , and such that  $X = \{x^1, \dots, x^n\}$ . Let  $P'$  be the order associated to  $P$  as previously defined, by construction the connected components of  $\mathcal{UC}(P')$  are exactly the sets  $A(x)$ . We also denote by  $A(x)$  the element of  $\mathcal{C}(P')$  corresponding to the set  $A(x)$ . Any upper-contiguity extension of  $P'$  can be written  $(x_0^1, \pi(x^{i_1}), \pi(x^{i_2}), \dots, \pi(x^{i_n}))$  where  $(A(x^{i_1}), \dots, A(x^{i_n}))$  is any linear extension of  $\mathcal{C}(P')$  and  $\pi(x^{i_j})$  is any permutation of the set  $A(x^{i_j})$ . Every linear extension of  $P$  induces exactly  $\prod_{x \in X} |UC_P(x)|!$  distinct upper-contiguity

extensions of  $P'$ . Moreover two different linear extensions of  $P$  induce two disjoint sets of upper-contiguity extensions of  $P'$ . So the number of upper-contiguity extensions is exactly the product  $|\mathcal{L}(P)| \prod_{x \in X} |UC_P(x)|!$ . The

result follows since the factor  $\prod_{x \in X} |UC_P(x)|!$  is polynomially tractable in the size of  $P$  and since the computation of the number of linear extensions of an order is  $\#P$ -complete (see [6]).

□

The problem of counting the number of linear extensions of an height one order remaining an open problem, our previous result is only available for height four orders.

### 3.4 Dimension of height one upper–contiguity orders

Height one upper–contiguity orders can be seen as bipartite graphs having the consecutive one’s property for their adjacency matrix. That is, an height one order is an upper–contiguity order iff its upper cover hypergraph is an interval hypergraph. The class of height one upper–contiguity orders is a strict generalization of that of height one 2–dimensional orders. Indeed, 2–dimensional orders have been characterized, by Spinrad et al. in [21], as being orders whose adjacency matrix has both the consecutive one’s property and the enclosure property. That is there exists an ordering  $\tau$  of  $Max(P)$  such that for every  $x$ ,  $UC_P(x)$  is an interval of  $\tau$  and if  $UC_P(x) \subseteq UC_P(y)$  then  $UC_P(y) - UC_P(x)$  is also an interval of  $\tau$ . Height one upper–contiguity orders are also called convex graphs in [5], moreover the jump number problem is polynomially solved for this class in [7].

**Theorem 4** *The dimension of height one upper–contiguity orders is at most 3.*

**Proof:** Let  $P = (X, \leq_P)$  be an upper–contiguity order of height one, let  $\tau$  be an upper–contiguity extension of  $P$  putting  $Min(P)$  below  $Max(P)$ . We denote  $UC_P(x)$  by the interval  $[L_x, R_x]$ . We associate  $R_{x'}$  to  $R_x$ , which is the rightmost element  $R_z$  according to  $\tau$  in  $Max(P)$ , such that there exists  $z$  in  $Min(P)$  with  $[L_x, R_x] \subseteq [L_z, R_z]$ . We define, on the same underlying set as  $P$ , the order  $P_1$  by setting  $x <_{P_1} y$  iff  $y \in [L_x, R_{x'}]$ . Clearly  $P_1$  is an extension of  $P$ .  $P_1$  is a two dimensional order since  $UC_{P_1}(x)$  is an interval of  $\tau$  for every  $x$  in  $Min(P_1)$  and  $UC_{P_1}(y) - UC_{P_1}(x)$  is also an interval whenever  $UC_{P_1}(x) \subseteq UC_{P_1}(y)$ . In order to do this, we prove that  $R_{x'} = R_{y'}$ . By assumption we have  $R_{x'} \leq_\tau R_{y'}$ . Moreover, by definition there exists  $z$  in  $Min(P_1)$  such that  $[L_y, R_{y'}] \subseteq [L_z, R_z]$  with  $R_{y'} = R_z$  ( $z$  can be  $y$ ), but then  $[L_x, R_x]$  is also contained in  $[L_z, R_z]$  so  $R_{x'} \geq_\tau R_{y'}$ . We now define  $P_2$  an other extension of  $P$  by adding the comparabilities  $z <_{P_2} x$  for every  $z \in [R_x, R_{x'}]$  and this for every  $x \in Min(P)$ . That is, all comparabilities added to  $P$  in order to obtain  $P_1$  are reversed in  $P_2$ . We claim that  $P_2$  is acyclic. Indeed, assume that  $(x_1, y_1, x_2, y_2, \dots, x_k, y_k, x_1)$  is a directed cycle



of  $P_2$  with  $x_i \in \text{Min}(P)$  and  $y_i \in \text{Max}(P)$  for  $1 \leq i \leq k$ . Since  $(y_i, x_{i+1})$  has been added in  $P_2$  then we have  $y_{i+1} <_\tau y_i$  for  $1 \leq i \leq k$  (where  $x_{k+1} = x_1$  and  $y_{k+1} = y_1$ ), that is a cycle in  $\tau$ : a contradiction. We get a realizer of  $P$  by taking a realizer of  $P_1$  and any linear extension of  $P_2$ , so  $\dim(P) \leq 3$ .

□

Since orders of dimension two are polynomially recognizable (Spinrad gives in [20] an  $\mathcal{O}(n^2)$  recognition algorithm), we get the following corollary.

**Corollary 1** *The dimension of height one upper-contiguity orders is polynomially tractable.*

### 3.5 Total-upper-contiguity orders

We will now strengthen the property of upper-contiguity to all the linear extensions of an order. We obtain a new class of orders which appears to be a slight generalization of weak-orders, where a *weak-order* is an order which does not contain a suborder isomorphic to the  $1 \oplus 2$  (see Figure 1.(c)).

**Definition 6** *An order  $P = (X, \leq_P)$  is a total-upper-contiguity order, iff every linear extension of  $P$  is an upper-contiguity extension of  $P$ .*

**Theorem 5** *An order  $P = (X, \leq_P)$  is a total-upper-contiguity order iff it does not contain four elements  $\{a, b, x, y\}$  such that  $a, b \in UC_P(x)$ ,  $y \notin UC_P(x)$  and  $(y \parallel_P a \text{ or } y \parallel_P b)$ .*

**Proof:** Assume that there exists  $a, b, x, y \in X$  such that  $a, b \in UC_P(x)$ ,  $y \notin UC_P(x)$  and  $wlog\ y \parallel_P a$ . Then, if  $y <_P b$  (resp.  $b <_P y$ ) we can construct a linear extension of  $P$  having  $xayb$  (resp.  $xyba$ ) as subsequence. Let  $L \in \mathcal{L}(P)$  be a non contiguity extension of  $P$ . Then there exists four elements  $a, b, x, y \in X$  such that  $wlog\ a, b \in UC_P(x)$ ,  $y \notin UC_P(x)$  and  $x <_L b <_L y <_L a$ . Assume that  $y \sim_P a$  and  $y \sim_P b$  then we have  $b <_P y <_P a$  which is in contradiction with  $a, b \in UC_P(x)$ .

□

An order  $P$  is a *tree* if  $D_P(x)$  is a chain in  $P$ . An *anti-tree* is the dual of a tree.

**Theorem 6**  $P$  is a total-upper-contiguity order iff  $P = \sum_{x \in C_k} P_x$  where  $C_k$  is a  $k$ -element chain and every  $P_x$  is the disjoint sum of anti-trees.

**Proof:** The sufficient condition is trivial. For the necessary condition, assume that for each  $x$  in  $X$  we have  $|UC_P(x)| \leq 1$ , then  $P$  is the disjoint sum of anti-trees. If there exists  $x$  in  $X$  such that  $|UC_P(x)| > 1$  then  $P = \sum_{x \in C_2} P_x$  with  $P_{x_1} = P|_{X_1}$  and  $P_{x_2} = P|_{X_2}$  where  $X_2 = U_P(x) - \{x\}$  and  $X_1 = X - X_2$ . Indeed, by Theorem 5, for every  $y$  in  $X_1$  and for every  $z$  in  $X_2$  we have  $y <_P z$ . The proof follows by induction.  $\square$

The property of total-upper-contiguity is not stable by duality (see Figure 1.(d)). Since the  $1 \oplus 2$  is a total upper-contiguity order and the  $1 \oplus V$  is not one, we have the following strict inclusions: weak-orders  $\subset$  total-contiguity orders  $\subset$  series-parallel orders.

## 4 Bi-contiguity orders

In order to capture the membership comparability invariance we restrict the contiguity property to both upper covers and lower covers. However, this restriction is not drastic since interval orders and  $N$ -free orders still belong to this class.

**Definition 7** An order  $P = (X, \leq_P)$  is a *bi-contiguity order* iff it is both an upper-contiguity and a lower-contiguity order.

### 4.1 Comparability invariance of bi-contiguity orders

A subset  $H$  of  $X$  is an *autonomous set* of  $P$  if  $\forall h_1, h_2 \in H$ ,  $U_P(h_1) - H = U_P(h_2) - H$  and  $D_P(h_1) - H = D_P(h_2) - H$ .

Given an autonomous set  $H$  of  $P$ ,  $P = P^H = \sum_{x \in P'} Q_x$  with  $P' = (X', \leq_{P'})$  where  $X' = \{z\} \cup (X - H)$ ,  $\forall x, y \in X' - \{z\}$ ,  $(x \leq_{P'} y) \iff (y \leq_P x)$ ,  $(z <_{P'} t) \iff (\forall h \in H, h <_P t)$  and  $(t <_{P'} z) \iff (\forall h \in H, t <_P h)$ . The order family is  $\{Q_x = (Y_x, \leq_{Q_x}), x \in X'\}$  where  $Q_z = (H, \leq_{P|_H})$  and otherwise  $Y_x = \{x\}$ . We denote by  $P^{H-}$  the order obtained by substituting  $Q_z$  by its dual  $Q_z^-$  in the previous order family.

**Lemma 2** *Let  $P = (X, \leq_P)$  be an upper-contiguity order and  $H$  be an autonomous set of  $P$ , then there exists an upper-contiguity extension  $L$  of  $P$  such that:*

- (i)  $Min(P|_H)$  is an interval of  $L$ .
- (ii)  $H - Min(P|_H)$  is an interval of  $L$ .

That is,  $L$  can be written as  $L^s < L|_{Min(P|_H)} < L^m < L|_{(H - Min(P|_H))} < L^e$ .

**Proof:** (i) Let  $L$  be an upper-contiguity extension of  $P$  starting with all minimal elements of  $P$ . If  $Min(P|_H) \subseteq Min(P)$  there is nothing to prove. Otherwise we write  $L = (x_1, \dots, x_n, a_1, \dots, a_k, y_1, \dots, y_m)$  with  $Min(P|_H) \subseteq \{a_1, \dots, a_k\}$  and  $a_1, a_k \in Min(P|_H)$ . If  $Min(P|_H) = \{a_1, \dots, a_k\}$  we are done. If not, let  $a_i \in \{a_2, \dots, a_{k-1}\} - Min(P|_H)$ ,  $i$  being minimal with this property. We denote by  $l$  the smallest subscript greater than  $i$  such that  $a_{l+1}$  is in  $Min(P|_H)$ . Assume that  $L' = (x_1, \dots, x_n, a_i, \dots, a_l, a_1, \dots, a_{i-1}, a_{l+1}, \dots, a_k, y_1, \dots, y_m)$  is not an upper-contiguity extension of  $P$ . There are only two cases. First, the sequence  $a_i, \dots, a_l$  is carried into an interval representing an upper cover set  $UC_P(y)$ . Clearly  $a_1 \in UC_P(y)$  and since  $H$  is an autonomous set,  $a_k \in UC_P(y)$ . Thus,  $a_i, \dots, a_l \in UC_P(y)$  because  $L$  is an upper-contiguity extension of  $P$ , which is a contradiction. Secondly, the sequence  $a_i, \dots, a_l$  is carried out of an interval representing an upper cover set  $UC_P(z)$ . Clearly  $a_{l+1} \in UC_P(z)$  and since  $H$  is an autonomous set  $a_1, \dots, a_{i-1} \in UC_P(y)$ , which is a contradiction.

(ii) Let  $L$  be an upper-contiguity extension of  $P$  satisfying (i), then we can write

$$L = L^s < L_{Min(H,P)} < L^m < L_1 < L'_1 < L_2 \cdots < L'_{k-1} < L_k < L'' \text{ with} \\ \bigcup_{i \in \{1, \dots, k\}} L_i = H - Min(P|_H), \left( \bigcup_{i \in \{1, \dots, k-1\}} L'_i \right) \cap H = \emptyset \text{ and } L'' \cap H = \emptyset.$$

Since  $H$  is an autonomous set of  $P$ , then on one hand elements of any  $L'_i$  are incomparable with elements of  $H$  and on the other hand there is no element  $x$  in  $X$  having upper covers in both  $\bigcup_{i \in \{1, \dots, k\}} L_i$  and  $L'' \cup \left( \bigcup_{i \in \{1, \dots, k-1\}} L'_i \right)$ . So  $L' = L^s < L_{\text{Min}(H,P)} < L^m < L_1 \cdots < L_k < L'_1 \cdots < L'_{k-1} < L''$  is an upper-contiguity extension of  $P$ .

□

**Theorem 7** *Belonging to the bi-contiguity order class is a comparability invariant.*

**Proof:** Given a bi-contiguity order  $P$ , using the result given in [8], it is sufficient to prove that  $P^{H^-}$  is a bi-contiguity order for any autonomous set  $H$  of  $P$ . Let  $L = L^s < L_{|\text{Min}(P|_H)} < L^m < L_{|H-\text{Min}(P|_H)} < L^e$  be an upper-contiguity extension of  $P$  satisfying Lemma 2. Since  $P$  is a bi-contiguity order, it is clear that  $P|_H$  is also a bi-contiguity order. Thus, there exists  $\Gamma$  a lower-contiguity extension of  $H$  ending with all the maximal elements of  $H$ . We claim that  $L' = L^s < \Gamma_{|\text{Max}(P|_H)}^- < L^m < \Gamma_{|H-\text{Max}(P|_H)}^- < L^e$  is an upper-contiguity extension of  $P^{H^-}$ . The claim is achieved since  $\Gamma^-$  is an upper-contiguity extension of  $H^-$  and

(i)  $\forall x \in X - H$ , for  $x$  such that  $UC_P(x) \cap \text{Min}(P|_H) = \emptyset$ , we have  $UC_{P^{H^-}}(x) = UC_P(x)$ ; otherwise, we have  $UC_{P^{H^-}}(x) = (UC_P(x) - \text{Min}(P|_H)) \cup \text{Max}(P|_H)$

(ii)  $\forall x \in H$ , for  $x \notin \text{Min}(P|_H)$ , we have  $UC_{P^{H^-}}(x) = LC_P(x)$ ; otherwise, we have  $UC_{P^{H^-}}(x) = UC_P(\text{Max}(P|_H))$ .

A dual construction ensures the existence of a lower-contiguity extension for the order  $P^{H^-}$ .

□

## 4.2 Some classes of bi-contiguity orders

The first relevant class we speak about is that of  $N$ -free orders which yields that the computation of the dimension is an  $NP$ -complete problem for bi-contiguity orders. The  $NP$ -completeness of the  $N$ -free orders dimension has been shown in [16].

$P = (X, \leq_P)$  is an  $N$ -free order iff it has no " $N$ " as subdiagram, where " $N$ " is the ordered set on four elements  $\{a, b, c, d\}$  defined by  $a <_P c$ ,  $b <_P c$  and  $b <_P d$  (see Figure 1.(b)). Equivalent characterizations are given by Habib and Jégou in [11]. Let us recall some of them.

**Definition 8** Let  $P_1 = (X_1, \leq_{P_1})$  and  $P_2 = (X_2, \leq_{P_2})$  be orders where  $X_1 \cap X_2 = \emptyset$ . We define the parallel composition, denoted by  $P_1 \oplus P_2$ , as being the disjoint sum of  $P_1$  and  $P_2$ . If  $A \subseteq \text{Max}(P_1)$  and  $B \subseteq \text{Min}(P_2)$ , the quasi-série composition, denoted by  $(P_1, A) \otimes (P_2, B)$ , generates the order  $P$  on  $X_1 \cup X_2$  obtained by setting  $UC_P(x) = B \ \forall x \in A$ ,  $UC_P(x) = UC_{P_1}(x) \ \forall x \in X_1 - A$  and  $UC_P(x) = UC_{P_2}(x) \ \forall x \in X_2$ .

**Theorem 8** Let  $P = (X, \leq_P)$  be an order, then the following properties are equivalent:

- (i)  $P$  is an  $N$ -free order.
- (ii) For all  $x, y \in X$ ,  $(UC_P(x) = UC_P(y))$  or  $(UC_P(x) \cap UC_P(y) = \emptyset)$ .
- (iii) For all  $x, y \in X$ ,  $(LC_P(x) = LC_P(y))$  or  $(LC_P(x) \cap LC_P(y) = \emptyset)$ .
- (iv)  $P$  can be obtained by a sequence of parallel and quasi-série compositions from the one-element order.

Using Theorem 1, we are now able to show that  $N$ -free orders are bi-contiguity orders giving by the same way a new proof of the lower-contiguity of  $N$ -free orders stated in [4].

**Theorem 9**  $N$ -free orders are bi-contiguity orders.

**Proof:** Let  $P = (X, \leq_P)$  be an  $N$ -free order. By duality it is sufficient to establish the upper-contiguity of  $P$ . We proceed by induction using condition (iv) of Theorem 8. The only point to check is the compatibility of the upper-contiguity property with the quasi-série composition. Assume that  $P = (P_1, A) \otimes (P_2, B)$  and let  $L_1$  be an upper-contiguity extension of  $P_1$  and let  $L_2$  be an upper-contiguity extension of  $P_2$  starting with  $B$ . Then  $L_1 < L_2$  is a suitable upper-contiguity extension for  $P$ .

□

**Remark 1** We obtain a linear time algorithm for constructing an upper-contiguity extension of any  $N$ -free order  $P$  without using the algorithm of Theorem 2 but the simpler linear time decomposition algorithm given in [11].

The second class we are concerned with is that of interval orders. Indeed, this implies that the computation of jump number is  $NP$ -complete for bi-contiguity orders. The  $NP$ -completeness of the jump number problem for interval orders has been shown in [17].

$P = (X, \leq_P)$  is an *interval order* iff we can associate to  $X$  a collection  $(I_x)_{x \in X}$  of intervals of the real line such that  $x \leq_P y \iff I_x$  lies strictly on the left of  $I_y$  (i.e.  $\forall a \in I_x, \forall b \in I_y$ , we have  $a \leq_{\mathbb{R}} b$  where " $\leq_{\mathbb{R}}$ " is the usual order on real numbers). Let us recall some equivalent characterizations given by Möhring in [18].

**Theorem 10** *Let  $P = (X, \leq_P)$  be an order, then the following properties are equivalent:*

- (i)  $P$  is an interval order.
- (ii)  $P$  does not contain a suborder isomorphic to the  $2 \oplus 2$  (see Figure 1.(a)).
- (iii) There exists a linear extension of  $P$  such that, according to this numbering, the downsets are linearly ordered with regard to inclusion.
- (iv) There exists a linear extension of  $P$  such that, according to this numbering, the upsets are linearly ordered with regard to inclusion.

The next proposition shows that the upset inclusion property and the lower-contiguity property are closely related.

**Proposition 2** *A linear extension of an order  $P$ , such that according to this numbering, the downsets are linearly ordered with regard to inclusion, is an upper-contiguity extension of  $P$ .*

**Proof:** Let  $L$  be a linear extension of  $P = (X, \leq_P)$  satisfying:  $x \leq_L y \implies D_P(x) \subseteq D_P(y)$ . Assume that  $L$  is not an upper-contiguity extension of  $P$ . Then, there exists  $x, y \in X$  such that  $x \notin UC_P(y)$  and  $a <_L x <_L b$  where  $a$  is the leftmost (in  $L$ ) upper cover of  $y$  in  $P$  while  $b$  is its rightmost one. Since  $a <_L x$ ,  $D_P(a) \subseteq D_P(x)$  then  $y <_P x$  and therefore  $\exists z \in UC_P(y)$  such that  $z \leq_P x$ . So,  $a \leq_L z <_L x <_L b$  and we have a contradiction with  $b \in UC_P(y)$  because  $z \in D_P(x)$  and  $D_P(x) \subseteq D_P(b)$ .

□

With Proposition 2 and condition (iv) of Theorem 10, we get a new proof of the lower-contiguity of interval orders stated in [4]. Moreover, adding condition (iii), we immediately have:

**Theorem 11** *Interval orders are bi-contiguity orders.*

Lattices and 2-dimensional orders are not bi-contiguity orders, for the first class take the eight element boolean algebra and for the second one see the poset given in Figure 2.(c). However, 2-dimensional lattices are bi-contiguity orders. Recall that a lattice is 2-dimensional iff it is planar. We will use definitions on planar orders and lattices mentioned by Kelly and Rival in [14].

$P = (X, \leq_P)$  is a *lattice* iff any two element subset of  $X$  has an *infimum* and a *supremum* in  $P$ . The infimum (resp. the supremum) of a subset  $A$  of  $X$  is, when it exists, the greatest (resp. smallest) element of  $U_P(A)$  (resp.  $D_P(A)$ ). The minimal (resp. maximal) element of a lattice is called the *bottom* (resp. the *top*).

An element  $y$  in  $P$  is said to be *doubly irreducible* if it has both an unique upper cover and an unique lower cover in  $P$ .

$P$  is *planar* if it has a planar embedding, where a planar embedding  $e(P)$  of  $P$  consists of:

- (i) An injection  $x \mapsto \bar{x}$  from  $X$  to  $\mathbb{R}^2$  such that
- (ii) Straight line segments  $\overline{\bar{x}\bar{y}}$ , connecting  $\bar{x}$  and  $\bar{y}$  with  $\bar{x}$  strictly below  $\bar{y}$  whenever  $x <_P y$ . These segments do not intersect, except possibly at their endpoints.

For each  $x \in X$ , a planar embedding  $e(P)$  induces a strict linear ordering  $L_x$  on the set  $UC_P(x)$  defined by  $z <_{L_x} t$  iff  $\widehat{\bar{x}z} > \widehat{\bar{x}t}$ , for  $z, t \in UC_P(x)$ . As usual,  $\widehat{\bar{x}z}$  is the angle taken counterclockwise when the Euclidean plane is centered on  $\bar{x}$ .

We summarize some properties stated in [14] in the following theorem.

**Theorem 12**

- (i) *Every finite planar lattice with at least 3 elements has a doubly irreducible element distinct from the top and the bottom on the right boundary of any planar embedding.*
- (ii) *Let  $y$  be a doubly irreducible element in a planar lattice  $P$ , then  $P - \{y\}$  is still a planar lattice.*
- (iii) *Every non infinite face of  $e(P)$  consists in two paths of  $e(P)$  whose bounds are the only elements common to both its left and right boundary and whose interior is empty.*

If  $P$  is planar lattice with planar embedding  $e(P)$  and  $y$  is a doubly irreducible element in  $P$  such that  $x \prec_P y \prec_P z$ , then  $P - \{y\}$  has a planar embedding denoted  $e^*(P - \{y\})$  where  $e^*(P - \{y\})$  is  $e(P)$  in which  $\overline{y}$ ,  $\overline{xy}$  and  $\overline{yz}$  have been deleted and  $\overline{xz}$  has been added if necessary.

We are now able to establish the following lemma:

**Lemma 3** *If  $P$  is a planar lattice, then for any planar embedding  $e(P)$  there exists an upper-contiguity extension of  $P$  such that the ordering induced on each upper cover set is exactly the strict linear ordering induced by  $e(P)$ .*

**Proof:** Let  $P = (X, \leq_P)$  be a planar lattice. If  $|X| \leq 3$  the lemma is clearly true. Assume that  $|X| = n$  with  $n \geq 4$ . From Theorem 12, there exists a doubly irreducible element  $y$  on the right boundary of the planar embedding  $e(P)$  of  $P$ . We denote by  $x$  and  $z$  its lower and upper covers. By induction hypothesis, let  $L$  be an upper-contiguity extension of  $P - \{y\}$  satisfying the lemma with  $e(P - \{y\}) = e^*(P - \{y\})$ . If  $UC_P(x) = \{y\}$ , we obtain  $L'$  by inserting  $y$  immediately after  $x$  in  $L$ . It is sufficient to prove that  $L'$  is an upper-contiguity extension. This property comes from the upper-contiguity of  $L'$  on  $P - \{y\}$  and from that  $x$  is necessarily the rightmost element of  $UC_P(t)$  in  $L$  whenever  $x \in UC_P(t)$ . Now if  $|UC_P(x)| \geq 2$ , consider the non infinite face of  $e(P)$  having  $y$  in its right boundary and let  $t$  be the upper cover of  $x$  on its left boundary.  $L'$  is obtained by inserting  $y$  immediately after  $t$  in  $L$ . Since  $L$  is an upper-contiguity extension of  $P - \{y\}$  and  $t$  is the rightmost element of  $UC_{P-\{y\}}(u)$  in  $L$  whenever  $t \in UC_{P-\{y\}}(u)$  then  $L'$  is an upper-contiguity extension of  $P$ .

□



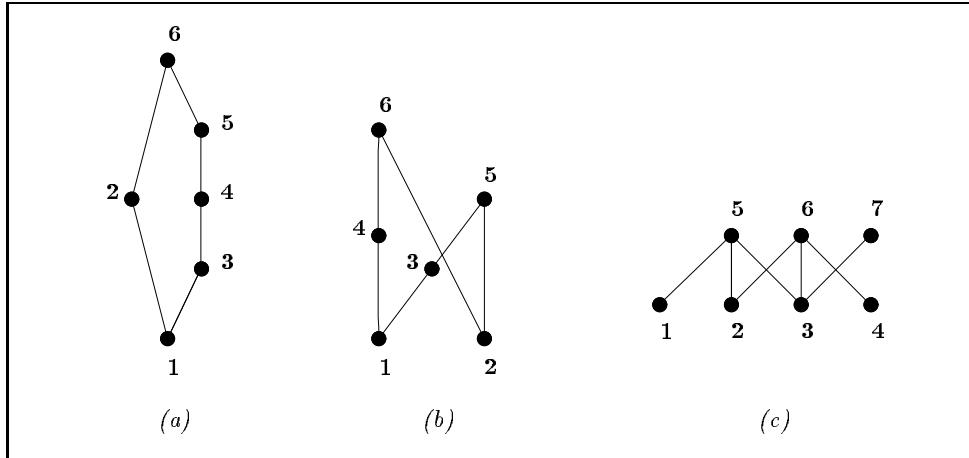


Figure 5: (a) and (b) two non strong-contiguity orders; (c) a 3-dimensional strong-contiguity order.

Since a planar lattice is stable by duality, we can immediately deduce:

**Theorem 13** *Planar lattices are bi-contiguity orders.*

## 5 Strong contiguity orders

In order to obtain a better algorithmic behaviour, we reinforce the contiguity property. Nevertheless, the dimension problem remains  $NP$ -complete on this new class and the jump number problem is still open.

**Definition 9** *An order  $P = (X, \leq_P)$  is a strong-contiguity order iff there exists  $L \in \mathcal{L}(P)$  which is both an upper-contiguity and lower-contiguity extension of  $P$ . Such an extension is called a strong-contiguity extension of  $P$ .*

The order given in Figure 5.(a) is a planar lattice and also a series-parallel interval order which proves that most of the known classes of orders have not the strong-contiguity property. The property of strong-contiguity is stable by duality but belonging to the strong-contiguity order class is not a comparability invariant as shown in Figure 5.(a) where the autonomous set

is  $\{2, 3, 4, 5, 6\}$ . Figure 5.(b) shows a non strong-contiguity order of height two which is a bi-contiguity order while Figure 5.(c) shows a 3-dimensional strong-contiguity order of height one.

Since strong-contiguity orders are defined in a similar way than semiorders (see condition (iii) of Theorem 14), it is natural to check if semiorders are strong-contiguity orders. *Semiorder* are interval orders which doesn't contain the  $1 \oplus 3$  as suborder. Equivalent characterizations can be found in [18]. Let us recall some of them.

**Theorem 14** *Let  $P = (X, \leq_P)$  be an order, then the following properties are equivalent:*

- (i)  *$P$  is a semiorder.*
- (ii)  *$P$  has an interval representation with unit length intervals.*
- (iii) *There exists a linear extension of  $P$  such that, according to this numbering, the downsets and the upsets are linearly ordered with regard to inclusion.*

With condition (iii) of Theorem 14, using Property 2 and duality we can deduce the following theorem.

**Theorem 15** *Semiorders are strong-contiguity orders.*

Height one bi-contiguity orders is a strict subclass of height one upper-contiguity orders (see Figure 2). However since an upper-contiguity order has a upper-contiguity extension starting with all its minimal elements, we can immediately deduce the following.

**Proposition 3** *The class of bi-contiguity orders of height one and the class of strong-contiguity orders of height one are identical.*

**Remark 2** *By Theorem 4, we can deduce that the dimension of height one strong-contiguity orders is at most 3. This bound is tight since the order given in Figure 5.(c) is a 3-dimensional strong-contiguity order. A linear time algorithm for the jump number of height one strong-contiguity orders is given in [5] under the name of biconvex graphs.*

## 5.1 Recognition of strong-contiguity orders

As we have done for the upper cover hypergraph we introduce the upper lower cover hypergraph associated to an order  $P$ , say  $\mathcal{ULC}(P) = (X, \mathcal{E})$ , where  $\mathcal{E} = \{UC_P(x), x \in X\} \cup \{LC_P(x), x \in X\}$  and its associated connexion graph  $\mathcal{C}(P)$ . Thus we can obtain the following characterization theorem for strong-contiguity orders whose proof follows from that of Theorem 1.

**Theorem 16** *Let  $P = (X, \leq_P)$  be an order, the two following conditions are equivalent:*

- (i)  $P$  is a strong-contiguity order
- (ii)  $P$  satisfies the properties:
  - (a) Each connected component  $\mathcal{H}_i = (X_i, \mathcal{E}_i)$  of  $\mathcal{ULC}(P)$ , is an interval hypergraph, and there exists a total order on  $X_i$  induced by  $\mathcal{H}_i$  which is an extension of  $P|_{X_i}$ .
  - (b)  $\mathcal{C}(P)$  is acyclic.

As in Theorem 2, we can state:

**Theorem 17** *Strong-contiguity orders are linearly recognizable.*

## 5.2 Dimension of strong-contiguity orders

Let  $P = (X, \leq_P)$  be an order, we define the order  $\tilde{P} = (\tilde{X}, \leq_{\tilde{P}})$  by  $\tilde{X} = X \cup X'$  where  $X' = \{x', x \in X\}$  is a disjoint copy of  $X$ ,  $x -<_{\tilde{P}} x'$  and  $LC_{\tilde{P}}(x) = \{y', y \in LC_P(x)\}$ . Then we have  $UC_{\tilde{P}}(x') = \{y, y \in UC_P(x)\}$ .

**Lemma 4** *If  $P$  is an  $N$ -free order then  $\tilde{P}$  is a strong-contiguity  $N$ -free order.*

**Proof:** By construction  $\tilde{P} = \sum_{x \in P} C_x$  where  $C_x$  is a two element chain. Then

$\tilde{P}$  is  $N$ -free. Let  $G = (V, E)$  be the directed graph where  $V = \{\{x\}, x \in \text{Min}(\tilde{P})\} \cup \{UC_{\tilde{P}}(x'), x' \in X'\} \cup \{\{x'\}, x' \in \text{Max}(\tilde{P})\} \cup \{LC_{\tilde{P}}(x), x \in X\}$  and  $(u, v) \in E$  iff  $\exists y \in u, \exists z \in v$  such that  $y -<_{\tilde{P}} z$ . Now we prove that  $G$  is acyclic. By construction an elementary cycle of  $G$  has an even number of vertices alternatively in  $X$  and  $X'$ , that is it consists of alternatively upper

cover sets and lower cover sets. Let  $u$  be a lower cover set and  $v$  be an upper cover set with  $(u, v) \in E$ , then there exists  $y \in u$  and  $z \in v$  such that  $y \prec_{\tilde{P}} z$ . Since  $\tilde{P}$  is an  $N$ -free order and  $v$  is an upper cover set of  $\tilde{P}$  we have  $y \prec_{\tilde{P}} t$  for any  $t \in v$ . Moreover for any  $t \in v$ , since  $t$  covers  $y$ ,  $t$  covers all elements in  $u$ . Thus  $u, v$  induce a complete bipartite subgraph in  $\tilde{P}$ . Let  $(u_1, u_2, \dots, u_{2k}, u_1)$  be an elementary cycle of  $G$  with  $u_1 \subseteq X$ . By definition of  $G$ ,  $(u_i, u_{i+1}) \in E$  implies there exists  $x_i \in u_i$ ,  $y_{i+1} \in u_{i+1}$  such that  $x_i \prec_{\tilde{P}} y_{i+1}$  and so  $(x_1, y_2, x_3, y_4, \dots, x_{2k-1}, y_{2k}, x_1)$  is a directed cycle in  $\tilde{P}$  which is a contradiction.

Let  $(u_1, \dots, u_k)$  be a linear extension of  $G$ , we get a linear extension  $L$  of  $\tilde{P}$  by substituting each  $u_i$  by its elements. Since each upper cover set and each lower cover set of  $\tilde{P}$  is either an  $u_i$  or reduced to a single element,  $L$  is a strong-contiguity extension.

□

**Theorem 18** *The problem of determining whether a strong-contiguity order has dimension  $\leq 3$  is NP-complete.*

**Proof:** Let  $P$  be an  $N$ -free order and let  $\tilde{P}$  be the order associated to  $P$  defined previously:  $\tilde{P} = \sum_{x \in P} C_x$  where  $C_x$  is a two element chain. Thus

$\dim(\tilde{P}) = \dim(P)$  and since  $\tilde{P}$  is a strong-contiguity order the result follows from [16].

□

**Remark 3** *Since the isomorphism problem for  $N$ -free orders is isomorphism-complete (stated by Habib and Möhring in [12]) and since  $P \simeq Q \iff \tilde{P} \simeq \tilde{Q}$  when  $P$  and  $Q$  are two  $N$ -free orders, we can deduce that the isomorphism problem for strong-contiguity order is isomorphism-complete.*

## 6 Conclusion

In this paper we have introduced a new linearly recognizable class of orders, namely the contiguity orders, and some natural subclasses. These classes are interesting in the sense that they strictly generalizes some well-known classes ( $N$ -free orders, interval orders, planar lattices for example) and give to them some new highlights, particularly a very simple encoding of their

Hasse diagram. However some questions remain opened as the classification of the jump number problem on strong contiguity orders and the number of strong-contiguity extensions.

We have obtained a characterization of contiguity orders in terms of interval hypergraphs which leads to a linear time algorithm. In order to get more structural properties on contiguity orders, the hypergraph theory seems to be fruitful. Furthermore contiguity orders are defined by the existence of a particular linear extension which seems to be closely related to a breadth-first extension i.e. obtained by a breadth-first search respecting the constraints of the order. Can this analogy gives some new highlights on these orders?

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Éditeur

INRIA, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)

ISSN 0249-6399