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# RATE OF CONVERGENCE OF A STOCHASTIC PARTICLE METHOD FOR THE KOLMOGOROV EQUATION WITH VARIABLE COEFFICIENTS

# VITESSE DE CONVERGENCE D'UNE METHODE PARTICULAIRE STOCHASTIQUE POUR L'EQUATION DE KOLMOGOROV A COEFFICIENTS NON CONSTANTS

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## Abstract

In a recent paper [12], E.G. Puckett proposed a stochastic particle method for the non linear diffusion–reaction P.D.E. in $[0,T] \times \boldsymbol{R}$ (the so–called "KPP" (Kolmogorov-Petrovskii-Piskunov) equation):

$$\begin{cases} \dfrac{\partial u}{\partial t} = Au = \Delta u + f(u) \\ \\ u(0,\cdot) = u_0(\cdot) \end{cases}$$

where $1 - u_0$ is the cumulative function, supposed to be smooth enough, of a probability distribution, and $f$ is a function describing the reaction. His justification of the method and his analysis of the error were based on a splitting of the operator $A$. He proved that, if $h$ is the time discretization step and $N$ the number of particles used in the algorithm, one can obtain an upper bound of the norm of the random error on $u(T,x)$ in $L^1(\Omega \times \boldsymbol{R})$ of order $1/N^{\frac{1}{4}}$, provided $h = \mathcal{O}\left(1/N^{\frac{1}{4}}\right)$, but conjectured, from numerical experiments, that it should be of order $\mathcal{O}(h) + \mathcal{O}(1/\sqrt{N})$, without any relation between $h$ and $N$.

We prove that conjecture. We also construct a similar stochastic particle method for more general non linear diffusion–reaction–convection P.D.E.'s

$$\begin{cases} \dfrac{\partial u}{\partial t} = Lu + f(u) \\ \\ u(0,\cdot) = u_0(\cdot) \end{cases}$$

where $L$ is a strongly elliptic second order operator with smooth coefficients, and prove that the preceding rate of convergence still holds when the coefficients of $L$ are constant, and is in the other case: $\mathcal{O}(\sqrt{h}) + \mathcal{O}(1/\sqrt{N})$.

The construction of the method and the analysis of the error are based on a stochastic representation formula of the exact solution $u$.



**AMS(MOS) classification:** 35-04, 35K57, 60J15, 60J65, 65-04, 65C05, 65C20, 65M15, 65N15.

**Résumé**

Dans un article récent [12], E.G. Puckett a proposé une méthode particulaire stochastique pour l'équation de diffusion–réaction non linéaire dans $[0,T] \times \boldsymbol{R}$ (dite équation "KPP", ou de Kolmogorov-Petrovskii-Piskunov):

$$\begin{cases} \dfrac{\partial u}{\partial t} = Au = \Delta u + f(u) \\ u(0,\cdot) = u_0(\cdot) \end{cases}$$

où $1 - u_0$ est la fonction de répartition, supposée suffisamment régulière, d'une loi de probabilité, et $f$ est une fonction décrivant la réaction. Sa justification de la méthode et son analyse de l'erreur sont fondées sur une décomposition ("splitting") de l'opérateur $A$. Il montre que, si $h$ est le pas de discrétisation en temps et $N$ le nombre de particules utilisées dans l'algorithme, une borne supérieure de la norme de l'erreur aléatoire sur $u(T,x)$ dans $L^1(\Omega \times \boldsymbol{R})$ est: $1/N^{\frac{1}{4}}$, pourvu qu'on ait la relation $h = \mathcal{O}\left(1/N^{\frac{1}{4}}\right)$, mais il conjecture aussi, à partir de résultats numériques, qu'elle devrait être d'ordre $\mathcal{O}(h) + \mathcal{O}(1/\sqrt{N})$, sans relation entre $h$ et $N$.

Nous démontrons cette conjecture. Nous construisons également une méthode particulaire similaire pour des équations de diffusion–reaction–convection plus générales

$$\begin{cases} \dfrac{\partial u}{\partial t} = Lu + f(u) \\ u(0,\cdot) = u_0(\cdot) \end{cases}$$

où $L$ est un opérateur différentiel du second ordre fortement elliptique à coefficients réguliers, nous montrons que la vitesse de convergence $\mathcal{O}(h) + \mathcal{O}(1/\sqrt{N})$ est conservée quand les coefficients de $L$ sont constants, et nous obtenons pour le cas général: $\mathcal{O}(\sqrt{h}) + \mathcal{O}(1/\sqrt{N})$ (sans relation entre $N$ et $h$).

La construction de la méthode et l'analyse de l'erreur reposent sur une représentation probabiliste de la solution exacte $u$.

**AMS(MOS) classification:** 35-04, 35K57, 60J15, 60J65, 65-04, 65C05, 65C20, 65M15, 65N15.

## Contents

# 1 Introduction

## 1.1 Setting of the problem

In a recent paper [12], E.G. Puckett proposed a stochastic particle method for the non linear P.D.E. in $[0,T] \times \mathbb{R}$:

$$\begin{cases} \dfrac{\partial u}{\partial t} = Au = \Delta u + f(u) \\ \\ u(0,\cdot) = u_0(\cdot) \end{cases}$$

where $1 - u_0$ is the cumulative function, supposed to be smooth enough, of a probability distribution, and $f$ is a function satisfying properties ensuring, in particular, that the solution $u(t,x)$ takes values in $[0,1]$.

His justification of the method and his analysis of the error were based on a splitting of the operator $A$; a rough presentation of the algorithm is the following:

**(A) initialization:** one locates $N$ particles on the real axis at positions $x_0^i$ with weights $\omega_0^i$ $(i = 1,\ldots,N)$ of order $\frac{1}{N}$, such that the function $\bar{u}(0,x) = \sum_{i=1}^{N} \omega_0^i H(X_0^i - x)$[1] is a good approximation of $u_0$ in $L^1(\mathbb{R})$;

**(B) approximation of the reaction:** one numerically solves the O.D.E.

$$\begin{cases} \dfrac{\partial v}{\partial t} = f(v) \\ \\ v(0,\cdot) = \bar{u}(0,\cdot) \end{cases}$$

on a time interval of length $h$ (this operation changes the weights of the particles);

**(C) approximation of the diffusion:** one numerically solves

$$\begin{cases} \dfrac{\partial w}{\partial t} = \Delta w \\ \\ w(0,\cdot) = v(h,\cdot) \end{cases}$$

by randomly and independently moving the particles, considered as independent Brownian particles, during a time interval equal to $h$, each particle keeping its own weight;

**(D) computation of the approximate solution:** the value at time $h$ and point $x$ of the approximate solution, $\bar{u}(h,x)$, is obtained by adding the weights of all the particles which are at the right hand-side of $x$;

**(E) iteration:** at each time step, one performs the operations $(B)$ (using $\bar{u}(ph,\cdot)$ instead of $\bar{u}(0,\cdot)$), $(C)$ and $(D)$.

[1] in all the sequel, $H$ will denote the Heaviside function: $H(y) = 0$ for $y < 0$, $H(y) = 1$ for $y \geq 0$.

The upper bound of the random error on $u(T,x)$ in $L^1(\Omega \times \mathbb{R})$ is shown to be of order $1/N^{\frac{1}{4}}$, provided $h = \mathcal{O}\left(1/N^{\frac{1}{4}}\right)$.

In the last Section of the paper, Puckett presents numerical results which obviously show that this estimation is very pessimistic, and conjectured that the rate of convergence should be of order $\mathcal{O}(h) + \mathcal{O}(1/\sqrt{N})$, without any relation between $h$ and $N$.

We tried to prove this conjecture by keeping the idea of the splitting but changing the technique used by Puckett to obtain some of his estimations. We could obtain a better rate of convergence than $1/N^{\frac{1}{4}}$ (we got $1/N^{\frac{2}{7}}$ provided $h$ is of order $1/N^{\frac{2}{7}}$), but we can neither get the right one, nor avoid a relation between $h$ and $N$, mainly because we had to sum up the approximation errors made at each step on the solution of the following P.D.E., where the initial condition $\bar{u}(ph,\cdot)$ is the approximate solution computed at step $p = 1, \ldots, \frac{T}{h}$:

$$\begin{cases} \dfrac{\partial w}{\partial t} = \Delta w \\ \\ w(0,\cdot) = \bar{u}(ph,\cdot) \end{cases}$$

and these local errors appear to be of order $\frac{h^{\frac{1}{4}}}{\sqrt{N}}$.

Besides, the notion of splitting does not represent the basic operation of the algorithm, which is the approximation of the measure $\frac{\partial}{\partial x}u(t,x)dx$ by a linear combination of Dirac measures at points defined by the current positions of the particles, and coefficients in the combination equal to the respective weights.

Thus we were lead to change our point of view.

Our objective was also to extend the algorithm to more general non-linear reaction–diffusion–convection P.D.E.'s, namely

$$\begin{cases} \dfrac{\partial u}{\partial t} = Au = Lu + f(u) \\ \\ u(0,\cdot) = u_0(\cdot) \end{cases} \tag{1.1}$$

where $L$ is a strongly elliptic second order operator with smooth coefficients. A natural question is then: WHAT MUST BE THE LAW OF THE MOTION OF THE PARTICLES ? A natural attempt is to move the particles according to the law of a diffusion process whose infinitesimal generator is $L$, but then one can see the computed solution (considered as a wave) propagate in the opposite direction to the propagation direction of the exact solution !

The answers to this question and to the determination of the rate of convergence of the algorithm that we will construct (which reduces to the Puckett algorithm when $L = \Delta$) are based upon an interpretation of the method completely different from the splitting of $A$ in (1.1), and an analysis of the error completely different from Puckett's one. The main tool will be a probabilistic representation formula of the exact solution, which will be deeply used to get estimations on the rate of convergence. Let us underline that, applied to the Puckett algorithm for the KPP equation, our estimations below prove Puckett's conjecture.

We also stress that the stochastic particle algorithm we analyse is not the only one that can be developed for non linear reaction–convection–diffusion equations. In particular, Sherman and Peskin have proposed a numerical method (without proving the convergence) in [14], based upon the simulation of branching Brownian Motions ; the term $f(u)$ is used to describe the law of the branching. For the convergence and the analysis of this algorithm, see the papers of Chauvin & Rouault ([5], [4], [3]). The main difference between the two algorithms is the following : the Sherman–Peskin particles have constant weights, but are highly dependent (they are the living particles of the branching process) ; the Puckett particles are independent, but the weights are dependent. For a finite horizon problem, the Puckett method seems to be simpler to implement, and easier to use on a parallel computer ; but if the problem is, for example, to study the asymptotic propagation velocity of a wave, then the Puckett algorithm cannot be efficient, because there is no reason at all for which it would be stable (see our results on the rate of convergence) ; in that case, the Sherman–Peskin method must be prefered, since it is naturally related to the evolution of the solution, the particles concentration being large where the gradient of the solution is large.

Our paper is organized as follows: in the second Section, we state our hypotheses and we present a collection of elementary results, which are frequently used in the sequel; then, in the third Section, we establish an original stochastic representation of the solution of the above non linear P.D.E. ; this formula permits to construct a stochastic particle method, which reduces to the splitting method of Puckett when the coefficients of $L$ are constant; in Section 4, we state our result on the rate of convergence; before proving it (Sections 6 and 7), we need to study in an accurate way how dependent are the weights of the particles: this is done in Section 5. Finally (Section 8) we consider the special case of constant coefficients.

Our numerical experiments for non constant coefficients examples do not add information to the excellent last Section of the paper of Puckett (devoted to the KPP equation), so that we refer to it.

### 1.2 Notation, conventions

In all the sequel, $C$ will denote any **deterministic** strictly positive constant independent from the time discretization step $h$ and the number of particles $N$ (but, most often, it will depend on $T$).

We always will suppose: $h \in ]0,1[$, of the form $\frac{T}{M}$, where $M$ is an integer.

When a stochastic process $(X_t)$ is such that $X_0 = y \;\; a.s.$ for some real number $y$, we will often write $(X_t(y))$.

When we write $\mathcal{O}(h)$ or $\mathcal{O}\left(\frac{1}{N}\right)$, etc, it must be understood that the concerned quantity (which may be random) can be bounded, uniformly in $\omega$ if it is random, by, respectively, $Ch$ or $\frac{C}{N}$, the constant $C$ being deterministic and uniform with respect to $h$ **and** $N$.

## 2 Hypotheses and elementary results

### 2.1 Hypotheses

Let us assume

**(H1)** $f$ is a $C^2$ function on $[0,1]$ such that $f(0) = f(1) = 0$, $f(u) \geq 0$ for $u \in [0,1]$, $\frac{f(u)}{u}$ is bounded in $]0,1]$ and continuous in 0;

**(H2)** $b$, $\sigma$ are two bounded $C^\infty$ functions; any derivative of any order is supposed to be a bounded function; $\sigma$ is bounded below by a strictly positive constant;

**(H3)** $1 - u_0$ is the cumulative function of a probability distribution.

In the Appendix, we will recall that, under **(H1)**, **(H2)**, **(H3)**, for any $T > 0$, there exists a unique classical solution in $]0,T] \times \mathbb{R}$, taking values in $[0,1]$, to the problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = L\, u + f(u) \\ \lim_{t \to 0} u(t, \cdot) = u_0(\cdot) \ \ \textit{at every continuity point of} \ \ u_0 \end{cases} \tag{2.1}$$

where

$$L = b(x)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(x)\frac{\partial^2}{\partial x^2}$$

In the sequel, we will often need an additional assumption on $u_0$:

**(H4)** $u_0$ is of class $C_b^\infty(\mathbb{R})$, and there exists strictly positive constants $C_1$, $C_2$ such that, for any $x$ in $\mathbb{R}$: $|u_0'(x)| \leq C_1 e^{-C_2 x^2}$,

or

**(H5)** $u_0$ is of the form

$$u_0(x) = \sum_{i=1}^N \omega_0^i \, H(x_0^i - x)$$

where the $\omega_0^i$'s are positive and such that:

$$\sum_{i=1}^N \omega_0^i = 1$$

## 2.2 Elementary results

In this short section, we will state easy consequences of quite classical results, needed in the sequel.

We begin with the obvious (but useful) inequality:

$$\forall x > 0, \quad \int_x^{+\infty} e^{-y^2} dy \leq C e^{-x^2} \tag{2.2}$$

Then, let us see some consequences of **(H2)**.

Let $(\zeta_t)$ a diffusion process whose infinitesimal generator has bounded and $C_b^\infty(\mathbb{R})$ coefficients $b_0$ and $\sigma_0$, and is strictly elliptic.

We have the well known property (see for instance Friedman [6]): under **(H2)**, there exists $\lambda > 0$ such that, for the density $p_t(y,z)$ of the law of $\zeta_t(y)$, we have, for any $y, z \in \mathbb{R}$, for any $0 < t \le T$:

$$p_t(y,z) \le \frac{C}{\sqrt{t}} \exp\left(-\frac{(z-y)^2}{2\lambda t}\right) \tag{2.3}$$

Therefore (for $(iii)$ we apply (2.2)):

**Corollary 2.1** *Let $(\zeta_t)$ a diffusion process whose infinitesimal generator has bounded and $C_b^\infty(\mathbb{R})$ coefficients, and is strictly elliptic.*

*Then there exist $C > 0$ and $\lambda > 0$ such that, $\forall 0 < t \le T$, $\forall x, y \in \mathbb{R}$, we have:*

$(i)$
$$\mathbb{P}(\zeta_t(y) < x) \le \frac{C}{\sqrt{t}} \int_{-\infty}^{x} \exp\left(-\frac{(z-y)^2}{2\lambda t}\right) dz$$

$(ii)$
$$\mathbb{P}(\zeta_t(y) > x) \le \frac{C}{\sqrt{t}} \int_{x}^{+\infty} \exp\left(-\frac{(z-y)^2}{2\lambda t}\right) dz$$

$(iii)$
$$\mathbb{P}(\zeta_t(y) < x)\mathbb{P}(\zeta_t(y) > x) \le C \exp\left(-\frac{(y-x)^2}{2\lambda t}\right)$$

Besides we observe that, under the above hypotheses, there exists a constant $C > 0$ such that:

$$\forall 0 < t \le T, \quad \forall y \in \mathbb{R}, \quad \mathbb{E}|\xi_t(y) - y| \le C\sqrt{t} \tag{2.4}$$

and ([2])

$$\forall\, t > 0, \qquad \lim_{y \to +\infty} \zeta_t(y) = +\infty \quad , \quad a.s. \tag{2.5}$$

because the function $y \to \xi_t(y)$ is a.s. increasing since its derivative is an exponential.

**Lemma 2.2** *Under the above hypotheses on $(\xi_t)$, for some $C > 0$, for any $T \ge t > 0$, the probability density $p_t(x,y)$ of the law of $\xi_t(x)$ verifies:*

$$\left| \int_{\mathbb{R}} p_t(x,y)\, dx - 1 \right| \le C\, t$$

*for any $y \in \mathbb{R}$.*

[2]see Kunita [8], chapter 2, e.g. for the diffeomorphism property of stochastic flows associated to stochastic differential equations.

**Proof**

Let $L$ be the infinitesimal generator of $(\xi_t)$; then the function

$$q(t,y) = \int_R p_t(x,y)\,dx$$

verifies the equation (cfr. [6], chapter 6, Problem 10):

$$\begin{cases} \dfrac{\partial q}{\partial t}(t,y) = L^* q\,(t,y) = \dfrac{1}{2}\dfrac{\partial^2}{\partial y^2}(\sigma_0^2(y)q(t,y)) - \dfrac{\partial}{\partial y}(b_0(y)q(t,y)) \\ \lim\limits_{t\to 0} q(t,y) = 1 \end{cases}$$

Denote by $(\alpha_t(y))$ the diffusion issued from $y$ with infinitesimal generator $(\sigma_0(x)\sigma_0'(x) - b_0(x))\frac{\partial}{\partial x} + \frac{1}{2}\sigma_0^2(x)\frac{\partial^2}{\partial x^2}$. We have, denoting $a_0(x) = \sigma_0^2(x)$ (see again [6], chapter 6, e.g.):

$$q(t,y) = \boldsymbol{E}\exp\left(\int_0^t \left(\frac{1}{2}a_0''(\alpha_s(y)) - b_0'(\alpha_s(y))\right) ds\right)$$

∎

We recall that we denote by $b_0(\cdot)$ and $\sigma_0(\cdot)$ respectively the drift and the diffusion coefficients of $(\zeta_t)$. Let $(B_t)$ be a standard real Brownian motion.

The Euler scheme is defined by

$$\hat{\zeta}_{p+1} = \hat{\zeta}_p + b_0(\hat{\zeta}_p)\,h + \sigma_0(\hat{\zeta}_p)(B_{(p+1)h} - B_{ph}) \tag{2.6}$$

and the Milshtein scheme is defined by

$$\bar{\zeta}_{p+1} = \bar{\zeta}_p + b_0(\bar{\zeta}_p)\,h + \sigma_0(\bar{\zeta}_p)(B_{(p+1)h} - B_{ph}) + \frac{1}{2}\sigma_0(\bar{\zeta}_p)\sigma_0'(\bar{\zeta}_p)\left((B_{(p+1)h} - B_{ph})^2 - h\right) \tag{2.7}$$

Now, let us recall a result of convergence rate (the part (i) is easy to show, the second is due to Milshtein [9]):

**Proposition 2.3** *Let us suppose that the functions $b_0$ and $\sigma_0$ are of class $C^\infty$, and that any derivative is uniformly bounded. For the Milshtein scheme, we have:*

(i) *for any $k \in \mathbb{N}^*$, for any initial condition $\zeta_0$ such that $\boldsymbol{E}|\zeta_0|^{2k} < \infty$, there exists a strictly positive constant $C$ (depending only on $T$, and the bounds of $b_0$, $\sigma_0$ and their two first derivatives) such that:*

$$\forall M \in \mathbb{N}^*, \quad \forall\, h = \frac{T}{M} \quad ,\forall\, p = 1,\ldots,M, \quad \boldsymbol{E}|\zeta_{ph}|^{2k} + \boldsymbol{E}|\bar{\zeta}_p|^{2k} < C(1 + \boldsymbol{E}|\zeta_0|^{2k}) \tag{2.8}$$

(ii) *there exists a positive constant $C$ such that, for any initial condition $y$:*

$$\boldsymbol{E}|\zeta_h - \bar{\zeta}_1|^2 \leq C\,h^3 \tag{2.9}$$

*and, for any $p = 1,\ldots,M = \frac{T}{h}$:*

$$\boldsymbol{E}|\zeta_{ph} - \bar{\zeta}_p|^2 \leq C\,h^2 \tag{2.10}$$

**Remark 2.4** *When $\sigma_0$ is not a constant function, for the Euler scheme, one generically has*

$$\mathbb{E}|\zeta_{ph} - \hat{\zeta}_p|^2 \leq Ch$$

*When $b_0$ and $\sigma_0$ are constant functions, there is no approximation error.*

Now, let us remark some consequences of the hypothesis **(H4)** or **(H5)**.

**Remark 2.5** *Under* **(H3)**, **(H4)**, *one has:*

$$\begin{aligned} \int_0^{+\infty} u_0(x)\,dx + \int_{-\infty}^0 (1-u_0(x))\,dx &= -\int_0^{+\infty} x\,du_0(x) + \int_{-\infty}^0 x\,du_0(x) \\ &= \int_{\mathbb{R}} |x|\,d(1-u_0)(x) < +\infty \end{aligned}$$

*The hypothesis* **(H5)** *instead of* **(H4)** *implies:*

$$\int_0^{+\infty} u_0(x)\,dx + \int_{-\infty}^0 (1-u_0(x))\,dx \leq \sup_{i=1,\ldots,N} |\omega_0^i| \sum_{i=1}^N |x_0^i|$$

**Lemma 2.6** *Let us assume that $u_0$ verifies hypotheses* **(H3)**, **(H4)**; *then there exists $C > 0$ such that:*

$$\int_0^{+\infty} u(t,x)\,dx + \int_{-\infty}^0 (1-u(t,x))\,dx \leq C$$

*for any $t \in [0,T]$.*

*If $u_0$ satisfies* **(H5)**, *then:*

$$\int_0^{+\infty} u(t,x)\,dx + \int_{-\infty}^0 (1-u(t,x))\,dx \leq C\left(1 + \sup_{i=1,\ldots,N} |\omega_0^i| \sum_{i=1}^N |x_0^i|\right)$$

### Proof

We will only treat the first case, the **(H5)** case asking just local and easy modifications.

Let us introduce the process $(Z_t(x))$ solution of:

$$dZ_t = b(Z_t)\,dt + \sigma(Z_t)\,dB_t \quad , \qquad Z_0(x) = x \tag{2.11}$$

The function $\frac{f\circ u}{u}$ being bounded, the Feynman–Kac formula

$$u(t,x) = \mathbb{E}u_0(Z_t(x)) \exp\left(\int_0^t \frac{f\circ u}{u}(s, Z_s(x))\,ds\right)$$

yields (using (2.3)):

$$\begin{aligned}
\int_0^{+\infty} u(t,x)\,dx &\le C \int_0^{+\infty} \boldsymbol{E} u_0(Z_t(x))\,dx \\
&\le C \int_0^{+\infty} \int_{\boldsymbol{R}} u_0(y) \frac{1}{\sqrt{t}} \exp\left(-\frac{(y-x)^2}{2\lambda t}\right) dy\,dx \\
&\le C \int_{-\infty}^{0} \int_{-\frac{y}{\sqrt{\lambda t}}}^{+\infty} e^{-\frac{\theta^2}{2}}\,d\theta\,dy + C \int_0^{+\infty} \frac{u_0(y)}{\sqrt{t}} \int_0^{+\infty} \exp\left(-\frac{(y-x)^2}{2\lambda t}\right) dx\,dy \\
&\le C \int_{-\infty}^{0} \exp\left(-\frac{y^2}{2\lambda t}\right) dy + C \int_0^{+\infty} u_0(y)\,dy
\end{aligned}$$

and we apply the preceding Remark.

For the term $\int_{-\infty}^{0}(1-u(t,x))dx$, let us denote $v(t,x) = 1 - u(t,x)$ and observe:

$$\begin{cases} \dfrac{\partial v}{\partial t} = Lv - f(u) = Lv + \dfrac{f(1)-f(u)}{1-u} v \\ v_0 = 1 - u_0 \end{cases}$$

and, by observing that the function $\frac{f(1)-f(u)}{1-u}$ is bounded, we get:

$$\int_{-\infty}^{0} v(t,x)\,dx \le C \int_{-\infty}^{0} (1 - \boldsymbol{E} u_0(Z_t(x))\,dx$$

from which we proceed as before.

■

The next lemma gives a control of the error due to the permutation of the expectation and a non-linear function:

**Lemma 2.7** *Let $g$ a function of $C^2$ class with bounded second derivative; then for any square integrable random variable $X$ we have*

$$|\boldsymbol{E} g(X) - g(\boldsymbol{E} X)| \le C\, \boldsymbol{E}(X - \boldsymbol{E} X)^2 \tag{2.12}$$

**Proof**

It follows easily from the Taylor formula

$$g(x) - g(y) = g'(y)(x-y) + \frac{1}{2} g''(y + \theta(x-y))(x-y)^2$$

with $\theta \in ]0,1[$ and $y = \boldsymbol{E} X$.

■

Finally, we point out that the next equality will be used in several subsequent computations: for any $y, z \in \boldsymbol{R}$:

$$\int_{\boldsymbol{R}} |H(y-x) - H(z-x)|dx = |y - z| \tag{2.13}$$

## 3 Representation of the solution of (1.1) and construction of the algorithm

### 3.1 A probabilistic representation of the solution

We introduce a probability space $(\Omega, \mathcal{F}, \boldsymbol{P})$ equipped with a Brownian motion $(B(t))$ ; for $0 \leq s < t \leq T$, $\mathcal{F}_s^t$ will denote the least complete $\sigma$–field for which all the $B_v - B_u$ $(s \leq u < v \leq t)$ are measurable.

**Theorem 3.1** *Under $\{(Hi), i = 1, \ldots, 3\}$, if $u_0$ is of class $C_b^\infty(\boldsymbol{R})$, we have the following representation:*

$$\begin{aligned} u(t,x) &= -\int_{-\infty}^{+\infty} \widetilde{\boldsymbol{E}}_t^z \left[ H(\xi_{0,t}^{-1}(z) - x) \exp\left(\int_0^t f' \circ u(s, \xi_{t-s,t}^{-1}(z))\, ds\right)\right] u_0'(z)\, dz \\ &= \int_{-\infty}^{+\infty} \widetilde{\boldsymbol{E}}_t^z \left[ H(\xi_{0,t}^{-1}(z) - x) \exp\left(\int_0^t f' \circ u(s, \xi_{t-s,t}^{-1}(z))\, ds\right)\right] d(1 - u_0(z)) \quad (3.1) \end{aligned}$$

*where $\widetilde{\boldsymbol{E}}_t^z$ denotes the expectation computed with respect to a probability measure $\widetilde{\boldsymbol{P}}_t^z$ equivalent to $\boldsymbol{P}$, and where $(\xi_{\theta,t}^{-1}(z))$ denotes the solution to the following stochastic differential equation, backward in time:*

$$\begin{cases} d\xi_{\theta,t}^{-1}(z) = -\sigma(\xi_{\theta,t}^{-1}(z))\, \hat{d}\tilde{B}_\theta - \{b(\xi_{\theta,t}^{-1}(z)) - \sigma(\xi_{\theta,t}^{-1}(z))\, \sigma'(\xi_{\theta,t}^{-1}(z))\}\, d\theta \\ \xi_{t,t}^{-1}(z) = z \end{cases} \quad (3.2)$$

*$(\tilde{B}_\theta)$ being a $(\mathcal{F}_\theta^t)_{\theta<t}$-$\widetilde{\boldsymbol{P}}_t^z$ Brownian motion (in backward time), and $\hat{d}$ denoting the backward stochastic integral* ([3]).

**Proof**

The function $v(t,x) := \frac{\partial u}{\partial x}(t,x)$ satisfies the following equation:

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) = \dfrac{1}{2}\sigma^2(x) \dfrac{\partial^2 v}{\partial x^2}(t,x) + (b(x) + \sigma(x)\,\sigma'(x)) \dfrac{\partial v}{\partial x}(t,x) \\ \qquad\qquad + (b'(x) + f' \circ u(t,x))\, v(t,x) \\ v(0,x) = u_0'(x) \end{cases}$$

By applying the Feynman-Kac formula we obtain:

$$v(t,x) = \boldsymbol{E}\left[u_0'(Y_t(x)) \exp\left\{\int_0^t [b'(Y_s(x)) + f' \circ u(t-s, Y_s(x))]\, ds\right\}\right] \quad (3.3)$$

[3] For a definition, cfr. Kunita [8], end of chapter I.

where $(Y_t)$ is the solution to:

$$dY_t = (b(Y_t) + \sigma(Y_t)\,\sigma'(Y_t))\,dt + \sigma(Y_t)\,dB_t \tag{3.4}$$

Consequently, recalling that $1 - u_0$ is increasing and that $u(t,x) \to 1$ as $x \to -\infty$ (see Lemma (2.6)):

$$u(t,x) = -\mathbb{E}\int_x^{+\infty} u_0'(Y_t(y))\,\exp\left\{\int_0^t [b'(Y_s(y)) + f'\circ u(t-s, Y_s(y))]\,ds\right\}\,dy$$

As we will see below, the particles algorithm is based on approximating the measure $u_0'(y)\,dy$ by a measure of type $\sum_{i=1}^N \omega_0^i\,\delta_{x_0^i}$; that suggests to perform the change of variable $z = \xi_{0,t}(y)$, where $\xi_{0,t}(\cdot)$ is the flow associated to the stochastic differential equation (3.4); hence we set $y = \xi_{0,t}^{-1}(z)$.

Using results of the second chapter of Kunita [8] we have, for $\theta < t$:

$$\xi_{\theta,t}^{-1}(z) = z - \int_\theta^t \sigma(\xi_{s,t}^{-1}(z))\,\hat{d}B_s - \int_\theta^t b(\xi_{s,t}^{-1}(z))\,ds$$

where $\hat{d}B_\theta$ denotes the "backward" stochastic integral.

One can infer:

$$\frac{\partial}{\partial z}\xi_{0,t}^{-1}(z) = 1 - \int_0^t \sigma'(\xi_{\theta,t}^{-1}(z))\frac{\partial}{\partial z}\xi_{\theta,t}^{-1}(z)\,\hat{d}B_\theta - \int_0^t b'(\xi_{\theta,t}^{-1}(z))\frac{\partial}{\partial z}\xi_{\theta,t}^{-1}(z)\,d\theta$$

from which

$$\frac{\partial}{\partial z}\xi_{0,t}^{-1}(z) = \exp\left(\int_0^t \left\{-b'(\xi_{\theta,t}^{-1}(z)) - \frac{1}{2}\sigma'^2(\xi_{\theta,t}^{-1}(z))\right\}d\theta - \int_0^t \sigma'(\xi_{\theta,t}^{-1}(z))\,\hat{d}B_\theta\right)$$

Hence, taking in account (2.5), we have:

$$\begin{aligned} u(t,x) \;=\; -\mathbb{E}\Bigg[\int_{\xi_{0,t}(x)}^{+\infty} u_0'(z)\,\exp\left\{\int_0^t \Big(b'(\xi_{0,s}(\alpha)) + f'\circ u(t-s,\xi_{0,s}(\alpha))\Big)\,ds\Big|_{\alpha=\xi_{0,t}^{-1}(z)}\right\} \\ \exp\left\{\int_0^t\Big[-b'(\xi_{s,t}^{-1}(z)) - \frac{1}{2}\sigma'^2(\xi_{s,t}^{-1}(z))\Big]\,ds - \int_0^t \sigma'(\xi_{s,t}^{-1}(z))\hat{d}B_s\right\}\,dz\Bigg] \end{aligned}$$

One now uses the Lemma 6.2 of chapter II of Kunita [8]: for any continuous function $g(s,x)$ we have

$$\int_0^t g(s,\xi_{0,s}(\alpha))\Big|_{\alpha=\xi_{0,t}^{-1}(z)}\,ds = \int_0^t g(s,\xi_{s,t}^{-1}(z))\,ds$$

Thus

$$u(t,x) = -\mathbb{E}\left[\int_{\xi_{0,t}(x)}^{+\infty} u_0'(z)\,\exp\left\{\int_0^t f'\circ u(t-s,\xi_{s,t}^{-1}(z))\,ds\right\}M_0^t(z)\,dz\right]$$

where $(M_\theta^t(z))_{\theta\leq t}$ is the exponential (backward) $(\mathcal{F}_\theta^t)_{\theta\leq t}$-martingale defined by

$$M_\theta^t(z) = \exp\left\{-\int_\theta^t \frac{1}{2}\sigma'^2(\xi_{s,t}^{-1}(z))\,ds - \int_\theta^t \sigma'(\xi_{s,t}^{-1}(z))\hat{d}B_s\right\}$$

On $(\Omega, \mathcal{F}, \boldsymbol{P}, (\mathcal{F}_\theta^t)_{\theta\leq t})$, one performs the Girsanov transformation defined by

$$\widetilde{\boldsymbol{P}}_t^z(A) := \boldsymbol{E}\left[1_A\, M_0^t(z)\right] \quad , \quad A \in \mathcal{F}_0^t$$

then

$$u(t,x) = -\int_{-\infty}^{+\infty} u_0'(z)\,\widetilde{\boldsymbol{E}}_t^z\left[H(-\xi_{0,t}(x)+z)\exp\left\{\int_0^t f'\circ u(s,\xi_{t-s,t}^{-1}(z))\,ds\right\}\right]\,dz$$

The application $x \to \xi_{0,t}(x)$ is a.s. increasing (its derivative is an exponential). Hence

$$u(t,x) = -\int_{-\infty}^{+\infty} \widetilde{\boldsymbol{E}}_t^z\left[H(\xi_{0,t}^{-1}(z)-x)\,\exp\left\{\int_0^t f'\circ u(s,\xi_{t-s,t}^{-1}(z))\,ds\right\}\right]\,u_0'(z)\,dz$$

where $(\xi_{\theta,t}^{-1}(z))$ is, on $(\Omega, \mathcal{F}, \widetilde{\boldsymbol{P}}_t^z, (\mathcal{F}_\theta^t)_{\theta\leq t})$, the solution to

$$\begin{cases} d\xi_{\theta,t}^{-1}(z) = -\sigma(\xi_{\theta,t}^{-1}(z))\hat{d}\tilde{B}_\theta - (b(\xi_{\theta,t}^{-1}(z)) - \sigma(\xi_{\theta,t}^{-1}(z))\,\sigma'(\xi_{\theta,t}^{-1}(z)))\,d\theta \\ \xi_{t,t}^{-1}(z) = z \end{cases}$$

where $(\tilde{B}_\theta)$, defined by

$$\tilde{B}_\theta = B_\theta + \int_\theta^t \sigma'(\xi_{s,t}^{-1}(z))\,ds,$$

is a $(\mathcal{F}_\theta^t)_{\theta\leq t}$-$\widetilde{\boldsymbol{P}}_t^z$-Brownian motion.

∎

One can deduce the same type of the preceding result for a piecewise constant initial data $u_0$:

**Proposition 3.2** *If $u_0$ is of the form $\sum_{i=1}^N \omega_0^i H(x_0^i - x)$, then we have:*

$$u(t,x) = \sum_{i=1}^N \omega_0^i\,\widetilde{\boldsymbol{E}}_t^{x_0^i}\left[H(\xi_{0,t}^{-1}(x_0^i)-x)\,\exp\left\{\int_0^t f'\circ u(s,\xi_{t-s,t}^{-1}(x_0^i))\,ds\right\}\right] \tag{3.5}$$

## Proof

Set $u_0^{(\varepsilon)} = u(\varepsilon, x)$, where $u$ is the solution to the nonlinear problem (2.1) with $u_0$ as initial datum; from the Appendix we know that $1 - u_0^{(\varepsilon)}$ is a regular cumulative function (of a probability distribution) such that $u_0^{(\varepsilon)} \to u_0$ for all continuity points of $u_0$. Hence, for any $t > 0$, using the representation (3.1), we have:

$$\begin{aligned} u(t+\varepsilon, x) &= u^{(\varepsilon)}(t,x) \\ &= \int_{-\infty}^{+\infty} \underbrace{\widetilde{\boldsymbol{E}}_t^z \left[ H(\xi_{0,t}^{-1}(z) - x) \exp\left( \int_0^t f' \circ u^{(\varepsilon)}(s, \xi_{t-s,t}^{-1}(z))\, ds \right) \right]}_{\psi_\varepsilon(z)} d(1 - u_0^{(\varepsilon)}(z)) \end{aligned} \tag{3.6}$$

We can write:

$$u(t+\varepsilon, x) = \underbrace{\int_{-\infty}^{+\infty} \Big(\psi_\varepsilon(z) - \psi_0(z)\Big)\, d(1 - u_0^{(\varepsilon)}(z))}_{A_\varepsilon} + \underbrace{\int_{-\infty}^{+\infty} \psi_0(z)\, d(1 - u_0^{(\varepsilon)}(z))}_{B_\varepsilon}$$

Because of the continuity of the flow, we remark: $z \to \exp\left( \int_0^t f' \circ u(s, \xi_{t-s,t}^{-1}(z))\, ds \right) M_0^t(z)$ is $\boldsymbol{P}$-almost surely a continous function; besides, using (2.3), we observe:

$$\boldsymbol{E}|H(z - \xi_{0,t}(x)) - H(z' - \xi_{0,t}(x))|^2 < C(t)|z - z'|$$

Therefore $\psi_0(z)$ is continuous in $z$, and also, as $u_0^{(\varepsilon)}(z) \to u_0(z)$ at the continuity points of $u_0(z)$, $B_\varepsilon \to \sum_{i=1}^N \omega_0^i\, \psi_0(x_0^i)$.

Moreover $A_\varepsilon \to 0$ for, for any $0 < \delta < t$:

$$\begin{aligned} \Big|\psi_\varepsilon(z) - \psi_0(z)\Big| &\le \widetilde{\boldsymbol{E}}_t^z \left| \exp\left( \int_0^t f' \circ u^{(\varepsilon)}(s, \xi_{t-s,t}^{-1}(z))\, ds \right) - \exp\left( \int_0^t f' \circ u(s, \xi_{t-s,t}^{-1}(z))\, ds \right) \right| \\ &\le C \widetilde{\boldsymbol{E}}_t^z \int_0^t \Big| u^{(\varepsilon)}(s, \xi_{t-s,t}^{-1}(z)) - u(s, \xi_{t-s,t}^{-1}(z)) \Big|\, ds \\ &\le C \left\{ \widetilde{\boldsymbol{E}}_t^z \int_0^\delta \Big| u^{(\varepsilon)}(s, \xi_{t-s,t}^{-1}(z)) - u(s, \xi_{t-s,t}^{-1}(z)) \Big|\, ds + \|u^{(\varepsilon)} - u\|_{L^\infty([\delta,t]\times \boldsymbol{R})} \right\} \\ &\le C \left\{ \delta + \|u^{(\varepsilon)} - u\|_{L^\infty([\delta,t]\times \boldsymbol{R})} \right\}, \end{aligned}$$

from which

$$A_\varepsilon \le \int_{-\infty}^{+\infty} \Big|\psi_\varepsilon(z) - \psi_0(z)\Big| d(1 - u_0^{(\varepsilon)}(z)) \le C \left\{ \delta + \|u^{(\varepsilon)} - u\|_{L^\infty([\delta,t]\times \boldsymbol{R})} \right\}$$

We apply now the last remark of the Appendix to bound the right–hand side by $C\delta$ for any $\varepsilon$ small enough.

■

### 3.2 Principle of the algorithm

Let $T > 0$ be fixed, and $h$ a time discretization step of type $\frac{T}{M}$, for some integer $M$. We want to approximate $u(T,x)$.

Approximating $-u_0'(z)\,dz$ by $\sum_{i=1}^N \omega_0^i\,\delta_{x_0^i}$, one gets the following approximating formula:

$$u(T,x) \simeq \sum_{i=1}^N \omega_0^i\, \widetilde{\boldsymbol{E}}_T^{x_0^i} \left[ H(\xi_{0,T}^{-1}(x_0^i) - x) \exp\left\{ \int_0^T f' \circ u(s, \xi_{T-s,T}^{-1}(x_0^i))\, ds \right\} \right]$$

Now, on $(\Omega, \mathcal{F}, \boldsymbol{P})$, we are given N independent Brownian motions $\{(B_\theta^i), i = 1, \ldots, N\}$ with respect to the filtration $(\mathcal{F}_0^\theta)_{\theta > 0}$, simply denoted by $(\mathcal{G}_\theta)$ in the sequel.

Let $(X_\theta^i)$ be the solutions (independent) to the following S.D.E.'s (in forward time):

$$\begin{cases} dX_\theta^i = -\sigma(X_\theta^i)\, dB_\theta^i - \{b(X_\theta^i) - \sigma(X_\theta^i)\,\sigma'(X_\theta^i)\}\, d\theta \\ X_0^i = x_0^i \end{cases}$$

We remark that the law of the process $(\xi_{T-\theta,T}^{-1}(x_0^i))_{0 \le \theta \le T}$, on $(\Omega, \mathcal{F}, \widetilde{\boldsymbol{P}}_T^{x_0^i}, (\mathcal{F}_\theta^T)_{0 \le \theta \le T})$, is identical to the law of the process $(X_\theta^i)_{0 \le \theta \le T}$ on $(\Omega, \mathcal{F}, \boldsymbol{P}, (\mathcal{G}_\theta)_{0 \le \theta \le T})$.

The particle algorithm replaces the expectation by a point estimation:

$$u(T,x) \simeq \sum_{i=1}^N \omega_0^i\, H(X_T^i - x) \exp\left\{ \int_0^T f' \circ u(s, X_s^i)\, ds \right\}$$

Then we approximate $\exp\left\{ \int_0^T f' \circ u(s, X_s^i)\, ds \right\}$ by $\exp\left\{ h \sum_0^{M-1} f' \circ u(ph, X_{ph}^i) \right\}$ and if we define by induction:

$$\rho_0^i = \omega_0^i \quad , \quad \rho_{(k+1)h}^i = \rho_{kh}^i\, \exp\{h\, f' \circ u(kh, X_{kh}^i)\}$$

we get, for any $p = 0, 1, \ldots, M = T/h$:

$$u(ph, x) \simeq \sum_{i=1}^N \rho_{ph}^i\, H(X_{ph}^i - x)$$

In fact the $(X_{ph}^i)$'s will be, to their turn, approximated by the Milshtein scheme (2.7) applied to (3.2):

$$\boxed{\begin{aligned} \bar{X}_{p+1}^i = \bar{X}_p^i - \left(b(\bar{X}_p^i) - \sigma(\bar{X}_p^i)\sigma'(\bar{X}_p^i)\right) h - \sigma(\bar{X}_p^i)(B_{(p+1)h}^i - B_{ph}^i) \\ + \frac{1}{2}\sigma(\bar{X}_p^i)\sigma'(\bar{X}_p^i)\left((B_{(p+1)h}^i - B_{ph}^i)^2 - h\right) \end{aligned}} \tag{3.7}$$

Thus, if we define:

$$\bar{\rho}_0^i = \omega_0^i \quad , \quad \bar{\rho}_{(k+1)h}^i = \bar{\rho}_{kh}^i \exp\{h\, f' \circ u(kh, \bar{X}_k^i)\}$$

we have:

$$u(ph, x) \simeq \sum_{i=1}^{N} \bar{\rho}_{ph}^i \, H(\bar{X}_p^i - x)$$

Actually one considers the weights in a slightly different way in order that the sum of the weights is equal to 1 (this fact will be used in the sequel):

$$\begin{aligned} \bar{\rho}_{kh}^i \exp\{h\, f' \circ u(kh, \bar{X}_k^i\} &\simeq \bar{\rho}_{kh}^i + h\, f' \circ u(kh, \bar{X}_k^i)\, \bar{\rho}_{kh}^i \\ &\simeq \bar{\rho}_{kh}^i + h\left(f \circ u(kh, \bar{X}_k^i) - f \circ u(kh, \bar{X}_k^{\pi_k(i)})\right) \end{aligned}$$

where $\pi_k(i)$ denotes the label number of the particle just at the right of the particle of label $i$ at the time $kh$ (if the considered particle is at the most right position at time $kh$, we just have: $\bar{\rho}_{kh}^i \exp\{h\, f' \circ u(kh, \bar{X}_k^i\} \simeq \bar{\rho}_{kh}^i + h f \circ u(kh, \bar{X}_k^i))$. That transformation of the weights corresponds to the step "$R_{k\Delta t}$" of the splitting method of Puckett (cfr. [12]).

## 3.3 Algorithm

Finally, the algorithm will be the following: let us define the initial weights and the initial approximation by:

$$\boxed{\omega_0^i = \frac{1}{N}, \text{ for } i = 1, \ldots, N \quad , \qquad \bar{u}_0(x) = \sum_{i=1}^{N} \omega_0^i \, H(x_0^i - x)}$$

where

$$\boxed{\forall i < N \quad : \quad x_0^i = u_0^{-1}\left(1 - \frac{i}{N}\right) \quad ; \quad x_0^N = u_0^{-1}\left(\frac{1}{2N}\right)} \tag{3.8}$$

Evidently $\bar{u}_0(\cdot)$ is a piecewise constant approximation to the inital datum $u_0(\cdot)$.

We recall that we define the approximating process by (3.7).

Let us define now, in a recurrent way (and using the same convention for the particle at the most right position as previously):

$$\boxed{\omega_p^i = \omega_{p-1}^i \left(1 + h \frac{f \circ \bar{u}_{p-1}(\bar{X}_{p-1}^i) - f \circ \bar{u}_{p-1}(\bar{X}_{p-1}^{\pi_{p-1}(i)})}{\omega_{p-1}^i}\right)} \tag{3.9}$$

and

$$\boxed{\bar{u}_p(x) = \sum_{i=1}^{N} \omega_p^i \, H(\bar{X}_p^i - x)} \tag{3.10}$$

for $p = 1, 2, \ldots, M = T/h$.

**Remark 3.3** *We remark that all the weights, for some constant $C$ uniform on $h$, $N$, $i$ and $k$, are bounded by*

$$0 \leq \omega_k^i \leq \frac{C}{N} \tag{3.11}$$

*and, for any $k = 1, \ldots, M = T/h$, the weights $\omega_k^i$ ($i = 1, \ldots, N$) are $\mathcal{G}_{(k-1)h}$-measurable (this will play an important role in the sequel). Moreover it is easy to check from the definition (3.9) that, for any $p = 1, \ldots, M = T/h$,*

$$\sum_{i=1}^{N} \omega_p^i = 1 \tag{3.12}$$

*By using the fact that $f'$ and $f''$ are bounded and that the $\omega_{ph}$'s are bounded by $C/N$, we have*

$$\omega_p^i = \omega_{p-1}^i \left(1 + h\, f' \circ \bar{u}_{p-1}(\bar{X}_{p-1}^i)\right) + \mathcal{O}(h)\mathcal{O}\left(\frac{1}{N^2}\right) \tag{3.13}$$

**Proposition 3.4** *Under the hypotheses $\{(Hi), i = 1, \ldots, 4\}$, we have:*

$$\frac{1}{N}\sum_{i=1}^{N} |x_0^i|^2 \leq C \tag{3.14}$$

*Besides:*

$$\frac{1}{N}\sum_{i=1}^{N} \boldsymbol{E}|\bar{X}_p^i|^2 \leq C \tag{3.15}$$

## Proof

Let $i_0$ such that $u_0^{-1}(\frac{i_0-1}{N}) > 0$ and $u_0^{-1}(\frac{i_0}{N}) \leq 0$. Then (the factor $\frac{1}{2N}$ instead of $\frac{1}{N}$ is due to the definition of $x_0^N$):

$$\frac{1}{2N}\sum_{i=1}^{i_0-1} |x_0^i|^2 \leq \int_0^{\frac{i_0-1}{N}} |u_0^{-1}(s)|^2\, ds$$

and

$$\frac{1}{N}\sum_{i=i_0}^{N} |x_0^i|^2 \leq \int_{\frac{i_0}{N}}^{T} |u_0^{-1}(s)|^2\, ds$$

We deduce, using **(H4)**:

$$\frac{1}{N}\sum_{i=1}^{N} |x_0^i|^2 \leq 2\int_0^1 |u_0^{-1}(s)|^2\, ds \leq -2 \int_{\boldsymbol{R}} u_0'(y)\, y^2\, dy \leq C$$

To get the other part of the statement, we just apply (2.8).

∎

# 4 Main result

The main result of the paper is the following Theorem.

**Theorem 4.1** *(i) Under the hypotheses $\{(Hi), i = 1, \ldots, 4\}$, there exist strictly positive constants $C$ and $h_0 < 1$ such that, for any $h < h_0$ and any $N \geq 1$:*

$$\|u(T,\cdot) - \bar{u}(T,\cdot)\|_{L^1(\mathbb{R}\times\Omega)} \leq C\left(\frac{1}{\sqrt{N}} + \sqrt{h}\right)$$

*(ii) When the functions $b$ and $\sigma$ are constant, then the rate of convergence is given by:*

$$\|u(T,\cdot) - \bar{u}(T,\cdot)\|_{L^1(\mathbb{R}\times\Omega)} \leq C\left(\frac{1}{\sqrt{N}} + h\right)$$

When $f \equiv 0$, the estimate $(i)$ can be improved. Indeed, if $\mu_0$ denotes the probability measure whose $1 - u_0$ is the cumulative function, and $(X_t)$ is defined by:

$$dX_t = -\sigma(X_t)\, dB_t - \{b(X_t) - \sigma(X_t)\,\sigma'(X_t)\}\, dt$$

then, from (3.1): $u(t,x) = \mathbb{E}_{\mu_0} H(X_t - x)$ and, to the error $\|u(T,\cdot) - \bar{u}(T,\cdot)\|_{L^1(\mathbb{R}\times\Omega)}$ contribute a statistical error

$$\|u(T,\cdot) - \frac{1}{N} H(X_T^i - \cdot)\|_{L^1(\mathbb{R}\times\Omega)}$$

which is of order $\frac{1}{\sqrt{N}}$, and an approximation error

$$\|\frac{1}{N} H(X_T^i - \cdot) - \frac{1}{N} H(\bar{X}_T^i - \cdot)\|_{L^1(\mathbb{R}\times\Omega)}$$

which generically is of order $h$ when the Milshtein scheme is used. The non linearity of the P.D.E. induced by $f$ changes the order of convergence, at least in our proofs. Our numerical experiments have not permitted to check whether $\sqrt{h}$ is the best estimate: typically, the algorithm was extremely sensitive to $h$; when $h$ was small, it was difficult to isolate the error due to the discretization from the statistical error (we could not choose $N$ so large as it would have been necessary), and for different but not small $h$, some numerical instabilities produced statistical and discretization errors of comparable magnitude. In any case, the important point seems to us that the behavior of the error can be described without supposing a relation between $h$ and $N$.

The gain in accuracy when $b$ and $\sigma$ are constant is not mysterious: to give the feeling of what happens, let us suppose: $b \equiv 0$ and $\sigma \equiv 1$; in that case, the particles are Brownian, and the law of the $X_{(p+1)h}^i - X_{ph}^i$'s can be simulated exactly (one just has to simulate independent gaussian variables), whereas, when $\sigma$ is not constant, one has to approximate the processes $(X_t^i)$; the passage from $h$ to $\sqrt{h}$ is due to this approximation (see the Proposition 2.3).

Let us remark also that, when the coefficients are not constant, we obtained the above estimates after having used the Milshtein scheme, not the Euler scheme (compare Remark (2.4) and (2.10)). Finally, we stress that the Euler and Milshtein schemes are the only schemes reasonable from the point of view of numerical efficiency (see Talay [16]).

The three next Sections are devoted to the proof of the part $(i)$ of this Theorem. In Section 8, we will explain what must be changed in the proof in order to obtain the better estimate in part $(ii)$.

## 5 The weights are not far from being independent

The $\omega_p^i$'s are not independent, but we can choose others weights that are independent and approximate the $\omega_p^i$'s in order to get, in the sequel, useful estimates.

Let us define $\rho_p^i$ by

$$\rho_0^i = \omega_0^i \quad , \quad \rho_p^i = \rho_{p-1}^i \left(1 + h\, f' \circ u((p-1)h, X^i_{(p-1)h})\right) \tag{5.1}$$

The $\rho_p^i$'s ($i = 1, \ldots, N$) are independent, and it is easy to show there exists a $C > 0$ such that $|\rho_p^i| \leq \frac{C}{N}$.

Set $\alpha_p^i := \boldsymbol{E}|\omega_p^i - \rho_p^i|^2$, and $\alpha_p := \sup_i \alpha_p^i$.

The objective of this Section is to prove (cf. Proposition (5.8)) :

$$\boxed{\forall h, \quad \forall p = 1, \ldots, M = \frac{T}{h} \quad : \quad \alpha_p \leq \frac{C\,h}{N^2} + \frac{C}{N^3}}$$

**Remark 5.1** *We observe:*

$$\begin{aligned} \omega_{p+1}^i - \rho_{p+1}^i \;=\;& \omega_p^i - \rho_p^i + h\, \omega_p^i\, \{f' \circ \bar{u}_p(\bar{X}_p^i)) - f' \circ u(ph, \bar{X}_p^i)\} + h\, (\omega_p^i - \rho_p^i)\, f' \circ u(ph, \bar{X}_p^i) \\ &+ h\, \rho_p^i\, \{f' \circ u(ph, \bar{X}_p^i)) - f' \circ u(ph, X_{ph}^i)\} + O\left(\frac{h}{N^2}\right) \end{aligned}$$

As $f' \circ u$ is Lipschitz, and as $\bar{X}$ is defined by the Milshtein scheme (cfr. Proposition 2.3), we get:

$$\alpha_{p+1}^i \leq \alpha_p^i + \frac{C\,h}{N}\sqrt{\alpha_p^i}\sqrt{\boldsymbol{E}|\bar{u}_p(\bar{X}_p^i) - u(ph, \bar{X}_p^i)|^2} + C\,h\,\alpha_p^i + C\,\frac{h^2}{N^2} + C\,\frac{h}{N^3} \tag{5.2}$$

We need now to get a precise estimate of $\sqrt{\boldsymbol{E}|\bar{u}_p(\bar{X}_p^i) - u(ph, \bar{X}_p^i)|^2}$. Having defined:

$$u_p^*(x) = \sum_{j=1}^{N} \rho_p^j\, H(X_{ph}^j - x)$$

we have:

$$\sqrt{\boldsymbol{E}|\bar{u}_p(\bar{X}_p^i) - u(ph, \bar{X}_p^i)|^2} \leq \sqrt{\boldsymbol{E}|\bar{u}_p(\bar{X}_p^i) - u_p^*(\bar{X}_p^i)|^2} + \sqrt{\boldsymbol{E}|u_p^*(\bar{X}_p^i) - u(ph, \bar{X}_p^i)|^2} \tag{5.3}$$

An upper bound for the first term of the right-hand side will be given in Section 5.1 below, an upper bound for the second term in Section 5.2, and finally we will come back to the inequality (5.2) in Section 5.3.

### 5.1 An upper bound for $\boldsymbol{E}|\bar{u}_p(\bar{X}_p^i) - u_p^*(\bar{X}_p^i)|^2$

**Proposition 5.2**

$$\boldsymbol{E}|\bar{u}_p(\bar{X}_p^i) - u_p^*(\bar{X}_p^i)|^2 \leq 2\,N^2\alpha_p + \frac{C}{N} + C\,h \tag{5.4}$$

## Proof

$$\begin{aligned}
\boldsymbol{E}|\bar{u}_p(\bar{X}_p^i) - u_p^*\bar{X}_p^i)|^2 &\le 2\boldsymbol{E}\left[\sum_{j=1}^{N}|\omega_p^j - \rho_p^j|\, H(\bar{X}_p^j - \bar{X}_p^i)\right]^2 \\
&\quad +2\boldsymbol{E}\left[\sum_{j=1}^{N}\rho_p^j\,|H(\bar{X}_p^j - \bar{X}_p^i) - H(X_{ph}^j - \bar{X}_p^i)|\right]^2 \\
&\le 2\sum_{j,k=1}^{N}\sqrt{\alpha_p^j\,\alpha_p^k} \\
&\quad +\frac{C}{N^2}\sum_{j=1}^{N}\boldsymbol{E}|H(\bar{X}_p^j - \bar{X}_p^i) - H(X_{ph}^j - \bar{X}_p^i)|^2 \\
&\quad +\frac{C}{N^2}\sum_{\substack{j,k=1\\ j\neq k\\ j\neq i,k\neq i}}^{N}\begin{array}{l}\boldsymbol{E}|H(\bar{X}_p^j - \bar{X}_p^i) - H(X_{ph}^j - \bar{X}_p^i)|\\ \quad\cdot|H(\bar{X}_p^k - \bar{X}_p^i) - H(X_{ph}^k - \bar{X}_p^i)|\end{array} + \frac{C}{N} \\
&\le 2\sum_{j,k=1}^{N}\sqrt{\alpha_p^j\,\alpha_p^k} + \frac{C}{N} + \frac{C}{N^2}\sum_{\substack{j,k=1\\ j\neq k\\ j\neq i,k\neq i}}^{N}\begin{array}{l}\boldsymbol{E}\,|H(\bar{X}_p^j - \bar{X}_p^i) - H(X_{ph}^j - \bar{X}_p^i)|\\ \quad\cdot|H(\bar{X}_p^k - \bar{X}_p^i) - H(X_{ph}^k - \bar{X}_p^i)|\end{array}
\end{aligned}$$

Therefore, to get the conclusion it just remains to prove :

**Lemma 5.3** *For $i \neq j \neq k$ :*

$$\boldsymbol{E}\,|H(\bar{X}_p^j - \bar{X}_p^i) - H(X_{ph}^j - \bar{X}_p^i)|\cdot|H(\bar{X}_p^k - \bar{X}_p^i) - H(X_{ph}^k - \bar{X}_p^i)| \le Ch$$

## Proof

For $j \neq i$, $k \neq i$, $j \neq k$, the pairs $((X_\cdot^i),(\bar{X}_\cdot^i))$, $((X_\cdot^j),(\bar{X}_\cdot^j))$ and $((X_\cdot^k),(\bar{X}_\cdot^k))$ are mutually independent; hence

$$\begin{aligned}
&\boldsymbol{E}|H(\bar{X}_p^j - \bar{X}_p^i) - H(X_{ph}^j - \bar{X}_p^i)|\cdot|H(\bar{X}_p^k - \bar{X}_p^i) - H(X_{ph}^k - \bar{X}_p^i)| \\
&\quad = \boldsymbol{E}\Big(\boldsymbol{E}(|H(\bar{X}_p^j - y) - H(X_{ph}^j - y)|\cdot|H(\bar{X}_p^k - y) - H(X_{ph}^k - y)|)\Big|_{y=\bar{X}_p^i}\Big) \\
&\quad = \boldsymbol{E}\Big(\{\boldsymbol{E}(|H(\bar{X}_p^j - y) - H(X_{ph}^j - y)|)\cdot\boldsymbol{E}(|H(\bar{X}_p^k - y) - H(X_{ph}^k - y)|)\}\Big|_{y=\bar{X}_p^i}\Big)
\end{aligned}$$

Let $A^j$ denote :

$$A^j := \boldsymbol{E}|H(\bar{X}_p^j - y) - H(X_{ph}^j - y)|$$

and let us suppose we have shown :

$$\exists C>0,\quad \forall h<1,\quad \forall p,\quad \forall j,\quad \forall y\ : A^j \le C\sqrt{h}(1+|y|) \tag{5.5}$$

Then we would have :

$$\begin{aligned}\mathbb{E}\,|H(\bar{X}^j_p-\bar{X}^i_p)-H(X^j_{ph}-\bar{X}^i_p)||H(\bar{X}^k_p-\bar{X}^i_p)-H(X^k_{ph}-\bar{X}^i_p)| &\le \int_{\mathbb{R}} A^j A^k\, d\mathbb{P}_{\bar{X}^i_p}(y)\\ &\le Ch\int_{\mathbb{R}}(1+|y|^2)\,d\mathbb{P}_{\bar{X}^i_p}(y)\end{aligned}$$

Thus we could conclude by applying the inequality (2.8).

Let us now show (5.5).

Let $\beta := -b+\sigma\sigma'$ (cf. (3.7)) and set :

$$\begin{aligned}\psi(x,z) &:= x+\beta(x)h+\sigma(x)z\\ \phi(x,z) &:= \psi(x,z)+\frac{1}{2}\sigma(x)\sigma'(x)(z^2-1)\end{aligned}$$

We remark : $\bar{X}^j_p = \phi(\bar{X}^j_{p-1}, B^j_p - B^j_{p-1})$ and :

$$\begin{aligned}A^j &\le \mathbb{E}|H(\bar{X}^j_p-y)-H(\psi(\bar{X}^j_{p-1},B^j_p-B^j_{p-1})-y)|\\ &\quad+\mathbb{E}|H(\psi(\bar{X}^j_{(p-1)},B^j_p-B^j_{p-1})-y)-H(\psi(X^j_{(p-1)h},B^j_p-B^j_{p-1})-y)|\\ &\quad+\mathbb{E}|H(\psi(X^j_{(p-1)h},B^j_p-B^j_{p-1})-y)-H(\phi(X^j_{(p-1)h},B^j_p-B^j_{p-1})-y)|\\ &\quad+\mathbb{E}|H(\phi(X^j_{(p-1)h},B^j_p-B^j_{p-1})-y)-H(X^j_{ph}-y)|\\ &:= A^j_1+A^j_2+A^j_3+A^j_4\end{aligned}$$

Let us first show : $A^j_1 \le C\sqrt{h}$.

We have (using the fact that the function $H$ is increasing and takes value equal to 0 or 1) :

$$\begin{aligned}A^j_1 &\le C\mathbb{E}\sum_n\int_n^{n+1}|H(\phi(\bar{X}^j_{p-1},\sqrt{h}z)-y)-H(\psi(\bar{X}^j_{p-1},\sqrt{h}z)-y)|e^{-\frac{z^2}{2}}\,dz\\ &\le C\mathbb{E}\sum_n e^{-\frac{n^2}{2}}\int_n^{n+1}\Big\{H(\psi(\bar{X}^j_{p-1},\sqrt{h}z)+C(n+1)^2h-y)-H(\psi(\bar{X}^j_{p-1},\sqrt{h}z)-y)\Big\}\\ &\qquad\qquad 1_{[\psi(\bar{X}^j_{p-1},\sqrt{h}z)-y<0]}\,dz\\ &\quad+C\mathbb{E}\sum_n e^{-\frac{n^2}{2}}\int_n^{n+1}\Big\{H(\psi(\bar{X}^j_{p-1},\sqrt{h}z)-y)-H(\psi(\bar{X}^j_{p-1},\sqrt{h}z)-C(n+1)^2h-y)\Big\}\\ &\qquad\qquad 1_{[\psi(\bar{X}^j_{p-1},\sqrt{h}z)-y>0]}\,dz\\ &\le C\mathbb{E}\sum_n e^{-\frac{n^2}{2}}\int_{\mathbb{R}}|H(\psi(\bar{X}^j_{p-1},\sqrt{h}z)+C(n+1)^2h-y)-H(\psi(\bar{X}^j_{p-1},\sqrt{h}z)-y)|\,dz\\ &\quad+C\mathbb{E}\sum_n e^{-\frac{n^2}{2}}\int_{\mathbb{R}}|H(\psi(\bar{X}^j_{p-1},\sqrt{h}z)-C(n+1)^2h-y)-H(\psi(\bar{X}^j_{p-1},\sqrt{h}z)-y)|\,dz\end{aligned}$$

Then we divide the arguments of the function $H$ by the positive (see **(H2)**) quantity $\sqrt{h}\sigma(\bar{X}^j_{p-1})$, and we use (2.13) to conclude:

$$A_1^j \leq \boldsymbol{E}\sum_n e^{-\frac{n^2}{2}}\frac{C(n+1)^2 h}{\sigma(\bar{X}^j_{p-1})\sqrt{h}} \leq C\sqrt{h}$$

Let us now show : $A_2^j \leq C\sqrt{h}(1+|y|)$.

$$A_2^j \leq \boldsymbol{E}\int |H(\psi(\bar{X}^j_{p-1},\sqrt{h}z)-y) - H(\psi(X^j_{(p-1)h},\sqrt{h}z)-y)|\,dz$$

We remark that the function $\frac{1}{\sigma}$ is Lipschitz ; dividing the arguments of $H$ by, respectively, $\sqrt{h}\sigma(\bar{X}^j_{p-1})$ and $\sqrt{h}\sigma(X^j_{p-1})$, and using again the equality (2.13), we get :

$$A_2^j \leq C\sqrt{h} + \frac{C}{\sqrt{h}}(1+|y|)\boldsymbol{E}|\bar{X}^j_{p-1} - X^j_{(p-1)h}|$$

We use the Proposition (2.3) to conclude.

To show : $A_3^j \leq C\sqrt{h}$, we repeat the inequalities for $A_1^j$, substituting $X^j_{(p-1)h}$ to $\bar{X}^j_{p-1}$.

Finally, it remains to check : $A_4^j \leq C\sqrt{h}$.

For any $h > 0$ and any $y$ :

$$\begin{aligned}&\{(x_1,x_2)\in\boldsymbol{R}^2 : |H(x_1-y)-H(x_2-y)| = 1\} \subset\\ &\quad\{(x_1,x_2)\in\boldsymbol{R}^2 : |x_1-x_2| > h\} \cup \{(x_1,x_2)\in\boldsymbol{R}^2 : |x_2-y|\leq h\}\end{aligned}$$

Besides, a Taylor expansion shows :

$$X^j_{ph} = \phi(X^j_{(p-1)h}, B^j_{ph} - B^j_{(p-1)h}) + r^j_p$$

with $\boldsymbol{E}|r^j_p|^2 \leq Ch^3$.

Therefore :

$$A_4^j \leq \boldsymbol{P}(|X^j_{ph} - y| \leq h) + \boldsymbol{P}(|r^j_p| \geq h)$$

We use the fact that the density of the law of $X^j_{(p-1)h}$ is bounded from above by $\frac{C}{\sqrt{(p-1)h}} < \frac{C}{\sqrt{h}}$ (see (2.3)) for the first term of the right-hand side, and the Bienaymé-Tschebychev inequality for the second, to get the conclusion.

∎

## 5.2 An upper bound for $\boldsymbol{E}|u_p^*(\bar{X}_p^i) - u(ph, \bar{X}_p^i)|^2$

For brevity, we will denote:

$$U_p^i := \boldsymbol{E}|u_p^*(\bar{X}_p^i) - u(ph, \bar{X}_p^i)|^2$$

The objective of this Section is to prove (see (5.6) below):

**Proposition 5.4**

$$U_p^i \leq \frac{C}{N} + C\,h^2$$

Let $u^N(t,x) := \sum_{j=1}^N \omega_0^j\,\boldsymbol{E}\Big[H(X_t^j - x)\,\exp\Big(\int_0^t f' \circ u(s, X_s^j)\,ds\Big)\Big]$. Then:

$$U_p^i \leq C\,\boldsymbol{E}|u(ph, \bar{X}_p^i) - u^N(ph, \bar{X}_p^i)|^2 + C\boldsymbol{E}\left\{\sum_{j=1}^N H(X_{ph}^j - \bar{X}_p^i)\left[\omega_0^j\, e^{\int_0^{ph} -f' o u(s, X_s^j)\,ds} - \rho_p^j\right]\right\}^2$$

$$+C\,\boldsymbol{E}\left\{\sum_{j=1}^N \omega_0^j\left[\boldsymbol{E}H(X_{ph}^j - \bar{X}_p^i)\,e^{\int_0^{ph} -f' o u(s, X_s^j)\,ds} - H(X_{ph}^j - \bar{X}_p^i)\,e^{\int_0^{ph} -f' o u(s, X_s^j)\,ds}\right]\right\}^2$$

Each of the three following Lemmae will concern a term of the right–hand side of the preceding inequality.

**Lemma 5.5** *There exists a $C > 0$ such that for any $t \in [0,T]$ we have:*

$$\|u(t,\cdot) - u^N(t,\cdot)\|_{L^\infty(\boldsymbol{R})} \leq \frac{C}{N}$$

### Proof

Let us consider $v^N$ the solution of (2.1) with an initial condition equal to $\bar{u}_0$ (see the Appendix for the existence), and let $v := u - v^N$; $v$ is a solution to:

$$\begin{cases} \dfrac{\partial v}{\partial t} = L\,v + \dfrac{f(u) - f(v^N)}{u - v^N} v \\ \\ v_0 = u_0 - \bar{u}_0 \end{cases}$$

By the Feynman-Kac formula, $(Z_t)$ being defined as in (2.11) :

$$|v(t,x)| \leq C\,\boldsymbol{E}|(u_0 - \bar{u}_0)(Z_t(x))| \leq C\,\|u_0 - \bar{u}_0\|_{L^\infty(\boldsymbol{R})} \leq \frac{C}{N}$$

Therefore, as by Proposition 3.2, $v^N(t,x) = \sum_{j=1}^N \omega_0^j \boldsymbol{E}\Big[H(X_t^j - x)\,\exp\Big(\int_0^t f'\circ v^N(s,X_s^j)\,ds\Big)\Big]$:

$$\|u(t,\cdot) - u^N(t,\cdot)\|_{L^\infty(\boldsymbol{R})} \le \frac{C}{N} + \|v^N(t,\cdot) - u^N(t,\cdot)\|_{L^\infty(\boldsymbol{R})}$$

$$\le \frac{C}{N} + \left\|\sum_{j=1}^N \omega_0^j \boldsymbol{E}\left[H(X_t^j - \cdot)\left(\exp\Big(\int_0^t f'\circ v^N(s,X_s^j)\,ds\Big) - \exp\Big(\int_0^t f'\circ u(s,X_s^j)\,ds\Big)\right)\right]\right\|_{L^\infty(\boldsymbol{R})}$$

$$\le \frac{C}{N} + C \sup_{0\le s\le t} \|v(s,\cdot)\|_{L^\infty(\boldsymbol{R})} \le \frac{C}{N}$$

■

**Lemma 5.6**

$$T_p^N := \boldsymbol{E}\left|\sum_{j=1}^N H(X_{ph}^j - \bar{X}_p^i)\,\{\omega_0^j \exp\int_0^{ph} f'\circ u(s,X_s^j)\,ds - \rho_p^j\}\right|^2 \le Ch^2$$

## Proof

From the definition (5.1), it comes:

$$T_p^N \le \frac{C}{N^2}\boldsymbol{E}\left[\sum_{j=1}^N |\exp\int_0^{ph} f'\circ u(s,X_s^j)\,ds - \prod_{k=0}^{p-1}(1 + h\,f'\circ u(kh,X_{ph}^j))|\right]^2$$

Now, we write (for any $h$ small enough):

$$\prod_{k=0}^{p-1}(1 + h\,f'\circ u(kh,X_{ph}^j)) = \exp\left[\sum_{k=0}^{p-1}\log(1 + h\,f'\circ u(kh,X_{ph}^j))\right] \le C$$

and, using the inequality $|e^a - e^b| \le e^{\max\{|a|,|b|\}}|b-a|$, we get:

$$T_p^N \le \frac{C}{N^2}\boldsymbol{E}\left[\sum_{j=1}^N\left|\int_0^{ph} f'\circ u(s,X_s^j)\,ds - h\sum_{k=1}^{p-1} f'\circ u(kh,X_{ph}^j)\right|\right]^2 + C\,h^2$$

We expand the right-hand side, and we remark that, for some uniformly bounded function $\psi$:

$$\boldsymbol{E}\left|\int_{kh}^{(k+1)h} f'\circ u(s,X_s^j)\,ds - h\,f'\circ u(kh,X_{ph}^j)\right|^2 \le \boldsymbol{E}\left|\int_{kh}^{(k+1)h}\psi(s,X_s^j)\,(W_s - W_{kh})\,ds\right|^2 + Ch^2$$

so that, using Cauchy–Schwarz inequality in each term of the expansion:

$$T_p^N \le C\,h^2$$

■

**Proposition 5.7**

$$S_p^N := \boldsymbol{E}|u^N(ph,\bar{X}_p^i) - \sum_{j=1}^N \omega_0^j\,H(X_{ph}^j - \bar{X}_p^i)\,\exp\int_0^{ph} f'\circ u(s,X_s^j)\,ds|^2 \le \frac{C}{N}$$

## Proof

We have:

$$S_p^N \leq 2\,\mathbb{E}\left|\sum_{\substack{j=1\\ j\neq i}}^{N} \omega_0^j \left[H(X_{ph}^j - \bar{X}_p^i)\, e^{\int_0^{ph} -f'\circ u(s,X_s^j)\,ds} - \mathbb{E} H(X_{ph}^j - \bar{X}_p^i)\, e^{\int_0^{ph} -f'\circ u(s,X_s^j)\,ds}\right]\right|^2 + \frac{C}{N^2}$$

and, by the independence property of the $(X^j)$'s:

$$S_p^N \leq \frac{C}{N^2}\mathbb{E}\sum_{\substack{j=1\\ j\neq i}}^{N} H(X_{ph}^j - \bar{X}_p^i)\ \exp\left(2\int_0^{ph} f'\circ u(s,X_s^j)\,ds\right) + \frac{C}{N^2} \leq \frac{C}{N}$$

■

Finally, we obtain, collecting Lemmae 5.5, 5.7, 5.6:

$$U_p^i = \mathbb{E}|u_p^*(\bar{X}_p^i) - u(ph,\bar{X}_p^i)|^2 \leq \frac{C}{N} + C\,h^2 \qquad (5.6)$$

### 5.3 An upper bound for $\alpha_p$

From the two previous subsections, we obtain, considering (5.2), (5.3), (5.4) and (5.6):

$$\alpha_{p+1} \leq \alpha_p + \frac{C\,h}{N}\sqrt{\alpha_p}\left(N\sqrt{\alpha_p} + \frac{1}{\sqrt{N}} + \sqrt{h}\right) + C\,h\,\alpha_p + \frac{C\,h^2}{N^2} + \frac{C\,h}{N^3}$$

**Proposition 5.8**

$$\boxed{\forall p, \qquad \alpha_p \leq \frac{C\,h}{N^2} + \frac{C}{N^3}} \qquad (5.7)$$

## Proof

We have:

$$\alpha_{p+1} \leq (1 + C\,h)\,\alpha_p + \frac{C\,h}{N}\sqrt{\alpha_p}\left(\frac{1}{\sqrt{N}} + \sqrt{h}\right) + \frac{C\,h^2}{N^2} + \frac{C\,h}{N^3}$$

Let us define by induction $\tau_0 = \alpha_0$ and :

$$\tau_{p+1} = (1 + C\,h)\,\tau_p + \frac{C\,h}{N}\sqrt{\tau_p}\left(\frac{1}{\sqrt{N}} + \sqrt{h}\right) + \frac{C\,h^2}{N^2} + \frac{C\,h}{N^3}$$

We observe that, for any $p$: $\alpha_p \leq \tau_p$.

If, for any $p$, we have $\sqrt{\tau_p} \leq \frac{\sqrt{h}}{N} + \frac{1}{N\sqrt{N}}$, then we have obtained (5.7), otherwise there exists a $j$ such that:

$$\begin{cases} \sqrt{\tau_{j+1}} & > & \frac{\sqrt{h}}{N} + \frac{1}{N\sqrt{N}} \\ \\ \sqrt{\tau_j} & \leq & \frac{\sqrt{h}}{N} + \frac{1}{N\sqrt{N}} \end{cases}$$

As $(\tau_p)$ is increasing, we would then have that, for any $p > j$, $\sqrt{\tau_p} > \frac{\sqrt{h}}{N} + \frac{1}{N\sqrt{N}}$; then, for any $p > j$, we would also have:

$$\tau_{p+1} \leq (1 + C\,h)\,\tau_p + C\,h\,\tau_p + \frac{C\,h^2}{N^2} + \frac{C\,h}{N^3}.$$

from which we deduce that:

$$\tau_M \leq (1 + C\,h)^{M-j}\tau_j + \left(\frac{C\,h^2}{N^2} + \frac{C\,h}{N^3}\right)\frac{(1 + C\,h)^{M-j} - 1}{C\,h} \leq \frac{C\,h}{N^2} + \frac{C}{N^3}$$

Hence (5.7) is true for any $p$.
∎

# 6 Local expansion of the solution $u(t,x)$

For the sequel we need to compare the solution $u(t,x)$ to problem (2.1) with the solution $v(t,x)$ to the problem

$$\begin{cases} \dfrac{\partial v}{\partial t} = L\,v \\ \\ v(0,\cdot) = u_0(\cdot) \end{cases} \tag{6.1}$$

for small values of $t$. We can represent

$$v(t,x) = \mathbb{E}(u_0(Z_t(x)))$$

where again $(Z_t(x))$ is the solution to the following equation:

$$dZ_t = b(Z_t)\,dt + \sigma(Z_t)\,dB_t \quad , \qquad Z_0(x) = x. \tag{6.2}$$

Let $P_\theta(x,dy)$ the transition probability associated to $(Z_t)$.

**Theorem 6.1** *Let us suppose the hypotheses* (**H1**), (**H2**) *and* (**H3**) *verified; then for any* $0 < h < 1$ *and any* $x \in \mathbb{R}$, *we have:*

$$u(h,x) = \mathbb{E}\,u_0(Z_h(x)) + hf(\mathbb{E}\,u_0(Z_h(x))) + R_h(x)$$

*with the following estimate:*

- *if* $u_0$ *verifies* (**H4**) *then:*

$$\|R_h(\cdot)\|_{L^1(\mathbb{R})} \leq C\, h^2 \tag{6.3}$$

- *if* $u_0$ *belongs to a family of functions verifying* (**H5**) *with weights bounded by* $\frac{C}{N}$*, the constant* $C$ *being uniform on the family, then:*

$$\|R_h(\cdot)\|_{L^1(\mathbb{R})} \leq C\, h\sqrt{h} + C\frac{h^2}{N}\sum_{i=1}^{N}|x_0^i| \tag{6.4}$$

**The proof is obtained by collecting the Propositions of this section, and Remark 6.6: Proposition (6.3) expands $u(h,x)$, the sequel gives estimates of the norm of the remaining terms in $L^1(\mathbb{R})$.**

**Remark 6.2** *The proof will make appear that the constant in (6.4) can be somewhat explicited (see the footnotes in the sequel):*

$$\|R_h(\cdot)\|_{L^1(\mathbb{R})} \leq C\, h\sqrt{h}\left(\frac{1}{N} + \frac{C}{N^2}\sum_{i,j,i<j}\exp\left(-\frac{(x_0^i - x_0^j)^2}{8\lambda h}\right)\right) + C\frac{h^2}{N}\sum_{i=1}^{N}|x_0^i| \tag{6.5}$$

*This will be used to treat the special case of constant $b$ and $\sigma$ (see Section 8).*

**Proposition 6.3** *Let us suppose the hypotheses* (**H1**), (**H2**) *and* (**H3**) *verified; then for any* $0 < h < 1$ *and any* $x \in \mathbb{R}$ *we have:*

$$u(h,x) = \mathbb{E}\, u_0(Z_h(x)) + hf(\mathbb{E}\, u_0(Z_h(x))) + R_h(x)$$

*with the following estimate:*

$$\begin{aligned}
|R_h(x)| \;\leq\; & C\mathbb{E}\left\{u_0(Z_h(x))\left\{\left[\int_0^h \frac{f\circ u(h-s, Z_s(x))}{u(h-s, Z_s(x))}ds\right]^2 + \int_0^h\int_0^s \frac{f\circ u(\theta, Z_{h-\theta}(x))}{u(\theta, Z_{h-\theta}(x))}d\theta ds\right\}\right\} \\
& + C\int_0^h\int_{\mathbb{R}} \mathbb{E}\Big[u_0(Z_s(y)) - \mathbb{E}u_0(Z_s(y))\Big]^2 P_{h-s}(x,dy)ds \\
& + Ch\mathbb{E}\Big[u_0(Z_h(x)) - \mathbb{E}u_0(Z_h(x))\Big]^2
\end{aligned}$$

For the proof of this proposition we need the following Lemma.

**Lemma 6.4**

$$\int_{\mathbb{R}} f(\mathbb{E}u_0(Z_s(y)))\, P_{h-s}(x,dy) = f(\mathbb{E}u_0(Z_h(x))) + R$$

*with*

$$|R| \leq C\int_{\mathbb{R}} \mathbb{E}\Big[u_0(Z_s(y)) - \mathbb{E}u_0(Z_s(y))\Big]^2 P_{h-s}(x,dy) + C\,\mathbb{E}\Big[u_0(Z_h(x)) - \mathbb{E}u_0(Z_h(x))\Big]^2$$

**Proof of Lemma 6.4**

Using Lemma 2.7 we have:

$$\int_{\mathbb{R}} f(\mathbb{E}u_0(Z_s(y)))\, P_{h-s}(x,dy) = \int_{\mathbb{R}} \mathbb{E} f \circ u_0(Z_s(y))\, P_{h-s}(x,dy) + R_1$$

with

$$|R_1| \le C \int_{\mathbb{R}} \mathbb{E}\Big[u_0(Z_s(y)) - \mathbb{E}u_0(Z_s(y))\Big]^2 P_{h-s}(x,dy)$$

Let us remark now:

$$\int_{\mathbb{R}} \mathbb{E} f \circ u_0(Z_s(y))\, P_{h-s}(x,dy) = \mathbb{E} f \circ u_0(Z_h(x))$$

from which, by applying once again Lemma 2.7, we get the conclusion.
∎

**Proof of Proposition 6.3**

Hypothesis (**H1**) implies that $|f(y)/y| \le C$ for a suitable $C$ and $0 < y \le 1$; moreover $f(y)/y$ is continuous in 0.

By Feynman-Kac formula we have:

$$\begin{aligned} u(h,x) &= \mathbb{E}\left[u_0(Z_h(x)) \exp\left\{\int_0^h \frac{f \circ u(h-s, Z_s(x))}{u(h-s,Z_s(x))} ds\right\}\right] && (6.6)\\ &= \mathbb{E}u_0(Z_h(x)) + \mathbb{E}u_0(Z_h(x)) \left\{\exp\left\{\int_0^h \frac{f \circ u(h-s, Z_s(x))}{u(h-s,Z_s(x))} ds\right\} - 1\right\} \\ &=: \mathbb{E}u_0(Z_h(x)) + A_h && (6.7) \end{aligned}$$

By applying the Taylor formula, we can write:

$$\begin{aligned} A_h &= \mathbb{E}u_0(Z_h(x)) \int_0^h \frac{f \circ u(h-s, Z_s(x))}{u(h-s,Z_s(x))} ds \\ &+ \frac{1}{2}\mathbb{E}u_0(Z_h(x)) \left[\int_0^h \frac{f \circ u(h-s, Z_s(x))}{u(h-s,Z_s(x))} ds\right]^2 \exp\left(\theta_h \int_0^h \frac{f \circ u(h-s, Z_s(x))}{u(h-s,Z_s(x))} ds\right) && (6.8) \end{aligned}$$

from which we have $A_h = B_h + R_h^0$ with

$$|R_h^0| \le C\, \mathbb{E}u_0(Z_h(x)) \left[\int_0^h \frac{f \circ u(h-s, Z_s(x))}{u(h-s,Z_s(x))} ds\right]^2$$

and

$$\begin{aligned}
B_h &= \mathbb{E}u_0(Z_h(x))\int_0^h \frac{f\circ u(h-s,Z_s(x))}{u(h-s,Z_s(x))}ds \\
&= \mathbb{E}\int_0^h u_0(Z_h(x))\frac{f\circ u(s,Z_{h-s}(x))}{u(s,Z_{h-s}(x))}ds \\
&= \int_0^h\int_{\mathbb{R}} \mathbb{E}u_0(Z_s(y))\frac{f\circ u(s,y)}{u(s,y)}\,P_{h-s}(x,dy)\,ds
\end{aligned}$$

where, in the last passage, we used the transition property of $P_\theta(x,dy)$.

On the other hand, by the same argument used to obtain (6.6) we have:

$$\mathbb{E}u_0(Z_s(y)) = u(s,y) + D(s,y)$$

with, for some $C>0$ large enough:

$$|D(s,y)| \le C\ \mathbb{E}u_0(Z_s(y))\int_0^s \frac{f\circ u(s-\theta,Z_\theta(y))}{u(s-\theta,Z_\theta(y))}\,d\theta \tag{6.9}$$

Hence we can write:

$$B_h = \int_0^h\int_{\mathbb{R}} f\circ u(s,y)\,P_{h-s}(x,dy)\,ds + E_h$$

with (remembering that $\frac{f(y)}{y}$ is uniformly bounded on $[0,1]$):

$$\begin{aligned}
|E_h| &\le C\int_0^h\int_{\mathbb{R}} \mathbb{E}u_0(Z_s(y))\int_0^s \frac{f\circ u(s-\theta,Z_\theta(y))}{u(s-\theta,Z_\theta(y))}\,d\theta\,P_{h-s}(x,dy)\,ds \\
&= C\int_0^h \mathbb{E}u_0(Z_h(x))\int_0^s \frac{f\circ u(s-\theta,Z_{\theta+h-s}(x))}{u(s-\theta,Z_{\theta+h-s}(x))}\,d\theta\,ds
\end{aligned} \tag{6.10}$$

Finally, by collecting together (6.7), (6.8), (6.10), we have:

$$u(h,x) = \mathbb{E}u_0(Z_h(x)) + \int_0^h\int_{\mathbb{R}} f\circ u(s,y)\,P_{h-s}(x,dy)\,ds + R_h^1$$

with

$$\begin{aligned}
|R_h^1| \le\ & C\mathbb{E}u_0(Z_h(x)) \\
& \left\{\left[\int_0^h \frac{f\circ u(h-s,Z_s(x))}{u(h-s,Z_s(x))}\,ds\right]^2 + \int_0^h\int_0^s \frac{f\circ u(s-\theta,Z_{\theta+h-s}(x))}{u(s-\theta,Z_{\theta+h-s}(x))}\,d\theta\,ds\right\}
\end{aligned}$$

Therefore, it remains to treat $\int_0^h\int_{\mathbb{R}} f\circ u(s,y)P_{h-s}(x,dy)ds$. We remark:

$$\begin{aligned}
\int_0^h\int_{\mathbb{R}} f\circ u(s,y)\,P_{h-s}(x,dy)\,ds &= \int_0^h\int_{\mathbb{R}} f(\mathbb{E}u_0(Z_s(y)) - D(s,y))\,P_{h-s}(x,dy)\,ds \\
&= \int_0^h\int_{\mathbb{R}} f(\mathbb{E}u_0(Z_s(y)))\,P_{h-s}(x,dy)\,ds + C_h
\end{aligned}$$

with (using (6.9)):

$$
\begin{aligned}
|C_h| &\leq C \int_0^h \int_{\boldsymbol{R}} |D(s,y)|\, P_{h-s}(x,dy) \\
&\leq C \int_0^h \int_{\boldsymbol{R}} \boldsymbol{E} u_0(Z_s(y)) \int_0^s \frac{f \circ u(s-\theta, Z_\theta(y))}{u(s-\theta, Z_\theta(y))}\, d\theta\, P_{h-s}(x,dy)\, ds \\
&= C \int_0^h \boldsymbol{E} u_0(Z_h(x)) \int_0^s \frac{f \circ u(s-\theta, Z_{\theta+h-s}(x))}{u(s-\theta, Z_{\theta+h-s}(x))}\, d\theta\, ds
\end{aligned}
$$

We conclude by applying Lemma 6.4.

∎

**Proposition 6.5**

(i) *Let us suppose that* **(H1)**, **(H2)**, **(H3)** *are verified, and that* $u_0$ *belongs to a family of functions verifying* **(H5)** *with weights bounded by* $\frac{C}{N}$*, the constant* $C$ *being uniform on the family; then, for any* $0 < s < h$*, we have:*

$$\int_{\boldsymbol{R}} \int_{\boldsymbol{R}} \boldsymbol{E}\Big[u_0(Z_s(y)) - \boldsymbol{E} u_0(Z_s(y))\Big]^2 P_{h-s}(x,dy)\, dx \leq C\, \sqrt{h}$$

*where the constant* $C$ *depends only on* $T$ *and the coefficients of the differential operator* $L$.

(ii) *If* $u_0$ *satisfies* **(H4)** *instead of the above condition, then we have:*

$$\int_{\boldsymbol{R}} \int_{\boldsymbol{R}} \boldsymbol{E}\Big[u_0(Z_s(y)) - \boldsymbol{E} u_0(Z_s(y))\Big]^2 P_{h-s}(x,dy)\, dx \leq C\, h\, \|\sigma u_0'\|_{L^2(\boldsymbol{R})} + C\, h^2\, \|L u_0\|_{L^2(\boldsymbol{R})}$$

**Proof**

In the case of Hypothesis **(H5)** we have:

$$
\begin{aligned}
\boldsymbol{E}\Big[u_0(Z_s(y)) - \boldsymbol{E} u_0(Z_s(y))\Big]^2 &= \sum_{i=1}^N \boldsymbol{E}\{\omega_0^i H(x_0^i - Z_s(y)) - \boldsymbol{E}[\omega_0^i H(x_0^i - Z_s(y))]\}^2 \\
&\quad + 2 \sum_{i<j} \boldsymbol{E}\Big[\{\omega_0^i H(x_0^i - Z_s(y)) - \boldsymbol{E}[\omega_0^i H(x_0^i - Z_s(y))]\} \\
&\qquad \{\omega_0^j H(x_0^j - Z_s(y)) - \boldsymbol{E}[\omega_0^j H(x_0^j - Z_s(y))]\}\Big] \\
&=: S_N(y) + 2\, T_N(y)
\end{aligned}
$$

Thus, using (3.11) and Corollary 2.1(iii) we have:

$$S_N(y) = \sum_{i=1}^N (\omega_0^i)^2 \Big(\boldsymbol{P}(x_0^i > Z_s(y)) - [\boldsymbol{P}(x_0^i > Z_s(y))]^2\Big)^2$$

$$
\begin{aligned}
&= \sum_{i=1}^{N} (\omega_0^i)^2 \, \boldsymbol{P}(x_0^i > Z_s(y)) \, \boldsymbol{P}(x_0^i \le Z_s(y)) \\
&\le \frac{C}{N^2} \sum_{i=1}^{N} \exp\left(-\frac{(y - x_0^i)^2}{2\lambda s}\right)
\end{aligned}
\tag{6.11}
$$

Therefore:

$$
\begin{aligned}
\int_{\boldsymbol{R}} S_N(y) \, P_{h-s}(x, dy) &\le \frac{C}{N^2} \sum_{i=1}^{N} \int_{\boldsymbol{R}} \exp\left(-\frac{(y - x_0^i)^2}{2\lambda s}\right) \frac{1}{\sqrt{h-s}} \exp\left(-\frac{(y-x)^2}{2\lambda(h-s)}\right) dy \\
&\le \frac{C}{N^2} \frac{\sqrt{s}}{\sqrt{h}} \sum_{i=1}^{N} \exp\left(-\frac{(x - x_0^i)^2}{2\lambda h}\right) \\
&\le \frac{C}{N^2} \sum_{i=1}^{N} \exp\left(-\frac{(x - x_0^i)^2}{2\lambda h}\right)
\end{aligned}
$$

from which:

$$
\int_{\boldsymbol{R}} \int_{\boldsymbol{R}} S_N(y) \, P_{h-s}(x, dy) \, dx \le \frac{C}{N} \sqrt{h}
$$

On the other hand:

$$
\begin{aligned}
T_N(y) \le \quad & \sum_{i<j} \sqrt{\boldsymbol{E}\{\omega_0^i \, H(x_0^i - Z_s(y)) - \boldsymbol{E}[\omega_0^i H(x_0^i - Z_s(y))]\}^2} \\
& \qquad \sqrt{\boldsymbol{E}\{\omega_0^j \, H(x_0^j - Z_s(y)) - \boldsymbol{E}[\omega_0^j H(x_0^j - Z_s(y))]\}^2}
\end{aligned}
$$

By using again (3.11) and Lemma 2.1(iii) we have:

$$
T_N(y) \le \frac{C}{N^2} \sum_{i<j} \exp\left(-\frac{(y - x_0^i)^2}{4\lambda s}\right) \exp\left(-\frac{(y - x_0^j)^2}{4\lambda s}\right)
\tag{6.12}
$$

hence:

$$
\begin{aligned}
&\int_{\boldsymbol{R}} T_N(y) \, P_{h-s}(x, dy) \\
&\le \frac{C}{N^2} \sum_{i<j} \int_{\boldsymbol{R}} \exp\left(-\frac{(y - x_0^i)^2}{4\lambda s}\right) \exp\left(-\frac{(y - x_0^j)^2}{4\lambda s}\right) P_{h-s}(x, dy) \\
&\le \frac{C}{N^2} \sum_{i<j} \left[\int_{\boldsymbol{R}} \exp\left(-\frac{(y - x_0^i)^2}{2\lambda s}\right) P_{h-s}(x, dy) \int_{\boldsymbol{R}} \exp\left(-\frac{(y - x_0^j)^2}{2\lambda s}\right) P_{h-s}(x, dy)\right]^{\frac{1}{2}} \\
&\le \frac{C}{N^2} \sum_{i<j} \left[\frac{\sqrt{s}}{\sqrt{h}} \exp\left(-\frac{(x - x_0^i)^2}{2\lambda s}\right)\right]^{\frac{1}{2}} \left[\frac{\sqrt{s}}{\sqrt{h}} \exp\left(-\frac{(x - x_0^j)^2}{2\lambda s}\right)\right]^{\frac{1}{2}} \\
&\le \frac{C}{N^2} \sum_{i<j} \exp\left(-\frac{(x - x_0^i)^2}{4\lambda h}\right) \exp\left(-\frac{(x - x_0^j)^2}{4\lambda h}\right)
\end{aligned}
$$

from which[4]:

$$\begin{aligned}\int_{\mathbf{R}}\int_{\mathbf{R}} T_N(y)\, P_{h-s}(x,dy)\, dx &\leq \frac{C}{N^2}\sum_{i<j}\int_{\mathbf{R}} \exp\left(-\frac{(x-x_0^i)^2}{4\lambda h}\right)\exp\left(-\frac{(x-x_0^j)^2}{4\lambda h}\right)dx \\ &\leq \frac{C}{N^2}\sqrt{h}\sum_{i<j}\exp\left(-\frac{(x_0^i-x_0^j)^2}{8\lambda h}\right) \\ &\leq C\sqrt{h}\end{aligned}$$

In the case of hypothesis (**H4**) we can apply Itô formula:

$$\begin{aligned}u_0(Z_s(y)) - \mathbf{E}u_0(Z_s(y)) &= \int_0^s \left[Lu_0(Z_\theta(y)) - \mathbf{E}Lu_0(Z_\theta(y))\right] d\theta \\ &\quad + \int_0^s \sigma(Z_\theta(y))u_0'(Z_\theta(y))\, dW_\theta \end{aligned} \tag{6.13}$$

hence

$$\begin{aligned}&\int_{\mathbf{R}} \mathbf{E}\Big[u_0(Z_s(y)) - \mathbf{E}u_0(Z_s(y))\Big]^2 P_{h-s}(x,dy) \\ &\qquad \leq 2\int_{\mathbf{R}} \mathbf{E}\left[\int_0^s \left[Lu_0(Z_\theta(y)) - \mathbf{E}Lu_0(Z_\theta(y))\right] d\theta\right]^2 P_{h-s}(x,dy) \\ &\qquad\qquad +2\int_{\mathbf{R}} \mathbf{E}\left[\int_0^s \sigma^2(Z_\theta(y))\, u_0'^2(Z_\theta(y))\, d\theta\right] P_{h-s}(x,dy) \\ &\qquad =: 2A + 2B\end{aligned}$$

We estimate $A$ in the following way:

$$A \leq \int_{\mathbf{R}} s\int_0^s \mathbf{E}\Big[Lu_0(Z_\theta(y))\Big]^2 d\theta\, P_{h-s}(x,dy) = s\int_{h-s}^h \mathbf{E}\Big[Lu_0(Z_\theta(x))\Big]^2 d\theta$$

Using again (2.3):

$$\int_{\mathbf{R}} A\, dx \leq C\int_{\mathbf{R}} s\int_{h-s}^h\int_{\mathbf{R}} \Big[Lu_0(z)\Big]^2 \frac{1}{\sqrt{\theta}}\exp\left(-\frac{(z-x)^2}{2\lambda\theta}\right) dz\, d\theta\, dx = C\, s^2\int_{\mathbf{R}} \Big[Lu_0(z)\Big]^2 dz$$

In the same way we have

$$B \leq C\, s\int_{\mathbf{R}} \sigma^2(z)\, {u_0'}^2(z)\, dz$$

∎

**Remark 6.6** *From (6.11) and (6.12) in the preceding proof, we have also shown that in the case* (**H5**)*, we have:*

$$\int_{\mathbf{R}} \mathbf{E}\Big[u_0(Z_h(x)) - \mathbf{E}u_0(Z_h(x))\Big]^2 dx \leq C\sqrt{h}$$

[4] see Remark (6.2).

*or, more precisely*[5]*:*

$$\int_{\mathbb{R}} \mathbb{E}\left[u_0(Z_h(x)) - \mathbb{E}u_0(Z_h(x))\right]^2 dx \le C\sqrt{h}\left(\frac{1}{N} + \frac{C}{N^2}\sum_{i<j}\exp\left(-\frac{(x_0^i - x_0^j)^2}{8\lambda h}\right)\right)$$

*and from (6.13), in the case of hypothesis* **(H4)**, *we have:*

$$\int_{\mathbb{R}} \mathbb{E}\left[u_0(Z_h(x)) - \mathbb{E}u_0(Z_h(x))\right]^2 dx \le C\, h$$

**Proposition 6.7** *For $\theta \in [0,h]$, let us define:*

$$\psi_{h,\theta}(x) := \mathbb{E}\left[u_0(Z_h(x))\frac{f \circ u(\theta, Z_{h-\theta}(x))}{u(\theta, Z_{h-\theta}(x))}\right]$$

*Then under* **(H1)**, **(H2)**, **(H3)***:*

*(i) if $u_0$ satisfies* **(H4)**, *there exists a $C > 0$ such that $\forall h < 1$, $\forall\theta \in ]0,h[$ we have:*

$$\|\psi_{h,\theta}(\cdot)\|_{L^1(\mathbb{R})} \le C$$

*(ii) if $u_0$ belongs to a family of functions verifying* **(H5)** *with weights bounded by $\frac{C}{N}$, the constant $C$ being uniform on the family, there exists a $C > 0$ such that $\forall h < 1$, $\forall\theta \in ]0,h[$ we have:*

$$\|\psi_{h,\theta}(\cdot)\|_{L^1(\mathbb{R})} \le C + \frac{C}{N}\sum_{i=1}^{N}|x_0^i|$$

**Proof**

The result will be an easy consequence of Lemma 2.6. Indeed: first we integrate between 0 and $+\infty$, and we use the equality (6.6) which implies: $\mathbb{E}u_0(Z_h(x)) \le Cu(h,x)$. We thus have:

$$\int_0^{+\infty} \psi_{h,\theta}(x)\, dx \le C\int_0^{+\infty} u(h,x)\, dx$$

For the case of integration between $-\infty$ and 0, we remark:

$$\frac{f(y)}{y(1-y)} \le C$$

[5] see Remark (6.2).

hence

$$\begin{aligned}
\int_{-\infty}^{0} \psi_{h,\theta}(x)\,dx &\le C \int_{-\infty}^{0} \boldsymbol{E}(1-u(\theta, Z_{h-\theta}(x)))\,dx \\
&\le C \int_{-\infty}^{0}\int_{\boldsymbol{R}}(1-u(\theta,y))\frac{1}{\sqrt{h-\theta}}\exp\left(-\frac{(y-x)^2}{2\lambda(h-\theta)}\right)dy\,dx \\
&\le C \int_{\boldsymbol{R}}(1-u(\theta,y))\int_{\frac{y}{\sqrt{\lambda(h-\theta)}}}^{+\infty}\exp\left(-\frac{\alpha^2}{2}\right)d\alpha\,dy \\
&\le C \int_{-\infty}^{0}(1-u(\theta,y))\,dy + C\int_{0}^{+\infty}\exp\left(-\frac{y^2}{2\lambda(h-\theta)}\right)dy
\end{aligned}$$

from which by Lemma 2.6 we get the conclusion.
■

Now, let us remark:

$$\left[\int_0^h \frac{f\circ u(h-s,Z_s(x))}{u(h-s,Z_s(x))}ds\right]^2 \le Ch\int_0^h \frac{f\circ u(h-s,Z_s(x))}{u(h-s,Z_s(x))}ds$$

As the preceding Proposition, one can show:

**Proposition 6.8** *Let us define:*

$$\psi_h(x) := \boldsymbol{E}\left\{u_0(Z_h(x))\left[\int_0^h \frac{f\circ u(h-s,Z_s(x))}{u(h-s,Z_s(x))}ds\right]^2\right\}$$

*Then, under the hypotheses of Proposition 6.5:*

*(i) if $u_0$ satisfies* **(H4)**, *there exists a $C>0$ such that $\forall h<1$:*

$$\|\psi_h(\cdot)\|_{L^1(\boldsymbol{R})} \le Ch^2$$

*(ii) if $u_0$ belongs to a family of functions verifying* **(H5)** *with weights bounded by $\frac{C}{N}$, the constant $C$ being uniform on the family, there exists a $C>0$ such that $\forall h<1$:*

$$\|\psi_h(\cdot)\|_{L^1(\boldsymbol{R})} \le Ch^2\left(1+\frac{1}{N}\sum_{i=1}^{N}|x_0^i|\right)$$

# 7 Estimate of the global error

We recall that we denote: $M = T/h$.

We are now in position of proving the first part of our main Theorem 4.1.

First, we write:

$$\|u(T,\cdot)-\bar{u}_M(\cdot)\|_{L^1(\mathbb{R}\times\Omega)} \le \|\mathbb{E}\bar{u}_M(\cdot)-\bar{u}_M(\cdot)\|_{L^1(\mathbb{R}\times\Omega)} + \|u(T,\cdot)-\mathbb{E}\bar{u}_M(\cdot)\|_{L^1(\mathbb{R})}$$

In Section 7.1, we will bound the first term of the right-hand side by $\frac{C}{\sqrt{N}}+C\sqrt{h}$; in Section 7.3, the second term will be bounded by $\frac{C}{N}+C\sqrt{h}$, so that the announced convergence rate (for general functions $b$ and $\sigma$) will be established.

## 7.1 Estimate of $\|\mathbb{E}\bar{u}_M(\cdot)-\bar{u}_M(\cdot)\|_{L^1(\mathbb{R}\times\Omega)}$

Our objective is to show :

**Proposition 7.1** *There exists a constant $C>0$ such that, for any $M\in\mathbb{N}^*$, any $p\le M$, any $N$:*

$$\|\mathbb{E}\bar{u}_p(\cdot)-\bar{u}_p(\cdot)\|_{L^1(\mathbb{R}\times\Omega)} \le \frac{C}{\sqrt{N}}+C\sqrt{h}$$

Let us define

$$\hat{u}_p(x)=\sum_{i=1}^{N}\omega_p^i\,H(X_{ph}^i-x)$$

Using (2.13) and (2.10), we have:

$$\|\bar{u}_p(\cdot)-\mathbb{E}\bar{u}_p(\cdot)-\hat{u}_p(\cdot)+\mathbb{E}\hat{u}_p(\cdot)\|_{L^1(\mathbb{R}\times\Omega)}=\mathcal{O}(h) \tag{7.1}$$

Therefore it is sufficient to prove:

**Lemma 7.2** *There exists a $C>0$ such that, for any $N$, $h\le 1$ and $p\le M=T/h$:*

$$\|\hat{u}_p(\cdot)-\mathbb{E}\hat{u}_p(\cdot)\|_{L^1(\mathbb{R}\times\Omega)} \le \frac{C}{\sqrt{N}}+C\sqrt{h}$$

**Proof**

We have (using the fact that the sum of the weights is equal to 1):

$$\begin{aligned}
&\int_{\mathbb{R}}\left|\sum_{i=1}^{N}\left(\mathbb{E}\omega_p^iH(X_{ph}^i-x)-\omega_p^iH(X_{ph}^i-x)\right)\right|dx\\
&=\int_0^{+\infty}\left|\sum_{i=1}^{N}\left(\mathbb{E}\omega_p^iH(X_{ph}^i-x)-\omega_p^iH(X_{ph}^i-x)\right)\right|dx\\
&+\int_{-\infty}^{0}\left|\sum_{i=1}^{N}\left(\mathbb{E}\omega_p^iH(x-X_{ph}^i)-\omega_p^iH(x-X_{ph}^i)\right)\right|dx
\end{aligned}$$

We will only consider the first term of the right hand side of the previous equality, the second one being treated in the same way. We use the independent weights of Section 5.

$$\begin{aligned}
&\int_0^{+\infty}\left|\sum_{i=1}^N\Big(\boldsymbol{E}\omega_p^i H(X_{ph}^i-x)-\omega_p^i H(X_{ph}^i-x)\Big)\right|dx\\
&\leq \int_0^{+\infty}\left|\sum_{i=1}^N\Big(\boldsymbol{E}[\rho_p^i H(X_{ph}^i-x)]-\rho_p^i H(X_{ph}^i-x)\Big)\right|dx\\
&\quad+\int_0^{+\infty}\left|\sum_{i=1}^N\boldsymbol{E}[(\omega_p^i-\rho_p^i)H(X_{ph}^i-x)]\right|dx\\
&\quad+\int_0^{+\infty}\sum_{i=1}^N\left|\omega_p^i-\rho_p^i\right|H(X_{ph}^i-x)\,dx
\end{aligned}$$

Using the independence of the $\rho^i$'s and of the $(X_\cdot^i)$'s, and bounding the variance by the 2$^{\text{nd}}$ moment, one gets:

$$\begin{aligned}
&\boldsymbol{E}\int_0^{+\infty}\left|\sum_{i=1}^N\Big(\boldsymbol{E}\omega_p^i H(X_{ph}^i-x)-\omega_p^i H(X_{ph}^i-x)\Big)\right|dx\\
&\leq \int_0^{+\infty}\sqrt{\sum_{i=1}^N\boldsymbol{E}(\rho_p^i)^2 H(X_{ph}^i-x)}\,dx+2\sum_{i=1}^N\boldsymbol{E}|\omega_p^i-\rho_p^i|\,|X_{ph}^i|\\
&\leq \int_0^{+\infty}\frac{C}{N}\sqrt{\sum_{i=1}^N\boldsymbol{P}(X_{ph}^i>x)}\,dx+2\sqrt{\alpha_p}\sum_{i=1}^N\sqrt{\boldsymbol{E}|X_{ph}^i|^2}
\end{aligned}$$

Now let us remark that the first term of the right hand side can be upper bounded by:

$$\frac{C}{\sqrt{N}}\int_0^{+\infty}\sqrt{\frac{1}{N}\sum_{i=1}^N\int_{\frac{x-x_0^i}{\sqrt{\lambda ph}}}^{+\infty}\exp\left\{-\frac{y^2}{2}\right\}dy}\;\;dx$$

For $x\in[0,+\infty[$ the function

$$s\xmapsto{\psi_x}\frac{1}{\sqrt{2\pi}}\int_{\frac{x-u_0^{-1}(s)}{\sqrt{\lambda ph}}}^{+\infty}\exp\left\{-\frac{y^2}{2}\right\}dy$$

is decreasing from $]0,1[$ to $]0,1[$; therefore, the definition of the $x_0^i$ implies:

$$\begin{aligned}
&\frac{1}{2N}\sum_{i=1}^N\int_{\frac{x-x_0^i}{\sqrt{\lambda ph}}}^{+\infty}\exp\left\{-\frac{y^2}{2}\right\}dy\\
&\leq\int_0^1\int_{\frac{x-u_0^{-1}(s)}{\sqrt{\lambda ph}}}^{+\infty}\exp\left\{-\frac{y^2}{2}\right\}dy\,ds
\end{aligned}$$

$$= - \int_{R} \int_{\frac{x-z}{\sqrt{\lambda ph}}}^{+\infty} \exp\left\{-\frac{y^2}{2}\right\} dy\, u_0'(z)\, dz$$

$$\leq - \int_{-\infty}^{x} u_0'(z) \int_{\frac{x-z}{\sqrt{\lambda ph}}}^{+\infty} \exp\left\{-\frac{y^2}{2}\right\} dy\, dz - C \int_{x}^{+\infty} u_0'(z)\, dz$$

$$\leq - \int_{-\infty}^{+\infty} u_0'(z) \exp\left\{-\frac{(x-z)^2}{2\lambda ph}\right\} dz + C\, u_0(x)$$

Using **(H4)** we deduce for suitable $\lambda_0 > 0$:

$$\frac{1}{N} \sum_{i=1}^{N} \int_{\frac{x-x_0^i}{\sqrt{\lambda ph}}}^{+\infty} \exp\left\{-\frac{y^2}{2}\right\} dy \leq C \exp\left(-\frac{x^2}{2\lambda_0(1+ph)}\right) + C\, u_0(x)$$

so that, by Remark 2.5:

$$\frac{1}{N} \int_0^{+\infty} \sqrt{\sum_{i=1}^{N} \boldsymbol{P}(X_{ph}^i > x)}\, dx \leq \frac{C}{\sqrt{N}}$$

Now, by (5.7):

$$\sqrt{\alpha_p} \sum_{i=1}^{N} \sqrt{\boldsymbol{E}|X_{ph}^i|^2} \leq \left(\frac{C\sqrt{h}}{N} + \frac{C}{N\sqrt{N}}\right) \sum_{i=1}^{N} (1 + \boldsymbol{E}|X_{ph}^i|^2)$$

But (see (2.8)):

$$\frac{1}{N} \sum_{i=1}^{N} \boldsymbol{E}|X_{ph}^i|^2 \leq \frac{C}{N} \sum_{i=1}^{N} (1 + |x_0^i|^2) \leq C + \frac{1}{N} \sum_{i=1}^{N} |x_0^i|^2$$

Then we apply (3.14).

■

## 7.2 A Corollary

As a Corollary of the previous Section, we have the following result :

**Lemma 7.3** *Define*

$$A_p^h(N) := \int_{R} \boldsymbol{E}\left[\int_{R} \left(\bar{u}_p(y) - \boldsymbol{E}\bar{u}_p(y)\right) p_h(x,y)\, dy\right]^2 dx$$

*There exists a* $C > 0$ *such that*

$$A_p^h(N) \leq \frac{C}{\sqrt{N}} + C\sqrt{h}$$

*for any* $N$, $h < 1$ *and* $p \leq M = T/h$.

**Proof**

First we remark:

$$A_p^h(N) \le \int_{\mathbf{R}}\int_{\mathbf{R}} \mathbf{E}\Big[\bar{u}_p(y) - \mathbf{E}\bar{u}_p(y)\Big]^2 p_h(x,y)\,dy\,dx$$

and by Lemma 2.2:

$$A_p^h(N) \le (1 + C\,h)\int_{\mathbf{R}} \mathbf{E}\Big[\bar{u}_p(y) - \mathbf{E}\bar{u}_p(y)\Big]^2 dy$$

from which, by the boundedness of the function $\bar{u}_p$:

$$A_p^h(N) \le C\int_{\mathbf{R}} \mathbf{E}\Big|\bar{u}_p(y) - \mathbf{E}\bar{u}_p(y)\Big|\,dy.$$

We then apply Lemma 7.2.

■

### 7.3 Estimate of $\|u(T,\cdot) - \mathbf{E}\bar{u}_M(\cdot)\|_{L^1(\mathbf{R})}$

For any $p = 1, \ldots, M$, let us define $\bar{\nu}_p(t,x)$ as the solution to:

$$\begin{cases} \dfrac{\partial \bar{\nu}_p}{\partial t} = L\bar{\nu}_p + f \circ \bar{\nu}_p \\ \\ \bar{\nu}_p(0,x) = \bar{u}_{p-1}(x) \end{cases}$$

and let us consider:

$$\beta_p := \|u(ph,\cdot) - \mathbf{E}\bar{u}_p(\cdot)\|_{L^1(\mathbf{R})} \le \underbrace{\|u(ph,\cdot) - \mathbf{E}\bar{\nu}_p(h,\cdot)\|_{L^1(\mathbf{R})}}_{\gamma_p} + \underbrace{\|\mathbf{E}\bar{\nu}_p(h,\cdot) - \mathbf{E}\bar{u}_p(\cdot)\|_{L^1(\mathbf{R})}}_{\delta_p} \quad (7.2)$$

We are going to show (cf. proposition (7.6) below):

$$\boxed{\forall p\ , \qquad \beta_p \le C\left(\sqrt{h} + \frac{1}{N}\right)}$$

Let us first treat:

$$\delta_p = \int_0^{+\infty} |\mathbf{E}\bar{\nu}_p(h,x) - \mathbf{E}\bar{u}_p(x)|dx + \int_{-\infty}^0 |\mathbf{E}(1 - \bar{\nu}_p(h,x)) - \mathbf{E}(1 - \bar{u}_p(x))|dx \qquad (7.3)$$

Our objective is to show:

**Proposition 7.4**

$$\forall p, \quad \delta_p \le C\left(h^{\frac{3}{2}} + \frac{h}{N}\right) \qquad (7.4)$$

**Proof**

We will only consider the first term of the right hand side of (7.3).

Let $(B_\theta)$ a $(\mathcal{G}_\theta)$-Brownian motion (see the beginning of Section (3.2)), and $(\eta_\theta(y))$ the solution to:

$$\begin{cases} d\eta_\theta = -\sigma(\eta_\theta)\, d(B_{ph+\theta} - B_{ph}) - \{b(\eta_\theta) - \sigma(\eta_\theta)\,\sigma'(\eta_\theta)\}\, d\theta \\ \\ \eta_0 = y \end{cases} \tag{7.5}$$

We stress that, for each $\theta \geq 0$, $\eta_\theta(y)$ is independent of $\mathcal{G}_{ph}$.

We will denote by $\bar{\eta}_h(y)$ the approximation of $\eta_h(y)$ obtained by applying the Milshtein scheme (2.7) to the stochastic differential equation (7.5).

We first remark, using (3.13) and the conventions described in (1.2):

$$\begin{aligned} \boldsymbol{E}\bar{u}_{p+1}(x) &= \boldsymbol{E}\sum_{i=1}^{N} \omega_{p+1}^{i} H(\bar{X}_{p+1}^{i} - x) \\ &= \boldsymbol{E}\sum_{i=1}^{N} \left[\omega_p^i \left(1 + h\, f' \circ \bar{u}_p(\bar{X}_p^i)\right) + \mathcal{O}(h)\mathcal{O}(\frac{1}{N^2})\right] H(\bar{\eta}_h(\bar{X}_p^i) - x) \qquad (7.6) \\ &= \boldsymbol{E}\sum_{i=1}^{N} \omega_p^i H(\bar{\eta}_h(\bar{X}_p^i) - x) + h\, \boldsymbol{E}\sum_{i=1}^{N} \omega_p^i f' \circ \bar{u}_p(\bar{X}_p^i) H(\bar{X}_p^i - x) + \\ &\quad + \boldsymbol{E}\; \mathcal{O}(h)\mathcal{O}(\frac{1}{N^2}) \sum_{i=1}^{N} H(\bar{\eta}_h(\bar{X}_p^i) - x) + \qquad (7.7) \\ &\quad + h\, \boldsymbol{E}\sum_{i=1}^{N} \omega_p^i f' \circ \bar{u}_p(\bar{X}_p^i)(H(\bar{\eta}_h(\bar{X}_p^i) - x) - H(\bar{X}_p^i - x)) \end{aligned}$$

Therefore:

$$\boldsymbol{E}\bar{u}_{p+1}(x) = \boldsymbol{E}\sum_{i=1}^{N} \omega_p^i H(\bar{\eta}_h(\bar{X}_p^i) - x) + h\boldsymbol{E}\sum_{i=1}^{N} \omega_p^i f' \circ \bar{u}_p(\bar{X}_p^i) H(\bar{X}_p^i - x) + \bar{R}(h, p, x) \tag{7.8}$$

with (we apply (2.4) and (2.8)):

$$\begin{aligned} \|\bar{R}(h,p,\cdot)\|_{L^1(\boldsymbol{R}_+)} &\leq Ch\sqrt{h} + C\frac{h}{N^2}\sum_{i=1}^{N} \boldsymbol{E}|\bar{\eta}_h(\bar{X}_p^i)| \\ &\leq Ch\sqrt{h} + C\frac{h}{N^2}\sum_{i=1}^{N} \boldsymbol{E}|\bar{X}_p^i| \\ &\leq Ch\sqrt{h} + C\frac{h}{N^2}\sum_{i=1}^{N}(1 + |x_0^i|) \end{aligned}$$

Applying (3.14), we deduce[6] :

$$\|\bar{R}(h,p,\cdot)\|_{L^1(\boldsymbol{R}_+)} \leq Ch\sqrt{h} + C\frac{h}{N} \tag{7.9}$$

[6] when $f \equiv 0$, this term is absent; this permits to justify a remark we made in Section 4.

Now, let us perform a similar expansion of $\boldsymbol{E}\bar{\nu}_{p+1}(h,x)$.

Using the representation (3.5), we can also write:

$$\boldsymbol{E}\bar{\nu}_{p+1}(h,x) = \boldsymbol{E}\sum_{i=1}^{N}\omega_p^i\psi(\bar{X}_p^i,x) \tag{7.10}$$

where

$$\psi(y,x) = \boldsymbol{E}H(\eta_h(y)-x)\exp\left(\int_0^h f'\circ\bar{\nu}_{p+1}(s,\eta_s(y))ds\right)$$

Using again (2.13) and the estimate (2.9), one can check:

$$\psi(y,x) = \boldsymbol{E}H(\bar{\eta}_h(y)-x)\exp\left(\int_0^h f'\circ\bar{\nu}_{p+1}(s,\eta_s(y))ds\right) + \psi_0(h,y,x)$$

with

$$\exists C>0,\quad \forall y\in\boldsymbol{R},\quad \|\psi_0(h,y,\cdot)\|_{L^1(\boldsymbol{R})}\le Ch^{\frac{3}{2}}$$

Therefore:

$$\begin{aligned}\boldsymbol{E}\bar{\nu}_{p+1}(h,x) &= \boldsymbol{E}\sum_{i=1}^{N}\omega_p^iH(\bar{\eta}_h(\bar{X}_p^i)-x)\\ &\quad+\boldsymbol{E}\sum_{i=1}^{N}\omega_p^i\int_0^h f'\circ\bar{\nu}_{p+1}(s,\eta_s(\bar{X}_p^i))ds\,H(\bar{X}_p^i-x)+\tilde{R}(h,p,x)\end{aligned} \tag{7.11}$$

with

$$\|\tilde{R}(h,p,\cdot)\|_{L^1(\boldsymbol{R}_+)}\le Ch\sqrt{h} \tag{7.12}$$

Therefore, combining (7.11) and (7.8), in view of (7.12) and (7.9), we see that it remains to treat :

$$\phi(h,x) := \boldsymbol{E}\sum_{i=1}^{N}\omega_p^i\Big(hf'\circ\bar{u}_p(\bar{X}_p^i)-\int_0^h f'\circ\bar{\nu}_{p+1}(s,\eta_s(\bar{X}_p^i))ds\Big)\,H(\bar{X}_p^i-x)$$

and to show that its norm in $L^1(\boldsymbol{R}_+)$ can be bounded by $Ch\sqrt{h}+\frac{Ch}{N}$.

But:

$$\|\phi(h,\cdot)\|_{L^1(\boldsymbol{R}_+)}\le\boldsymbol{E}\sum_{i=1}^{N}\omega_p^i|\bar{X}_p^i|\Big|\boldsymbol{E}^{\mathcal{G}_{ph}}[hf'\circ\bar{u}_p(\bar{X}_p^i)-\int_0^h f'\circ\bar{\nu}_{p+1}(s,\eta_s(\bar{X}_p^i))ds]\Big|$$

Besides:

$$\boldsymbol{E}^{\mathcal{G}_{ph}}f'\circ\bar{\nu}_{p+1}(s,\eta_s(\bar{X}_p^i)) = f'\Big(\boldsymbol{E}^{\mathcal{G}_{ph}}\bar{\nu}_{p+1}(s,\eta_s(\bar{X}_p^i))\Big)+r_1(i,p,h,s) \tag{7.13}$$

with

$$|r_1(i,p,h,s)|\le C\boldsymbol{E}^{\mathcal{G}_{ph}}\left|\bar{\nu}_{p+1}(s,\eta_s(\bar{X}_p^i))-\boldsymbol{E}^{\mathcal{G}_{ph}}\bar{\nu}_{p+1}(s,\eta_s(\bar{X}_p^i))\right|^2$$

Let us perform an expansion of $f'\Big(\mathbb{E}^{\mathcal{G}_{ph}}\bar{\nu}_{p+1}(s,\eta_s(\bar{X}_p^i))\Big)$.

Let $(\eta_s^i)$ $N$ independent copies of the process $(\eta_s)$. The representation (3.5) permits to justify:

$$f'\Big(\mathbb{E}^{\mathcal{G}_{ph}}\bar{\nu}_{p+1}(s,\eta_s(\bar{X}_p^i))\Big) = f'\Big(\mathbb{E}^{\mathcal{G}_{ph}}\sum_{k=1}^{N}\omega_p^k H(\eta_s^k(\bar{X}_p^k)-\eta_s^i(\bar{X}_p^i))\Big) + \mathcal{O}(h) + \mathcal{O}\left(\frac{1}{N}\right)$$

Then, if we define:

$$r_2(i,p,h,s) := f'\Big(\mathbb{E}^{\mathcal{G}_{ph}}\sum_{k=1}^{N}\omega_p^k H(\eta_s^k(\bar{X}_p^k)-\eta_s^i(\bar{X}_p^i))\Big) - f'\Big(\mathbb{E}\sum_{k=1}^{N}\omega_p^k\, H(\bar{X}_p^k-\bar{X}_p^i)\Big)$$

we get:

$$\begin{aligned} f'\Big(\mathbb{E}^{\mathcal{G}_{ph}}\bar{\nu}_{p+1}(s,\eta_s(\bar{X}_p^i))\Big) &= f'\Big(\sum_{k=1}^{N}\omega_p^k\, H(\bar{X}_p^k-\bar{X}_p^i)\Big) + \mathcal{O}(h) + r_2(i,p,h,s) + \mathcal{O}\left(\frac{1}{N}\right) \\ &= f'\circ\bar{u}_p(\bar{X}_p^i) + \mathcal{O}(h) + r_2(i,p,h,s) + \mathcal{O}\left(\frac{1}{N}\right) \end{aligned}$$

Thus, we have got:

$$\|\phi(h,\cdot)\|_{L^1(\mathbb{R}_+)} \le \mathbb{E}\sum_{i=1}^{N}\omega_p^i|\bar{X}_p^i|\Big(Ch^2 + \int_0^h |r_1(i,p,h,\theta)|\,d\theta + \int_0^h |r_2(i,p,h,\theta)|\,d\theta + C\,\frac{h}{N}\Big)$$

As above (see (3.15)), we deduce:

$$\begin{aligned} &\|\phi(h,\cdot)\|_{L^1(\mathbb{R}_+)} \le C\,h^2 + \\ &+ C\,\mathbb{E}\sum_{i=1}^{N}\omega_p^i|\bar{X}_p^i|\Big(\int_0^h |r_2(i,p,h,\theta)|d\theta + \int_0^h \mathbb{E}^{\mathcal{G}_{ph}}\left|\bar{\nu}_{p+1}(s,\eta_s(\bar{X}_p^i)) - \mathbb{E}^{\mathcal{G}_{ph}}\bar{\nu}_{p+1}(s,\eta_s(\bar{X}_p^i))\right|^2 ds + C\,\frac{h}{N}\Big) \\ &\le C\,h^2 + C\,\underbrace{\mathbb{E}\sum_{i=1}^{N}\omega_p^i|\bar{X}_p^i|\int_0^h |r_2(i,p,h,\theta)|\,d\theta}_{A} + \\ &+ \underbrace{C\,\mathbb{E}\sum_{i=1}^{N}\omega_p^i|\bar{X}_p^i|\int_0^h \Big|\sum_{k=1}^{N}\omega_p^k\Big\{\Big(\mathbb{E}H(\eta_s(\bar{X}_p^k)-y)\Big)_{y=\eta_s(\bar{X}_p^i)} - \mathbb{E}^{\mathcal{G}_{ph}}\Big(\mathbb{E}H(\eta_s(\bar{X}_p^k)-y)\Big)_{y=\eta_s(\bar{X}_p^i)}\Big\}\Big|^2 ds}_{B} \\ &+ C\,\frac{h}{N^2}\sum_{i=1}^{N}\mathbb{E}|\bar{X}_p^i| \end{aligned}$$

For $A$ we have:

$$\begin{aligned} A &\le C\mathbb{E}\sum_{i=1}^{N}\omega_p^i|\bar{X}_p^i|\int_0^h \Big|\mathbb{E}^{\mathcal{G}_{ph}}\sum_{k=1}^{N}\omega_p^k H(\eta_s^k(\bar{X}_p^k)-\eta_s^i(\bar{X}_p^i)) - \mathbb{E}\sum_{k=1}^{N}\omega_p^k\, H(\bar{X}_p^k-\bar{X}_p^i)\Big|\,ds \\ &\le C\mathbb{E}\sum_{i=1}^{N}\omega_p^i|\bar{X}_p^i|\int_0^h\sum_{k=1}^{N}\omega_p^k A_k\,ds \end{aligned}$$

where

$$A_k := \boldsymbol{E}^{\mathcal{G}_{ph}}|H(\bar{X}_p^k - \bar{X}_p^i) - H(\eta_s^k(\bar{X}_p^k) - \eta_s^i(\bar{X}_p^i))|$$

* for $k = i$ we have $A_k \leq 1$

* for $k \neq i$ let us suppose (without any loss of generality) that the particles have been labelled in the increasing order of their positions (at time $ph$); we just look at $k < i$, the other case being treated by symmetry; therefore $H(\bar{X}_p^k - \bar{X}_p^i) = 0$, and we consider:

$$\begin{aligned}
\boldsymbol{E}^{\mathcal{G}_{ph}} H(\eta_s^k(\bar{X}_p^k) - \eta_s^i(\bar{X}_p^i)) &\leq \frac{C}{s}\int_{\boldsymbol{R}^2} H(y-z)\exp\left(-\frac{(y-\bar{X}_p^k)^2 + (z-\bar{X}_p^i)^2}{2\lambda s}\right) dy\, dz \\
&= C\int_{y-z>\frac{\bar{X}_p^i - \bar{X}_p^k}{\sqrt{\lambda s}}} \exp\left(-\frac{y^2+z^2}{2}\right) dy\, dz \\
&\leq C\exp\left(-\frac{(\bar{X}_p^i - \bar{X}_p^k)^2}{4\lambda s}\right)
\end{aligned}$$

thanks to the following easy inequality:

$$\int_{x_1 - x_2 > \rho} \exp\left(-\frac{x_1^2 + x_2^2}{2}\right) dx_1\, dx_2 \leq C\, e^{-\frac{\rho^2}{4}}$$

Therefore:

$$\begin{aligned}
A &\leq C\, h\, \boldsymbol{E}\sum_{i=1}^N \omega_p^i |\bar{X}_p^i| \left\{\sum_{k=1}^N \omega_p^k \exp\left(-\frac{(\bar{X}_p^k - \bar{X}_p^i)^2}{4\lambda h}\right) + \frac{C}{N}\right\} \\
&\leq \frac{C\, h}{N^2}\boldsymbol{E}\sum_{\substack{i,k=1\\ i\neq k}}^N |\bar{X}_p^i|\exp\left(-\frac{(\bar{X}_p^k - \bar{X}_p^i)^2}{4\lambda h}\right) + \frac{C\, h}{N^2}\sum_{i=1}^N \boldsymbol{E}|\bar{X}_p^i|
\end{aligned}$$

From the following elementary inequality:

$$\left|e^{-\frac{x^2}{2h}} - e^{-\frac{y^2}{2h}}\right| \leq \frac{|x-y|}{\sqrt{h}}$$

and (2.10), (3.15), we get:

$$A \leq \frac{C\, h}{N^2}\boldsymbol{E}\sum_{\substack{i,k=1\\ i\neq k}}^N |X_p^i|\exp\left(-\frac{(X_p^k - X_p^i)^2}{4\lambda h}\right) + C\, h\sqrt{h} + \frac{C\, h}{N}$$

Now, we use the independence of $X^i$ and $X^k$ for $i \neq k$, and (2.3), to get:

$$\begin{aligned}
A &\leq \frac{C\, h}{N^2}\sum_{\substack{i,k=1\\ i\neq k}}^N \int_{\boldsymbol{R}^2} \frac{|x|}{\lambda ph} e^{-\frac{(y-x)^2}{4\lambda h}} e^{-\frac{(x-x_0^k)^2}{2\lambda ph}} e^{-\frac{(y-x_0^i)^2}{2\lambda ph}}\, dx\, dy + C\, h\sqrt{h} + \frac{C\, h}{N} \\
&\leq \frac{C\, h\sqrt{h}}{N^2}\sum_{\substack{i,k=1\\ i\neq k}}^N \int_{\boldsymbol{R}} \frac{|x|}{\lambda h\sqrt{p(p+2)}}\exp\left(-\frac{(x-x_0^k)^2}{2\lambda ph}\right)\exp\left(-\frac{(x-x_0^i)^2}{2\lambda(p+2)h}\right) dx + C\, h\sqrt{h} + \frac{C\, h}{N}
\end{aligned}$$

Using the definition of the $x_0^k$'s and **(H4)**, we observe (see the proof of (3.14) and of Lemma (7.2) for analogous computations):

$$\frac{1}{N}\sum_{k=1}^{N}\frac{1}{\sqrt{\lambda ph}}\exp\left(-\frac{(x-x_0^k)^2}{2\lambda ph}\right)\leq C$$

so that, using again the same arguments, we obtain:

$$A\leq\frac{C\,h\sqrt{h}}{N}\sum_{i=1}^{N}\int_{\mathbb{R}}\frac{|x|}{\sqrt{\lambda(p+2)h}}\exp\left(-\frac{(x-x_0^i)^2}{2\lambda(p+2)h)}\right)\,dx+C\,h\sqrt{h}+\frac{C\,h}{N}\leq C\,h\sqrt{h}+\frac{C\,h}{N}$$

For the term $B$ we observe, using the independence of the particles:

$$\begin{aligned}
&\boldsymbol{E}^{\mathcal{G}_{ph}}\Big|\sum_{k=1}^{N}\omega_p^k\left\{(\boldsymbol{E}H(\eta_s(\bar{X}_p^k)-y))_{y=\eta_s(\bar{X}_p^i)}-\boldsymbol{E}^{\mathcal{G}_{ph}}(\boldsymbol{E}H(\eta_s(\bar{X}_p^k)-y))_{y=\eta_s(\bar{X}_p^i)}\right\}\Big|^2\\
=\;&\boldsymbol{E}^{\mathcal{G}_{ph}}\sum_{\substack{k=1\\k\neq i}}^{N}(\omega_p^k)^2\Big|(\boldsymbol{E}H(\eta_s(\bar{X}_p^k)-y))_{y=\eta_s(\bar{X}_p^i)}-\boldsymbol{E}^{\mathcal{G}_{ph}}(\boldsymbol{E}H(\eta_s(\bar{X}_p^k)-y))_{y=\eta_s(\bar{X}_p^i)}\Big|^2+\mathcal{O}\left(\frac{1}{N}\right)\\
\leq\;&\frac{C}{N}
\end{aligned}$$

Hence, using (3.15):

$$B\leq\frac{Ch}{N}\,\boldsymbol{E}\sum_{i=1}^{N}\omega_p^i|\bar{X}_p^i|\leq\frac{C\,h}{N}$$

■

Let us now treat $\gamma_p$.

**Proposition 7.5**

$$\forall p,\quad \gamma_{p+1}\leq(1+Ch)\beta_p+Ch^{3/2}+\frac{Ch}{\sqrt{N}}\tag{7.14}$$

**Proof**

We use the local expansion of $u(ph,\cdot)$ and $\bar{\nu}_p(\cdot)$ deduced from Theorem 6.1. Here, Hypothesis (H2) implies that $P_t(x,dy)=p_t(x,y)\,dy$. Besides, we apply twice Proposition 6.1:

- we substitute $u(ph,\cdot)$ to $u_0$; then we are in the case where the initial condition satisfies **(H4)** (from (3.3), (2.3), and **(H4)**, it is easy to obtain the condition on the spatial derivative), and we have:

  $$u((p+1)h,x)=\boldsymbol{E}\,u(ph,Z_h(x))+hf(\boldsymbol{E}\,u(ph,Z_h(x)))+R^{h,1}_{(p+1)h}(x)$$

  with $\|R^{h,1}_{(p+1)h}(\cdot)\|_{L^1(\mathbb{R})}\leq C\,h^2$ ;

- we substitute $\bar{u}_p(\cdot)$ to $u_0$; then we are in the case where the initial condition satisfies (H5), and we have:

$$\bar{\nu}_{p+1}(x) = \int_{\boldsymbol{R}} \bar{u}_p(y)\, p_h(\cdot, y)\, dy + hf\left(\int_{\boldsymbol{R}} \bar{u}_p(y)\, p_h(\cdot, y)\, dy\right) + R^{h,2}_{(p+1)h}(x)$$

with

$$\|R^{h,2}_{(p+1)h}(\cdot)\|_{L^1(\boldsymbol{R})} \leq C\, h\sqrt{h} + Ch^2 \frac{1}{N} \sum_{i=1}^{N} |\bar{X}^i_p|$$

so that, using (3.15): $\boldsymbol{E}\|R^{h,2}_{(p+1)h}(\cdot)\|_{L^1(\boldsymbol{R})} \leq C\, h\sqrt{h}$.

Thus, using again (2.12):

$$\begin{aligned}
\gamma_{p+1} &= \left\| \int_{\boldsymbol{R}} u(ph, y)\, p_h(\cdot, y)\, dy + h\, f\left(\int_{\boldsymbol{R}} u(ph, y)\, p_h(\cdot, y)\, dy\right) \right. \\
&\quad \left. - \boldsymbol{E} \int_{\boldsymbol{R}} \bar{u}_p(y)\, p_h(\cdot, y)\, dy - h\, \boldsymbol{E}\, f\left(\int_{\boldsymbol{R}} \bar{u}_p(y)\, p_h(\cdot, y)\, dy\right) + R^{h,1}_{(p+1)h}(\cdot) - \boldsymbol{E} R^{h,2}_{(p+1)h}(\cdot) \right\|_{L^1(\boldsymbol{R})} \\
&\leq \int_{\boldsymbol{R}} |u(ph, y) - \boldsymbol{E}\bar{u}_p(y)| \int_{\boldsymbol{R}} p_h(x, y)\, dx\, dy + Ch^{\frac{3}{2}} + \\
&\quad + C\, h \int_{\boldsymbol{R}} |u(ph, y) - \boldsymbol{E}\bar{u}_p(y)| \int_{\boldsymbol{R}} p_h(x, y)\, dx\, dy + \\
&\quad + C\, h \int_{\boldsymbol{R}} \boldsymbol{E}\left(\int_{\boldsymbol{R}} \bar{u}_p(y)\, p_h(x, y)\, dy - \boldsymbol{E} \int_{\boldsymbol{R}} \bar{u}_p(y)\, p_h(x, y)\, dy\right)^2 dx
\end{aligned}$$

Applying the Lemmae 2.2 and 7.3, one gets:

$$\gamma_{p+1} \leq (1 + Ch)\beta_p + Ch^{3/2} + \frac{Ch}{\sqrt{N}}$$

■

Finally, we can prove:

**Proposition 7.6**

$$\forall p \quad , \quad \beta_p \leq C\left(\sqrt{h} + \frac{1}{\sqrt{N}}\right)$$

## Proof

We use the definition (7.2), and the estimates (7.14) and (7.4), to get:

$$\beta_{p+1} \leq (1 + Ch)\beta_p + Ch^{3/2} + \frac{Ch}{\sqrt{N}}$$

In the proof of Lemma 5.5, we remarked that:
$\gamma_1 = \|u(h,\cdot) - v^N(h,\cdot)\|_{L^1(\mathbb{R})}$ can be bounded by $C\|\mathbb{E}|(u_0 - \bar{u}_0)(Z_h(\cdot))|\|_{L^1(\mathbb{R})}$; as $\bar{u}_0 \geq u_0$ for $x < x_0^{N-1} < C(1+\sqrt{\log N})$, using $u(x) = -\int_x^{+\infty} u'(y)dy$, (2.2) and **(H4)**, we get, for some $C > 0$ large enough:

$$\begin{aligned}
\gamma_1 &\leq C\int_{|x|<C+C\sqrt{\log N}} (\bar{u}_0(x) - u_0(x))dx \\
&\quad + C\int_{x\leq -C-C\sqrt{\log N}} (1-u_0(x))dx + C\int_{x\geq C+C\sqrt{\log N}} u_0(x)dx \\
&\leq \frac{C\sqrt{\log N}}{N} + \frac{C}{N}
\end{aligned}$$

Thus, $\beta_1 \leq C\left(\frac{1}{\sqrt{N}} + h^{\frac{3}{2}} + \frac{h}{N}\right)$, and we can proceed by induction to end the proof.
∎

# 8 The case of constant coefficients

In this section, we explain what must be changed in the proof to get the better estimate $\frac{C}{\sqrt{N}} + Ch$ for the error when the coefficients of $L$ are constant.

Without losing in generality, we can suppose that $b \equiv 0$ and $\sigma \equiv 1$. In that case we have that $\bar{X}_p^i = X_p^i = x_0^i + W_{ph}^i$.

First, one remarks that the expansion in Remark 5.1 can be then changed in:

$$\omega_{p+1}^i - \rho_{p+1}^i = \omega_p^i - \rho_p^i + h\,\omega_p^i\,\{f'\circ\bar{u}_p(\bar{X}_p^i)) - f'\circ u(ph,\bar{X}_p^i)\} + h\,(\omega_p^i - \rho_p^i)\,f'\circ u(ph,\bar{X}_p^i) + O\left(\frac{h}{N^2}\right)$$

so that the inequality (5.2) can be modified in:

$$\alpha_{p+1}^i \leq (1+Ch)\alpha_p^i + \frac{C\,h}{N}\sqrt{\alpha_p^i}\sqrt{\mathbb{E}|\bar{u}_p(\bar{X}_p^i) - u(ph,\bar{X}_p^i)|^2} + C\,\frac{h^2}{N^2}\mathbb{E}|\bar{u}_p(\bar{X}_p^i) - u(ph,\bar{X}_p^i)|^2 + C\,\frac{h}{N^3}$$

One can readily show that (5.4) can be reduced in:

$$\mathbb{E}|\bar{u}_p(\bar{X}_p^i) - u_p^*(\bar{X}_p^i)|^2 \leq N^2\alpha_p$$

Therefore, with the same arguments as in the proof of Proposition 5.8, one can show that the inequality (5.7) can be modified in:

$$\forall p, \qquad \alpha_p \leq \frac{C\,h^2}{N^2} + \frac{C}{N^3} \tag{8.1}$$

This remark permits to change the last lines of the proof of Lemma 7.2, so that one gets:

$$\|\hat{u}_p(\cdot) - \mathbb{E}\hat{u}_p(\cdot)\|_{L^1(\mathbb{R}\times\Omega)} \leq \frac{C}{\sqrt{N}} + Ch.$$

Consequently, the conclusion of Lemma 7.3 becomes: there exists a constant $C > 0$ such that

$$A_p^h(N) \leq \frac{C}{\sqrt{N}} + Ch$$

for any $N$, $h \leq 1$ and $p \leq M = T/h$.

Now, we recall the inequality (6.5). This permits to modify the beginning of the proof of Proposition 7.5 in the following way:

$$\begin{aligned}
\gamma_{p+1} &= \Bigg\| \int_{\boldsymbol{R}} u(ph,y)\, p_h(\cdot,y)\, dy + h\, f\left(\int_{\boldsymbol{R}} u(ph,y)\, p_h(\cdot,y)\, dy\right) \\
&\quad - \boldsymbol{E}\int_{\boldsymbol{R}} \bar{u}_p(y)\, p_h(\cdot,y)\, dy - h\, \boldsymbol{E}\, f\left(\int_{\boldsymbol{R}} \bar{u}_p(y)\, p_h(\cdot,y)\, dy\right) + R^{h,1}_{(p+1)h}(\cdot) + \boldsymbol{E} R^{h,2}_{(p+1)h}(\cdot)\Bigg\|_{L^1(\boldsymbol{R})} \\
&\leq \int_{\boldsymbol{R}} |u(ph,y) - \boldsymbol{E}\bar{u}_p(y)| \int_{\boldsymbol{R}} p_h(x,y)\, dx\, dy + Ch^2 + C\frac{h\sqrt{h}}{N} + \\
&\quad + C\, h \int_{\boldsymbol{R}} |u(ph,y) - \boldsymbol{E}\bar{u}_p(y)| \int_{\boldsymbol{R}} p_h(x,y)\, dx\, dy + \\
&\quad + C\, h \int_{\boldsymbol{R}} \boldsymbol{E}\left(\int_{\boldsymbol{R}} \bar{u}_p(y)\, p_h(x,y)\, dy - \boldsymbol{E}\int_{\boldsymbol{R}} \bar{u}_p(y)\, p_h(x,y)\, dy\right)^2 dx
\end{aligned}$$

Thus, it remains to check that we can improve the estimate for $\delta_p$. Namely, instead of (7.4), we have:

$$\forall p, \quad \delta_p \leq C\left(h^2 + \frac{h}{N}\right)$$

Actually, one just has to consider (7.6) and (7.10): now $\bar{\eta}_h(\bar{X}_p^i)$ and $\eta_h(\bar{X}_p^i)$ are equal, thus the conclusion is straightforward.

# 9 Conclusion

We have constructed a stochastic particle algorithm for general one–dimensional reaction–diffusion–convection P.D.E.'s, by establishing a convenient probabilistic representation of the solution and discretising it in space and time.

We have given its rate of convergence, what also proves a conjecture of Puckett concerning this method for the KPP equation.

# A Appendix

In this Section, we suppose $(H_i, i = 1, \ldots, 3)$. Moreover let us consider the differential operator $L$, defined in §2, as an abstract unbounded closed operator in a suitable Banach space (of functions) $\mathbf{X}$.

## A.1 A result for linear equations

Let $g(\cdot)$ be a function in $L^\infty(0,T;L^\infty(\mathbb{R}))$, and let us consider the following abstract linear equation in $\mathbf{X} = L^\infty(\mathbb{R})$:

$$\begin{cases} u'(t) = L\,u(t) + g(t) \\ \\ u(0) = u_0 \end{cases} \tag{A.1}$$

It is well known that, when $\boxed{g \equiv 0}$, $L$ is the infinitesimal generator of an analytic semigroup $e^{Lt}$ in $L^2_\pi(\mathbb{R})$, the space of (classes of equivalence of) functions that are square integrable with respect to a weight $\pi(x)$([7]) and also in $C_{b.u}(\mathbb{R})$ ([8]) (cfr. e.g. Stewart [15], Cannarsa-Vespri [2]). Hence $e^{Lt}$ is well defined for any $u_0 \in L^\infty(\mathbb{R})$, but is not strongly continuous, that is $\|e^{Lt}u_0 - u_0\|_{L^\infty(\mathbb{R})}$ does not tend to 0 for any $u_0 \in L^\infty(\mathbb{R})$. One only has: $e^{Lt}u_0$ is a smooth function of $x$ and $e^{Lt}u_0\,(x) \to u_0(x)$ almost everywhere (see also Hida [7], Theorem 2.11).

In the non-homogeneous case $\boxed{g \neq 0}$ the following function

$$u(t) = e^{Lt}u_0 + \int_0^t e^{L\,(t-s)}g(s)\,ds \tag{A.2}$$

is called the *mild* solution to (A.1). In fact we need some regularity on $g$ (for example $g$ satisfies a Holder condition, i.e $g \in C^\theta([0,T], L^\infty(\mathbb{R}))$) to ensure that (A.2) is the classical solution to (A.1) (see, for example, Da Prato-Sinestrari [11], or Pazy [10], Theorem 3.5).

(Remark: in our case, we can also use the results of the variational theory, as illustrated in Bensoussan -Lions [1], chapter 6, in the weighted spaces $L^2_\pi(\mathbb{R})$).

## A.2 Existence and uniqueness of the solution of the non linear equation

Consider now the non linear problem with initial datum in $L^\infty(\mathbb{R})$:

$$\begin{cases} \dfrac{\partial u}{\partial t} = L\,u + f(u) \\ \\ u(0) = u_0 \end{cases} \tag{A.3}$$

**Theorem A.1** *Suppose $(H_1, H_2, H_3)$. Then there exists a unique solution $u \in C^{1,2}(]0,T] \times \mathbb{R})$ to (A.3). Moreover, for $t \to 0$, $u(t)$ tends to $u_0$ in the continuity points of $u_0$.*

We only outline the proof; we refer to Rothe [13]. Actually in [13] the results are stated for problems in bounded domains $I$ of $\mathbb{R}$; but, as the proofs are based on semigroup results that are still true in $\mathbb{R}$ — as claimed in the preceding section —, they can be extended to cover the case of problems in the whole space.

[7] for instance, $\pi(x) = (1+x^2)^{-s}$, with $s > 0$.

[8] i.e. the space of bounded and uniformly continuous functions.

Indeed, let us consider the following integral equation:

$$u(t) = e^{Lt}u_0 + \int_0^t e^{L(t-s)} f(u(s))\,ds \;=: \mathcal{G}(u)(t) \quad , \qquad t \in [0,T] \tag{A.4}$$

Then, *under* $(H_1, H_2, H_3)$, *there exists a unique solution* $u \in L^\infty(0,T; L^\infty(\mathbb{R}))$ *to (A.4).* For this, we refer to Theorem 1 of [13] (page 111); the proof uses the classical Picard method: the sequence $u_n = \mathcal{G}(u_{n-1})$ converges to the solution (obviously unique) $u$ of (A.4) as $f(\cdot)$ is Lipschitz.

Moreover, *for any* $\delta > 0$, $u(t)$ *is in* $C^{1+\frac{\alpha}{2},2+\alpha}([\delta,T] \times \mathbb{R})$ *and satisfies the first equation of (A.3).* We refer to Theorem 2 of [13] (page 120).

**Remark A.2** Theorem 3 of [13] (pag. 123) gives a comparison result, based on the maximum principle, for "smooth" initial data; but, even when $u_0$ is only supposed to satisfy Hypothesis (**H5**), it is easy to verify that, as $0 \le u_0 \le 1$, we also have $0 \le u(t) \le 1$ by an approximation argument, as in the proof of Theorem 2 of [13], and taking in account that $u \equiv 0$ and $u \equiv 1$ are solutions to (A.3).

Moreover, when $t \to 0$, $u(t,x) \to u_0(x)$ in all continuity points of $u_0$: we use $e^{Lt}u_0 \to u_0$ in all continuity points of $u_0$ (see Hida [7], Theorem 2.11, e.g.) and one can easily show:

$$\|u(t) - e^{Lt}u_0\|_{L^\infty(\mathbb{R})} = \mathcal{O}(t)$$

Finally, if the initial data $u_{0n}, u_0$ are in $C_{b,u}(\mathbb{R})$ and $\|u_{0n} - u_0\|_{L^\infty(\mathbb{R})} \to 0$ then we also have:

$$\sup_{t \in [0,T]} \|u_n(t) - u(t)\|_{L^\infty(\mathbb{R})} \to 0$$

where, by $u_n(t)$ and $u(t)$, we mean the solutions to (A.3) with initial data respectively $u_{0n}$ and $u_0$.

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