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Boniface Nkonga, Hervé Guillard

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Boniface Nkonga
Hervé Guillard

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Institut National
de Recherche
en Informatique
et en Automatique

2004 route des Lucioles
B.P. 93
06902 Sophia-Antipolis
France

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Boniface Nkonga, Hervé Guillard.
INRIA, centre de Sophia-Antipolis, B.P. 93 , 06902 Sophia-Antipolis Cedex (France)
Tel : 93.65.77.95. *e–mail :* nkonga@cosinus.inria.fr
Tel : 93.65.77.96. *e–mail :* guillard@mingus.inria.fr

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Abstract: This paper presents a numerical method for the computation of compressible flows in domains whose boundaries move in a well defined predictable manner. The method uses the space-time formulation by Godunov while the discretization is conducted on non-structured tetrahedral meshes, using Roe's approximate Riemann solver, an implicit time stepping and a MUSCL-type interpolation. The computation of the geometrical parameters required to take into account the movement of the boundaries is described. Examples including the calculation of the flow in the cylinder of an internal combustion engine illustrates the possibilities of the method.

METHODE DE TYPE GODUNOV EN NON STRUCTURE POUR DES PROBLEMES TRIDIMENSIONNELS A FRONTIERE MOBILE

Boniface Nkonga, Hervé Guillard.
INRIA, centre de Sophia-Antipolis, B.P. 93 , 06902 Sophia-Antipolis Cedex (France)
Tel : 93.65.77.95. *e–mail :* nkonga@cosinus.inria.fr
Tel : 93.65.77.96. *e–mail :* guillard@mingus.inria.fr

Mots clés : Problèmes Hyperboliques, Dynamique des Fluides, Domaines Déformables, Maillages non Structurés, Méthodes type Godunov, Ecoulements Tridimensionnels, Moteur à Piston.

Résumé: Nous présentons dans cet article une méthode numérique pour la résolution des écoulements compressibles dans un domaine à frontière mobile. Dans le cadre d'une formulation mixte Volume finis/Eléments finis sur des maillages non structurés, on utilise la discrétisation dans l'espace spatio-temporel introduite par Godunov. Ceci nous permet de généraliser les solveurs de Riemann et en particulier le solveur approché de Roe, de construire des méthodes implicites par linéarisation des flux et d'utiliser une interpolation de type MUSCL pour obtenir des schémas précis à l'ordre deux. Nous décrivons le calcul des paramètres géométriques nécessaire pour prendre en compte le mouvement des frontières. Pour illustrer les possibilités de la méthode, cette dernière est appliquée à plusieurs type d'écoulements parmi lesquels les écoulements dans le cylindre d'un moteur à piston.

1 INTRODUCTION

This work deals with the computation of the flows of compressible fluids in moving geometries such as those occurring for instance in reciprocating engines and more generally in many industrial devices. As the geometry of such industrial devices is inherently complex, the versatility and flexibility of non-structured meshes is very appealing. Accordingly, the present method is based on the use of tetrahedral meshes. Because compressible flows often exhibits strong compressions or discontinuities, the spatial approximation uses the robust second-order Euler solver developed in [1, 3, 4]. This solver is also attractive because of its nice matricial properties when coupled with an implicit time stepping. This paper explains how the movement of the boundaries is included into this framework in a very simple and natural way. It is organized as follows: Section 2 discusses the approximation of conservation laws on moving meshes. As stressed out by many authors (see e.g. [13] for a review), the key point here is the respect of the so-called geometrical conservation law that is the form taken by the consistency relation on a moving grid. The time-space formulation given by Godunov's method provides a very simple and elegant way to automatically satisfy this constraint. Then in Section 3 we provide the computation of the geometrical parameters when the intersection of two spatial control-volumes is a triangular face whose three nodes move with different velocities. As any planar polygonal surface can be split into triangular facets, the general case can be treated by repeated application of formula obtained on triangular facets. Actually in Section 4, we specialized these results to the finite volume method of [11] where the boundaries of the control volumes are composed of an union of plane quadrangular faces. The fifth section deals with modifications that are necessary to implement Roe's approximate Riemann solver on a moving grid. Finally, in the last section we conclude by presenting several test cases and examples illustrating the possibilities of the method.

2 CONSERVATION LAWS IN MOVING DOMAIN

Let $\mathcal{D}(t) \subset I\!R^m \ \ (m \leq 3)$ be the domain filled by the fluid at time t. The displacement of the domain from its position at time $t=0$ to its position at time t is given by a map $\vec{x} = \varphi(\vec{a})$ from $\mathcal{D}(0)$ onto $\mathcal{D}(t)$. The space time domain $\mathcal{E}$ occupied by the fluid between $t=0$ and $t=T$ is defined by

$$\mathcal{E} = \bigcup_{t=0}^{T} \mathcal{D}(t).$$

Except when the domain boundaries are fixed, the space-time domain $\mathcal{E}$ is not a simple product of a spatial domain by a time interval. This introduces an additional difficulty with respect to fixed boundary problems.

We will denote by $J(t) = \frac{D\vec{x}}{D\vec{a}}$ the jacobian matrix of the map φ and by $\kappa(\vec{x},t)$ the velocity displacement of the domain $\mathcal{D}(t)$ at $\vec{x} \in \mathcal{D}(t)$:

$$\vec{\kappa}(\vec{x},t) = \frac{\partial \varphi}{\partial t}\Big(\varphi^{-1}(\vec{x},t)\Big)$$

For all $(\vec{x}, t) \in \mathcal{E}$, we consider a first-order conservation law:

$$\frac{\partial W}{\partial t} + \nabla \cdot \vec{F}(W) = 0 \tag{1}$$

where W is a vector of $I\!R^p$ and $\vec{F}$ a smooth function from $I\!R^p$ into $I\!R^p$. Let $\mathcal{C}_i(t)_{i=1,\dots,ns}$ be a splitting of the domain $\mathcal{D}(t)$ into no intersecting cells such that:

$$\bigcup_{i=1}^{ns} \mathcal{C}_i(t) = \mathcal{D}(t) \qquad \text{and} \qquad \dot{\mathcal{C}}_i \cap \dot{\mathcal{C}}_j = \emptyset \quad \text{if } i \neq j$$

we will denote by $\partial\mathcal{C}_i(t)$ the boundary of the cell $\mathcal{C}_i(t)$ and by $\partial\mathcal{C}_{ij}(t)$ the intersection of the boundaries of the cells $\mathcal{C}_i(t)$ and $\mathcal{C}_j(t)$ such that :

$$\partial\mathcal{C}_i(t) \;=\; \bigcup_{j \in \mathcal{T}(i)} \partial\mathcal{C}_{ij}(t)$$

where $\mathcal{T}(i)$ is the set of the neighbors of i. We denote by $\mathcal{Q}_i$ the space time hyper volume defined by the displacement of $\mathcal{C}_i$ under the action of the map φ between times $t = t^n$ and $t = t^{n+1}$ and by $\mathcal{S}_{ij}$ the intersection of the two hyper volumes $\mathcal{Q}_i$ and $\mathcal{Q}_j$ during the same time interval. Obviously the union of $\mathcal{Q}_i$ fills $\mathcal{E}$ and we have:

$$\mathcal{S}_{ij} = \bigcup_{t=t^n}^{t^{n+1}} \partial\mathcal{C}_{ij}(t) \qquad \text{and} \qquad \partial\mathcal{Q}_i = \mathcal{C}_i(t^n) \cup \mathcal{C}_i(t^{n+1}) \bigcup \Big(\cup_{j\in\mathcal{T}(i)} \mathcal{S}_{ij} \Big)$$

Figure 1 provides a graphical representation of the space time and spatial cells $\mathcal{Q}_i$ and $\mathcal{C}_i$ in the two-dimensional case.

The discrete equations are obtained by integrating equation (1) over $\mathcal{Q}_i$ resulting in:

$$\int_{t^n}^{t^{n+1}} \iiint_{\mathcal{C}_i(t)} \frac{\partial}{\partial t} W \, d\vec{x}\, dt + \int_{t^n}^{t^{n+1}} \iint_{\partial\mathcal{C}_i(t)} \vec{F}(W) \cdot \vec{n}(t)\, dS\, dt = 0 \tag{2}$$

Using the following identity:

$$\frac{d}{dt}\left\{ \iiint_{\mathcal{C}_i(t)} W d\vec{x} \right\} = \iiint_{\mathcal{C}_i(t)} \frac{\partial}{\partial t} W d\vec{x} + \iint_{\partial\mathcal{C}_i(t)} W \; \vec{\kappa} \cdot \vec{n}(t)\, dS$$

equation (2) becomes:

$$Vol(\mathcal{C}_i^{n+1})\, W_i^{n+1} - Vol(\mathcal{C}_i^n)\, W_i^n + \sum_{j\in\mathcal{T}(i)} \int_{t^n}^{t^{n+1}} \iint_{\partial\mathcal{C}_i(t)} \Big(\vec{F}(W) - W\,\vec{\kappa} \Big) \cdot \vec{n}(t)\, dS\, dt = 0 \tag{3}$$

where W_i^{n+1} and W_i^n are defined as the mean value of W over the cell $\mathcal{C}_i(t^{n+1})$ and $\mathcal{C}_i(t^n)$. Similarly, if we define W_{ij} and $\vec{F}_{ij}$ as the mean value of W and $\vec{F}$ over $\mathcal{S}_{ij}$, we obtain:

$$Vol(\mathcal{C}_i^{n+1})\, W_i^{n+1} - Vol(\mathcal{C}_i^n)\, W_i^n + \Delta t \sum_{j\in\mathcal{T}(i)} \left(\vec{F}_{ij} \cdot \vec{\eta}_{ij} \; - \; \sigma_{ij} ||\vec{\eta}_{ij}|| W_{ij} \right) \; = 0 \tag{4}$$

where the mean value $\vec{\eta}_{ij}$ of the integral of the normal $\vec{n}$ is defined by:

$$\vec{\eta}_{ij} \;=\; \frac{1}{\Delta t}\iiint_{\mathcal{S}_{ij}} \vec{n}(t)\; dS\; dt \;\;=\;\; \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \iint_{\partial\mathcal{C}_{ij}(t)} \vec{n}(t)\; dS\; dt \tag{5}$$

and where we have introduced σ_{ij} the mean value of the normal velocity defined by:

$$\sigma_{ij} \;=\; \frac{1}{\Delta t\, ||\vec{\eta}_{ij}||}\iiint_{\mathcal{S}_{ij}} \vec{\kappa}\cdot\vec{n}(t)\; dS\; dt \;\;=\;\; \frac{1}{\Delta t\, ||\vec{\eta}_{ij}||}\int_{t^n}^{t^{n+1}} \iint_{\partial\mathcal{C}_{ij}(t)} \vec{\kappa}\cdot\vec{n}(t)\; dS\; dt \tag{6}$$

The correct evaluation of the above quantities is the key point in the design of a numerical scheme on a moving grid. This can be understood as follows : let us consider that at time $t = 0$ the variable W is equal to a constant state W_0. If the domain boundaries $\partial\mathcal{D}$ are fixed, the solution of any conservation law of the form (1) will be $W(\vec{x}, t) = W_0 \;\; \forall t$. Consequently the integration of (1) over any volume $C(t) \subset \mathcal{D}$ and for any map φ respecting the boundaries $\partial\mathcal{D}$ will degenerate into the identity:

$$\frac{d}{dt}\int_{C(t)} d\vec{x} = \int_{\partial C(t)} \vec{\kappa}\cdot\vec{n}\, dS \tag{7}$$

According to (5)-(6) the discrete counterpart of this identity takes the form:

$$Vol(\mathcal{C}_i^{n+1}) \;-\; Vol(\mathcal{C}_i^{n}) \;\;=\;\; \Delta t \sum_{j\in\mathcal{T}(i)} \Big(||\vec{\eta}_{ij}||\; \sigma_{ij} \Big) \tag{8}$$

and states that the increase of the volume of a cell must equal the summation of the volumes swept out by each face of the cell between t^n and t^{n+1}. Therefore the respect of (8) is a necessary condition for a numerical scheme to keep constant an uniform initial state and thus the evaluation of the geometrical parameters (5) and (6) must be carefully done. This fact was first pointed out in [12]. Given an approximation of $\vec{\eta}_{ij}$ and σ_{ij}, a sometimes used manner to enforce (8) is to update the volume of the cells by explicitly defining $Vol(\mathcal{C}_i^{n+1})$ by equation (8). However, in this case, the actual geometrical volume of the cells may differ from the value given by (8) and it is preferable to devise a direct geometrical way to respect (8). In the one and two-dimensional cases, the correct evaluation of (5) and (6) is described in Godunov's book [6]. Actually the original 1959 Godunov's method [7] was designed for a moving as well as for a fixed grid. We now give the computation of the geometrical parameters (5) and (6) for the case of a triangular facet moving in a three-dimensional space. As any plane polygonal surface can be split into a finite number of triangles a repeated application of these results will give the general solution (see Section 4 for an example).

3 GEOMETRICAL PARAMETERS

Let $T(t^n)$ be a triangular surface in $I\!R^3$ whose nodes are $I(t^n)$, $J(t^n)$ and $K(t^n)$. Our aim is to compute the parameters (5) and (6) under the action of the map φ. It may happen that φ is explicitely known for every $(\vec{x}, t)$ for instance by an analytical expression, however when the motion of the domain is complex, the map φ is generally computed by a numerical procedure and is only known on a discrete set of points. Therefore we make the assumption that the velocities of the three nodes I, J and K are constant (but different) during a time step Δt and that the velocity of any point P belonging at time t^n over $T(t^n)$ is a linear interpolation of the velocities of the nodes defining the facet. Let (α, β) denote the barycentric coordinate of $P \in T(t^n)$:

$$\vec{X}^n = \vec{X}_P(t^n) = (1-\alpha-\beta)\vec{X}_I(t^n) + \alpha\vec{X}_J(t^n) + \beta\vec{X}_K(t^n)$$

then the velocity of P is :

$$\vec{\kappa}(P) = \ (1-\alpha-\beta)\,\vec{\kappa}_I \ + \ \alpha\,\vec{\kappa}_J \ + \ \beta\,\vec{\kappa}_K$$

and thus $\vec{\kappa}(P)$ is a constant and for any $t \in [t^n, t^{n+1}]$ we have :

$$\begin{aligned} \vec{X}_P(t) &= \vec{X}^n + (t-t^n)\,\vec{\kappa}(P) \\ &= (1-\alpha-\beta)\vec{X}_I(t) + \alpha\vec{X}_J(t) + \beta\vec{X}_K(t) \end{aligned} \tag{9}$$

From this equation, we deduce that $P(t)$ belongs to the triangle $I(t)$, $J(t)$, $K(t)$ for any $t \in [t^n, t^{n+1}]$ (the facet remains plane), that the barycentric coordinates (α, β) of P remain constant during the time step and that $\vec{X}_P(t)$ varies linearly in time :

$$\vec{X}_P(t) = (1-s)\vec{X}^n + s\,\vec{X}^{n+1}; \qquad \text{with} \qquad s = \frac{t-t^n}{\Delta t}$$

With these relations it is now an easy task to compute $\vec{\eta}_{ij}$, we have first:

$$\vec{\mu}(t) = \iint_{T(t)} \vec{n}(t)\,dS = \frac{1}{2}(\vec{X}_J(t) - \vec{X}_I(t)) \wedge (\vec{X}_K(t) - \vec{X}_I(t))$$

and it is readily found that

$$\vec{\eta}_T = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \iint_{T(t)} \vec{n}(t)\,dS\,dt = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \vec{\mu}(t)\,dt = \frac{1}{3}(\vec{\mu}^n + \vec{\mu}^{n+1} + \vec{\mu}^*) \tag{10}$$

where $\vec{\mu}^{n+1}$ and $\vec{\mu}^n$ are respectively the integrals of the normal unit on $T(t^n)$ and $T(t^{n+1})$, and the vector $\vec{\mu}^*$ is defined by:

$$2\vec{\mu}^* = \frac{(\vec{X}_J^n - \vec{X}_I^n) \wedge (\vec{X}_K^{n+1} - \vec{X}_I^{n+1})}{2} + \frac{(\vec{X}_J^{n+1} - \vec{X}_I^{n+1}) \wedge (\vec{X}_K^n - \vec{X}_I^n)}{2}$$

The computation of σ_T proceeds in the same way. We first compute the volume δV swept out by the facet T (Figure 2) :

$$\begin{aligned} \delta V &= \int_{t^n}^{t^{n+1}} \iint_{T(t)} \vec{\kappa}\cdot\vec{n}(t)\,dS\,dt = \int_{t^n}^{t^{n+1}} \vec{n}(t)\cdot\left[\iint_{T(t)} \vec{\kappa}\,dS\right] dt \\ &= \vec{\kappa}_G \cdot \left(\int_{t^n}^{t^{n+1}} \vec{n}(t)\,area\{T(t)\}\,dt\right) \ = \ \Delta t\,\vec{\kappa}_G\cdot\vec{\eta}_T \end{aligned}$$

where $\vec{\kappa}_G$ is the velocity of the gravity center of the facet. We then obtain :

$$\sigma_T = \ \frac{\delta V}{\Delta t\,||\vec{\eta}_T||} = \left(\frac{\vec{\eta}_T}{||\vec{\eta}_T||}\right)\cdot\left(\frac{\vec{\kappa}_I + \vec{\kappa}_J + \vec{\kappa}_K}{3}\right) \tag{11}$$

This ends the computation of geometrical parameters for a triangular facet.

4 MUSCL-FEM METHOD

In this section we apply the previous results to the MUSCL-FEM method by [1, 4, 3]. The three-dimensional version of this method is developped in ([11]). This method can be understood either as a classical Galerkin finite element method stabilized by upwind terms [2] or as a finite volume method on non-structured simplicial meshes. For ease of presentation, we adopt here the second point of view. Given a non-structured tetrahedral mesh, to each node i of the mesh is associated a cell defined by the midpoints of the edges of the tetrahedra having i as a node, the gravity center of these tetrahedra and the gravity center of the faces of these tetrahedra. Figure 3 displays the intersection of such a control volume with a tetrahedra. Let us split the boundary of the i–cell by an union of partial boundaries associated to each edge:

$$\partial\mathcal{C}_i = \bigcup_{j\in\mathcal{T}(i)} \partial\mathcal{C}_{ij}$$

From Figure 3, it is seen that $\partial\mathcal{C}_{ij}$ is itself an union on T_{ij}, the set of tetrahedra having $[i,j]$ as an edge, of quadrangular surfaces :

$$\partial\mathcal{C}_{ij} = \bigcup_{\tau\in T_{ij}} (\partial\mathcal{C}_{ij}\cap\tau)$$

Let i,j,k,l be the four nodes defining such a tetrahedra τ, $\partial\mathcal{C}_{ij}\cap\tau$ is defined by the four points $\{M_1;\ G_1;\ G;\ G_3\}$ (see Figure 3) whose coordinates are :

$$\begin{aligned}
\vec{X}_{M_1} &= \frac{1}{2}(\vec{X}_i+\vec{X}_j), & \vec{X}_{G_1} &= \frac{1}{3}(\vec{X}_i+\vec{X}_j+\vec{X}_l),\\
\vec{X}_{G_3} &= \frac{1}{3}(\vec{X}_i+\vec{X}_j+\vec{X}_k), & \vec{X}_{G} &= \frac{1}{4}(\vec{X}_i+\vec{X}_j+\vec{X}_l+\vec{X}_k).
\end{aligned} \tag{12}$$

The following relations are easily obtained from the definition of $\{M_1;\ G_1;\ G;\ G_3\}$:

$$\begin{aligned}
\overrightarrow{M_1G_1} &= \frac{1}{6}(-\vec{X}_i-\vec{X}_j+2\vec{X}_l), & \overrightarrow{GG_1} &= \frac{1}{12}(\vec{X}_i+\vec{X}_j+\vec{X}_l-3\vec{X}_k),\\
\overrightarrow{M_1G_3} &= \frac{1}{6}(-\vec{X}_i-\vec{X}_j+2\vec{X}_k) \quad\text{and} & \overrightarrow{GG_3} &= \frac{1}{12}(\vec{X}_i+\vec{X}_j+\vec{X}_k-3\vec{X}_l).
\end{aligned}$$

and it is readily checked that these relations imply :

$$\overrightarrow{G_1M_1}\wedge\overrightarrow{G_3M_1} = 2\,(\overrightarrow{G_3G}\wedge\overrightarrow{G_1G}) \tag{13}$$

which states that $\partial\mathcal{C}_{ij}\cap\tau(t)$ is a *planar* quadrangle. A quite remarkable consequence of our assumption on the piecewise linearity of the velocity displacement of the mesh is that the relations (12) holds for all $t\in[t^n,t^{n+1}]$. Therefore, the planar quadrangular surface $\{M_1;\ G_1;\ G;\ G_3\}$ remains plane during the time interval $[t^n,t^{n+1}]$. We note that this feature is not so common for moving grid algorithms. In particular, the deformation of hexahedral meshes usually results in the formation of non-planar boundaries of the cells which introduces a certain arbitrariness in the definition of the volume of the cells.

To compute the geometrical parameters (5,6), let us consider $T_1(t) = \{M_1(t)\,;\, G_1(t)\,;\, G_3(t)\}$ and $T_2(t) = \{G_1(t)\,;\, G(t)\,;\, G_3(t)\}$, the two triangles that form the quadrangle $\partial\mathcal{C}_{ij} \cap \tau(t)$ and let us use the results of Section 3. We note $a^{\tau}_{ij}(t)$ the area of $\partial\mathcal{C}_{ij} \cap \tau(t)$, $a_1(t)$ and $a_2(t)$ respectively areas of T_1 and T_2. Equation (13) implies that:

$$3\, a_1(t) \;=\; 6\, a_2(t) \;=\; 2\, a^{\tau}_{ij}(t) \qquad \forall\, t \in\; [t^n, t^{n+1}] \tag{14}$$

We note $\vec{\mu}^{\tau}_{ij}(t)$ the value of the integral of the normal on $\partial\mathcal{C}_{ij} \cap \tau(t)$. We have:

$$\begin{aligned}
\vec{\mu}^{\tau}_{ij}(t) = \iint_{\partial\mathcal{C}_{ij}(t)\cap\tau(t)} \vec{n}(t)\; dS \;&=\; a^{\tau}_{ij}(t)\, \vec{n}^{\tau}(t) \\
&=\; \frac{3}{2} \iint_{T_1(t)} \vec{n}(t)\; dS \;=\; \frac{3}{2} a_1(t)\vec{n}^{\tau}(t) \\
&=\; 3 \iint_{T_2(t)} \vec{n}(t)\; dS \;=\; 3\, a_2(t)\vec{n}^{\tau}(t)
\end{aligned}$$

The application of the formula established for a triangular facet in Section 3 gives the contribution $\vec{\eta}^{\tau}_{ij}$ of the element τ to the mean normal associated to the segment $[i,j]$:

$$\vec{\eta}^{\tau}_{ij} = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \iint_{\partial\mathcal{C}_{ij}(t)\cap\tau(t)} \vec{n}(t)\; dS\; dt \;=\frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \vec{\mu}^{\tau}_{ij}(t)\; dt = \frac{1}{3} \Big\{ \vec{\mu}^{\tau,n+1}_{ij} + \vec{\mu}^{\tau,*}_{ij} + \vec{\mu}^{\tau,n}_{ij} \Big\} \tag{15}$$

With

$$\vec{\mu}^{\tau,n+1}_{ij} = \frac{3}{2} (\vec{X}^{n+1}_{G_1} - \vec{X}^{n+1}_{G}) \wedge (\vec{X}^{n+1}_{G_3} - \vec{X}^{n+1}_{G}), \qquad \vec{\mu}^{\tau,n}_{ij} = \frac{3}{2} (\vec{X}^{n}_{G_1} - \vec{X}^{n}_{G}) \wedge (\vec{X}^{n}_{G_3} - \vec{X}^{n}_{G})$$

$$\text{and} \quad 2\,\vec{\mu}^{\tau,*}_{ij} = \frac{3}{2} (\vec{X}^{n+1}_{G_1} - \vec{X}^{n+1}_{G}) \wedge (\vec{X}^{n}_{G_3} - \vec{X}^{n}_{G}) \;+\; \frac{3}{2} (\vec{X}^{n}_{G_1} - \vec{X}^{n}_{G}) \wedge (\vec{X}^{n+1}_{G_3} - \vec{X}^{n+1}_{G}).$$

and finally we obtain the mean value of the normal by summing the contribution of all tetrahedra having $[i,j]$ as an edge :

$$\vec{\eta}_{ij} = \sum_{\tau \in \mathcal{T}_{ij}} \vec{\eta}^{\tau}_{ij} \tag{16}$$

The contribution σ^{τ}_{ij} of the tetrahedra τ to the normal speed σ_{ij} associated with the edge $[i,j]$ is computed in the same way. For the two triangles T_1 and T_2, the assumption made on the mesh displacement implies that:

$$\iint_{T_{1,2}(t)} \vec{\kappa}\; dS = a_{1,2}(t)\; \vec{\kappa}(G_{T_{1,2}})$$

where $\vec{\kappa}(G_{T_1}), \vec{\kappa}(G_{T_2})$ are the (constant) speeds of the gravity center of T_1, T_2. Then

$$\begin{aligned}
\sigma^{\tau}_{ij} \;&=\frac{1}{\Delta t ||\vec{\eta}_{ij}||} \int_{t^n}^{t^{n+1}} \iint_{\partial\mathcal{C}_i(t)\cap\tau(t)} \vec{\kappa} \cdot \vec{n}^{\tau}(t)\; dS\; dt \\
&= \frac{1}{\Delta t ||\vec{\eta}_{ij}||} \int_{t^n}^{t^{n+1}} \Big\{ a_1(t)\vec{n}^{\tau}(t) \cdot \vec{\kappa}(G_{T_1}) + a_2(t)\vec{n}^{\tau}(t) \cdot \vec{\kappa}(G_{T_2}) \Big\}\; dt
\end{aligned}$$

Combining this relation with equation (15) we obtain:

$$\sigma_{ij}^{\tau} = \frac{1}{3||\vec{\eta}_{ij}||}\left\{ 2\vec{\kappa}(G_{T_1}) + \vec{\kappa}(G_{T_2})\right\} \cdot \vec{\eta}_{ij}^{\tau} = \frac{\vec{\kappa}_{ij}^{\tau} \cdot \vec{\eta}_{ij}^{\tau}}{||\vec{\eta}_{ij}||} \tag{17}$$

where the mean value velocity $\vec{\kappa}_{ij}^{\tau}$ is given by:

$$\vec{\kappa}_{ij}^{\tau} = \frac{2\vec{\kappa}(G_{T_1}) + \vec{\kappa}(G_{T_2}}{3} = \frac{1}{36}\Big(13\,\vec{\kappa}_i + 13\,\vec{\kappa}_j + 5\,\vec{\kappa}_k + 5\,\vec{\kappa}_l \Big)$$

and finally the mean value of the normal velocity is obtained by summing the contribution of all tetrahedra having $[i, j]$ as an edge:

$$\sigma_{ij} = \sum_{\tau \in \mathcal{T}_{ij}} \sigma_{ij}^{\tau} = \frac{1}{||\vec{\eta}_{ij}||} \sum_{\tau \in \mathcal{T}_{ij}} \vec{\kappa}_{ij}^{\tau} \cdot \vec{\eta}_{ij}^{\tau} \tag{18}$$

5 THE ROE SCHEME ON MOVING MESH

The previous scheme is now applied to the solution of the three-dimensional Euler's equations. In this case, the vector W and the flux $\vec{F}$ take the form:

$$W = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ E \end{pmatrix} \qquad \vec{F} = \begin{pmatrix} F(W) \\ G(W) \\ H(W) \end{pmatrix}$$

with

$$F(W) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u w \\ u(E+p) \end{pmatrix} \;;\; G(W) = \begin{pmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ \rho v w \\ v(E+p) \end{pmatrix} \;;\; H(W) = \begin{pmatrix} \rho w \\ \rho u w \\ \rho v w \\ \rho w^2 + p \\ w(E+p) \end{pmatrix}$$

The pressure p is related to the other variables by the perfect gases state law following:

$$p = (\gamma - 1)\left(E - \frac{1}{2}\rho(u^2 + v^2 + w^2) \right)$$

γ is the specific heat ratio.
In accordance with Godunov's method, the values W_{ij} and $\vec{F}_{ij}$ used to approximate W and the flux $\vec{F}$ on the hyper-surface $\mathcal{S}_{ij}$ will be obtained from the solution of a 1-D Riemann problem along the normal $\vec{\eta}_{ij}$ and between the two states W_j and W_i:

$$\begin{cases} \dfrac{\partial W}{\partial t} + \dfrac{\partial \mathcal{F}}{\partial n} & = 0 \\ W(n, t^n) & = \begin{cases} W_i \text{ for } n < 0 \\ W_j \text{ for } n > 0 \end{cases} \end{cases} \tag{19}$$

where n is the co-ordinate along $\vec{n} = \vec{\eta}_{ij}/||\vec{\eta}_{ij}||$ and $\mathcal{F} = \vec{F} \cdot \vec{n} = n^x F + n^y G + n^z H$. Using the notation $\vartheta = \vec{u} \cdot \vec{n}$ the flux $\mathcal{F}$ is given by:

$$\mathcal{F} = (\rho\vartheta, \quad \rho u\, \vartheta \;+\; p\, n^x, \quad \rho v\, \vartheta \;+\; p\, n^y, \quad \rho w\, \vartheta \;+\; p\, n^z, \quad \vartheta\,(E \;+\; p)) \tag{20}$$

The solution $\mathcal{W}^{exact}(n,t)$ of the Riemann problem (19) is constant on any curve defined by: $\dfrac{n}{t} = \sigma$. The approximate values of W and $\vec{F}.\vec{n}$ will then be chosen as the values of W and $\mathcal{F}$ on the straight line defined by the equation : $\dfrac{n}{t} = \sigma_{ij}$ where σ_{ij} is the average "slope" of the (hyper)-surface $\mathcal{S}_{ij}$ separating cells $\mathcal{Q}_i$ and $\mathcal{Q}_j$.
Of course the exact solution $\mathcal{W}^{exact}(\sigma)$ of the Riemann problem can be used to calculate the hyperbolic flux. However to reduce the computational cost, we use Roe's approximate Riemann solver [10] which is based on a linearization of the fluxes. In 1-D, the modification of Roe's scheme necessary to take into account the movement of the grid is detailed in [8]. The extension to the system (19)(20) is straightforward and we obtain:

$$\begin{aligned} \Phi_{ij} &= \vec{F}_{ij} \cdot \vec{\eta}_{ij} \;-\; \sigma_{ij}||\vec{\eta}_{ij}||W_{ij} \\ &\simeq \frac{||\vec{\eta}_{ij}||}{2} \Big\{ \mathcal{F}(W_i) + \mathcal{F}(W_j) \;-\; \sigma_{ij}(\,W_j + W_i\,) \;+\; |\tilde{A} - \sigma_{ij}\, I_d| \cdot (W_i - W_j) \Big\} \end{aligned} \tag{21}$$

where $\tilde{A} = A(\tilde{W})$ is the Roe's matrix, $\tilde{T}$ and $\tilde{\Lambda}$ respectively the matrix of the eigenvectors and eigenvalues of $\tilde{A}$, $|\tilde{A} - \sigma_{ij}\, I_d| = \tilde{T}^{-1}|\tilde{\Lambda} - \sigma_{ij} I_d|\tilde{T}$. The components of the vector $\tilde{W}$ known as Roe's average are defined by:

$$\tilde{u} = \frac{u_i\sqrt{\rho_i} + u_j\sqrt{\rho_j}}{\sqrt{\rho_i} + \sqrt{\rho_j}}, \qquad\qquad \tilde{v} = \frac{v_i\sqrt{\rho_i} + v_j\sqrt{\rho_j}}{\sqrt{\rho_i} + \sqrt{\rho_j}},$$

$$\tilde{w} = \frac{w_i\sqrt{\rho_i} + w_j\sqrt{\rho_j}}{\sqrt{\rho_i} + \sqrt{\rho_j}}, \qquad H = \frac{E + p}{\rho}, \qquad \text{and} \qquad \tilde{H} = \frac{H_i\sqrt{\rho_i} + H_j\sqrt{\rho_j}}{\sqrt{\rho_i} + \sqrt{\rho_j}}.$$

The previous scheme is first order accurate. It can easily be extended to second order accuracy using the MUSCL technique introduced by Van Leer [9] and extended to non structured mesh in [3]. In this case, the two states W_i^+ and W_j^- defining the Riemann problem are given by :

$$\begin{cases} W_i^+ &= W_i^n + \vec{P}_i \cdot \dfrac{\overrightarrow{ij}}{2} \\ W_j^- &= W_j^n - \vec{P}_j \cdot \dfrac{\overrightarrow{ij}}{2} \end{cases} \tag{22}$$

where $\vec{P}_i$ is an approximation of ∇W on the spatial cell $\mathcal{C}_i(t^n)$ defined as an average on the set of tetrahedra having i as a node of the constant values of ∇W given by P_1 interpolation on each simplex.

To complete the description of the numerical method, we note that it is sometimes useful to dispose of an implicit scheme for advancing in time the solution. We used here an implicit linearized scheme where the approximate linearization of the fluxes writes as:

$$\begin{aligned}
\Phi_{ij}(W^n, W^{n+1}) \;&= \tfrac{1}{2}\{ \quad A(W_i^n)\cdot W_i^{n+1} + A(W_j^n)\cdot W_j^{n+1} \\
&\qquad -\sigma_{ij}\, W_i^{n+1} - \sigma_{ij}\, W_j^{n+1} \\
&\qquad + |\tilde{A}(W_i^n, W_j^n) - \sigma_{ij} I_d| \cdot (W_i^{n+1} - W_j^{n+1}) \quad \}
\end{aligned} \tag{23}$$

Let $\delta W = W^{n+1} - W^n$, the previous equation can be re–cast in δ-scheme form:

$$\Phi_{ij}(W^n, W^{n+1}) = \Phi_{ij}(W^n, \delta W) + \Phi_{ij}(W^n, W^n) \tag{24}$$

In this case, the explicit part of the flux $\Phi_{ij}(W^n, W^n)$ will be calculated with a second order accurate method while the implicit part will be evaluated by first-order formula. Although this scheme is formally first order accurate, in the sequel the resulting scheme will be denoted by "implicit second order accurate" to distinguish it with a pure first-order implicit scheme. Indeed many experiments reveal that this scheme is much more accurate than a first order one.

Let us end with some indications on the numerical boundary conditions. We suppose that the whole boundary Γ of the spatial domain represent a solid wall. In the finite volume method used here, the boundary Γ is composed of an union of plane quadrilateral like (i, M_3, ,G_3 ,M_1) of Figure 3. Let Γ_i be equal to $\Gamma \cap \partial\mathcal{C}_i$, where i is a node belonging to the boundary. The geometrical parameters $\vec{n}_{\Gamma_i}$ and σ_{Γ_i} are obtain in the way that is described in section 3, taking into account the fact that the area of Γ_i is 1/3 the area of the triangle (i, j, k) (see figure 3). The slip condition on Γ_i then reads as follow:

$$\vec{v}_i \cdot \vec{n}_{\Gamma_i} = \sigma_{\Gamma_i} \tag{25}$$

This condition is imposed in a weak form by taking it into account in the computation of the flux crossing the boundary Γ_i. After appropriate linearization, the resulting boundary flux is given by:

$$\phi_{\Gamma_i}^{IL} = \phi_{\Gamma_i}(W^n) \; + \; \Delta t ||\vec{n}_{\Gamma_i}||\, A_{\Gamma_i}\, \delta W_i \tag{26}$$

where the explicit part ϕ_{Γ_i} is given by:

$$\phi_{\Gamma_i}^T(W^n) = \Delta t ||\vec{n}_{\Gamma_i}|| \left(0, \quad n_{\Gamma_i}^x p_i^n, \quad n_{\Gamma_i}^y p_i^n, \quad n_{\Gamma_i}^z p_i^n, \quad \sigma_{\Gamma_i} p_i^n\right) \tag{27}$$

and the implicit matrix $A_{\Gamma_i}(W^n)$ reads as:

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
\beta\dfrac{||\vec{v}_i^n||^2}{2} n_{\Gamma_i}^x & -u_i^n \beta n_{\Gamma_i}^x & -v_i^n \beta n_{\Gamma_i}^x & -w_i^n \beta n_{\Gamma_i}^x & \beta n_{\Gamma_i}^x \\
\beta\dfrac{||\vec{v}_i^n||^2}{2} n_{\Gamma_i}^y & -u_i^n \beta n_{\Gamma_i}^y & -v_i^n \beta n_{\Gamma_i}^y & -w_i^n \beta n_{\Gamma_i}^y & \beta n_{\Gamma_i}^y \\
\beta\dfrac{||\vec{v}_i^n||^2}{2} n_{\Gamma_i}^z & -u_i^n \beta n_{\Gamma_i}^z & -v_i^n \beta n_{\Gamma_i}^z & -w_i^n \beta n_{\Gamma_i}^z & \beta n_{\Gamma_i}^z \\
\dfrac{||\vec{v}_i^n||^2}{2} \sigma_{\Gamma_i} & -u_i^n \beta \sigma_{\Gamma_i} & -v_i^n \beta \sigma_{\Gamma_i} & -w_i^n \beta \sigma_{\Gamma_i} & \beta \sigma_{\Gamma_i}
\end{pmatrix} \tag{28}$$

with $\beta = (\gamma - 1)$.

6 NUMERICAL RESULTS

Solutions of half-Riemann problems

Let us consider a rectangular tube containing a fluid at rest, closed at the right end by a piston. We note $W_l = (\rho_l, u_l, p_l)$ the initial state of the fluid. At time $t = 0$, we move suddenly the piston with a constant velocity u_p. The position of the right extremity of the tube is then given by: $x_p(t) = x_p(0) + t\, u_p$. The exact solution of this (half)-Riemann problem consists of two states W_l and W_f connected by a rarefaction wave if $u_p > 0$ and by a shock wave if $u_p < 0$. For $u_p > 0$ (rarefaction wave) the state W_f is given by:

$$\begin{cases} \rho_f &= \rho_l \left(1 - \dfrac{(\gamma-1)\, u_p}{2} \sqrt{\dfrac{\rho_l}{\gamma p_l}}\right)^{\frac{2}{\gamma-1}} \\ u_f &= u_p \\ p_f &= p_l \left(1 + \dfrac{(\gamma-1)\, u_p}{2} \sqrt{\dfrac{\rho_l}{\gamma p_l}}\right)^{\frac{2\gamma}{\gamma-1}} \end{cases}$$

while if $u_p < 0$ (shock wave) and if we call σ the shock speed, the expression of W_f is given by:

$$\begin{cases} \rho_f &= \rho_l \dfrac{\sigma}{\sigma - u_p} \\ u_f &= u_p \\ p_f &= p_l + \sigma \rho_l u_p \end{cases} \qquad \mathit{with} \qquad \sigma = \frac{(1+\gamma)\, u_p}{4} \left(1 + \sqrt{1 + \frac{16 \gamma p_l}{\rho_l (1+\gamma)^2\, u_p^2}}\right)$$

Note that in the two cases, the piston plays the role of the contact discontinuity in the complete Riemann problem.

We begin to investigate the capabilities of the method by considering the previous two problems. The numerical code used here is a two dimensional one and in order to simulate a 1-D experiment, the mesh is 101 points in the x-direction by 3 points in the y-direction. A first-order scheme with a CFL number of 0.6 has been used. Initially, the pressure and density have constant values equal to unity while the velocity is zero everywhere. At time t=0, the piston (localized on the right in Figure 4 and Figure 5) is suddenly moved with a velocity $u_p = 1$ resulting in the development of a rarefaction wave in the cylinder or with a velocity $u_p = -1$ resulting in the creation of a shock wave. Figure 4 shows the velocity and density obtained for the rarefaction wave case while Figure 5 displays the results for the moving shock wave. For reference we have plotted in Figure 6 the same quantities in the case of the classical Sod shock tube problem where the contact discontinuity plays the role of the piston. It can be seen that the results obtained with the moving boundary algorithm is of the same quality than the ones obtained in the case of a fixed mesh. No oscillation are present in the shock and rarefaction waves profiles. Near the moving boundary (right boundary), a smearing of the contact discontinuity can be noticed in the results. A small overshot due to the boundary conditions can be seen on the density profiles at this location, this overshot is absent from the pressure and velocity (not displayed here) profiles, indicating that this problem is related to the contact discontinuity that remains attached to the piston in its displacement.

Isentropic compression

In this example, the experimental conditions are the same as previously but the piston is moved in a smooth manner in order to avoid the generation of shock waves and to keep the flow isentropic. The law giving the piston position is representative of the motion of a piston in an internal combustion engine and reads:

$$x_p(t) = -(l + \beta + \frac{\alpha}{2}) + \frac{\alpha}{2}\cos(\theta) + \sqrt{\beta^2 - \frac{\alpha^2}{4}\sin^2(\theta)} \tag{29}$$

where $\theta(t)$ ($\theta(t)$ is the Crank angle) equals $\theta(0) + 2\,\pi\,\omega t$, and the parameters l, α, β, $\theta(0)$ and ω take the values:

$$l = 0.132\,cm, \qquad \alpha = 8.9\,cm, \qquad \beta = 15.5\,cm, \qquad \theta(0) = -\pi, \qquad \omega = 2000\,tr/mn$$

An analytical solution of this problem can be found using a low Mach number approximation. Let us note $\dot{x}_p(t)$, $\ddot{x}_p(t)$ the velocity and the acceleration of the piston, then up to terms of order $\mathcal{M}a$ the solution of this problem writes:

$$\rho(\vec{x},t) = \rho_l \left(\frac{x_p(0)}{x_p(t)}\right), \qquad u(\vec{x},t) = x\left(\frac{\dot{x}_p(t)}{x_p(t)}\right),$$

$$p(\vec{x},t) = p_l \left(\frac{x_p(0)}{x_p(t)}\right)^{\gamma} + \frac{\rho_l\, x_p(0)\, \ddot{x}_p(t)}{2\, x_p^2(t)} \left(\frac{x_p^2(t)}{3} - x^2\right).$$

For this low Mach number experiment, the numerical results are obtained using the implicit second order scheme described in section 4. A (21×5) triangular mesh is used and the time step is constant corresponding in formula (29) to a variation $\delta\theta = \pi/360$. This corresponds to a CFL number in the interval [7,100]. Figure 7 displays the result at $\theta = -\pi/9$. It is seen that the agreement with the analytical solution is excellent: the velocity profile is linear and except some small oscillations near the boundaries, the quadratic behaviour of the dynamic pressure is well reproduced. These numerical results were obtained using a MUSCL approximation without the gradient limitation usually associated to TVD scheme for the capture of discontinuity waves. For this very smooth solution, gradient limitation have not been found necessary. Figure 8 show that first order implicit scheme failed to compute the quadratic perturbation of the pressure. Indeed it seems that for this type of flow where the spatial variation of pressure and density are very small, second order accuracy are necessary.

2-D model of a pitching NACA 0012 airfoil

We consider now an airfoil experiencing a periodic pitching between -1 and $+1$ degree. The angle of attack at time t is defined by $\theta(t) = \sin(\frac{\pi t}{2})$. The initial conditions are the converged solution of the steady problem with a null angle of attack. The numerical scheme is explicit and uses the MUSCL technique with gradient limitation to obtain second-order accuracy. A maximum CFL number of 0.5 and a mesh of 2280 vertices has been used. Figure 9 displays the results obtained after a half cycle calculation. We observe that the inertia of the movement induces several changes with respect to the steady state solutions. In particular, even for a null angle of attack, the solution is not symmetric as can be seen in Figure 9.b where the shock position on the extrados is shift with respect to the shock position on the intrados. Figure 9.d displays the concurent evolution of the angle of attack and lift defined by $\int_{(airfoil)} p\, n_y\, dl$. It can be seen that the two cuves are out of phase and that a negative lift can correspond to a positive angle of attack when this latter is diminishing while on the contrary a positive lift can be obtained for a negative angle of attack when this one is increasing.

Flow in a three-dimensional piston engine

We end this Section by some results obtained on a three-dimensional computation of the flow inside the cylinder of a piston engine. This type of application is characterized by a low or very low Mach number, complex geometries, intrinsically three-dimensional effects and strong deformation of the meshes due to the movement of the piston. The geometry under study is an industrial combustion chamber used in some Renault vehicles. Figures 10 and 11 displays two views of the chamber at differents time during the cycle. Only one half of the chamber is displayed as it is symmetric. The radius of the cylinder is 4 cm and at time t=0 (bottom dead center ; Figure 10) its height is 8.45 cm while at top dead center the height is reduced to 0.10 cm. The roof closing the cylinder head is 1.6 cm high. Figures 10 and 11 also show the mesh used in the computation. It consists of a 4032 nodes, 20166 elements non-structured tetraedral mesh built by the 3-D Delaunay-Voronoi algorithm by [5]. The motion of the piston is given by the law defined by (29) where now the parameters $l, \alpha, \beta, \theta(0), \omega$ have the value:

$$l = 0.100\, cm, \qquad \alpha = 8.35\, cm, \qquad \beta = 14.0\, cm, \qquad \theta(0) = -\pi, \qquad \omega = 1200\, tr/mn$$

In order to simulate the motion of the flow at the end of the induction stroke, at time $t = 0$ a recirculating motion of horizontal axis (called a tumble in engineer's litterature) is artificially created. The initial conditions are then described by the following relations:

$$\rho_l = 1.19\ 10^{-3}\, g/cm^3, \quad v_l = 0\, cm/s, \quad p_l = 10^6\, (cgs),$$

$$u_l(\vec{x}) = -K(r)\frac{z - 3.6}{r}, \qquad w_l(\vec{x}) = K(r)\frac{x}{r}$$

$$r = \sqrt{x^2 + (z - 3.6)^2} \qquad K(r) = C * \frac{\sin(0.14\pi r)\ \sin(0.16\pi r)}{(r + 10^{-3}\ )^{0.8}}$$

The corresponding velocity field on the symmetry plane is plotted in figure 12.
The computation uses the second order linearized implicit scheme described in Section 4 without gradient limitation. The time step is constant and equivalent to a variation of $\delta\theta = \frac{\pi}{180}$. The computation presented here extends from $\theta = -\pi$ (bottom dead center) to $\theta = \pi$. As can be seen in Figure 13, at $\theta = -\frac{\pi}{3}$, the piston motion have compress the initial vortex and the rotation axis has moved toward the right of the chamber. An explanation may be that the piston moving upward accelerates the left side of the initial recirculating zone while it decellerates the right side. Figure 14 displaying stream–surfaces (surfaces tangential to the velocity field) shows that the compressed tumble is accompanied by

an helicoidal motion from the side of the cylinder toward the symmetry plane. At top dead center ($\theta = 0.$) we notice an important change into the structure of the flow. The roof and the spherical part of the head chamber perturb significantly the tumble structure and change its rotation axis that becomes gradually more and more inclined as we move from the symmetry plane toward the spherical side of the chamber head (Figure 15). During the expansion phase after top dead center, the piston moves downward and thus accelerates now the right part of the flow. This destroys largely the tumble motion and no recirculating region can now be seen in the symmetry plane (Figures 17 and 18). Actually, a closer examination of the velocity field reveals that the flow has now an important horizontal component with a small recirculating motion of near vertical axis that moves slowly to the right of the chamber (Figure 19, 20 and 21). Finally we note that the flow intensity, measured by the maximum velocity norm, remains important during the computation : at $\theta = \pi/3$ the maximum velocity is still of the order of 14 m/s (note that at this time, the piston velocity is 5m/s).

Concluding remarks.

A numerical algorithm for the computation of three-dimensional flows on moving grids has been developped and tested. Due to the space-time formulation of Godunov's method, this technique incorporates the movement of the grid in a very simple way that automatically satisfy the geometrical conservation law. Beside that, one of the advantages of this technique is that its results from straightforward modifications of existing Eulerian codes. This has been examplified by the implementation of the technique into a three dimensional method designed to use non-structured tetrahedral meshes. Second-order and implicit versions of the method have also been built. Numerical examples have shown that when combined with unstructured meshes, this technique can be efficiently used for a broad range of applications from very subsonic flow to transonic flows. Works are presently undertaken to extend the method to true second-order (in time and space) accuracy and to add to it reggridding capabilities to deal with the appearance of large or small cells induced by the displacement of the boundaries.

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