Batch routing queuing networks with jump-over blocking

Richard J. Boucherie

UNITÉ DE RECHERCHE
INRIA-SOPHIA ANTIPOLIS

Institut National
de Recherche
en Informatique
et en Automatique

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BATCH ROUTING QUEUEING NETWORKS WITH JUMP-OVER BLOCKING

Richard J. BOUCHERIE

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Files d'attente avec routage par "batch" et blocage "jump-over"

Richard J. Boucherie
INRIA Centre Sophia Antipolis,
B.P. 93, 06902 Sophia Antipolis Cedex, France

Résumé

Cet article considère la distribution à l'équilibre d'un réseau de files d'attente avec blocage dans lequel plusieurs clients peuvent changer de station à chaque transition. Quand, pendant une transition, un client d'un groupe ne peut pas entrer dans une station, tous les clients de ce groupe choisissent une nouvelle destination. Cet article montre que la distribution à l'équilibre de ce réseau a une forme produit.

Batch routing queueing networks with jump-over blocking*

Richard J. Boucherie†
INRIA Centre Sophia Antipolis,
B.P. 93, 06902 Sophia Antipolis Cedex, France

Abstract

This paper shows that the equilibrium distribution of a queueing network with batch routing is of product-form if a batch which cannot enter the destination stations, for example as a consequence of capacity constraints, jumps over these stations and selects a new set of destination stations according to the routing probabilities, that is if also customers in the batch who arrive at a non-saturated station jump over that station.

Keywords: jump-over blocking, product-form, batch routing, queueing network.

1 Introduction

Jackson observed in [3] that the equilibrium distribution of a Jackson network in which the service-speed at a saturated station is set to infinity if a customer arrives at that station still possesses the product-form equilibrium distribution found for Jackson networks without capacity constraints. However, a rigorous proof of this phenomenon is not given in [3]. In [6] this notion of blocking is again discussed. In this reference the service-rate is not set to infinity, but a customer arriving at a saturated station jumps over the station and selects a new station according to the state-independent routing probabilities. In this case the mathematical problem related to infinite service speeds is avoided, and the product-form result is proven under the assumption that the state-independent routing probabilities are reversible. For Jackson-type queueing networks, in [2] the term *jump-over blocking* is introduced. In this reference a rigorous proof is given for the product-form equilibrium result under this blocking protocol for arbitrary routing probabilities (non-reversible routing).

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This paper generalizes these results to batch routing queueing networks. It will be shown that the equilibrium distribution of a batch routing queueing network with capacity constraints at the stations is of product-form if a batch of which at least one customer cannot enter the destination station selects new destination stations according to the batch routing probabilities, that is also the customers arriving at non-saturated stations jump over the station. Of course, this blocking protocol, referred to as the *complete jump-over blocking protocol*, may be very unrealistic in practical applications. However, the more realistic jump-over blocking protocol, in which only the customers who cannot enter a station jump over the station to choose a new destination station is shown not to preserve the product-form equilibrium distribution (cf. Remark 3.7). As a consequence, a closed form expression for the equilibrium distribution for a general queueing network under this partial jump-over blocking protocol should not be expected.

2 Model

Consider a continuous-time queueing network consisting of N queues or stations, labelled $1, 2, \ldots, N$, in which a single type of customers routes among the stations. *Assume* that the queueing network can be represented by a stable, regular, continuous-time Markov chain with state space $S \subseteq \mathbb{N}_0^N$. A state $\bar{n} = (n_1, \ldots, n_N)$ is a vector with components n_i, $i = 1, \ldots, N$, where n_i denotes the number of customers at station i, $i = 1, \ldots, N$. The transition rate from state $\bar{n}$ to state $\bar{n}'$ is denoted by $q(\bar{n}, \bar{n}')$. Such a transition may occur due to a group $\bar{g} = (g_1, \ldots, g_N)$ leaving the stations, that is due to g_i customers leaving station i, $i = 1, \ldots, N$, a group $\bar{g}' = (g_1', \ldots, g_N')$ entering the stations, while the customers $\bar{m} = (m_1, \ldots, m_N)$ remain at the stations. The transition rate for this particular transition is denoted by $q(\bar{g}, \bar{g}'; \bar{m})$. Observe that a transition $\bar{n} \to \bar{n}'$ may occur due to different groups $\bar{g}, \bar{g}'$ entering and leaving the stations. In particular, the relation between the two sets of transition rates defined above is given by

$$q(\bar{n}, \bar{n}') = \sum_{\{\bar{g}, \bar{g}', \bar{m} : \bar{m}+\bar{g}=\bar{n}, \bar{m}+\bar{g}'=\bar{n}'\}} q(\bar{g}, \bar{g}'; \bar{m}).$$

Furthermore, note that the transition rates defined above can be used to model both open and closed queueing networks (cf. [1]).

In the sequel, the restriction of the Markov chain to a set $V \subset S$ will be investigated. It will be *assumed* that the Markov chain is irreducible at V, and that there exists a unique equilibrium distribution $\pi = (\pi(\bar{n}),\ \bar{n} \in V)$ at V. Then this equilibrium distribution can be obtained as the unique solution to the global balance equations, that is π satisfies for all $\bar{n} \in V$

$$\pi(\bar{n}) \sum_{\bar{n}' \neq \bar{n}} q(\bar{n}, \bar{n}') = \sum_{\bar{n}' \neq \bar{n}} \pi(\bar{n}') q(\bar{n}', \bar{n}). \tag{2.1}$$

Note that the assumption that the Markov chain is irreducible is made only for simplicity. Without this assumption, a solution $m = (m(\bar{n}),\ \bar{n} \in V)$ to (2.1) is an invariant measure, which need not be unique or normalisable.

In [1] it is shown that a sufficient condition for π to be the unique *product-form* equilibrium distribution is that π satisfies the group-local-balance equations, that is π satisfies for all $\bar{m}$ and $\bar{n} = \bar{m} + \bar{g} \in V$

$$\pi(\bar{m}+\bar{g}) \sum_{\bar{g}' \neq \bar{g}} q(\bar{g}, \bar{g}'; \bar{m}) = \sum_{\bar{g}' \neq \bar{g}} \pi(\bar{m}+\bar{g}') q(\bar{g}', \bar{g}; \bar{m}). \tag{2.2}$$

Moreover, it is shown that group-local-balance is particularly suitable to analyse blocking phenomena in product-form queueing networks with batch movements.

Assume that the transition rates decompose in a service-part and a routing-part, that is assume that for all $\bar{g}, \bar{g}', \bar{m}$

$$q(\bar{g}, \bar{g}'; \bar{m}) = \frac{\psi(\bar{m})}{\phi(\bar{m}+\bar{g})} p(\bar{g}, \bar{g}'; \bar{m}). \tag{2.3}$$

Here $\frac{\psi(\bar{m})}{\phi(\bar{n})}$ is the state-dependent service-rate for service of a group $\bar{g}$ in state $\bar{n} = \bar{m} + \bar{g}$, chosen in accordance with the recent literature on product-form queueing networks (cf. [1], [4], [5], [7]), and shown to be the most general form available for the service-rates for which product-form results can be derived. Upon departure from the stations a group $\bar{g}$ routes to $\bar{g}'$ according to the state-dependent routing probabilities $p(\bar{g}, \bar{g}'; \bar{m})$. These routing probabilities will be studied in detail in Section 3.

The following result can immediately be concluded from [1], and will be the basis for the analysis of jump-over blocking.

Lemma 2.1 (Decomposition) *Assume that a positive solution $p = (p(\bar{n}),\ \bar{n} \in V)$ exists to*

$$p(\bar{m}+\bar{g}) \sum_{\bar{g}' \neq \bar{g} : \bar{m}+\bar{g}' \in V} p(\bar{g}, \bar{g}'; \bar{m}) = \sum_{\bar{g}' \neq \bar{g} : \bar{m}+\bar{g}' \in V} p(\bar{m}+\bar{g}') p(\bar{g}', \bar{g}; \bar{m}). \tag{2.4}$$

Then the Markov chain with transition rates (2.3) has a unique equilibrium distribution given by

$$\pi(\bar{n}) = B\phi(\bar{n})p(\bar{n}), \quad \bar{n} \in V, \tag{2.5}$$

where B is a normalising constant, defined as

$$B = \left[\sum_{\bar{n} \in V} \phi(\bar{n}) p(\bar{n}) \right]^{-1}.$$

Lemma 2.1 allows the routing and service parts of the Markov chain to be analysed separately. Since blocking effects are, by nature, routing effects, in the sequel the routing part will be studied only.

3 The complete jump-over blocking protocol

This section considers the *complete jump-over blocking protocol* for batch routing queueing networks. In principle, under the jump-over blocking protocol customers who cannot enter a station jump over that station to route to another station as if they have received service. Since customers in a batch are in general not independent, for this protocol to preserve the product-form equilibrium distribution it must be assumed that the whole batch $\bar{g}$ jumps over the stations, that is, if the g_i' customers arriving at station i cannot all enter station i, then all customers $\bar{g}'$ jump over the stations, also at stations $j \neq i$. Of course, this protocol need not be very realistic in a practical environment (also, cf. Remark 3.6).

Assume that the batch routing probabilities are state-independent, except for blocking phenomena, that is assume that customers who leave the stations and who are not blocked route among the stations according to the *stochastic* routing matrix P with $\bar{g}, \bar{g}'$-th entry $p(\bar{g}, \bar{g}')$. If not all customers of $\bar{g}'$ can be accepted, then the whole batch $\bar{g}'$ selects new destination stations according to P. For all $\bar{m}$ define the matrix $P(\bar{m})$ from P as

$$P(\bar{m})_{\bar{g},\bar{g}'} = p(\bar{g}, \bar{g}')\mathbf{1}(\bar{m} + \bar{g} \notin V).$$

$P(\bar{m})$ represents the routing probability of groups $\bar{g}$ who cannot enter the stations when $\bar{m}$ remain, that is the group $\bar{g}$ selects a new station according to $P(\bar{m})$. Obviously, $P(\bar{m})$ should not contain positive entries for $\bar{g}$ such that $\bar{m}+\bar{g} \in V$ since in this case the group $\bar{g}$ was accepted at the stations. Note that $P(\bar{m})$ has rows containing only 0's for all $\bar{g}$ such that $\bar{m} + \bar{g} \in V$, and row sums 1 if $\bar{m} + \bar{g} \notin V$. Then the routing probability of $\bar{g}$ to $\bar{g}'$ when $\bar{m}$ customers remain at the stations under the complete jump-over blocking protocol is given by, for $\bar{m}, \bar{g}, \bar{g}'$ such that $\bar{m} + \bar{g} \in V$, and $\bar{m} + \bar{g}' \in V$

$$\begin{aligned} p(\bar{g}, \bar{g}'; \bar{m}) &= P_{\bar{g},\bar{g}'} + (PP(\bar{m}))_{\bar{g},\bar{g}'} + (PP(\bar{m})^2)_{\bar{g},\bar{g}'} + \cdots \\ &= \sum_{n=0}^{\infty} (PP^n(\bar{m}))_{\bar{g},\bar{g}'}, \end{aligned} \tag{3.1}$$

where $P^0(\bar{m}) = I$, the identity matrix, and $P^n(\bar{m}) = P(\bar{m})^n$.

Assume that an up to normalisation unique positive solution y exists to the state-independent routing version of (2.4), that is assume that y satisfies for all $\bar{g}$:

$$y(\bar{g}) \sum_{\bar{g}'} p(\bar{g}, \bar{g}') = \sum_{\bar{g}'} y(\bar{g}')p(\bar{g}', \bar{g}), \tag{3.2}$$

and *assume* that a function p exists such that for all $\bar{m}$, and $\bar{g}, \bar{g}'$ such that $\bar{m}+\bar{g}, \bar{m}+\bar{g}' \in S$

$$\frac{p(\bar{m} + \bar{g})}{p(\bar{m} + \bar{g}')} = \frac{y(\bar{g})}{y(\bar{g}')}. \tag{3.3}$$

Observe that p is the solution to (2.4) for the system in which all stations have infinite capacity, i.e. for the system in which blocking does not occur. From Lemma 2.1, with $V = S$, we now immediately obtain that the Markov chain at S has a product-form invariant measure $m = (m(\bar{n}),\ \bar{n} \in S)$ given by

$$m(\bar{n}) = \phi(\bar{n})p(\bar{n}), \quad \bar{n} \in S.$$

The following theorem shows that p is also a solution to the state-dependent traffic equations at V under the complete jump-over blocking protocol. As a consequence the equilibrium distribution at V is the normalised version of the invariant measure at S.

Theorem 3.1 (Complete jump-over blocking protocol) *Consider the Markov chain with transition rates (2.3) under the complete jump-over blocking protocol, that is the routing probabilities have the form (3.1). Then the routing function $p(\bar{g},\bar{g}';\bar{m})$ defined in (3.1) is properly normalized, and the unique equilibrium distribution at V is*

$$\pi(\bar{n}) = B\phi(\bar{n})p(\bar{n}),$$

where p is given in (3.3).

Proof From Lemma 2.1 it is sufficient to prove that p satisfies the state-dependent traffic equations (2.4) for all $\bar{m}$, $i=1,\ldots,k(\bar{m})$, and $\bar{g},\bar{g}'$ such that $\bar{m}+\bar{g},\bar{m}+\bar{g}'\in V_i(\bar{m})$, where $V_i(\bar{m})$, $i=1,\ldots,k(\bar{m})$, are the *local irreducible sets* of the Markov chain at V with transition rates $q(\bar{g},\bar{g}';\bar{m})$ for fixed $\bar{m}$ (cf. [1]).

Choose arbitrary, but fixed $\bar{m}$ such that $V(\bar{m})=\cup_i V_i(\bar{m})\neq\emptyset$, that is choose $\bar{m}$ such that $\exists\ \bar{g}$ for which $\bar{m}+\bar{g}\in V$.

We will first show that for all $\bar{g},\bar{g}'$ it must be that

$$\lim_{n\to\infty} P^n(\bar{m})_{\bar{g},\bar{g}'} = 0. \tag{3.4}$$

To this end, observe that the rows of $P(\bar{m})$ for which $\bar{m}+\bar{g}\in V$ contain only 0's. Furthermore, since (3.2) possesses an up to normalisation unique positive solution, and $\exists\ \bar{g}'$ such that $\bar{m}+\bar{g}'\in V$, it must be that for all $\bar{g}$ such that $\bar{m}+\bar{g}\notin V$ there exists a k and a sequence $\bar{g}_0=\bar{g},\bar{g}_1,\bar{g}_2,\ldots,\bar{g}_k,\bar{g}_{k+1}=\bar{g}'$ such that $\bar{m}+\bar{g}_i\notin V$, $i=0,\ldots,k$, $p(\bar{g}_i,\bar{g}_{i+1})>0$, $i=0,\ldots,k$. As a consequence, the transition matrix $\tilde{P}(\bar{m})$ of the Markov chain with state space $\{\bar{g}|\bar{m}+\bar{g}\notin V\}\cup\{*\}$:

$$\begin{aligned}
\tilde{P}(\bar{m})_{\bar{g},\bar{g}'} &= P(\bar{m})_{\bar{g},\bar{g}'}\mathbf{1}(\bar{m}+\bar{g}'\notin V),\\
\tilde{P}(\bar{m})_{\bar{g},*} &= \sum_{\bar{g}':\bar{m}+\bar{g}'\in V} P(\bar{m})_{\bar{g},\bar{g}'},\\
\tilde{P}(\bar{m})_{*,*} &= 1,
\end{aligned}$$

is the transition matrix of an absorbing Markov chain. Since for $n\geq 2$ we have for $\bar{m}+\bar{g},\bar{m}+\bar{g}'\notin V$ that

$$P^n(\bar{m})_{\bar{g},\bar{g}'} = \tilde{P}^n(\bar{m})_{\bar{g},\bar{g}'},$$

$P^n(\bar{m})$ satisfies (3.4).

From (3.4) we obtain that the routing function $p(\bar{g},\bar{g}';\bar{m})$ is properly normalized. To this end, note that for all $n\geq 1$ and for all $\bar{g}$

$$\begin{aligned}
\sum_{\bar{g}':\bar{m}+\bar{g}'\in V}(PP^n(\bar{m}))_{\bar{g},\bar{g}'} &= \sum_{\bar{g}':\bar{m}+\bar{g}'\in V}\ \sum_{\bar{g}'':\bar{m}+\bar{g}''\notin V}(PP^{n-1}(\bar{m}))_{\bar{g},\bar{g}''}P(\bar{m})_{\bar{g}'',\bar{g}'}\\
&= \sum_{\bar{g}'':\bar{m}+\bar{g}''\notin V}(PP^{n-1}(\bar{m}))_{\bar{g},\bar{g}''}\left[1-\sum_{\bar{g}':\bar{m}+\bar{g}'\notin V}P(\bar{m})_{\bar{g}'',\bar{g}'}\right]\\
&= \sum_{\bar{g}'':\bar{m}+\bar{g}''\notin V}(PP^{n-1}(\bar{m}))_{\bar{g},\bar{g}''}-\sum_{\bar{g}':\bar{m}+\bar{g}'\notin V}(PP^n(\bar{m}))_{\bar{g},\bar{g}'}.
\end{aligned}$$

This gives

$$
\begin{aligned}
\sum_{\bar g':\bar m+\bar g'\in V} p(\bar g,\bar g';\bar m) &= \sum_{\bar g':\bar m+\bar g'\in V}\left\{P_{\bar g,\bar g'}+\sum_{n=1}^{\infty}(PP^n(\bar m))_{\bar g,\bar g'}\right\}\\
&= 1-\sum_{\bar g':\bar m+\bar g'\notin V} P_{\bar g,\bar g'}\\
&\quad +\sum_{n=1}^{\infty}\left\{\sum_{\bar g'':\bar m+\bar g''\notin V}(PP^{n-1}(\bar m))_{\bar g,\bar g''}-\sum_{\bar g':\bar m+\bar g'\notin V}(PP^n(\bar m))_{\bar g,\bar g'}\right\}\\
&= 1-\lim_{n\to\infty}\sum_{\bar g':\bar m+\bar g'\notin V}(PP^n(\bar m))_{\bar g,\bar g'}\\
&= 1.
\end{aligned}
$$

Insertion of this result in (2.4) gives that it is sufficient to prove that p satisfies for all $\bar m,\bar g$ such that $\bar m+\bar g\in V$

$$
p(\bar m+\bar g)=\sum_{\bar g':\bar m+\bar g'\in V} p(\bar m+\bar g')\sum_{n=0}^{\infty}(PP^n(\bar m))_{\bar g',\bar g}. \tag{3.5}
$$

Evaluation of the right-hand side gives:

$$
\begin{aligned}
&\sum_{\bar g':\bar m+\bar g'\in V} p(\bar m+\bar g')\sum_{n=0}^{\infty}(PP^n(\bar m))_{\bar g',\bar g}\\
&= \sum_{\bar g':\bar m+\bar g'\in V} p(\bar m+\bar g')\left[p(\bar g',\bar g)+\sum_{\bar g'':\bar m+\bar g''\notin V}\sum_{n=1}^{\infty}p(\bar g',\bar g'')P^n(\bar m)_{\bar g'',\bar g}\right]\\
&= \sum_{\bar g':\bar m+\bar g'\in V} p(\bar m+\bar g')p(\bar g',\bar g)\\
&\quad +\sum_{\bar g'':\bar m+\bar g''\notin V}\sum_{\bar g':\bar m+\bar g'\in V} p(\bar m+\bar g')p(\bar g',\bar g'')\sum_{n=1}^{\infty}P^n(\bar m)_{\bar g'',\bar g}\\
&= \sum_{\bar g':\bar m+\bar g'\in V} p(\bar m+\bar g')p(\bar g',\bar g)\\
&\quad +\sum_{\bar g'':\bar m+\bar g''\notin V}\left\{\sum_{\bar g':\bar m+\bar g'\in S} p(\bar m+\bar g')p(\bar g',\bar g'')-\sum_{\bar g':\bar m+\bar g'\notin V} p(\bar m+\bar g')p(\bar g',\bar g'')\right\}\sum_{n=1}^{\infty}P^n(\bar m)_{\bar g'',\bar g}\\
&= \sum_{\bar g':\bar m+\bar g'\in V} p(\bar m+\bar g')p(\bar g',\bar g)\\
&\quad +\sum_{\bar g'':\bar m+\bar g''\notin V}\left\{p(\bar m+\bar g'')-\sum_{\bar g':\bar m+\bar g'\notin V} p(\bar m+\bar g')p(\bar g',\bar g'')\right\}\sum_{n=1}^{\infty}P^n(\bar m)_{\bar g'',\bar g}\\
&= \sum_{\bar g':\bar m+\bar g'\in V} p(\bar m+\bar g')p(\bar g',\bar g)\\
&\quad +\sum_{\bar g'':\bar m+\bar g''\notin V} p(\bar m+\bar g'')P(\bar m)_{\bar g'',\bar g}+\sum_{\bar g'':\bar m+\bar g''\notin V} p(\bar m+\bar g'')\sum_{n=2}^{\infty}P^n(\bar m)_{\bar g'',\bar g}\\
&\qquad -\sum_{\bar g'':\bar m+\bar g''\notin V}\sum_{\bar g':\bar m+\bar g'\notin V} p(\bar m+\bar g')p(\bar g',\bar g'')\sum_{n=1}^{\infty}P^n(\bar m)_{\bar g'',\bar g}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{\bar g':\bar m+\bar g'\in S} p(\bar m+\bar g')p(\bar g',\bar g) \\
&= p(\bar m+\bar g),
\end{aligned}
$$

which completes the proof. □

Remark 3.2 (Proof) Observe that the last step of the proof can also be obtained as follows:
From (3.4) we obtain that

$$\sum_{n=0}^{\infty} P^n(\bar m)(I-P(\bar m)) = I - \lim_{n\to\infty} P^n(\bar m) = I,$$

which proves that $I-P(\bar m)$ is the inverse of $\sum_{n=0}^{\infty} P^n(\bar m)$. The state-dependent routing probabilities can now be written as

$$p(\bar g,\bar g';\bar m) = (P(I-P(\bar m))^{-1})_{\bar g,\bar g'}. \tag{3.6}$$

It is now sufficient to prove the following relation for all $\bar m+\bar g\in S$

$$p(\bar m+\bar g) = \sum_{\bar g':\bar m+\bar g'\in V} p(\bar m+\bar g')(P(I-P(\bar m))^{-1})_{\bar g',\bar g}.$$

Multiplication of this relation on the right-hand side by $I-P(\bar m)$ gives that it is sufficient to prove that

$$\sum_{\bar g} p(\bar m+\bar g)(I-P(\bar m))_{\bar g,\bar g^\star} = \sum_{\bar g}\sum_{\bar g':\bar m+\bar g'\in V} p(\bar m+\bar g')(P(I-P(\bar m))^{-1})_{\bar g',\bar g}(I-P(\bar m))_{\bar g,\bar g^\star}.$$

This relation is equivalent to

$$p(\bar m+\bar g^\star) - \sum_{\bar g:\bar m+\bar g\notin V} p(\bar m+\bar g)P(\bar m)_{\bar g,\bar g^\star} = \sum_{\bar g':\bar m+\bar g'\in V} p(\bar m+\bar g')p(\bar g',\bar g^\star),$$

which holds true as a consequence of (3.2). □

Remark 3.3 (Single changes, reference [2]) The result of Theorem 3.1 is a generalisation to batch routing queueing networks of the result obtained in [2] for Jackson-type queueing networks with single changes. The structure of the proof given above is very similar to the proof given in this reference. Note that the result of Theorem 3.1 is more general than the result obtained in [2]. This is obvious since batch routing is allowed, but also if single changes are allowed only, the result of Theorem 3.1 is more general, since the service-rate is more general, but more important, since the solution $p(\bar n)$ to the state-dependent traffic equations is more general. □

Remark 3.4 (Product-form result, intuitive justification) Observe that the product-form result (2.5) under the complete jump-over protocol is the same as the result under the *stop-protocol* (cf. [1]). This is not surprising since the jump-over protocol can intuitively be justified by setting the service-speed at saturated stations to infinity. If we just consider the ratio of the service-speed between saturated and non-saturated stations, then it is clear that increasing the service-speed to infinity at saturated stations is intuitively equivalent to decreasing the service-speed at non-saturated stations to zero.

Observe that the intuitive justification of the jump-over blocking protocol can serve as intuition only as increasing of the service-speed to infinity in general is not allowed (also cf. Remark 2.2 of [2]). □

Remark 3.5 (State space) Observe that there are no restrictions imposed on the state space V of the Markov chain. An obvious choice would be

$$V = \{\bar{n} \in S | n_i \leq B_i,\ i = 1, \dots, N\},$$

corresponding to capacity constraints at the stations. □

Remark 3.6 (Departures from one station) If the batch routing probabilities have the form

$$p(\bar{g}, \bar{g}') = p(ge_i, ge_j),$$

that is in each transition customers can depart from only one station, then the complete jump-over protocol seems to be a reasonable protocol, since in this case it may be argued that the customers in the batch g must remain together. Note that in general we will not have this structure. In this case the complete jump-over protocol can be seen as an alternative for the stop-protocol. □

Remark 3.7 (The partial jump-over blocking protocol) The more realistic jump-over blocking protocol in which only customers who cannot enter a station jump over the station to select a new destination station does not preserve the product-form for the equilibrium distribution. This can immediately be seen by observing a single station, or equivalently, a tandem line of N stations with capacity constraint at the last station, only. Since this partial jump-over behaviour will imply that only some customers in a batch select a new station, it will be assumed that customers in a batch route independently. Furthermore, it will be assumed that both the service-probability and the arrival probability are of Bernoulli type. For the tandem line with capacity constraint B_N at station N we then have

$$q(\bar{g}, \bar{g}'; \bar{m}) = \begin{cases} \lambda^k & \text{if } \bar{g} = 0,\ \bar{g}' = ke_1,\ k = 1, 2, \dots, \\ \mu_i^k & \text{if } \bar{g} = ke_i,\ \bar{g}' = ke_{i+1},\ k = 1, \dots, n_i,\ i = 1, \dots, N-2, N, \\ \mu_{N-1}^k & \text{if } \bar{g} = ke_{N-1},\ \bar{g}' = \min(ke_N, (B_N - m_N)e_N),\ k = 1, \dots, n_{N-1}, \end{cases}$$

were for simplicity it is assumed that a batch of customers can depart at only one station in each transition.

The global balance equations now take the form

$$\sum_{k=1}^{\infty} \pi(\bar{n})\lambda^k + \sum_{k=1}^{n_1} \pi(\bar{n})\mu_1^k + \sum_{i=2}^{N} \sum_{k=1}^{n_i} \pi(\bar{n})\mu_i^k$$
$$= \sum_{k=1}^{B_N - n_N} \pi(\bar{n} + ke_N)\mu_N^k + \sum_{k=1}^{n_1} \pi(\bar{n} - ke_1)\lambda^k + \sum_{i=2}^{N} \sum_{k=1}^{n_i} \pi(\bar{n} + ke_{i-1} - ke_i)\mu_{i-1}^k,$$

for $\bar{n}$ such that $n_N < B_N$, and if $n_N = B_N$,

$$\sum_{k=1}^{\infty} \pi(\bar{n})\lambda^k + \sum_{k=1}^{n_1} \pi(\bar{n})\mu_1^k + \sum_{i=2}^{N}\sum_{k=1}^{n_i} \pi(\bar{n})\mu_i^k$$
$$= \sum_{k=1}^{n_1} \pi(\bar{n} - ke_1)\lambda^k + \sum_{i=2}^{N-1}\sum_{k=1}^{n_i} \pi(\bar{n} + ke_{i-1} - ke_i)\mu_{i-1}^k + \sum_{k=1}^{B_N}\sum_{r\geq k} \pi(\bar{n} + re_{N-1} - ke_N)\mu_{N-1}^r.$$

Insertion of the product-form equilibrium distribution for $\bar{n}$ such that $n_N < B_N$ immediately shows that local balance is preserved at stations $1, \ldots, N$, and at the 'outside-station' for $n_N < B_N$. However, if $n_N = B_N$ the outside station does not satisfy local balance. In fact, the equilibrium distribution of this tandem line with partial jump-over blocking is

$$\pi(\bar{n}) = C \prod_{i=1}^{N} \left(\frac{\lambda}{\mu_i}\right)^{n_i} \left[1 + \frac{\lambda}{1-\lambda}\mathbf{1}(n_N = B_N)\right].$$

Since local balance is not preserved under the partial jump-over blocking protocol, for capacity constraints at other stations than the last station closed form expressions based on product-form distributions such as given above for the equilibrium distribution will not be easy to find. In fact, if the first station in a tandem line is the station with capacity constraint, then I was unable to find an equilibrium distribution based on a product-form distribution. □

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