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Contrôle de Flux dans un Systèmes d'Assemblage à Multi-Niveaux: cas mono-produit

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RESUME

Dans ce papier, nous nous intéressons au contrôle de flux dans un système d'assemblage dans lequel un produit fini unique est fabriqué à partir de composants par assemblages successifs. La capacité de chaque machine reste constante et la demande est connue sur l'horizon du problème. Les ruptures de stocks sont interdites. Le problème du contrôle de flux consiste à ajuster la production de chaque machine de manière à minimiser la somme des coûts engendrés par les stocks intermédiaires et le stock de produit fini. Nous établissons des propriétés des solutions optimales et en déduisons une solution analytique.

Flow Control of Multi-Level Assembly Systems with A Unique Finished Product *

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ABSTRACT

The paper addresses the flow control problem of muti-level assembly production systems in which a unique finished product is obtained from the initial components by successive assembly operations. The manufacturing process of the finished product can be represented as a tree-like graph. We assume that the production capacity of each machine is constant and that the demand is known over the whole problem horizon. Backlogging is not allowed. The flow control problem consists in adjusting the production of the machines in order to minimize the total cost incurred by holding components and the finished product. Properties of the optimal solutions are established. We then propose a simple analytical solution based on these properties.

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+ This work was done while the first author was visiting INRIA

1. INTRODUCTION

Consider a L-level assembly system manufacturing a unique finished product. Some initial components are first assembled to obtain level-L sub-assemblies. These sub-assemblies are then combined with components or other sub-assemblies to obtain level-(L-1) sub-assemblies and so on. Finally, the level-2 sub-assemblies are combined with some initial components or other sub-assemblies to obtain the finished product. We assume that each assembly operation is performed on a particular machine and the parts manufactured by this machine are stored in a particular buffer.

The rate at which a machine produces is called production rate. We assume that the maximal production rate (or capacity) of each machine is constant and that the demand for the finished product is known over the whole problem horizon. Backlogging is not allowed. The flow control problem consists in adjusting the production of the machines in order to minimize the total cost incurred by holding components and the finished product.

Due to its importance in the control of manufacturing systems, flow control has been widely addressed for various types of production systems both in the deterministic case and the stochastic case (see [1-13]). In most work, mathematical programming models were proposed. Linear programming methods were used to find optimal flow control policies in deterministic case and dynamic programming approaches were used in the stochastic case.

This paper addresses the flow control in multi-level assembly systems. Only the deterministic case is considered. The objective is to establish some properties of the optimal control policies. In particular, we establish some conditions under which the intermediate buffers are always empty, i.e. zero work-in-process. Conditions under which a particular buffer is always empty are also established. Finally, a simple algorithm is proposed for computing the optimal control policy.

The flow control of multi-level assembly systems is not a trivial problem. Some naive control policies are not optimal. To see this, let us consider two policies. Under the first policy, all machines produce at the same speed and the production is limited by the capacity of the bottleneck machine (i.e. the machine with the smallest capacity). Under the second policy, the control of the bottleneck machine is determined as in the single machine case and the other machines produce as late as possible in order to follow the production of the bottleneck machine and to meet the demand. Let us consider a two-level production line composed an upstream machine M1 with buffer B1 and a downstream machine M2 with buffer B2. It was proved in [8] that the first policy is optimal if the M2 is the bottleneck machine or B2 has the smallest inventory holding cost, and the second policy is optimal if the inventory holding cost is increasing from upstream to diwnstream. However, none of these two policies is optimal for some general multi-level assembly systems.

To the best of our knowledge, the results presented in this paper are new. As a matter of fact, analytical solutions were proposed in [3, 11] for the single machine case, in [7] for the transfer line case and in [8] for the two-level assembly system case. However, no analytical solution has been proposed for the multi-level assembly system case.

This paper is organized as follows. Section 2 describes the flow control model. Sections 3 and 4 present the results for the single machine case and the two-level assembly system case which are needed for solving the general case. Section 5 first addresses the demand feasibility, and some characteristics of the optimal control policies. It then proposes optimal control policies and establishes sufficient conditions under which the intermediate buffers are always empty. Section 6 presents a numerical example and Section 7 is a conclusion.

2. PROBLEM SETTING

We consider a multi-level assembly system in which the unique finished product is obtained from initial components by several consecutive assembly operations.

More precisely, we assume that the manufacturing process of the finished product can be described as a tree-like graph $G(\mathcal{N}, \mathcal{A})$. In this graph, $\mathcal{N} = \{0, 1, \ldots, n\}$ is the set of nodes which correspond to the set of machines $\{M_0, M_1, \ldots, M_n\}$ and their related output buffers $\{B_0, B_1, \ldots, B_n\}$. $\mathcal{A}$ is the set of arcs and the arc (i,j) belong to $\mathcal{A}$ if machine M_j needs the contents of buffer B_i to start the related assembly operation.

We assume that each intermediate buffer feeds exactly one downstream machine. This implies that the graph is an acyclic graph which one sink node in which each node except the sink node has exactly one successor. Figure 1 illustrates an assembly system of 12 machines.

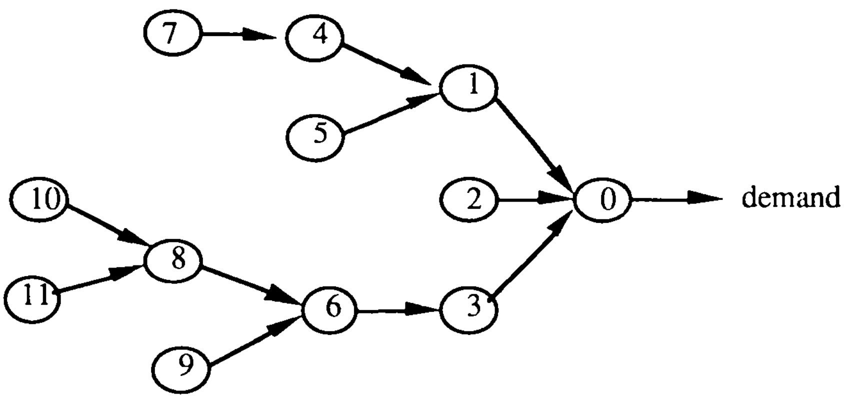

Figure 1 : An 12-machines assembly system

Without loss of generality, let 0 be the sink node which implies that B_0 is the finished product buffer and that M_0 is the final assembly machine.

We notice that there is a unique path from any node i to the sink node. In the following, the number of nodes containing in this path (including nodes i and 0) is called the level of the node i and is denoted as L(i). We call the machine M_i a machine of level L(i) and the buffer B_i a buffer of level L(i). We also call the assembly system an assembly system of L(G) levels with:

$$L(G) = \underset{i \in \mathcal{N}}{\mathrm{Max}}\, L(i). \tag{1}$$

Of course, the node 0 is the unique node of level 1. In the example illustrated in figure 1, there are one machine of level 1, three machines of level 2, three machines of level 3, three machines of level 4 and 2 machines of level 5. This system is an assembly system of 5 levels.

A discrete time model is used in this paper. Let H be the number of elementary periods considered. It is commonly called the problem horizon.

The following notations will be used throughout the paper :

$\sigma(i)$: unique successor of node i $\forall i \in \mathcal{N}\setminus\{0\}$,
$\pi(i)$: set of predecessors of node i $\forall i \in \mathcal{N}$,
$q(i)$: number of parts in buffer B_i needed to start an operation on machine $M_{\sigma(i)}$, $\forall i \in \mathcal{N}\setminus\{0\}$,
W_i : production capacity of machine M_i during each period,
c_i : cost incurred by keeping in B_i one unit of product at the end of a period,
$u_{i,t}$: production of M_i during period t,
$s_{i,t}$: buffer level of B_i at the end of period t,
d_t : demand during period t,
$\underline{W}$: minimal production capacity, i.e. $\underline{W} = \mathrm{Min}_{i\in\mathcal{N}} W_i$,
$\underline{c}$: minimal inventory holding cost, i.e. $\underline{c} = \mathrm{Min}_{i\in\mathcal{N}} c_i$.

Without loss of generality, we assume that each machine M_i needs exactly one part from each of its input buffers B_j for all $j \in \pi(i)$ to start an assembly operation, i.e.

$$q(i) = 1, \forall i \in \mathcal{N}\setminus\{0\}. \tag{2}$$

We assume that the demand $(d_1, \ldots, d_H)$ is known over the whole horizon. The control variables to be determined are $u_{i,t}$. The vector $[s_{0,t}, s_{1,t}, \ldots, s_{n,t}]$ describes the state of the system at the end of period t.

We further assume that the buffers are initially empty, i.e.

$$s_{i,0} = 0, \forall i \in \mathcal{N}. \tag{3}$$

The production capacity constraints can be expressed as follows :

$$0 \leq u_{i,t} \leq W_i, \forall i \in \mathcal{N}, \forall 1 \leq t \leq H. \tag{4}$$

The buffer levels can be determined as follows :

$$s_{i,t} = s_{i,t-1} + u_{i,t} - u_{\sigma(i),t}, \forall i \in \mathcal{N}\backslash\{0\}, \forall 1 \leq t \leq H, \tag{5}$$

and

$$s_{0,t} = s_{0,t-1} + u_{0,t} - d_t, \forall 1 \leq t \leq H. \tag{6}$$

Since the levels of the intermediate buffers are positive and since backlogging is not allowed, we have :

$$s_{i,t} \geq 0, \ \forall i \in \mathcal{N}, \forall 1 \leq t \leq H. \tag{7}$$

The total cost incurred by the buffer levels is given by :

$$\sum_{i \in \mathcal{N}} \sum_{t=1}^{H} c_i s_{i,t}.$$

The flow control problem consists in choosing $u_{i,t}$ $\forall i$ and $\forall t$ so as to

$$\text{minimize } \sum_{i \in \mathcal{N}} \sum_{t=1}^{H} c_i s_{i,t} \tag{8}$$

subject to the constraints (4) - (7), knowing that the initial state is given by (3).

In the following, we present the solution of the single machine case and the two-level case, from which we derive solutions of the general case.

3. SINGLE MACHINE CASE

This section summarizes some results presented in [7]. In the single machine case, we can neglect the index concerning the machines and the buffers. The problem becomes:

$$\text{minimize } \sum_{t=1}^{H} c\, s_t \tag{9}$$

subject to the following constraints :

$$0 \leq u_t \leq W, \forall 1 \leq t \leq H, \tag{10}$$

$$s_t = s_{t-1} + u_t - d_t, \forall 1 \le t \le H, \tag{11}$$
$$s_t \ge 0, \ \forall 1 \le t \le H, \tag{12}$$
$$s_0 = 0. \tag{13}$$

As it can be noticed, the optimal control policy is independent of the inventory holding cost c and we denote the problem (9) as **SMP(W, [d_t])** in the following.

Let us consider a mapping $[\xi_t] = \Phi(W,[d_t])$: IR x $IR^H \to IR^{H+1}$ defined as follows:

$$\xi_{t-1} = (\xi_t + d_t - W)^+, \quad \forall 1 \le t \le H \tag{14}$$

where

$$\xi_H = 0$$

Consider also another mapping $[\upsilon_t] = \Psi(W,[d_t])$: IR x $IR^H \to IR^H$ defined as follows :

$$\upsilon_t = \mathrm{Min}\{W, \xi_t + d_t\}, \quad \forall 1 \le t \le H \tag{15}$$

Theorem 1.

The demand is feasible iff $\xi_0 = 0$. In this case, $\Psi(W,[d_t])$ and $\Phi(W,[d_t])$ provide the optimal control policy and the optimal inventory trajectory respectively.

As it can be noticed, the optimal control policy consists of producing as late as possible, and this policy leads to lowest inventory levels over the whole horizon. Furthermore, this control policy has the following properties:

Property 1.

(a) $\xi_{t-1} = \mathrm{Max}\left\{0, \ \underset{t \le \tau \le H}{\mathrm{Max}}\left\{\sum_{s=t}^{\tau} d_s - (\tau - t + 1)W\right\}\right\}, \ \forall 1 \le t \le H$

(b) $\xi_t = \xi_{t-1} + \upsilon_t - d_t, \ \forall 1 \le t \le H$

(c) $0 \le \upsilon_t \le W, \qquad \forall 1 \le t \le H$

Property 2.

In case of feasible demand, let $[s_t]$ be the inventory trajectory of a feasible solution to the problem SMP(W, $[d_t]$). Then,

$$s_t \ge \xi_t, \ \forall 0 \le t \le H$$

From Property 1.a., it is obvious that the mapping $\Phi(W,[d_t])$ is non-increasing in the machine capacity W.

Property 3.

Consider two production capacities W_1 and W_2 with $W_1 \geq W_2 \geq 0$. The inventory level is always higher under the control $\Psi(W_2,[d_t])$ than under the control $\Psi(W_1,[d_t])$. Furthermore, the machine produces under $\Psi(W_2,[d_t])$ whenever it produces under $\Psi(W_1,[d_t])$. That is :

(a) $\Phi(W_1,[d_t]) \leq \Phi(W_2,[d_t])$

(b) $\upsilon_t^2 > 0$ whenever $\upsilon_t^1 > 0$

where $\left[\upsilon_t^1\right] = \Psi(W_1,[d_t])$ and $\left[\upsilon_t^2\right] = \Psi(W_2,[d_t])$

From Property 1, it can be easily shown that if the demand is always lower than the capacity, then the inventory is always empty and the machine follows the demand.

Property 4.

If $d_t \leq W$ for all t, then $\Phi(W,[d_t]) = [0]$ and $\Psi(W,[d_t]) = [d_t]$.

Property 5.

For any two positive numbers W_1 and W_2, it holds that.

$$\Psi(W_2,\Psi(W_1,[d_t])) = \Psi(\mathrm{Min}\{W_1,W_2\},[d_t])$$

4. TWO-LEVEL CASE

This section summarizes some results presented in [8]. Let us consider a two-level assembly system consisting of n upper level machines (M_1, M_2, ..., M_n) and a unique assembly machine (M_0). There are n component buffers (B_1, B_2, ..., B_n) which are located between the upper level machines and the assembly machine. B_0 is the buffer of the finshed product.

Without loss of generality, we assume that :

$$W_1 \leq W_2 \dots \leq W_n \tag{16}$$

The feasibility of the demand depends on the machine with the smallest production capacity called bottleneck machine. Theorem 2 claims that the demand is feasible iff it is feasible in the case of a single machine whose capacity is equal to the one of the bottleneck machine. The demand feasibility condition established in the single machine case can be used to check the demand feasibility of the general case.

Theorem 2.

The demand is feasible iff the single machine problem SMP($\underline{W}$, $[d_t]$) has at least one feasible solution.

In the following, we assume that the demand is feasible. Let us notice that whenever the control of the assembly machine is known, the optimal control of the upper level machines can be determined as in the single machine case with demand replaced by the the control of the assembly machine. That is :

Theorem 3.

Let $\left[u_{i,t}^*\right]$ be an optimal control policy and $\left[s_{i,t}^*\right]$ the related inventory trajectories. Then,

(a) $\left[u_{i,t}^*\right] = \Psi\left(W_i, \left[u_{0,t}^*\right]\right), \qquad \forall 1 \le i \le n$

(b) $\left[s_{i,t}^*\right] = \Phi\left(W_i, \left[u_{0,t}^*\right]\right), \qquad \forall 1 \le i \le n$

(c) $\left[s_{i,t}^*\right] \ge \left[s_{i+1,t}^*\right], \qquad \forall 1 \le i \le n-1$

In order to present the optimal control policy, let us distinguish two cases: (i) $W_0 \le W_1$ or $c_1 \ge c_0$; and (ii) $c_0 < c_1$ and $W_0 > W_1$. Theorems 4 and 5 give the optimal control policies in these two cases.

Theorem 4.
If $W_0 \le W_1$ or $c_0 \ge c_1$, then the optimal control policy is given by :

$$\left[u_{0,t}^*\right] = \left[u_{1,t}^*\right] = \ldots = \left[u_{n,t}^*\right] = \Psi\left(\underline{W}, [d_t]\right)$$

The component buffers are always empty under the optimal control policy, i.e.

$$s_{i,t}^* = 0, \qquad \forall 1 \le i \le n, \forall t$$

Let us consider the second case and define two quantities N* and W* as follows :

$$N^* = \mathrm{Max}\left\{N \in \{1,2,\ldots,n\} \Big/ \; W_N < W_0 \quad \text{and} \quad \sum_{i=1}^{N} c_i < c_0\right\} \tag{17}$$

$$W^* = \begin{cases} W_0, & \text{if } N^* = n; \\ \mathrm{Min}\{W_0, W_{N^*+1}\}, & \text{otherwise.} \end{cases} \tag{18}$$

Theorem 5.
If $c_0 > c_1$ and $W_0 > W_1$, the optimal control policy is given by :

$$\left[u_{i,t}^*\right] = \left[u_{0,t}^*\right] = \Psi\left(W^*, [d_t]\right), \qquad \forall N^* < i \le n$$

and

$$\left[u_{i,t}^*\right] = \Psi\left(W_i, [d_t]\right), \quad \forall 1 \le i \le N^*$$

where N* and W* are defined as in equations (17) and (18). The related inventory trajectories can be determined as follows :

$$\left[s^*_{0,t}\right] = \Phi(W^*,[d_t])$$
$$\left[s^*_{i,t}\right] = \Phi(W_i,[d_t]) - \Phi(W^*,[d_t]), \quad \forall 1 \le i \le N^*$$
$$\left[s^*_{i,t}\right] = [0], \qquad \forall N^* < i \le n$$

From Theorems 4 and 5, the following results can be easily proved.

Corollary 1.

$$\left[u^*_{i,t}\right] = \left[u^*_{0,t}\right] \text{ and } \quad \left[s^*_{i,t}\right] = [0], \quad \text{if } W_i \ge W_0$$

Corollary 2.

The criterion value of the optimal control policy is given by :

$$\sum_{i=0}^{n}\sum_{t=1}^{H} c_i s^*_{i,t} = \begin{cases} c_0 e^T \Phi(\underline{W},[d_t]), & \text{if } c_0 \le c_1 \text{ or } W_0 \le W_1; \\ \sum_{i=1}^{N^*} c_i e^T \Phi(W_i,[d_t]) + \left(c_0 - \sum_{i=1}^{N^*} c_i\right) e^T \Phi(W^*,[d_t]), & \text{otherwise} \end{cases}$$

where e^T is a (H+1)-dimension row vector (1, 1, ..., 1).

5. GENERAL CASE

In this section, we consider an assembly system of L levels with $L \ge 3$. We first address the demand feasibility and some characteristics of the optimal control policies. Based on these characteristics, we present a transformation which yields an equivalent but simple assembly system. An optimal control policy is then obtained by successively using this transformation. Finally, we establish some sufficient conditions under which the intermediate buffers are always empty.

5.1. Demand feasibility

Similarly to the two-level assembly system case, the feasibility of the demand depends on the machine with the smallest production capacity called bottleneck machine. Theorem 6 claims that the demand is feasible iff it is feasible in the case of single machine whose capacity is equal to the one of the bottleneck machine. The demand feasibility condition established in the single machine case can be used to check the demand feasibility of the general case.

Theorem 6.

The demand is feasible iff the single machine problem SMP($\underline{W}$, $[d_t]$) has at least one feasible solution.

The proof is similar to the one of Theorem 2. In the following, we assume that the demand is feasible.

5.2. Characteristics of the optimal control policies

The purpose of this section is to establish some characteristics of the optimal control policy which will be used to derive the optimal control policy.

First, let us notice that the optimal control of any machine without predecessor can be determined as in the single machine case with demand replaced by the the control of its downstream machine whenever the control of its downstream machine is known. That is :

Theorem 7.

Let $\left[u^*_{i,t}\right]$ be an optimal control policy and $\left[s^*_{i,t}\right]$ the related inventory trajectories. Then,

(a) $\left[u^*_{i,t}\right] = \Psi\left(W_i, \left[u^*_{\sigma(i),t}\right]\right)$, $\quad \forall$i such that $\pi(i) = \varnothing$

(b) $\left[s^*_{i,t}\right] = \Phi\left(W_i, \left[u^*_{\sigma(i),t}\right]\right)$, $\quad \forall$i such that $\pi(i) = \varnothing$

The proof is similar to that of Theorem 3. From Theorem 7 and Property 4, the following result can be easily proved.

Corollary 3.

$\left[u^*_{i,t}\right] = \left[u^*_{0,t}\right]$ and $\left[s^*_{i,t}\right] = [0]$, $\quad$ if $W_i \geq W_{\sigma(i)}$ and $\pi(i) = \varnothing$

Remark that since any node of level L(G) does not have any predecessor, Theorem 7 holds for all nodes of level L(G).

Let us consider now a node i of level (L(G)-1), i.e. L(i) = L(G) -1. Let $\{\nu(i,1), \nu(i,2), \ldots, \nu(i, |\pi(i)|)\}$ be the set of predecessors of node i arranged in such a way that :

$$W_{\nu(i,1)} \leq W_{\nu(i,2)} \cdots \leq W_{\nu(i,|\pi(i)|)} \tag{19}$$

Clearly, whenever the control for the machine $M_{\sigma(i)}$ is known, the optimal control policy for the machine M_i and its upstream machines M_j for all $j \in \pi(i)$ can be determined as in the two-level assembly system case with the demand replaced by the control of machine $M_{\sigma(i)}$. To present this characteristic, we distinguish two cases: (i) $W_i \leq W_{\nu(i,1)}$ or $c_i \leq c_{\nu(i,1)}$; and (ii) $c_i > c_{\nu(i,1)}$ and $W_i > W_{\nu(i,1)}$. From Theorems 4 and 5, Theorems 8 and 9, which give characteristics of the optimal control policies in these two cases, can be easily proved.

Theorem 8.
If $W_i \le W_{v(i,1)}$ or $c_i \le c_{v(i,1)}$, then the optimal control policy satisfies the following relations :

$$\left[u^*_{j,t}\right]=\left[u^*_{i,t}\right]=\Psi\left(\underline{W}(i),\left[u^*_{\sigma(i),t}\right]\right),\quad \forall j\in\pi(i)$$

where $\underline{W}(i) = \mathrm{Min}\{W_i, W_{v(i,1)}\}$. The optimal inventory trajectories are given by :

$$\left[s^*_{i,t}\right]=\Psi\left(\underline{W}(i),\left[u^*_{\sigma(i),t}\right]\right)\ \text{ and }\ \left[s^*_{j,t}\right]=[0],\quad \forall j\in\pi(i)$$

Let us consider the second case and define two numbers N(i) and W(i) as follows :

$$N(i)=\mathrm{Max}\left\{N\in\{1,2,\ldots,|\pi(i)|\}\Big/\ W_{\nu(i,N)}<W_i\ \text{ and }\ \sum_{j=1}^{N}c_{\nu(i,j)}<c_i\right\}\tag{20}$$

$$W(i)=\begin{cases}W_i, & \text{if } N(i)=|\pi(i)|;\\ \mathrm{Min}\left\{W_i,W_{\nu(i,N(i)+1)}\right\}, & \text{otherwise.}\end{cases}\tag{21}$$

Theorem 9.
If $c_i > c_{v(i,1)}$ and $W_i > W_{v(i,1)}$, the optimal control policy satisfies the following relations :

$$\left[u^*_{\nu(i,j),t}\right]=\left[u^*_{i,t}\right]=\Psi\left(W(i),\left[u^*_{\sigma(i),t}\right]\right),\qquad \forall N(i)<j\le|\pi(i)|$$

and

$$\left[u^*_{\nu(i,j),t}\right]=\Psi\left(W_{\nu(i,j)},\left[u^*_{\sigma(i),t}\right]\right),\quad \forall 1\le j\le N(i)$$

where N(i) and W(i) are defined as in equations (20) and (21). The related inventory trajectories are given by :

$$\left[s^*_{i,t}\right]=\Phi\left(W(i),\left[u^*_{\sigma(i),t}\right]\right)$$

$$\left[s^*_{\nu(i,j),t}\right]=\Phi\left(W_{\nu(i,j)},\left[u^*_{\sigma(i),t}\right]\right)-\Phi\left(W(i),\left[u^*_{\sigma(i),t}\right]\right),\qquad \forall 1\le j\le N(i)$$

$$\left[s^*_{\nu(i,j),t}\right]=[0],\qquad \forall N(i)<j\le|\pi(i)|$$

5.3. An equivalent assembly system
The purpose of this subsection is to show that an equivalent assembly system can be obtained by removing a node of level (L(G) - 1) as well as all its predecessors and by replacing them by a set of nodes of level (L(G) - 1).

For this purpose, for any node i of level (L(G) - 1), we define a mapping Γ(G, W, c, i) which gives a new assembly system (G°, W, c) = Γ(G, W, c, i). $G°=(\mathcal{N}°, \mathcal{A}°)$ is the structure, W is the vector of production capacity and c is the vector of the inventory holding costs of the new assembly system.

To present this mapping, we distinguish two cases: (i) $W_i \le W_{v(i,1)}$ or $c_i \le c_{v(i,1)}$; and (ii) $c_i > c_{v(i,1)}$ and $W_i > W_{v(i,1)}$.

If $W_i \le W_{v(i,1)}$ or $c_i \le c_{v(i,1)}$, the new assembly system is obtained by removing the nodes i and $\pi(i)$ and by adding an additional L(G) -1 node ω with capacity $\underline{W}(i)$ and holding cost c_i as follows :

$$\begin{aligned}
&\mathcal{N}^\circ = (\mathcal{N} - \{i\} - \pi(i)) \cup \{\omega\} \\
&\mathcal{A}^\circ = R(\mathcal{A}, \mathcal{N} - \{i\} - \pi(i)) \cup \{(\omega, \sigma(i))\} \\
&W_\omega = \underline{W}(i) \\
&c_\omega = c_i
\end{aligned}$$

where $R(\mathcal{A}, \mathcal{N} - \{i\} - \pi(i))$ is the restriction of the relation $\mathcal{A}$ to $\mathcal{N} - \{i\} - \pi(i)$.

If $c_i > c_{v(i,1)}$ and $W_i > W_{v(i,1)}$, the new assembly system is obtained by removing the nodes i and $\pi(i)$ and by adding N(i) + 1 additional L(G) -1 nodes as follows :

$$\begin{aligned}
&\mathcal{N}^\circ = (\mathcal{N} - \{i\} - \pi(i)) \cup \{\omega_0, \omega_1, \ldots, \omega_{N(i)}\} \\
&\mathcal{A}^\circ = R(\mathcal{A}, \mathcal{N} - \{i\} - \pi(i)) \cup \{(\omega_0, \sigma(i)), (\omega_1, \sigma(i)), \ldots, (\omega_{N(i)}, \sigma(i))\} \\
&W_{\omega_0} = W(i),\ W_{\omega_1} = W_{v(i,1)}, \ldots, W_{\omega_{N(i)}} = W_{v(i,N(i))} \\
&c_{\omega_0} = c_i - \sum_{j=1}^{N(i)} c_{v(i,j)},\ c_{\omega_1} = c_{v(i,1)}, \ldots, c_{\omega_{N(i)}} = c_{v(i,N(i))}
\end{aligned}$$

Theorem 10.
The two assembly systems (G, W, c) and (G°, W, c) are equivalent in the sense that the control is identical for all common nodes, i.e.

$$\left[u^*_{j,t}\right] = \left[u^\circ_{j,t}\right] \text{ and } \left[s^*_{j,t}\right] = \left[s^\circ_{j,t}\right], \qquad \forall j \in \mathcal{N}^\circ \cap \mathcal{N}$$

where $\left[u^*_{j,t}\right]$ and $\left[u^\circ_{j,t}\right]$ are the optimal control of the original and new assembly systems respectively. Moreover, the criterion value of the optimal control policy is the same for both systems.

Proof :
Only the proof for the case in which $c_i > c_{v(i,1)}$ and $W_i > W_{v(i,1)}$ is given and the proof for the case in which $W_i \le W_{v(i,1)}$ or $c_i \le c_{v(i,1)}$.

First, let us notice that the inventory trajectories of the common nodes for all $j \in \mathcal{N}^\circ \cap \mathcal{N}$ only depend on the control of these nodes. As a result, let us consider a partial control $[u_{j,t}]$ and the related inventory trajectories $[s_{j,t}]$ for all $j \in \mathcal{N}^\circ \cap \mathcal{N}$ such that

$$0 \le u_{j,t} \le W_j \text{ and } s_{j,t} \ge 0 \quad \forall j \in \mathcal{N}^\circ \cap \mathcal{N}, \forall t \tag{22}$$

Consider also two controls defined as follows :

$$\left[u^*_{j,t}\right] = \left[u^\circ_{j,t}\right] = \left[u_{j,t}\right], \qquad \forall j \in \mathcal{N}^\circ \cap \mathcal{N}$$
$$\left[u^\circ_{j,t}\right] = \Psi\left(W_j, \left[u_{\sigma(i),t}\right]\right), \qquad \forall j \in \mathcal{N}^\circ \backslash \mathcal{N}$$
$$\left[u^*_{\nu(i,j),t}\right] = \left[u^*_{i,t}\right] = \Psi\left(W(i), \left[u_{\sigma(i),t}\right]\right), \qquad \forall N(i) < j \leq |\pi(i)|$$
$$\left[u^*_{\nu(i,j),t}\right] = \Psi\left(W_{\nu(i,j)}, \left[u_{\sigma(i),t}\right]\right), \quad \forall 1 \leq j \leq N(i)$$

Thanks to assumption (22) and Theorem 1, $\left[u^\circ_{j,t}\right]$ is a feasible control for the new assembly system iff each single machine problem SMP(W_j, $[u_{\sigma(i),t}]$) for all $j \in \mathcal{N}^\circ \backslash \mathcal{N}$ has at least one feasible solution. Since $\underset{j \in \mathcal{N}^\circ \backslash \mathcal{N}}{\mathrm{Min}} W_j = W_{\nu(i,1)}$, $\left[u^\circ_{j,t}\right]$ is a feasible control for the new assembly system iff the single machine problem SMP($W_{\nu(i,1)}$, $[u_{\sigma(i),t}]$) has at least one feasible solution.

Meanwhile, thanks to assumption (22) and Theorems 9 and 2, $\left[u^*_{j,t}\right]$ is a feasible control for the original assembly system iff the single machine problem SMP($W_{\nu(i,1)}$, $[u_{\sigma(i),t}]$) has at least one feasible solution. This implies that $\left[u^*_{j,t}\right]$ is a feasible control for the original assembly system iff $\left[u^\circ_{j,t}\right]$ is a feasible control for the new assembly system.

Furthermore, from Theorems 7 and 9, $\left[u^\circ_{j,t}\right]$ is the optimal control for the new assembly system and $\left[u^*_{j,t}\right]$ is the optimal control for the original assembly system if the control for the common nodes is given. Hereafter we conclude the proof by showing that they have the same criterion value.

First, from assumption (22) and Corollary 2, we have :

$$\sum_{j \in \mathcal{N}} \sum_{t=1}^{H} c_j s^*_{j,t} = \sum_{j \in \mathcal{N}^\circ \cap \mathcal{N}} \sum_{t=1}^{H} c_j s_{j,t} + \sum_{j=1}^{N(i)} c_{\nu(i,j)} e^T \Phi\left(W_{\nu(i,j)}, \left[u_{\sigma(i),t}\right]\right)$$
$$+ \left(c_i - \sum_{j=1}^{N(i)} c_{\nu(i,j)}\right) e^T \Phi\left(W(i), \left[u_{\sigma(i),t}\right]\right)$$

where e^T is a (H+1)-dimension row vector (1, 1, ..., 1).

From assumption (22) and Theorem 7, we have

$$\sum_{j\in\mathcal{N}}\sum_{t=1}^{H} c_j \overset{\circ}{s}_{j,t} = \sum_{j\in\mathcal{N}^\circ\cap\mathcal{N}}\sum_{t=1}^{H} c_j s_{j,t} + \sum_{j\in\mathcal{N}^\circ\setminus\mathcal{N}}\sum_{t=1}^{H} c_j \overset{\circ}{s}_{j,t}$$
$$= \sum_{j\in\mathcal{N}^\circ\cap\mathcal{N}}\sum_{t=1}^{H} c_i s_{j,t} + \sum_{j=1}^{N(i)} c_{\nu(i,j)} e^T \Phi\left(W_{\nu(i,j)}, \left[u_{\sigma(i),t}\right]\right)$$
$$+\left(c_i - \sum_{j=1}^{N(i)} c_{\nu(i,j)}\right) e^T \Phi\left(W(i), \left[u_{\sigma(i),t}\right]\right)$$

Q.E.D.

5.4. Solving the multi-level assembly systems

In this subsection, we prove that the optimal control has the following form :

$$\left[u^*_{i,t}\right] = \Psi\left(\Omega_i, \left[u^*_{\sigma(i),t}\right]\right) \tag{23}$$

where Ω_i is called the effective capacity of node i. We also prove that Ω_i is not greater than W_i and is equal to the production capacity of one of its upstream nodes (immediate or not).

From Theorems 7, 8 and 9, these claims are clearly true for all nodes of levels L(G) and (L(G) -1). Furthermore, the effective capacity of any node of level L(G) is equal to its real capacity, i.e.

$$\Omega_i = W_i, \qquad \text{if } L(i) = L(G) \tag{24}$$

For the nodes of level (L(G) -1), the effective capacity is determined as follows :

$$\Omega_i = \begin{cases} W_i, & \text{if } \pi(i) = \varnothing \\ \underline{W}(i), & \text{if } \pi(i) \neq \varnothing \text{ and } \left(W_i \leq W_{\nu(i,1)} \text{ or } c_i \leq c_{\nu(i,1)}\right) \\ W(i), & \text{if } \pi(i) \neq \varnothing \text{ and } \left(W_i > W_{\nu(i,1)} \text{ and } c_i > c_{\nu(i,1)}\right) \end{cases} \tag{25}$$

Starting from the original assembly system, let us apply the mapping $\Gamma(G, W, c, i)$ to all nodes i of level (L(G) -1) and let (G^1, W, c) be the new assembly system. Clearly, (G^1, W, c) is an assembly system of (L(G) -1) levels, i.e. $L(G^1) = L(G) - 1$.

By iteratively applying Theorem 10, for all common nodes, the optimal control of this new assembly system is identical to that of the original assembly system. Applying Theorems 7, 8 and 9 to this new assembly system, we prove that the optimal control for all nodes of level (L(G) -2) has the form defined in equation (23) and that the effective capacity Ω_i is not greater than W_i and is equal to the production capacity of one of its upstream nodes (immediate or not). The exact value of the effective capacity can be obtained by applying equation (25) to the new assembly system.

Following the same way, Theorem 11 can be easily proved.

Theorem 11.
Let (G^k, W, c) be the assembly system obtained from the assembly system (G^{k-1}, W, c) by applying the mapping $\Gamma(\bullet)$ to all nodes i of level $(L(G^{k-1}) -1)$ where $G^0 = G$. The optimal control policy can be determined as follows :

$$\left[u^*_{i,t}\right] = \Psi\left(\Omega_i, \left[u^*_{\sigma(i),t}\right]\right), \qquad \forall i \in \mathcal{N} \setminus \{0\}$$

and

$$\left[u^*_{0,t}\right] = \Psi(\Omega_0, [d_t])$$

where Ω_i is given by equation (24) for all node i with $L(i) = L(G)$; otherwise, Ω_i is obtained by applying equation (25) to the assembly system (G^k, W, c) with $k = L(G) -1 - L(i)$.

From this theorem, the following corollary can be easily proved.

Corollary 4.
The optimal inventory trajectories can be determined as follows :

$$\left[s^*_{i,t}\right] = \Phi\left(\Omega_i, \left[u^*_{\sigma(i),t}\right]\right), \qquad \forall i \in \mathcal{N} \setminus \{0\}$$

and

$$\left[s^*_{0,t}\right] = \Phi(\Omega_0, [d_t])$$

Furthermore,

$$\left[s^*_{i,t}\right] = 0, \qquad \text{if } \Omega_i \geq \Omega_{\sigma(i)}$$

Combining Theorem 11 and Property 5, it can be shown that the optimal control policy can be rewritten in a simpler form.

Corollary 5.
The optimal control policy of a multi-level assembly system is given by :

$$\left[u^*_{i,t}\right] = \Psi\left(\operatorname*{Min}_{j \in P(i,0)} \Omega_j, [d_t]\right), \quad \forall i \in \mathcal{N} \setminus \{0\}$$

and

$$\left[u^*_{0,t}\right] = \Psi(\Omega_0, [d_t])$$

where P(i, 0) is the unique elementary path from i to 0.

Finally, we consider two particular cases in which the intermediate buffers are always empty under the optimal control policy.

Theorem 12.

If $W_0 = \underline{W}$ or $c_0 = \underline{c}$, then the optimal control is given by :

$$\left[u^*_{i,t}\right] = \Psi(\underline{W},[d_t]), \quad \forall i \in \mathcal{N}$$

and

$$\left[s^*_{0,t}\right] = \Phi(\underline{W},[d_t]) \text{ and } \left[s^*_{i,t}\right] = [0], \quad \forall i \in \mathcal{N} \setminus \{0\}$$

Proof :

The control $\left[u^*_{i,t}\right]$ is trivially a feasible control and the intermediate buffers are always empty, i.e.

$$\left[s^*_{0,t}\right] = \Phi(\underline{W},[d_t]) \text{ and } \left[s^*_{i,t}\right] = [0], \quad \forall i \in \mathcal{N} \setminus \{0\}$$

Let us prove the optimality. Consider a feasible control $[u_{i,t}]$ and its related inventory trajectories $[s_{i,t}]$.

Consider first the case in which $W_0 = \underline{W}$. The control of machine M_0, i.e. $[u_{0,t}]$ is a feasible control of the single machine case SMP(W_0, $[d_t]$). From Theorem 1, we have :

$$s_{0,t} \geq s^*_{0,t}, \quad \forall t$$

Furthermore, since the inventory levels $s_{i,t}$ are nonnegative, we have :

$$s_{i,t} \geq s^*_{i,t}, \quad \forall 1 \leq i \leq n, \forall t$$

The above two relations imply the optimality of $\left[u^*_{i,t}\right]$.

Consider now the case in which $c_0 = \underline{c}$ and $W_0 > \underline{W}$. The control of machine M_m, i.e. $[u_{m,t}]$ is a feasible control of the single machine case SMP($\underline{W}$, $[d_t]$) where $W_m = \underline{W}$. From Theorem 1, we have :

$$\sum_{i \in P(m,0)} s_{i,t} \geq \sum_{i \in P(m,0)} s^*_{i,t} = s^*_{0,t}, \quad \forall t$$

where P(m, 0) is the path from node m to node 0. Since $c_0 = \underline{c}$,

$$\sum_{i \in P(m,0)} c_i s_{i,t} \geq \sum_{i \in P(m,0)} c_i s^*_{i,t} = c_0 s^*_{0,t}, \quad \forall t$$

Furthermore, since the inventory levels $s_{i,t}$ are nonnegative, then :

$$s_{i,t} \geq s^*_{i,t}, \quad \forall i \notin P(m,0), \forall t$$

The above two relations imply the optimality of $\left[u^*_{i,t}\right]$.

Q.E.D.

6. A NUMERICAL EXAMPLE

Consider the example introduced in Figure 1. The production capacity and the inventory holding costs are given in Table 1.

i	0	1	2	3	4	5	6	7	8	9	10	11
W_i	10	11	10	11	11	11	8	8	5	10	8	9
c_i	10	10	7	7	7	9	10	1	3	9	5	9

Table 1: Production capacity and inventory holding cost

We consider 10 elementary periods, i.e. H = 10. The demand is $[d_t]$ = [2, 1, 3, 3, 7, 2, 2, 10, 12, 4]. The minimal production capacity is equal to 5 and the demand is feasible.

It is a assembly system of 5 levels, L(G) = 5. There are two nodes of level 5, namely nodes 10 and 11, and their effective capacity is given by :

$\Omega_{10} = 8$ and $\Omega_{11} = 9$.

There are three nodes of level 4, namely nodes 7, 8 and 9, and their effective capacity is given by :

$\Omega_7 = 8, \Omega_8 = 5$ and $\Omega_9 = 10$.

Applying the mapping G(•) to node 8, we obtain an equivalent assembly system (G^1, W, c) illustrated in figure 2. A new node (node 12) is introduced and its capacity and holding cost are given by :

$W_{12} = 5$ and $c_{12} = 3$, for node 12.

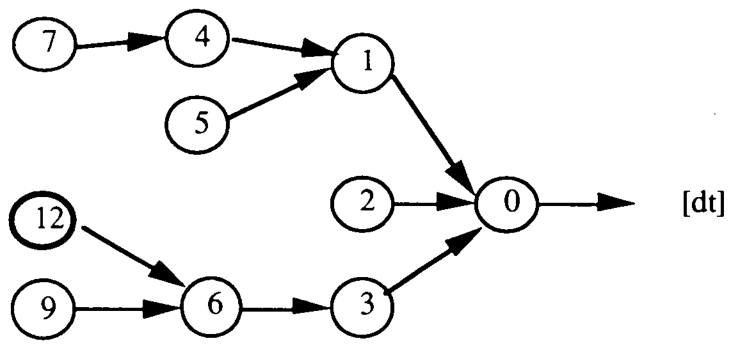

Figure 2 : An equivalent assembly system (G^1, W, c)

Consider now the nodes of level 3. These nodes are 4, 5 and 6. Applying equation (25) to this new assembly system, the effective capacity is given by :

$\Omega_4 = 11, \Omega_5 = 11$ and $\Omega_6 = 8$.

Applying the mapping G(•) to nodes 4 and 6, we obtain an equivalent assembly system (G^2, W, c) illustrated in figure 3. Four new nodes (nodes 13, 14, 15 and 16) are introduced and their capacity and holding costs are given by :

$W_{13} = 8$ and $c_{13} = 1$, for node 13,

$W_{14} = 11$ and $c_{14} = 6$, for node 14,

$W_{15} = 5$ and $c_{15} = 3$, for node 15,

$W_{16} = 8$ and $c_{16} = 7$, for node 16.

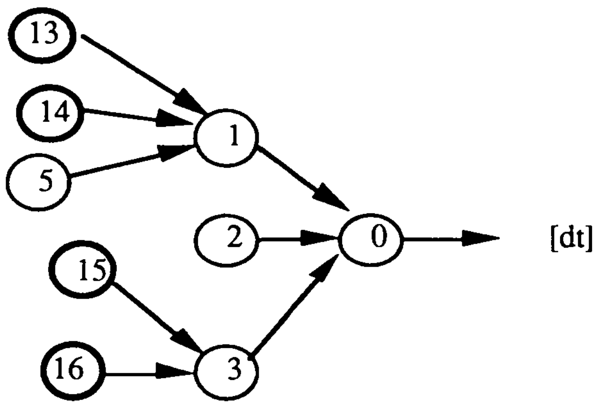

Figure 3 : An equivalent assembly system (G^2, W, c)

Consider now the nodes of level 2, i.e. nodes 1, 2 and 3. Applying equation (25) to this new assembly system, the effective capacity is given by :

$\Omega_1 = 11, \Omega_2 = 10$ and $\Omega_3 = 8$.

Applying the mapping $\Gamma(\bullet)$ to nodes 1 and 3, we obtain an equivalent assembly system (G^3, W, c) illustrated in figure 4. Four new nodes (nodes 17, 18, 19 and 20) are introduced and their capacity and holding costs are given by :

$W_{17} = 8$ and $c_{13} = 1$, for node 17,
$W_{18} = 11$ and $c_{18} = 9$, for node 18,
$W_{19} = 5$ and $c_{19} = 3$, for node 19,
$W_{20} = 8$ and $c_{20} = 6$, for node 20.

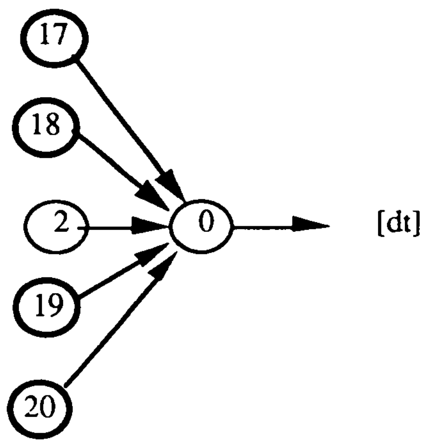

Figure 4 : An equivalent assembly system (G^3, W, c)

Consider now the sink node 0. Applying equation (25) to this new assembly system, the effective capacity is given by :

$\Omega_0 = 8$.

Thus, the optimal control and the related inventory trajectories are given by :

$$\left[u^*_{i,t}\right] = \Psi(8,[d_t]), \qquad \forall i \in \{0,1,2,3,4,5,6,7,9\}$$

$$\left[u^*_{i,t}\right] = \Psi(5,[d_t]), \qquad \forall i \in \{8,10,11\}$$

$$\left[s^*_{0,t}\right] = \Phi(8,[d_t])$$

$$\left[s^*_{8,t}\right] = \Phi(5,[d_t]) - \Phi(8,[d_t])$$

$$\left[s^*_{i,t}\right] = [0], \qquad \forall i \in \{1,2,3,4,5,6,7,9,10,11\}$$

The optimal control policy is given in Table 2.

t\i	0	8	10	11	1	2	3	4	5	6	7	9
0	0; 0	0; 0	0; 0	0; 0	0; 0	0; 0	0; 0	0; 0	0; 0	0; 0	0; 0	0;0
1	0; 2	0; 2	0; 2	0; 2	0; 2	0; 2	0; 2	0; 2	0; 2	0; 2	0; 2	0; 2
2	0; 1	4; 5	0; 5	0; 5	0; 1	0; 1	0; 1	0; 1	0; 1	0; 1	0; 1	0; 1
3	0; 3	6; 5	0; 5	0; 5	0; 3	0; 3	0; 3	0; 3	0; 3	0; 3	0; 3	0; 3
4	0; 3	8; 5	0; 5	0; 5	0; 3	0; 3	0; 3	0; 3	0; 3	0; 3	0; 3	0; 3
5	0; 7	6; 5	0; 5	0; 5	0; 7	0; 7	0; 7	0; 7	0; 7	0; 7	0; 7	0; 7
6	0; 2	9; 5	0; 5	0; 5	0; 2	0; 2	0; 2	0; 2	0; 2	0; 2	0; 2	0; 2
7	6; 8	6; 5	0; 5	0; 5	0; 8	0; 8	0; 8	0; 8	0; 8	0; 8	0; 8	0; 8
8	4; 8	3; 5	0; 5	0; 5	0; 8	0; 8	0; 8	0; 8	0; 8	0; 8	0; 8	0; 8
9	0; 8	0; 5	0; 5	0; 5	0; 8	0; 8	0; 8	0; 8	0; 8	0; 8	0; 8	0; 8
10	0; 4	0; 4	0; 4	0; 4	0; 4	0; 4	0; 4	0; 4	0; 4	0; 4	0; 4	0; 4

Table 2: The optimal control policy ($s_{i,t}$, $u_{i,t}$)

7. CONCLUSION

In this paper, we have addressed the flow control problem of muti-level assembly production systems. Properties of the optimal solutions were proposed. Based on these properties, we proposed a simple analytical solution. Sufficient conditions under which the intermediate buffers are always empty were established.

Future research work consists in extending the results to other manufacturing systems, manufacturing systems with machine set-ups for example.

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