A Genuinely multidimensional Riemann solver

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B.P. 93
06902 Sophia-Antipolis
France

Rapports de Recherche

N°1859

Programme 6

*Calcul scientifique, Modélisation
et Logiciel numérique*

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Février 1993

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UN SOLVEUR DE RIEMANN VRAIMENT MULTIDIMENSIONNEL

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Résumé

Une grande partie des méthodes modernes d'intégration des équations multidimensionnelles de la Mécanique des Fluides compressible reposent sur schémas schémas de type TVD utilisant un solveur de Riemann. Ces techniques sont issues directement de la géneralisation des schémas d'intégration des équations d'Euler unidimensionneles grâce aux méthodes de volume finis. Elles privilégient donc des directions de propagations, les directions normales aux facettes des volumes de contrôle, ce qui peut se révéler génant dans des cas où les mailles sont très étirées.

Dans ce rapport, on fait le calcul de la solution exacte du problème de Riemann pour une équation hyperbolique *linéaire* obtenue par linéarisation des équations d'Euler non linéaires. On esquisse ensuite comment peut être utilsée ce résulat pour bâtir un schéma du premier ordre de type volume fini vraiment multidimensionnel

Abstract

Many modern numerical method for the integration of the equation of compressible flows rely on TVD type schemes with Riemann solvers. These schemes are directly built from the generalisation of 1D numerical schemes for the Euler equations by mean of the finite volume method. Doing this, some propagation directions are priviledged, the directions orthogonal to the facets of the control volumes. This may lead to large numerical errors, especially when the mesh is distorted.

In this report, we compute the exact solution of the Riemann for a *linear* hyperbolic equation obtained from the Euler equation by linearisation. Then, we show how to apply this result to construct a first order finite volume scheme that is genuinely multidimensional.

A genuinely multidimensional Riemann Solver

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February 18, 1993

Many modern numerical schemes for compressible flows simulations are built on TVD and Riemann solvers techniques. These methods have been designed first for scalar nonlinear conservations equations and then, with the help of the finite volume formulation, extended to multidimensional flows. One must emphasis on the fact that though these schemes are devoted to multidimensional flows, the method is intrinsically one dimensional : at each interface of any control volume, the flux evaluation is done assuming a one dimensional structure of the acoustic and material waves. To cure to this default, a first idea is to design a mesh that follows the flow so that the interfaces of the control volumes are in some sense "orthogonal" to it and hence a 1D approximation is indeed valid. Another, and deeper, idea is to reexamine the problem of upwinding for multidimensional flows. A review of multidimensional upwinding may be found in [1]. A first try is to solve several Riemann problems in different directions at the interface, so that one may hope to better approximate the multidimensional character of the flow. Several attempts in that direction has been made, among them one may cite [2, 3, 4]. Another research direction is to derive truly multidimensional solvers. Among the most significant contribution in this topic, one has to consider the work by Deconinck, Roe and their coworkers [5, 6]. Their idea is to derive first a truly multidimensional solver for a linear convective equation. This has been carried out in [6] and their results are impressive. Then the idea is to use this convection scheme for compressible flows with the help of a wave decomposition. Several wave models has been yet considered by these researchers [7, 8].

To our opinion, the difficulty of this methodology is the wave decomposition because it carries arbitraryness in it. Moreover, this wave decomposition is often made with the help of gradient evaluations ; this may be to the detriment of the robustness of the scheme. Last the implementation of boundary conditions is not yet very clear.

This is why, in this paper, we try to restart from the original ideas of the finite volume methods and the derivation of Roe's Riemann solver where a linearization procedure of the Roe type is first applied to the Euler equations and second, with the *exact* solution of that linearized equation , a flux evaluation is given. We propose here to follow the same procedure.

The linearization of the Euler equation is done through the new linearization of Roe [5]. This leads to an hyperbolic problem of the form

$$\left\{ \begin{array}{l} U_t + A\,U_x + B\,U_y = 0 \\ U(x,y,t=0) = U_0(x,y) \end{array} \right. \tag{1}$$

which has to be solved. The exact solution is then used for flux evaluation through the interfaces of the control volume in a genuinely multidimensional way.

This paper will be divided into two parts : we first solve the problem (1) for a two dimensional approximation of the Euler equation. Then we use this solution to derive a flux evaluation, and hence a new Riemann solver.

1 Introduction

We assume that system (1) is hyperbolic, that is for any vector $\vec{n} = (n_x, n_y)$ the matrix $n_x A + n_y B$ is diagonalizable with real eigenvalues.

The initial condition U_0 is assumed to be :

$$U_0 = \left\{ \begin{array}{ll} U_1 & \text{if } \ (x,y) \in \mathcal{D}_1 \\ U_2 & \text{if } \ (x,y) \in \mathcal{D}_2 \\ U_3 & \text{if } \ (x,y) \in \mathcal{D}_3 \end{array} \right. \tag{2}$$

where the U_i's are constant states. and the sets $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ are angular sectors as defined on Figure 1. The point O is the origin.

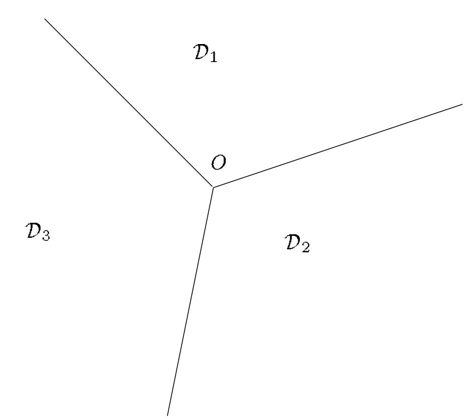

Figure 1: Definition of $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$.

It is very easy to see, from equation (1) and conditions (2) that the solution of the problem only depends on $\xi = x/t$ and $\nu = y/t$. For general matrices A and B, the solution of problem (1-2) may be very complicated. Here, we will specialize to the case where the matrices A and B are the Jacobian matrices evaluated at some state $\overline{U}$ of the Euler equations :

$$U_t + F(U)_x + G(U)_y = 0 \tag{3}$$

with

$$F(U) = (\rho, \rho u^2 + p, \rho uv, u(E+p))^T, \; G(U) = (\rho, \rho uv, \rho v^2 + p, u(E+p))^T \tag{4}$$

As usual, the symbol ρ denotes the density, (u, v) stands for the components of the velocity, p is the pressure and E the total energy. Assuming a calorically perfect gas, the pressure can be related to the conserved quantities by the state equation :

$$p = (\gamma - 1)\left(E - \frac{1}{2}\rho(u^2 + v^2)\right)$$

Here γ is set to 1.4.

To sum up, in all what follows, A and B will stand :

$$A = \frac{\partial F}{\partial U}(\overline{U}) \quad B = \frac{\partial G}{\partial U}(\overline{U}) \tag{5}$$

for some state $\overline{U} = (\overline{\rho}, \overline{\rho u}, \overline{\rho v}, \overline{E})^T$.

Remarks :

1. **Proposition 1.1** *Since the Euler equations are invariant under Galilean transformation, it follows easily that the vector*

$$W(U) = \begin{pmatrix} A \\ B - \overline{u}A \\ C - \overline{v}A \\ D - \frac{1}{2}(\overline{u}^2 + \overline{v}^2)A - \overline{u}B - \overline{v}C \end{pmatrix} \quad \text{for } \; U = \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix}$$

satisfies

$$W(U)_\tau + \tilde{A}W(U)_{x'} + \tilde{B}W(U)_{y'} = 0 \tag{6}$$

if U satisfies (1). In equation (6), the matrices $\tilde{A}$ and $\tilde{B}$ are given by

$$\tilde{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma - 1 \\ 0 & 0 & 0 & 0 \\ 0 & h & 0 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{B} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma - 1 \\ 0 & 0 & h & 0 \end{pmatrix}$$

with $h = H - \frac{1}{2}(u^2 + v^2)$. We also have use the new variables τ, $x' = x - \overline{u}t$ and $y' = y - \overline{v}t$.

Proof : One has : $U = W(U) + \mathcal{M}W(U)$ where $\mathcal{M}$ is the *constant* matrix

$$\mathcal{M} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \overline{u} & 0 & 0 & 0 \\ \overline{v} & 0 & 0 & 0 \\ \frac{1}{2}(\overline{u}^2 + \overline{v}^2) & \overline{u} & \overline{v} & 0 \end{pmatrix},$$

so that W satisfies :

$$(Id+\mathcal{M})W(U)_t + A(Id+\mathcal{M})W(U)_x + B(Id+\mathcal{M})W(U)_y = 0$$

or, which is equivalent,

$$W(U)_t + (Id+\mathcal{M})^{-1}A(Id+\mathcal{M})W(U)_x + (Id+\mathcal{M})^{-1}B(Id+\mathcal{M})W(U)_y = 0$$

It is straightforward to see that $(Id+\mathcal{M})^{-1}A(Id+\mathcal{M}) = \tilde{A}+\overline{u}Id$ and $(Id+\mathcal{M})^{-1}B(Id+\mathcal{M}) = \tilde{B}+\overline{v}Id$. The use ofthe new variables τ, x, y enable to get the result. •

2. It is also possible to simplify the problem as follows. Instead of considering problem (1-2), we consider equation (1) with the following three initial conditions :

   - Condition (a)
$$U_{0-a} = \begin{cases} U_1 & \text{if } (x,y)\in\mathcal{D}_1 \\ 0 & \text{elsewhere} \end{cases} \tag{7}$$
   - Condition (b)
$$U_{0-b} = \begin{cases} U_2 & \text{if } (x,y)\in\mathcal{D}_2 \\ 0 & \text{elsewhere} \end{cases} \tag{8}$$
   - Condition (c)
$$U_{0-c} = \begin{cases} U_3 & \text{if } (x,y)\in\mathcal{D}_3 \\ 0 & \text{elsewhere} \end{cases} \tag{9}$$

   Equation (1) is linear. Since $U_0 = U_{0-a}+U_{0-b}+U_{0-c}$, the solution of problem (1-2) is the sum of the solutions of the problems (1-7), (1-8), (1-9). We will specialize now to the solution of problem (1-7).

2 A simplified Riemann Problem

2.1 Geometrical description

Because of the first remark, we may assume that $\overline{U} = (\rho, 0, 0, \rho\epsilon)$. The structure of the solution is given in Figure 2 : each points of the lines D and D' are acoustic sources so that their influence domain is the envelop of the circles which center lie on D or D' and which radius is c, the speed of sound which square is : $c^2 = (\gamma-1)h$.

One can distinguish five zones in the $(x/t, y/t)$- plane :

- zones I and II where the solution is not modified,
- zone III where only the influence of D is visible,
- zone IV where only the influence of D' is visible,

- zone V where D and D' plays a role. It is limited by the circle $\mathcal{C}$: $(x/t)^2+(y/t)^2=c^2$.

The circle $\mathcal{C}$ must be divided in several parts :

- line $\mathcal{L}_1$ between zones I and V, limited by A and A',
- line $\mathcal{L}_2$ between points $A=(a)\cap\mathcal{C}$ and $B(b')\cap$ C.
- line $\mathcal{L}_3$, the symmetric line limited by A' and B',
- line $\mathcal{L}_4$ between B and B'.

3 Analytical description of the solution

We are looking for a self-similar solution, $W(x/t,y/t)$ of the problem

$$U_t+AU_x+BU_y=0$$

with

$$A=\begin{pmatrix}0&1&0&0\\0&0&0&\gamma-1\\0&0&0&0\\0&h&0&0\end{pmatrix},B=\begin{pmatrix}0&0&1&0\\0&0&0&0\\0&0&0&\gamma-1\\0&0&h&0\end{pmatrix}$$

If one set $\xi=x/t$ and $\nu=y/t$, the solution W must also fulfill :

$$-\xi W_\xi-\nu W_\nu+AW_\xi+BW_\nu=0 \tag{10}$$

In the following, the parameters r and θ will stand for :

$$r=\sqrt{\xi^2+\nu^2},\xi=r\cos\theta,\nu=r\sin\theta \tag{11}$$

with $\theta\in[0,2\pi]$.

Preliminary results Let us begin with some notations. For any angle θ, we consider the eigenvectors of $A\cos\theta+B\sin\theta$:

$$R_\epsilon=\begin{pmatrix}1\\\epsilon c\cos\theta\\\epsilon c\sin\theta\\H\end{pmatrix}\quad R_0=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\quad R_t=\begin{pmatrix}0\\-\sin\theta\\\cos\theta\\0\end{pmatrix} \tag{12}$$

The vectors R_ϵ ($\epsilon=\pm1$) are associated to the eigenvalue $\lambda=\epsilon c$ while R_0 and R_t are associated to the eigenvalue $\lambda=0$.

For any function f, regular enough, we have the identities :

$$\begin{cases} f_\xi = \cos\theta f_r - \dfrac{\sin\theta}{r} f_\theta \\ \\ f_\nu = \sin\theta f_r + \dfrac{\cos\theta}{r} f_\theta \end{cases} \tag{13}$$

so that, in particular,

$$r f_r = \xi f_\xi + \nu f_\nu. \tag{14}$$

From this relation, since the eigenvectors only depends on θ, we get

$$\xi R_\xi + \nu R_\nu = 0 \tag{15}$$

for $R = R_\epsilon, R_0, R_t$.

Moreover, because of (13), we have :

$$(A f_\xi + B f_\nu)\, R_\epsilon = \epsilon c f_r R_\epsilon + \frac{f_\theta}{r}\,(-A\sin\theta + B\cos\theta)\, R_\epsilon$$

A short calculation gives :

$$(-A\sin\theta + B\cos\theta)\, R_\epsilon = c^2 R_t$$

so that

$$(A f_\xi + B f_\nu)\, R_\epsilon = \epsilon c f_r R_\epsilon + R_\epsilon + \frac{c^2 f_\theta}{r} R_t \tag{16}$$

The same trick gives :

$$(A f_\xi + B f_\nu)\, R_t = \frac{f_\theta}{2r}\,(R_1 + R_{-1}) \tag{17}$$

because $(A\cos\theta + B\sin\theta)\, R_t = 0$ and last

$$(A f_\xi + B f_\nu)\, R_0 = 0 \tag{18}$$

since $(A\cos\theta + B\sin\theta)\, R_t = (-A\sin\theta + B\cos\theta)\, R_0 = 0$. There is another interesting equation on R_ϵ :

$$\frac{dR_\epsilon}{d\theta} = \epsilon c R_t\ ;$$

together with (17), it implies :

$$A R_{\epsilon\ \xi} + B R_{\epsilon\ \nu} = \epsilon c \frac{R_1 + R_{-1}}{2}. \tag{19}$$

Rewriting system (9) Any function W may be expressed as a linear combinations of the eigenvectors :

$$W(\xi,\nu) = f(\xi,\nu)R_1 + g(\xi,\nu)R_{-1} + h(\xi,\nu)R_t + k(\xi,\nu)R_0$$

where the functions f, g, h, k are assumed regular enough if W is. Putting this in equation (10) gives :

$$\begin{array}{lllllr}
-\left(\xi f_\xi + \nu f_\nu\right) R_1 & -f\left(\xi R_{1\ \xi} + \nu R_{1\ \nu}\right) & +\left(Af_\xi + Bf_\nu\right) R_1 & +f\left(AR_{1\ \xi} + BR_{1\ \nu}\right) & & (a) \\
-\left(\xi g_\xi + \nu g_\nu\right) R_{-1} & -g\left(\xi R_{-1\ \xi} + \nu R_{-1\ \nu}\right) & +\left(Ag_\xi + Bg_\nu\right) R_{-1} & +g\left(AR_{-1\ \xi} + BR_{-1\ \nu}\right) & & (b) \\
-\left(\xi h_\xi + \nu h_\nu\right) R_t & -h\left(\xi R_{t\ \xi} + \nu R_{t\ \nu}\right) & +\left(Ah_\xi + Bh_\nu\right) R_t & +h\left(AR_{t\ \xi} + BR_{t\ \nu}\right) & & (c) \\
-\underbrace{\left(\xi k_\xi + \nu k_\nu\right) R_0}_{I} & -\underbrace{k\left(\xi R_{0\ \xi} + \nu R_{0\ \nu}\right)}_{II} & +\underbrace{\left(Ak_\xi + Bk_\nu\right) R_0}_{III} & +\underbrace{k\left(AR_{0\ \xi} + BR_{0\ \nu}\right)}_{IV} & = 0 & (d)
\end{array}$$

Then we use the preliminary results of equations (14), (15), (16), (17), (18), (19) and obtain :

$$\begin{array}{lllllr}
-rf_rR_1 & +cf_rR_1 & +c^2\frac{f_\theta}{r}R_t & +\frac{c\ f}{2r}\left(R_1 + R_{-1}\right) & & (a) \\
-rg_rR_{-1} & -cg_rR_{-1} & +c^2\frac{g_\theta}{r}R_t & -\frac{c\ g}{2r}\left(R_1 + R_{-1}\right) & & (b) \\
-rh_rR_t & & +\frac{h_\theta}{2r}\left(R_1 + R_{-1}\right) & & & (c) \\
-rk_rR_0 & & & & = 0 & (d)
\end{array}$$

Since $\{R_1, R_{-1}, R_t, R_0\}$ is a basis, we get

$$\left\{\begin{array}{l}
2r(c-r)f_r + c(f-g) + h_\theta = 0 \\
-2r(r+c)g_r + c(f-g) + h_\theta = 0 \\
f_\theta + g_\theta - \frac{r^2}{c^2}h_r = 0 \\
rk_r = 0
\end{array}\right. \tag{20}$$

It is convenient to introduce $u = f - g$, $v = f + g$, $w = h/c$, and the variable $R = r/c$, so that (20) becomes :

$$\left\{\begin{array}{l}
(1-R^2)v_R + u + w_\theta = 0 \\
v_R - Ru_R = 0 \\
v_\theta - R^2 w_R = 0 \\
Rk_R = 0
\end{array}\right. \tag{21}$$

Until the end, we drop k, since the equation on k to totally decoupled from the others. It gives $k \equiv k(\theta)$.

Some algebraic manipulations of system (21) enable to write an equation on v only :

$$v_{\theta\theta} + R^2(1-R^2)v_{RR} + R(1-2R^2)v_R = 0 \tag{22}$$

This problem is elliptic for $R < 1$ and hyperbolic for $R > 1$. We now concentrate on the elliptic problem only. We will give the boundary conditions on the unit circle in an other paragraph. The only condition we impose on u, v and W is to remain bounded in the unit circle.

3.1 Solution of the elliptic problem in the unit circle

3.1.1 Boundary conditions

We have to set the following boundary condition on the unit circle :

1. On line $\mathcal{L}_1$, we have to impose the jump conditions associated with speed $\lambda = c$: $g = w = 0$,

2. On lines $\mathcal{L}_2$, $\mathcal{L}_3$ and $\mathcal{L}_4$ we impose the continuity of the solution across $\mathcal{C}$. In particular, on $\mathcal{L}_4$, the solution must be the sum of the solutions in part III and part IV, specialized in sub area 1 and 2 (see Figure 2).

3.1.2 Solution of the problem

We first look to particular solutions of the form $v(R,\theta) = a(R)b(\theta)$. The equation on a and b, obtained from equation (22) is :

$$\frac{a''}{a} + R^2(1-R^2)\frac{b''}{b} + R(1-2R^2)\frac{b'}{b} = 0$$

so that we get, for some constant λ :

$$\left\{\begin{array}{l} a'' + \lambda a = 0 \\ R^2(1-R^2)b'' + R(1-2R^2)b' - \lambda b = 0 \end{array}\right. \tag{23}$$

The function a must be periodic so that $\lambda = n^2$ for some integer n. The equation on b must be solved on $[0,1]$. Let us look for a change of variable $r \mapsto \rho(r)$, and introduce $\rho \mapsto B(\rho)$ such that $b(r) = B(\rho)$. Some algebraic manipulations give the equation that B must satisfy :

$$R^2(1-R^2)\rho_R^2 B'' + \left[R^2(1-R^2)\rho_{RR} + r(1-2R^2)\rho_R\right] B' - n^2 B = 0$$

We take ρ such that $\rho_R^2 R^2(1-R^2) = 1$. From that, we observe :

$$2\rho_R \left[R^2(1-R^2)\rho_{RR} + r(1-2R^2)\rho_R\right] = 0$$

so that the equation on B is

$$B'' - n^2 B = 0 \tag{24}$$

Since, we want bounded function, we obtain :

$$v(R,\theta) = \mathcal{R}e\left(a_n e^{in\theta}\right) e^{-n\rho(R)}$$

with $n \geq 0$ integer. We only have to precise the function ρ. A simple choice is

$$\rho(R) = \text{argch}\left(\frac{1}{R}\right)$$

and hence, we get

$$v(R,\theta)=\mathcal{R}e\left(\sum_{n\in\mathbf{Z}}a_ne^{in\theta}\left\{\frac{R}{1+\sqrt{1-R^2}}\right\}^{|n|}\right) \tag{25}$$

with, of course $\overline{a_n}=a_{-n}\in\mathbb{C}$.
Remark : This sum is indeed defined if the Fourier serie

$$\sum_{n\in\mathbf{Z}}a_ne^{in\theta}$$

converges.

Proof: Since $R<1$,

$$\frac{R}{1+\sqrt{1-R^2}}<1,$$

the sum converges •

Evaluation of u and w It is more simple to make this evaluation in the coordinates ρ, θ. In these variables, the system takes the form :

$$\begin{cases} -\mathrm{sh}\rho v_\rho+u+w_\theta=0\\ \mathrm{ch}\rho\ v_\rho-u_\rho=0\\ w_\rho+\mathrm{sh}\rho v_\theta=0\end{cases}$$

We want u,v and w bounded so that we finally get :

$$\begin{cases} v=\left(\sum_{|n|\neq 1}a_ne^{in\theta}\left\{\frac{R}{1+\sqrt{1-R^2}}\right\}^{|n|}\right)\\ u=-\frac{1}{2}\left(\sum_{|n|>1}|n|a_ne^{in\theta}\left(\frac{1}{1-|n|}\frac{1+\sqrt{1-R^2}}{R}-\frac{1}{1+|n|}\frac{R}{1+\sqrt{1-R^2}}\right)\left\{\frac{R}{1+\sqrt{1-R^2}}\right\}^{|n|}\right)+H(\theta)\\ w=-\frac{1}{2}\left(\sum_{|n|>1}ine^{in\theta}\left(\frac{1}{1-|n|}\frac{1+\sqrt{1-R^2}}{R}+\frac{1}{1+|n|}\frac{R}{1+\sqrt{1-R^2}}\right)\left\{\frac{R}{1+\sqrt{1-R^2}}\right\}^{|n|}\right)+G(\theta)\end{cases} \tag{26}$$

In equation (26),the complex a_n satisfy $\overline{a_n}=a_{-n}$, G and H are two arbitrary functions that fulfill :

$$H(\theta)+G'(\theta)=0. \tag{27}$$

Last, we have removed the terms with $n=1$ in order to fulfill the boundedness condition.
Remark : The equation (27) implies :

$$\int_0^{2\pi}H(\theta)d\theta=0$$

so that the constant term of its Fourier serie is nil.

3.1.3 The solution of the Riemann problem

In order to explicitly describe each part of the solution, we introduce θ_1 and θ_2 of $[0, 2\pi[$, the angles between a reference axis Ox and D (resp. D'). We also introduce the normal vector $\vec{n}_1$ to D, entering into $\mathcal{D}_1$ and $\vec{n}_2$, the normal vector to D' entering into $\mathcal{D}_1$.

The state $W_0 = (A, B, C, D)^T$ of $\mathcal{D}_1$ is decomposed in the for waves associated with direction $\vec{n}_1$:

$$W_0 = l_1^{\vec{n}_1}(W_0)R_0 + l_2^{\vec{n}_2}(W_0)R_1^{\vec{n}_1} + l_1^{\vec{n}_1}(W_0)R_{-1}^{\vec{n}_1} + l_4^{\vec{n}_1}(W_0)R_t^{\vec{n}_1}$$

with,

$$R_\epsilon^{\vec{n}} = \begin{pmatrix} 1 \\ \epsilon c\cos\theta \\ \epsilon c\sin\theta \\ H \end{pmatrix} \quad R_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad R_t^{\vec{n}} = \begin{pmatrix} 0 \\ -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}$$

for any vector $\vec{n} = \cos\theta\vec{i} + \sin\theta\vec{j}$.

With this setting, the solution, in III, writes :

- In zone III_1,
$$U(\xi,\nu) \equiv U_{III}^1(\xi,\nu) = l_2^{\vec{n}_1}(W_0)R_1^{\vec{n}_1} \tag{28}$$
- In zone III_2,
$$U(\xi,\nu) \equiv U_{III}^2(\xi,\nu) = W_0 - l_3^{\vec{n}_1}(W_0)R_{-1}^{\vec{n}_1} \tag{29}$$
- In zone IV_1,
$$U(\xi,\nu) \equiv U_{IV}^1(\xi,\nu) = l_2^{\vec{n}_2}(W_0)R_1^{\vec{n}_2} \tag{30}$$
- In zone IV_2,
$$vU(\xi,\nu) \equiv U_{IV}^2(\xi,\nu) = W_0 - l_3^{\vec{n}_2}(W_0)R_{-1}^{\vec{n}_2} \tag{31}$$
- In zone VI, the solution $U_{VI} = U_{III} + U_{IV}$ may be of mixted type depending on the angle Φ between D and D' (see Figure 2) :
  - If $\Phi \leq \frac{\pi}{2}$, $U_{VI} = U_{III}^2 + U_{IV}^2$,
  - If $\Phi < \frac{\pi}{2}$,
    * between B, C and (b') : $U_{VI} = U_{III}^1 + U_{IV}^2$,
    * between B', C' and $(\prime)$: $U_{VI} = U_{III}^1 + U_{IV}^2$,
    * elsewhere in VI, $U_{VI} = U_{III}^2 + U_{IV}^2$

We now give the expression of U in the circle. Let us first describe u, v and w on $\mathcal{C}$. For that, we use complex notations and set

$$\begin{aligned}
v(R=c,\theta) &\equiv v(\theta) = \left(\sum_{|n|\neq 1} a_n e^{i\ n\theta}\right)\\
u(R=c,\theta) &\equiv u(\theta) = \left(\sum_{|n|>1} \left\{\frac{n^2}{n^2-1}a_n + \alpha_n\right\} e^{i\ n\theta} + \alpha_1 e^{i\theta}\right)\\
w(R=c,\theta) &\equiv w(\theta) = \left(\sum_{|n|>1} i \left\{\frac{n^2}{n^2-1}a_n - \frac{\alpha_n}{n}\right\} - i\alpha_1 e^{i\theta}\right) + \beta_0
\end{aligned}$$

In these formula, the α_n's stand for the Fourier development of H, β_0 is the additional constant for G. We then get the Fourier development of f, g and h :

$$\begin{aligned}
f(R=c,\theta) &\equiv f(\theta) = \tfrac{1}{2}\left(\sum_{|n|>1} \left\{\frac{2n^2-1}{n^2-1}a_n + \alpha_n\right\} e^{i\ n\theta} + \alpha_1 e^{i\theta}\right) + \tfrac{a_0}{2}\\
g(R=c,\theta) &\equiv g(\theta) = \tfrac{1}{2}\left(\sum_{|n|>1} \left\{\frac{1}{n^2-1}a_n + \alpha_n\right\} e^{i\ n\theta} + \alpha_1 e^{i\theta}\right) - \tfrac{a_0}{2}\\
h(R=c,\theta) &\equiv h(\theta) = c\left(\sum_{|n|>1} i \left\{\frac{n^2}{n^2-1}a_n - \frac{\alpha_n}{n}\right\} e^{in\theta} - i\alpha_1 e^{i\theta}\right) + c\beta_0
\end{aligned} \tag{32}$$

It is necessary to compute the Fourier transform of g and h only as it can be seen on system (32). Then, as can be seen from Figure 2 because of the boundary conditions of paragraph 3.1.1, the g- and h- component of the solution are the sum of two independent contributions only, one from zone III and one from zone IV.

We start with giving the formula for the eigenvectors $R_\epsilon^{\vec{n}}$, for $\vec{n} = \vec{n}_1$ or $\vec{n}_2$. We have :

$$\begin{pmatrix} 1 \\ \epsilon c\cos\phi \\ \epsilon c\sin\phi \\ H \end{pmatrix} = \frac{1}{2}\left(1+\epsilon\cos(\phi-\theta)\right) R_1^\theta + \frac{1}{2}\left(1-\epsilon\cos(\phi-\theta)\right) R_{-1}^\theta + \epsilon c\sin(\phi-\theta) R_t^\theta$$

where ϕ is the angle between Ox and $\vec{n}_1$ (denoted ϕ_1 bellow) or the angle between Ox and $\vec{n}_2$ (denoted ϕ_2 bellow).

Last, we recall the expression of $l_1^{\vec{n}}, \ldots, l_4^{\vec{n}}$ of the linear forms associated to a direction $\vec{n} = \cos\theta\vec{i} + \sin\theta\vec{j}$:

$$l_1^{\vec{n}}(W_0) = A - \frac{D}{H}$$

$$l_2^{\vec{n}}(W_0) = \frac{1}{2}\left(\frac{D}{H} + \frac{B\cos\theta + C\sin\theta}{c}\right)$$

$$l_3^{\vec{n}}(W_0) = \frac{1}{2}\left(\frac{D}{H} - \frac{B\cos\theta + C\sin\theta}{c}\right)$$

$$l_4^{\vec{n}}(W_0) = -B\sin\theta + C\cos\theta$$

With all these notations, we see that $g = g_{III} + g_{IV}$ and $h = h_{III} + h_{IV}$ where g_{III} (resp.

h_{III}) and g_{IV} (resp. h_{IV}) are the contribution to g and h from zone III (resp. IV) :

$$
\begin{aligned}
g_{III} &= \frac{l_2^{\vec{n}_1}(W_0)}{2}(1-\cos(\phi_1-\theta)\chi_{[\theta_1-\pi/2,\theta_1]} \\
&+ \frac{1}{2}\left(\frac{D}{H}-\frac{B\cos\theta+C\sin\theta}{c}\right)\chi_{[\theta_1,\theta_1+\pi/2]} \\
&- \frac{l_3^{\vec{n}_1}(W_0)}{2}(1+\cos(\phi_1-\theta)\chi_{[\theta_1,\theta_1+\pi/2]} \qquad (a) \\
h_{III} &= c l_2^{\vec{n}_1}(W_0)\sin(\phi_1-\theta)\chi_{[\theta_1-\pi/2,\theta_1]}) \\
&+ (-B\sin\theta+C\cos\theta)\chi_{[\theta_1,\theta_1+\pi/2]} \\
&- c\frac{l_3^{\vec{n}_1}(W_0)}{2}\sin(\phi_1-\theta)\chi_{[\theta_1,\theta_1+\pi/2]} \qquad (b)
\end{aligned}
\tag{33}
$$

and

$$
\begin{aligned}
g_{IV} &= \frac{l_2^{\vec{n}_2}(W_0)}{2}(1-\cos(\phi_2-\theta)\chi_{[\theta_2-\pi/2,\theta_2]} \\
&+ \frac{1}{2}\left(\frac{D}{H}-\frac{B\cos\theta+C\sin\theta}{c}\right)\chi_{[\theta_2,\theta_2+\pi/2]} \\
&- \frac{l_3^{\vec{n}_2}(W_0)}{2}(1+\cos(\phi_2-\theta)\chi_{[\theta_2,\theta_2+\pi/2]} \qquad (a) \\
h_{IV} &= c l_2^{\vec{n}_2}(W_0)\sin(\phi_2-\theta)\chi_{[\theta_2-\pi/2,\theta_2]}) \\
&+ (-B\sin\theta+C\cos\theta)\chi_{[\theta_2,\theta_2+\pi/2]} \\
&- c\frac{l_3^{\vec{n}_2}(W_0)}{2}\sin(\phi_2-\theta)\chi_{[\theta_2,\theta_2+\pi/2]} \qquad (b)
\end{aligned}
\tag{34}
$$

Here, as usual, $\chi_{[a,b]}$ is the characteristic function of $[a,b]$.

From equations (33-a), (33-b), (34-a) and (34-b), it is straightforward to get the Fourier coefficients of f and g

$$
g = Re\left(\sum_{n\geq 0}\hat{g}_n e^{i\ n\theta}\right) \qquad h = Re\left(\sum_{n\geq 0}\hat{h}_n e^{i\ n\theta}\right)
$$

and then we get :

$$\begin{aligned}
a_n &= \frac{2(n+1)}{n^2+n+1}\left(\hat{g}_n - in\frac{\hat{h}_n}{c}\right),\ n>2 && (a)\\
\alpha_n &= -\frac{2(n+1)}{n^2+n+1}\left(\hat{g}_n + in\frac{\hat{h}_n}{c}\right),\ n>2 && (b)\\
\alpha_0 &= 2\hat{g}_0 && (c)\\
\beta_0 &= \frac{\hat{h}_n}{c} && (d)
\end{aligned} \tag{35}$$

Remark : In the case of a general system, with a non zero velocity $(\overline{u}, \overline{v})$, the solution is

$$W(x,y,t) = (Id + \mathcal{M})U(\frac{x}{t} - \overline{u}, \frac{y}{t} - \overline{v})$$

where U is the solution of the modified system 6.

4 A genuinely multidimensional finite first order volume scheme

In this section, we consider the Euler equations for a perfect gaz (3-4). Since we concentrate on the approximation of the flux term, we forget the important problem of the boundary condition so that the problem is set in $\mathbb{R}^2$. We consider a triangulation of $\mathbb{R}^2$ and around each node i, we construct a control volume $\mathcal{C}_i$ of boundary $\partial\mathcal{C}_i$.

The finite volume formulation for (3) gives :

$$W_i^{n+1} - W_i^n + \int_{\partial\mathcal{C}_i} F_{\vec{n}}(W(x,y,t))d\vec{l} = 0 \tag{36}$$

where as usual, W_i^n (resp.W_i^n+1) stands for the averaged value of W on $\mathcal{C}_i$ at time t_n (resp t_{n+1}), and $\vec{n}$ is the outgoing unit normal to $\mathcal{C}_i$.

In order to present the basic ideas of the method, we will concentrate on the case where the control volumes are the elements of the dual mesh, though this kind of mesh may not be the most appropriate as we will see. We adopt the notations of Figure 3 where T is the triangle (i,j,k), I_{ij} is the midpoint of $[i,j]$ and G_T the barycenter of T. We also introduce the midpoint $J_{G_T I_{ij}}$ of the segment between the barycenter of the triangle T and I_{ij}. We see that the flux evaluation (36) is the sum of the contribution on the segments $[I_{ij}, J_{G_T I_{ij}}$ (see Figure 4) that can be grouped by two : two for one point I_{ij} and two of one gravity center G_T. To simplify, we will work now with a segment $[J_1, J_2]$, as on Figure 4, that correspond to I_{ij}.

Now, we follow the idea of Lafon and Angrand [10], and we introduce two types of linearization :

- the I_{ij}-type : two constant states lies on each side of the interface, namely W_i^n and W_j^n, and we consider the Roe matrices $A(W_i, W_j)$ and $B(W_i, W_j)$ [9] defined by a single state $\overline{W}_{ij}$. The Euler equation (3) is then linearized around the Roe average $\overline{W}_{ij}$:

$$W_t + A(W_i, W_j)W_x + B(W_i, W_j)W_y \tag{37}$$

- the G_{ijk}-type : here one has to consider the three states, W_i, W_j, W_k. Then (3) is linearized around this new Roe average [5] :

$$W_t + A(W_i, W_j, W_k)W_x + B(W_i, W_j, W_k)w_y \tag{38}$$

Then we solve exactly the approximated problems (37) and (38), and evaluate the flux at the interface $[J_1, J_2]$ (see Figure 3) :

$$\int_{t_n}^{t_{n+1}} \int_{J_1}^{J_2} (An_x + Bn_y)\, W(x,y,t)dldt \tag{39}$$

We then use the autosimilarity of the solution where "ct" represents the intersection of the circle of center $I + (\overline{u}, \overline{v})t$ and $[I_{ij}, J1]$:

$$\int_{t_n}^{t_{n+1}} \int_{J_1}^{I} (An_x + Bn_y)\, W(x,y,t)dldt = \int_{t_n}^{t_{n+1}} \int_{ct}^{I} (An_x + Bn_y)\, W(x,y,t)dldt +$$

$$\int_{t_n}^{t_{n+1}} \int_{J_1}^{ct} (An_x + Bn_y)\, W(x,y,t)dldt$$

Then we get

$$\int_{t_n}^{t_{n+1}} \int_{J_1}^{I} (An_x + Bn_y)\, W(x/(t-t_n), y/(t-t_n))dldt = \int_{t_n}^{t_{n+1}} t \left\{ \int_1^I (An_\xi + Bn_\nu)\, W(\xi,\nu)dl'dt \right\} +$$

$$\int_{t_n}^{t_{n+1}} \left\{ \int_{J_1}^{ct} (An_x + Bn_y)\, W(x,y,t)dldt \right\}$$

The second part of the formula can be evaluated classically, because of the structure of the solution and because we aret far from the other corners of the control volume, while the first part is evaluated with the help of the exact solution of the Riemann problem. Note that a segment $[J_1, I]$ corresponds to one value of θ, so that one has only to evaluate integrals like :

$$\int_0^1 \left\{ \frac{R}{1 + \sqrt{1 - R^2}} \right\}^n dR$$

have to evaluated. This can be done exactly because :

$$\int_0^1 \left\{ \frac{R}{1 + \sqrt{1 - R^2}} \right\}^n dR = \int_0^1 \frac{4t}{1 + t^2} \left(\frac{1-t}{1+t} \right)^n dt$$

5 Conclusion

In this paper, we have solved the Riemann problem for a linear hyperbolic equations that comes form a linearization of the Euler equations around a constant state. This kind of linearization occurs very naturally in CFD because of the existence of the Roe average for two or three states in two dimension. We have indicated how to use this exact solution to derive a finite volume type scheme that is truly multidimensional.

Up to now, we have not been able to make any numerical tests for this new scheme. It should be evaluated in situations where the classical Riemann solvers are known to be uneffective, for example with highly stretched meshs. It is not clear how the well know entropy fix problem will occur and be solved with this new scheme. More over, additional hand calculations are needed to simplify, if possible, the expressions we have given. If this is not possible, one must try simplifications of this scheme to reduce the computational cost that is involved. Its extension to higher order of accuracy must be studied, and its coupling with ENO type methods [11] must be done. Last, a 3D version of this solver can probably be obtained with the same method.

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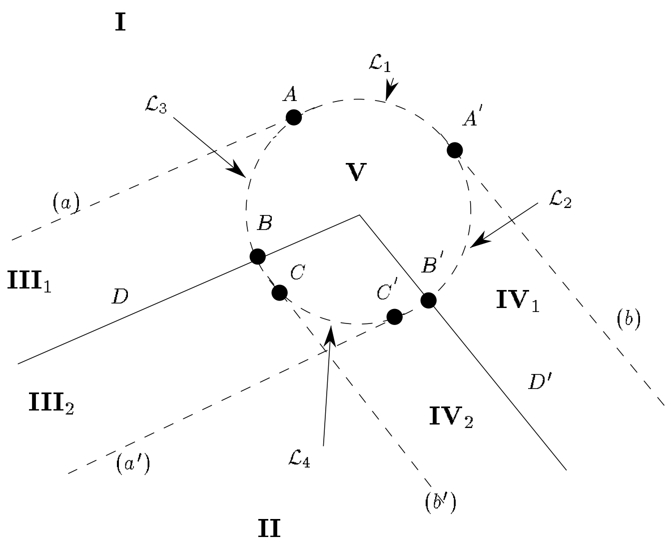

Figure 2: The solution of the Riemann problem, stationary case

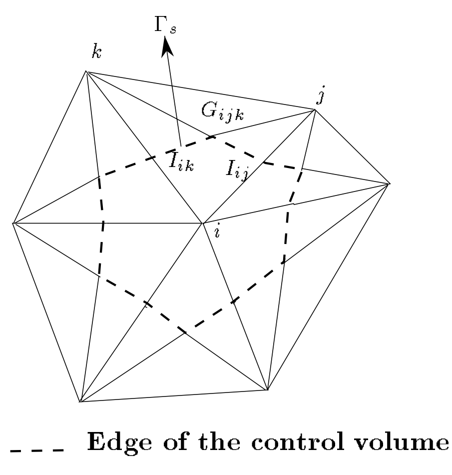

Figure 3: Element of the dual mesh

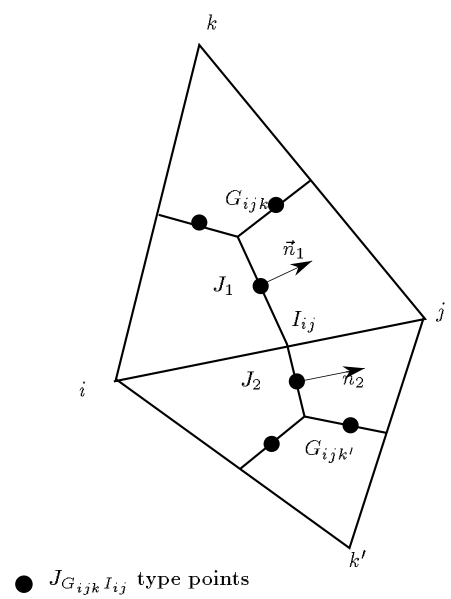

Figure 4: Segments for the flux evaluation